On the Group of Automorphisms of Finite Wreath Products

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Abstract

In this paper we study the structure of the group of automorphisms of the wreath product $A\, wr\, C_2$, where A is a finite nilpotent group, and C_2 is the cyclic group of order 2. In particular we give necessary and sufficient conditions for $\mathrm{Aut}(A\, wr\, C_2)$ to be supersolvable depending only on $\mathrm{Aut}(A)$ and on the Remak decomposition of A.

Introduction and statement of the main result

This work is a contribution to the study of the group of automorphisms of a (restricted) wreath product $A wr B$ of two non-trivial groups A and B. Throughout this paper we denote with G the group $A\, wr\, B$. Particular interest concerns the relationships among the structures and the group-theoretical properties of the groups A and B and the ones of $\mathrm{Aut}(G)$.

For instance, if $\mathrm{Aut}(G)$ is a nilpotent group much is known. In fact in this case G is clearly nilpotent and for a result of G. Baumslag ([2]) we have that both A and B are nilpotent p-groups (for the same prime p), where A has finite exponent and B has finite order. Moreover, if $\mathrm{Aut}(G)$ is supposed to be finite (and nilpotent), then $\mathrm{Aut}(A)$ and $\mathrm{Aut}(B)$ are finite p-groups (see [9]). Conversely, if A and B are finite p-groups (for $p \neq 2$) with $\mathrm{Aut}(A)$ and $\mathrm{Aut}(B)$ p-groups too, then $\mathrm{Aut}(A\, wr\, B)$ is a p-group (see M.V. Khoroshevskii [4]).

Concerning supersolvability, in a recent paper ([7]) G. Corsi Tani and R. Brandl proved the following

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Theorem. *If* $\mathrm{Aut}(G)$ *is supersolvable then* A *is nilpotent. Moreover if either* G *is supposed to be infinite or if* $(|A|, |B|) = 1$*, then* $B \simeq C_2$*, the cyclic group of order two.*

In this paper we study the "inverse problem" of the aforementioned result. We suppose that A is a finite nilpotent group and give necessary and sufficient conditions for $\mathrm{Aut}(A\, wr\, C_2)$ to be supersolvable.

Our main Theorem is proved in section 3 and it can be stated as follows.

Theorem 1 *Let* A *be a finite nilpotent group with* $\mathrm{Aut}(A)$ *supersolvable and let* $G := A\, wr\, C_2$*. Then* $\mathrm{Aut}(G)$ *is supersolvable if and only if either no two distinct factors in a Remak decomposition of* A *are isomorphic, or the only possible ones are* 2*-groups and in this case* $O_2(A)$ *is not isomorphic to* $C_2 \times C_2$*.*

Clearly the supersolvability of $\mathrm{Aut}(A)$ is a necessary condition since $\mathrm{Aut}(A)$ is, up to isomorphism, a subgroup of $\mathrm{Aut}(G)$.

The paper is organized as follows. In the first two sections we analyze what happens if A is a finite p-group, where p is any prime number. In the former we focus our attention on the inner structure of $\mathrm{Aut}(G)$, while in the latter on the problem of supersolvability. Finally, in the last section we generalize the results to any finite nilpotent group A.

1 The structure of $\mathrm{Aut}(G)$

Let A be a finite p-group, C_2 be the cyclic group of order two and $G := A\, wr\, C_2$ the restricted wreath product of A and C_2. Then, by [3], if A itself is not the group of order two, the base group F of G is characteristic in G and, if we fix a complement $C_2 = \langle t \rangle$, we can express $\mathrm{Aut}(G)$ as a product

$$\mathrm{Aut}(G) = K \cdot I,$$

where K is the subgroup of $\mathrm{Aut}(G)$ consisting of those automorphisms which fix t and I is the subgroup of $\mathrm{Aut}(G)$ consisting of those inner automorphisms which correspond to conjugations by elements of F. Moreover I is normal in $\mathrm{Aut}(G)$, and since I is isomorphic to a quotient of F, I is a p-group. Let us give now another interpretation of K. We identify the base subgroup F with $A \times A$, the direct product of two copies of A. Denote as usual the elements of F with couples (x, y), where $x, y \in A$, and with δ the involution of F defined by: $\delta(x, y) = (y, x)$ for each $x, y \in A$. Let $C_{\mathrm{Aut}(F)}(\delta)$ be the centralizer of δ in $\mathrm{Aut}(F)$.

Lemma 1 *The groups* K *and* $C_{\mathrm{Aut}(F)}(\delta)$ *are isomorphic.*

Proof. An isomorphism is given by the map that sends $\eta \in K$ into $\pi_F \eta i_F \in K$ where i_F is the canonical injection of F in G and π_F the canonical projection of G on F. The inverse of this map is the application that sends $\varphi \in K$ into the

automorphism Φ of G so defined: $\Phi(a_1 a_2 t^\epsilon) := i_G(\varphi(a_1, a_2))t^\epsilon$, for each $a_1 a_2 t^\epsilon \in G$.
$\square$

We note that in this situation the automorphism δ of F is indeed the conjugation by the element t that generates C_2 in G.

To study more in detail the structure of the group K, we give some more notation that will be used through all the paper.
We indicate with:

$\mathrm{Aut}_Z(A), \mathrm{Aut}_Z(F)$ the groups of the central automorphisms of A and of F respectively (that is the groups of the automorphisms which act trivially on the central factor groups $\frac{A}{Z(A)}$ and $\frac{F}{Z(F)}$ respectively).

