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On the commuting complex of finite metanilpotent groups

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Abstract. We study the commuting complex associated to the set of all non-trivial elements of a finite group. In particular we treat the case of metanilpotent groups, proving a wedge-decomposition formula for this simplicial complex and necessary and sufficient conditions for its contractibility.

Introduction

Given a non-empty subset U of a finite group G, the *commuting complex* $\Gamma(U)$ associated to U is defined as the simplicial complex with vertex set U and whose simplices are all finite non-empty subsets of U with elements that commute pairwise. Commuting complexes of various subsets U have been investigated (see for instance [2], [13] and [11]). We analyze the case in which U is the set of all non-trivial elements of G (namely $G^{\#} := G \backslash \{1\}$). In this situation we consider an alternative way of looking at $\Gamma(G^{\#})$, by using the fact (Proposition 5) that this complex is, up to homotopy equivalence, the same as the order complex of the poset $\mathscr{N}(G)$ consisting of all non-trivial nilpotent subgroups of G ordered by inclusion. (We remind the reader that if P is any partially ordered set ('poset' for short), then the order complex $\Delta(P)$ associated to P is the simplicial complex whose simplices are the totally ordered subsets of P). The aim of this work is to establish some results concerning the topology, and more specifically the homotopy type, of $\Delta(\mathscr{N}(G))$.

After some introductory results, in Section 2 we treat the case of finite metanilpotent groups in detail. Our main theorem is the following result:

Theorem 14. *If G is a finite metanilpotent group, then $\Delta(\mathscr{N}(G))$ has the same homotopy type as the wedge*

$$\bigvee_{|F|} \mathbf{S}(\Delta(C_{(F)})),$$

where $F = F(G)$ is the Fitting subgroup of G and

$$C_{(F)} = \{X \leqslant C \mid X \neq 1, C_F(X) \neq 1\},$$

where C is a fixed Carter subgroup of G. (The symbol $\mathbf{S}$ *denotes the suspension operator.)*

Recall that a Carter subgroup of a group G is a self-normalizing nilpotent subgroup of G (see Proposition 8 for some properties of Carter subgroups). Thus the homotopy type of the order complex of $\mathcal{N}(G)$ is determined by $\Delta(C_{(F)})$ and $|F|$. We obtain a complete result regarding the homotopy type of $\mathcal{N}(G)$ in the case when a Carter subgroup is a p-group (and hence a Sylow p-subgroup) of G. The results that we obtain (Propositions 15 and 17) resemble those concerning the Quillen complex at p for the same types of groups (see [12] and [3]). Our analysis allows us to answer a question in [11] (see Remark 11). Finally, in Section 3 we study the contractibility of $\mathcal{N}(G)$ for metanilpotent groups whose Carter subgroups are p-groups.

Notation. Our basic references are [4] for group theory and [15] for topology. The notation of the paper follows these books. For a poset $(P, \geqslant)$ and an element $r \in P$ we denote by $P_{\geqslant r}$ the subposet $\{q \in P \mid q \geqslant r\}$; the subposets $P_{>r}$, $P_{\leqslant r}$, $P_{<r}$ are defined similarly. A map $f : P \to Q$ between posets is said to be order-preserving if $f(x) \leqslant f(y)$ whenever $x \leqslant y$ in P. We denote by $\Delta(P)$ the order complex associated to the poset P, but, in order not to overburden the notation, we often use the same symbol P to denote both the poset and the associated simplicial complex. We reserve the symbol $\simeq$ to denote homotopy equivalence between topological spaces. The topological spaces that we are dealing with are always simplicial complexes, and in particular CW-complexes; the basic facts concerning these are taken for granted. For two simplicial complexes Δ_1, Δ_2, we define the *join* $\Delta_1 * \Delta_2$ and the *wedge* $\Delta_1 \vee \Delta_2$ as in [15]; in particular, $\Delta * \varnothing = \varnothing * \Delta = \Delta$, and $\Delta_1 * \Delta_2$ is a contractible space if and only if at least one of the two complexes is. The wedge operator is unambiguously defined if and only if the spaces involved are path-connected; otherwise we have to specify the points at which the two complexes are wedged. This will be crucial in our wedge-decomposition formulas (for instance formula (1) in Lemma 3). As usual, S^k denotes the k-dimensional sphere; we take S^{-1} to be the empty set and S^0 the set consisting of two disjoint points. The *suspension* $\mathbf{S}(\Delta)$ of a space Δ is defined to be $S^0 * \Delta$ and the *cone* $\mathbf{C}(\Delta)$ of Δ is defined to be $\{1\} * \Delta$.

For any finite group G, we denote by $\mathcal{N}(G)$, $\mathcal{A}(G)$, $\mathcal{E}(G)$ and $\mathcal{A}_p(G)$ respectively the posets of all non-trivial nilpotent subgroups of G, all non-trivial abelian subgroups of G, all non-trivial direct products of elementary abelian subgroups of G, and all non-trivial elementary abelian p-subgroups of G (if p is a prime divisor of $|G|$).

1 Preliminaries

In this section we collect some topological techniques and some basic facts about the complex $\mathcal{N}(G)$.

Lemma 1 (Fiber Lemma [12, (1.6)]). *Let $f : P \to Q$ be an order-preserving map between the finite posets P and Q. Suppose that for all $x \in Q$ the upper (resp. lower) fiber*

$f^{-1}(Q_