$K_Z := K \cap \mathrm{Aut}_Z(F)$ and similarly if X is any subgroup of $\mathrm{Aut}(F)$ we use X_Z for $X \cap \mathrm{Aut}_Z(F)$.

$\Delta(Y) := \{(y, y) \mid y \in Y\}$ for each Y subgroup of A
(we use simply Δ for $\Delta(A)$).

$\nabla(Y) := \langle (y, y^{-1}) \mid y \in Y \rangle$ for each Y subgroup of A
(we use simply ∇ for $\nabla(A)$).

$H := C_K(\Delta)$ the centralizer of Δ in K.

$L := C_K(\nabla)$ the centralizer of ∇ in K.

$A^* := \{\alpha_f \mid f \in \mathrm{Aut}(A)\}$ where α_f denotes the automorphism of F defined by $\alpha_f(a_1, a_2) = (f(a_1), f(a_2))$, for each $(a_1, a_2) \in F$.
(Clearly $A^* \simeq \mathrm{Aut}(A)$).

Given any element φ of $\mathrm{Aut}(F)$, define two endomorphisms φ_1, φ_2 of A by

$$\varphi(x, 1) := (\varphi_1(x), \varphi_2(x)), \forall x \in A.$$

As a Lemma we collect now some elementary facts. We omit the easy proofs.

Lemma 2 *(i) If $\varphi \in K$, then $\forall x, y \in A$*

$$\begin{aligned} &\varphi(1, x) = (\varphi_2(x), \varphi_1(x)) \quad \textit{and} \\ &\varphi(x, y) = (\varphi_1(x)\varphi_2(y), \varphi_2(x)\varphi_1(y)) \end{aligned}$$

(ii) $[\mathrm{Im}\,\varphi_1, \mathrm{Im}\,\varphi_2] = 1$ *and* $A = (\mathrm{Im}\,\varphi_1)(\mathrm{Im}\,\varphi_2)$
(in particular $\mathrm{Im}\,\varphi_1 \cap \mathrm{Im}\,\varphi_2 \leq Z(A)$ *and* $\mathrm{Im}\,\varphi_i$ *are normal subgroups of* A*).*

(iii) Any element of K fixes the subgroups Δ and ∇, in particular it permutes the generators (x, x^{-1}) of ∇.

(iv) *The subgroup H is the kernel of the group epimorphism $\Phi : \varphi \in K \longmapsto \varphi_1 + \varphi_2 \in \mathrm{Aut}(A)$ (where $\varphi_1 + \varphi_2$ is the map $a \mapsto \varphi_1(a)\varphi_2(a)$ for each $a \in A$) and $K = [H]A^*$ is the semidirect product of H and A^*.*

(v) *$\frac{H}{H_Z}$ is an elementary abelian 2-group.*

(vi) *The map $\tilde{}$: $\varphi \in H_Z \mapsto \tilde{\varphi} \in \mathrm{Aut}_Z(A)$ defined by $\tilde{\varphi}(a) := \varphi_1(a^2)a^{-1}$ is a homomorphism of groups.*

At this point we distinguish the two cases p even and p odd.
Let us suppose first that A is a finite 2-group.
In this situation the map $\tilde{}$ defined in Lemma 2 (vi) is in general not injective. In fact its kernel is constituted by the elements φ of H_Z such that $\varphi_1(a^2) = a^2$ for each $a \in A$, and so, since $\varphi_1 + \varphi_2 = id_A$, $\varphi_2(a^2) = 1$ for each $a \in A$. Then

$$\mathrm{Ker}\tilde{} = \{\varphi \in H_Z \mid \varphi_{|A^2 \times A^2} = id_{|A^2 \times A^2}\}.$$

Lemma 3 *The kernel of the map $\tilde{}$ is an elementary abelian 2-group.*

Proof. Let φ be a non trivial element of the kernel of $\tilde{}$ and a an element of A. Then

$$\varphi^2(a, 1) = (\varphi_1^2(a)\varphi_2^2(a), 1)$$

and so $(\varphi^2)_1 = \varphi_1^2 + \varphi_2^2$ and $(\varphi^2)_2 = 0$ the null endomorphism of A. Since $\varphi^2 \in H$, then $(\varphi^2)_1 + (\varphi^2)_2 = id_A$ and so $(\varphi^2)_1 = id_A$ and φ has order 2. $\square$

Lemmas 2, 3 and the fact that $\mathrm{Aut}(G) = K \cdot I$ imply the following

Theorem 2 *If A is a 2-group, then the composition factors of $\mathrm{Aut}(G)$ are either composition factors of $\mathrm{Aut}(A)$ or cyclic groups of order 2.*

The same result can be proved for $p \neq 2$, in fact if p is odd, the map $\tilde{}$ of Lemma 2 (vi) is a monomorphism, therefore H_Z is a normal subgroup of H that can be embedded in $\mathrm{Aut}(A)$. So, according to Theorem 2, we can state

Theorem 3 *If A is a p-group, with p any prime, then the composition factors of the group K are either composition factors of $\mathrm{Aut}(A)$, or cyclic groups of order 2.*

We next show that in the situation $p \neq 2$ we can lift the subgroup H_Z up to H, and this will be of fundamental importance for our purposes.
Fix a Remak decomposition of A, say $A = A_1 \times A_2 \times \cdots \times A_n$, where A_i are indecomposable direct factors of A. For each subset I of the set $\{1, 2, \ldots, n\}$, we define $X_I := \langle A_i \mid i \in I\rangle$, $Y_I := \langle A_j \mid j \notin I\rangle$, so that $A = X_I \times Y_I$, and denote with $\varphi_{(X_I,Y_I)}$ or φ_I the automorphism of F defined by

$$\varphi_{(X_I,Y_I)}(xy, x_1y_1) := \varphi_I(xy, x_1y_1) := (xy_1, x_1y)$$

for each $x, x_1 \in X_I$ $y, y_1 \in Y_I$.
It is immediate to verify that $\varphi_I \in H$. We can then consider the subset of H

$Q := \{\varphi_I \mid I \subseteq \{1, 2, \ldots, n\}\}$. It is not difficult to prove that Q is an elementary abelian 2-subgroup of H that can be generated by the elements $\varphi_{\{i\}}$, $i = 1, 2, \ldots, n$ (we leave the proofs to the reader).
Note that the subgroup Q depends on the Remak decomposition, say $\mathcal{A}$, of A chosen at first, so that it is convenient to write $Q^{\mathcal{A}} := \{\varphi_I^{\mathcal{A}} \mid I \subseteq \{1, 2, \ldots, n\}\}$ instead of $Q := \{\varphi_I \mid I \subseteq \{1, 2, \ldots, n\}\}$. If we choose another direct decomposition of A, say $\mathcal{B}$, then we obtain in general another elementary abelian 2-subgroup, say $Q^{\mathcal{B}}$. The relation between $Q^{\mathcal{A}}$ and $Q^{\mathcal{B}}$ is explicitated in the next

Lemma 4 *$Q^{\mathcal{A}}$ and $Q^{\mathcal{B}}$ are conjugate in K by an element of A_Z^*.*

Proof. Suppose the two different Remak decompositions of A are $\mathcal{A} := \{A_i\}_1^n$ and $\mathcal{B} := \{B_i\}_1^n$. By the Krull-Schmidt theorem there exists a central automorphism f of A such that for each $i = 1, 2, \ldots, n$, $f(A_i) = B_i$. Consider then the automorphism α_f of F. α_f is a central automorphism of F and for each $I \subseteq \{1, 2, \ldots, n\}$,

$$\alpha_f^{-1} \varphi_I^{\mathcal{B}} \alpha_f = \varphi_I^{\mathcal{A}},$$

and so $Q^{\mathcal{A}}$ and $Q^{\mathcal{B}}$ are conjugate by an element of A_Z^*. □

These arguments allow us to give the following description of the structure of the subgroup H.

Lemma 5 *Chosen a Remak decomposition $\mathcal{A}$ of A, the subgroup H of K is the product $H = H_Z \cdot Q^{\mathcal{A}}$.*

Proof. By Lemma 2 we already know that H_Z is normal in K and that $\frac{H}{H_Z}$ is an elementary abelian 2-group. Suppose that φ is an element of H of order a power of 2, say 2^m. If we call $M := \operatorname{Ker} \varphi_2$ and $N := \operatorname{Im} \varphi_2$, we have that $A = M \times N$. In fact by Lemma 2, M and N are both normal subgroups of A. In order to prove that their intersection is trivial, consider an element $a \in M \cap N$, say $a = \varphi_2(x)$. By an induction argument we have that $\varphi^k(x, 1) = (\varphi_1^k(x), \varphi_2(x)^k)$, for each $k \geq 1$. In particular for $k = 2^m$ we obtain $\varphi_2(x)^{2^m} = 1$ and since 2 does not divide $|A|$, we have that $x \in \operatorname{Ker} \varphi_2$, i.e. $a = 1$. So $A = M \times N$ and this decomposition of A allows us to consider the automorphism $\varphi_{(M,N)}$. A simple computation shows that $\varphi\varphi_{(M,N)} \in H_Z$. In this way, according to the fact that the order of $\frac{H}{H_Z}$ is a power of 2, we have that for any φ in H there exists an element of H of the form $\varphi_{(R,S)}$ (for some decomposition (R, S) of A) such that $\varphi\varphi_{(R,S)} \in H_Z$.
Now we fix a Remak decomposition $\mathcal{A}$ of A and call Q the subgroup $Q^{\mathcal{A}}$ of H. If φ is any element of H, we proved that there exist $\sigma \in H_Z$ and $\varphi_{(R,S)} \in Q^{\mathcal{B}}$, for some decomposition $\mathcal{B}$ of A, that is a refinement of (R, S), such that $\varphi = \sigma\varphi_{(R,S)}$. Call f the central automorphism of A such that $Q = \alpha_f^{-1} Q^{\mathcal{B}} \alpha_f$, then $\varphi^{\alpha_f} = \sigma^{\alpha_f} \varphi_{(R,S)}^{\alpha_f} \in H_Z Q$. But now we can write the element φ as $\varphi = [\varphi^{-1}, \alpha_f^{-1}]\varphi^{\alpha_f}$, and, since H and K_Z are normal subgroups of K, $[\varphi^{-1}, \alpha_f^{-1}] \in H_Z$ and so $\varphi \in H_Z Q$, i.e. $H = H_Z Q$.
□

2 Supersolvability of K

We now concentrate our study on the problem of the supersolvability of the group K. We of course assume that $\mathrm{Aut}(A)$ is supersolvable.
As before, we consider first the case $p = 2$.

With the considerations of section 2.1 it is not difficult now to establish when $\mathrm{Aut}(G)$ is supersolvable. Making use of [8], we know that the automorphism group of a 2-group, not $C_2 \times C_2$, is supersolvable if and only if it is itself a 2-group. Therefore we have the following

Theorem 4 *Let A be a finite 2-group. If A is not isomorphic to $C_2 \times C_2$, then $\mathrm{Aut}(G)$ is supersolvable if and only if $\mathrm{Aut}(A)$ is supersolvable. If $A \simeq C_2 \times C_2$, then $\mathrm{Aut}(G)$ is not supersolvable.*

Proof. For the case A not isomorphic to $C_2 \times C_2$, Theorem 2 and the results in [8] tell us that $\mathrm{Aut}(G)$ is a 2-group and so it is supersolvable.
If $A \simeq C_2 \times C_2$, $\mathrm{Aut}(G)$, being not a 2-group, is not supersolvable. (In this case the subgroup $I \cdot A$ of $\mathrm{Aut}(G)$ is isomorphic to the symmetric group S_4). □

The case p odd requires more work.
We recall that a group T is said to be *strictly p-closed* (where p is any prime number) if $T'T^{p-1}$ is a p-subgroup of T, or equivalently if $T = [O_p(T)]S$ with S an abelian group of exponent that divides $p-1$. Strictly p-closed groups are supersolvable (see Baer [1]), and the automorphism group of a finite p-group ($p \neq 2$) is supersolvable if and only if it is strictly p-closed (see G. Corsi Tani [8]).
Before making any other hypothesis than the supersolvability of $\mathrm{Aut}(A)$, we prove a relevant result concerning the subgroup K_Z of K.

Lemma 6 *If A is a p-group ($p \neq 2$), then, with the same notations as before:*

- *(i)* $K_Z = [H_Z]A^*_Z = [L_Z]A^*_Z$, *semidirect products.*
- *(ii)* $H_Z \cap L_Z = 1$.
- *(iii)* $H_Z \simeq L_Z \simeq$ *a subgroup of* A^*_Z *normal in* A^*.
- *(iv) If* $\mathrm{Aut}(A)$ *is supersolvable, then* K_Z *is supersolvable.*

Proof. *(i)* It is immediate by the fact that the applications Φ_1 and Φ_2 from K_Z to $\mathrm{Aut}_Z(G)$ defined respectively by $\Phi_1(\varphi) := \varphi_1 + \varphi_2$ and $\Phi_1(\varphi) := \varphi_1 - \varphi_2$ (where $\varphi_1 + \varphi_2(a) = \varphi_1(a)\varphi_2(a)$ and $\varphi_1 + \varphi_2^{-1}(a) = \varphi_1(a)(\varphi_2(a))^{-1}$ for each $a \in A$) are both epimorphisms of kernels respectively H_Z and L_Z.
(ii) Let $\varphi \in H \cap L$, then $\varphi(a^2, 1) = \varphi(a, a)\varphi(a, a^{-1}) = (a, a)(a, a^{-1}) = (a^2, 1)$ and since 2 does not divide $|A|$, φ is the identity map, and so $H \cap L = 1$.
(iii) We have already proved that H_Z can be embedded in A^*. Moreover, using a better argument, one can consider the subgroup $T := H_Z \times L_Z$. T is subgroup of

K_Z, which is normal in K, so $T \cap A^*$ is a subgroup of A_Z^* normal in A^*. Using *(i)* and Dedekind's modular law, we obtain that $T \cap A^* \simeq H_Z \simeq L_Z$.
(iv) By *(i)* we obtain that $\frac{K_Z}{H_Z} \simeq \frac{K_Z}{L_Z} \simeq A_Z^*$, and since $H_Z \cap L_Z = 1$, if $\mathrm{Aut}(A)$ is supersolvable, K_Z is a supersolvable group. □

In particular Lemma 6 applies when A is abelian and $\mathrm{Aut}(A)$ supersolvable. In this case $\mathrm{Aut}_Z(F) = \mathrm{Aut}(F)$ and so $K = K_Z$ is a supersolvable group.

Let us now concentrate on the case A *non*-abelian, and study more in detail the structure of the subgroup K_Z, with particular interest on the action of K on K_Z.

Lemma 7 *Let A be a non abelian finite p-group ($p \neq 2$), with* $\mathrm{Aut}(A)$ *supersolvable. Then:*

(i) $O_{p'}(K_Z) = 1$ *and* $K_Z = [O_p(K_Z)]S$*, where S is an abelian Hall $p'-$subgroup of K_Z.*

(ii) $K = K_Z C_K(S)$.

(iii) $[K_Z, K] \leq O_p(K_Z)$.

Proof. *(i)* Since $\mathrm{Aut}(A)$ is supersolvable, another result of G. Corsi ([8]) implies that $O_{p'}(\mathrm{Aut}(A)) = 1$, and so, as A_Z^* is normal in A^* and $A^* \simeq \mathrm{Aut}(A)$, $O_{p'}(A_Z^*) = 1$. Now the map Φ_1 defined in Lemma 6 *(i)* is a homomorphism with kernel H_Z, and so we deduce that $O_{p'}(K_Z) = O_{p'}(H_Z)$. By Lemma 6 again, H_Z is isomorphic to a normal subgroup of A_Z^* and so $O_{p'}(H_Z) = 1$, i.e. $O_{p'}(K_Z) = 1$ and the Fitting subgroup $\mathrm{Fit}(K_Z)$ coincides with $O_p(K_Z)$. The supersolvability of K and Schur-Zassenhaus theorem complete the proof of this step.
(ii) Take S a Hall $p'-$subgroup of K_Z. Using Frattini's argument we have that $K = K_Z N_K(S)$. In order to prove that $N_K(S) = C_K(S)$, consider a decomposition of A as $A = X \times Y$, where X and $Y \neq 1$ are respectively the product of the abelian and non-abelian direct factors of A. Since S is in particular a $p'-$subgroup of $\mathrm{Aut}_Z(F)$ we have, according to [5], $F = C_F(S) \times [F, S]$, where $C_F(S)$ is the centralizer of S in F and $[F, S] := \langle w^{-1}g(w) \mid g \in S, w \in F\rangle$ is abelian. Moreover, up to conjugation, we can suppose that S contains the automorphisms α and β defined by $\alpha(xy, \tilde{x}\tilde{y}) := (\tilde{x}y, x\tilde{y})$, $\beta(xy, \tilde{x}\tilde{y}) := (x^{-1}y, \tilde{x}^{-1}\tilde{y})$, for each $x, \tilde{x} \in X$, $y, \tilde{y} \in Y$. Consider now the subgroups $\Delta(X)$ and $\nabla(X)$. One can easily prove that $\Delta(X) \simeq \nabla(X) \simeq X$, $X \times X = \Delta(X) \times \nabla(X)$, $[F, \alpha] = \nabla(X)$, and $[F, \beta] = X \times X$. From this, we obtain that $[F, S] = X \times X$. Then $N_K(S)$ fixes the subgroups $\Delta(X), \nabla(X)$ and $Y \times Y$, and so any element f of $N_K(S)$ can be seen as a triple $f = (f_0, f_1, f_2)$, where $f_0 = f_{|Y \times Y}$ is an automorphism of $Y \times Y$ and $f_1 = f_{|\Delta(X)}$, $f_2 = f_{|\nabla(X)}$ are automorphisms of X. In particular the elements of S are of the form $\varphi = (id, \varphi_1, \varphi_2)$. Now taken any $f \in N_K(S)$ and any $\varphi \in S$ we have that $\varphi^f = (id, \varphi_1^{f_1}, \varphi_2^{f_2}) \in S$ and $\varphi_1, \varphi_1^{f_1}, \varphi_2$ and $\varphi_2^{f_2}$ can be seen as $p'-$elements of $\mathrm{Aut}(X)$. $\mathrm{Aut}(X)$ is a supersolvable group, since it is, up to isomorphism, a

subgroup of $\mathrm{Aut}(A)$, in particular $\mathrm{Aut}(X)$ is strictly p-closed. Therefore $[\varphi_1, f_1]$ lies in $O_p(\mathrm{Aut}(X))$, but φ_1 lies in a Hall $p'-$subgroup of $\mathrm{Aut}(X)$ and f_1 normalizes this subgroup and so $[\varphi_1, f_1]$ must be also a $p'-$element, therefore $[\varphi_1, f_1] = 1$. Similarly $[\varphi_2, f_2] = 1$, and so f centralizes S, and $K = K_Z C_K(S)$.
(iii) Using the previous two steps we obtain

$$[K_Z, K] = [O_p(K_Z)S, O_p(K_Z)C_K(S)] \leq O_p(K_Z).$$

$\square$

As a consequence of this Lemma we obtain the following result on the subgroup H.

Lemma 8 *The subgroup H of K is strictly p-closed.*

Proof. Using the previous Lemmas we obtain that

$$H' = [H_Z Q, H_Z Q] \leq O_p(H_Z) \leq O_p(H).$$

So we just have to prove that $\frac{H}{O_p(H)}$ has exponent that divides $p-1$.
Note that $\frac{H}{H_Z}$ is an elementary abelian 2-group and so $O_p(H) = O_p(H_Z)$. Using the facts that $H = H_Z Q$, Q is an elementary abelian 2-group, and $\frac{H}{O_p(H)}$ is abelian, we deduce that $(\frac{H}{O_p(H)})^{p-1} = (\frac{H_Z}{O_p(H_Z)})^{p-1}$, and since $H_Z \lesssim \mathrm{Aut}(A)$, H_Z is strictly p-closed and $(\frac{H}{O_p(H)})^{p-1} = 1$. $\square$

In particular we proved that $[H, H] \leq O_p(H)$. As $[A^*, A^*] \leq O_p(A^*)$, we now just have to consider the action of A^* on H. At this point we make hypotheses on the structure of A. Suppose first of all that *A has no pairs of isomorphic direct factors in any of its Remak decompositions.* Note that this hypothesis is consistent in the case A abelian, in fact a result of G. Corsi [8] shows that any abelian p-group $(p \neq 2)$ with supersolvable automorphism group necessary satisfies this condition. A first consequence of this assumption is contained in the next

Lemma 9 *Let A be a p-group $(p \neq 2)$, with* $\mathrm{Aut}(A)$ *supersolvable and such that A has no isomorphic direct factors in its Remak decompositions, then*

(i) $N_K(Q) = C_K(Q)$.

(ii) $K = K_Z C_K(Q)$.

Proof. *(i)* We let f be an element of $N_K(Q)$ and prove that f centralizes Q by showing that f centralizes the generators $\varphi_{\{i\}}$ of Q for $i = 1, 2, \ldots, n$.
Let us begin by observing that f permutes the elements $\varphi_{\{i\}}$. In fact if Q is associated to the following Remak decomposition of A, $A = X_1 \times X_2 \times \cdots \times X_n$, the automorphism $\varphi_{\{i\}}$ interchanges the elements of $X_i \times X_i$ in $A \times A$ and acts like

the identity on the others. Similarly $f^{-1}\varphi_{\{i\}}f$ changes the positions of the elements of $f^{-1}(X_i) \times f^{-1}(X_i)$ and acts trivially on the rest. Since X_i is indecomposable, $f^{-1}(X_i)$ is to be such, and so $f^{-1}\varphi_{\{i\}}f$ must be an element of Q that acts not trivially only on one pair of direct indecomposable factors, i.e. $f^{-1}\varphi_{\{i\}}f = \varphi_{\{j\}}$ for some $j = 1, 2, \dots, n$. Therefore f permutes the elements $\varphi_{\{i\}}$.
In order to prove that $i = j$, since A has no isomorphic direct factors, it is enough to show that $X_i \simeq X_j$. From the previous observation we immediately deduce that $|X_i| = |X_j|$. Let us prove now that $f(X'_j \times X'_j) = X'_i \times X'_i$.
We denote with $[\delta F, \varphi_{\{j\}}]$ the subgroup $\langle \delta(u^{-1})\varphi_{\{j\}}(u) \mid u \in F\rangle$; then

$$[\delta F, \varphi_{\{j\}}] = \langle (x, x^{-1}) \mid x \in X_j \rangle = \nabla(X_j)$$

and

$$X'_j \times X'_j \leq \nabla(X_j) \cap F' \leq (X_j \times X_j) \cap F' = X'_j \times X'_j$$

and so $\nabla(X_j) \cap F' = X'_j \times X'_j$. Similarly $\nabla(X_i) \cap F' = X'_i \times X'_i$. Now we claim that $f(\nabla(X_j)) = \nabla(X_i)$. In fact let $(x, x^{-1}) \in \nabla(X_j)$, then by Lemma 2 (iii), $f(x, x^{-1}) = (y, y^{-1})$ for some $y \in A$. Let $y = uv$ with $u \in X_i$, $v \in Y_i$, then

$$(uv, u^{-1}v^{-1}) = f(x, x^{-1}) = f\varphi_j(x, x^{-1}) = \varphi_i f(x, x^{-1}) = (uv^{-1}, u^{-1}v)$$

and so, since 2 does not divide the order of A, $f(x, x^{-1}) = (u, u^{-1})$. Since $|X_i| = |X_j|$, we have proved that $f(\nabla(X_j)) = \nabla(X_i)$, and from this we obtain that

$$\begin{aligned} X'_i \times X'_i &= F' \cap \nabla(X_i) = f(F') \cap f(\nabla(X_j)) = \\ &= f(F' \cap \nabla(X_j)) = f(X'_j \times X'_j) \end{aligned}$$

Now for the sake of simplicity we let

$X := \Delta(X_i) := \{(x, x) \mid x \in X_i\}$,

$Y := \Delta(Y_i) := \{(y, y) \mid y \in Y_i\}$,

$M := f(\Delta(X_j))$ and

$L := f(\Delta(Y_j))$.

Since f fixes $\Delta = \Delta(A)$, $M \leq \Delta \cap f(X_j \times X_j) = \Delta(f(X_j \times X_j))$ and by reasons of orders, $M = \Delta(f(X_j \times X_j))$. Similarly $L = \Delta(f(Y_j \times Y_j))$. Moreover, $A = X_i \times Y_i = X_j \times Y_j$ and this implies $M \simeq X_j, L \simeq Y_j$ and $\Delta = X \times Y = M \times L$.
Let now π_X and π_M be the projections from Δ respectively on X and on M, and set $\Phi := \pi_M \circ \pi_X$. Φ restricted to M is an endomorphism of M, so there exists $n \geq 1$ such that $\Phi^n(M) = \Phi^{n+1}(M) := R$. Now

$$M' = (\Delta f(X_j \times X_j))' = \Delta f(X_i \times X_i)' = \Delta(X'_i) = X',$$

and so

$$\Phi(M') = \pi_M \pi_X(X') = \pi_M(X') = \pi_M(M') = M'.$$

Therefore $R = \Phi^n(M) \geq \Phi^n(M') = M'$, in particular $R \trianglelefteq M$.
Now call $S = \text{Ker } \Phi^n$, then $S \trianglelefteq M$, $R \cap S = 1$ and $M = RS$. Therefore $M = R \times S$ and S is abelian as $R \geq M'$. Since $M \simeq X_j$, M is indecomposable, so either $M = R$ or $M = S$, in both cases we have $X_j \simeq X_i$. In fact if $M = R$, then $M = \Phi(M)$, so $M \simeq \pi_X(M)$. But $|M| = |X|$, and so $X = \pi_X(M) \simeq M$, i.e. $X_i \simeq X_j$. Otherwise if $M = S$ then M is cyclic, so X_j too. Using the fact that $f(X'_j \times X'_j) = X'_i \times X'_i$, we have that

$$X'_i \times X'_i = f(X'_j \times X'_j) = 1$$

so X_i is abelian and since it is indecomposable, it is cyclic. By the fact X_i and X_j are of the same order we deduce they are isomorphic.
(ii) As a consequence of Remak-Krull-Schmidt theorem, we have that for each $\alpha_f \in A^*$ there exists a $\alpha_{f_c} \in A^*_Z$ such that $Q^{\alpha_f} = Q^{\alpha_{f_c}}$, but this means that $\alpha_f \alpha_{f_c}^{-1} \in N_K(Q)$, and so $A^* = N_{A^*}(Q)A^*_Z$. Therefore $K = HA^* = H_Z Q N_{A^*}(Q) A^*_Z = N_K(Q)K_Z$, and by the previous Lemma $K = C_K(Q)K_Z$. □

Lemma 10 $[A^*, H] \leq O_p(H)$.

Proof. Using Lemmas 6 and 9, and the fact that D_Z centralizes H we obtain that $K = A^*_Z C_K(Q)$. In particular $A^* = A^*_Z C^*_A(Q)$. Then

$$\begin{aligned}[A^*, H] = [A^*, H_Z Q] \leq [A^*, H_Z]^Q [A^*, Q] \leq [K, K_Z][A^*, Q] \leq \\ \leq [K, K_Z][A^*_Z H_{A^*}(Q), Q] \leq [K, K_Z][A^*_Z, Q] \leq [K, K_Z] \leq O_p(K_Z),\end{aligned}$$

and since H is normal in K, $[A^*, H] \leq O_p(H)$. □

Now we are in condition to prove a result about K.

Theorem 5 *The group K is supersolvable if and only if* $\text{Aut}(A)$ *is supersolvable and either A has no isomorphic direct factors (and in this case K is strictly p-closed), or $A = A_1 \times A_2$ with $A_1 \simeq A_2$ indecomposable.*

Proof. Suppose first that in any Remak decomposition of A there are no pairs of isomorphic direct factors. Then from the previous Lemmas it is clear that $O_p(K) = O_p(H)O_p(A)$ and $K' \leq O_p(K)$. Since $\frac{K}{O_p(K)} = \frac{O_p(H)A^*}{O_p(K)} \times \frac{HO_p(A^*)}{O_p(K)}$, we have that the exponent of $\frac{K}{O_p(K)}$ divides the exponent of $\frac{\text{Aut}(A)}{O_p(\text{Aut}(A))}$ and so also $(p-1)$. It follows that K is strictly p-closed and thus supersolvable.

Suppose now that $A = A_1 \times A_2$ is the direct product of two isomorphic indecomposable groups. Since $\text{Aut}(A)$ is supersolvable, this implies that the A_i are not abelian ([8]). In particular F has no abelian direct factors too, and this implies that K_Z is a p-subgroup of K. In fact if S is any p'-subgroup of K_Z, using [5] we have that $F = C_F(S) \times [F, S]$, with $[F, S]$ central, then Krull-Schmidt theorem implies $[F, S] = 1$, i.e. $S = 1$. So we have that $K_Z \leq O_p(K)$ and $[K_Z, K] \leq O_p(K)$. Using the same notations as before, the subgroup Q now reduces to $\langle \varphi_1, \varphi_2 \rangle$, an elementary abelian group of order 4. We indicate with l any isomorphism from

A_1 to A_2; l induces the following involution in A $L(g_1g_2) := l^{-1}(g_2)l(g_1)$ for each $g_1 \in A_1, g_2 \in A_2$. Moreover, L induces $\alpha_L \in A^* \leq K$. We have that $|\alpha_L| = 2$ and $N_K(Q) = \langle \alpha_L \rangle C_K(Q)$. From these we obtain that $[A^*, H] \leq O_p(K)\langle \delta \rangle$, and so, since $[A^*, A^*] \leq O_p(A^*) \leq O_p(K)$ (as A^* is supersolvable with $O_{p'}(A^*) = 1$) and $[H, H] \leq [K_Z, K] \leq O_p(K)$, we have that $K' \leq O_p(K)\langle \delta \rangle \leq \mathrm{Fit}(K)$, the Fitting subgroup of K. Then we have that $\frac{K}{\mathrm{Fit}(K)} = \frac{\mathrm{Fit}(H)A^*}{\mathrm{Fit}(K)} \times \frac{HO_p(A^*)}{\mathrm{Fit}(K)}$ and so $\exp(\frac{K}{\mathrm{Fit}(K)})$ divides $(p-1)$. This is enough to deduce the supersolvability of K.

Finally we consider the case $A = A_1 \times A_2 \times B$ with $A_1 \simeq A_2$ indecomposable groups and $B \neq 1$. In this situation we prove that K is not supersolvable by showing that there is an element of order two in the Fitting subgroup of K which does not commute with all the elements of order p, and this is a contradiction, since p is the largest prime number that divides $|K|$.
Using the same notation as in the previous step, let us consider the following subgroup of K, $R := \langle \varphi_1\alpha_L, \varphi_2 \rangle$, (where α_L now acts like the identity automorphism on the elements of K). It is easy to see that R is a dihedral group of order 8 and that its centre consists of the subgroup generated by $\delta\varphi_{\{1,2\}} = [\varphi_1, \alpha_L]$. If K were supersolvable then, $O_p(K)$ would be a Sylow p-subgroup of K, moreover $K' \leq \mathrm{Fit}(K)$, and so $\delta\varphi_{\{1,2\}} \in \mathrm{Fit}(K)$ would commute with every element of order p. Now let θ be a non trivial homomorphism from $\frac{A_1}{A_1'}$ to $Z(K)$ (remember that in this situation $\mathrm{Aut}(A)$ supersolvable and $A_1 \simeq A_2$ imply that A_1 is non-abelian). θ defines the following automorphism of A: $\Theta(g) := g\theta(g)$. Θ has order p and it induces $\alpha_\Theta \in K$. Finally, a simple computation shows that $[\delta\varphi_{\{1,2\}}, \alpha_\Theta] \neq 1$ and this contradiction completes the proof. $\square$

3 Supersolvability of $\mathrm{Aut}(G)$

This section is devoted to prove Theorem 1.
We first consider the case that A is a finite p-group and then derive the result for finite nilpotent groups as a consequence.

Theorem 6 *Let A be a finite p-group with $\mathrm{Aut}(A)$ supersolvable.*
If $p = 2$, $\mathrm{Aut}(A\ wr\ C_2)$ is supersolvable if and only if $A \not\simeq C_2 \times C_2$.
If $p \neq 2$, $\mathrm{Aut}(A\ wr\ C_2)$ is supersolvable if and only if A has no isomorphic direct factors in any of its Remak decomposition.

Proof. If $p = 2$ the result follows from Theorem 4.
Let now $p \neq 2$ and suppose first that A has no isomorphic direct factors. Then $\mathrm{Aut}(A\ wr\ C_2) = K \cdot I$, by Theorem 5, is an extension of a strictly p-closed group by a normal p-group and so it is strictly p-closed and supersolvable.
If otherwise A has isomorphic direct factors, in order to have $\mathrm{Aut}(G)$ supersolvable, we must require that K is supersolvable, so Theorem 5 implies $A = A_1 \times A_2$ with $A_1 \simeq A_2$ indecomposable factors. Now the conjugation γ_a (where a is the generator of C_2) is not in the Fitting subgroup of $\mathrm{Aut}(A\ wr\ C_2)$, otherwise it will be a

central element. Therefore if we consider the dihedral subgroup $R := \langle \varphi_1 \alpha_L, \varphi_2 \rangle$ (with the same construction used in the second step of Theorem 5), we have that $C_R(I) = 1$. Then $R \cap \mathrm{Fit}(\mathrm{Aut}(A\, wr\, C_2)) = 1$, and so $\frac{\mathrm{Aut}(A wr C_2)}{\mathrm{Fit}(\mathrm{Aut}(A wr C_2))}$ is not abelian and $\mathrm{Aut}(A\, wr\, C_2)$ not supersolvable. $\square$

Proof of Theorem 1. The key idea of the proof consists in observing that $\mathrm{Aut}(G)$ is a central product of the groups $\mathrm{Aut}(A_p\ wr\ C_2)$, where A_p is the p-component of A and p varies on the set of prime divisors $\pi(A)$ of $|A|$. Use induction on $|\pi(A)|$.
If $|\pi(A)| = 1$, then A is a prime power group and there is nothing to prove. Let $|\pi(A)| \geq 2$ and write $A = P \times Q$, where $P = O_p(A)$ and $Q = O_{p'}(A)$ ($p \in \pi(A)$). Using the same notations as in the previous sections, we have again that F is characteristic in G and

$$\mathrm{Aut}(G) = I \cdot K.$$

Moreover $I = I_p \times I_{p'}$, where $I_p := \{\gamma_{(p_1,p_2)} \in \mathrm{Inn}(A\, wr\, C_2) \mid (p_1, p_2) \in P \times P\}$ ($I_{p'}$ is defined in a similar way), and $K := C_{\mathrm{Aut}(F)}(\delta)$. Since $F = A \times A$ is nilpotent $\mathrm{Aut}(F) = \mathrm{Aut}(P \times P) \times \mathrm{Aut}(Q \times Q)$ and we can write

$$K = K_p \times K_{p'}$$

where K_p is the subgroup of $\mathrm{Aut}(W)$ of all the elements which fix each element of $Q \times Q$ and commute with δ_p (δ_p is the involution of F that interchanges the elements of $P \times P$ and acts like the identity on the rest). In a similar way is defined $K_{p'}$. Finally we have that

$$[I_p, K_{p'}] = [I_{p'}, K_p] = 1.$$

So we obtain that $\mathrm{Aut}(G)$ is a central product of the groups $I_p \cdot K_p \simeq \mathrm{Aut}(P\, wr\, C_2)$ and $I_{p'} \cdot K_{p'} \simeq \mathrm{Aut}(Q\, wr\, C_2)$. The result follows from Theorem 6 and the inductive hypothesis. $\square$

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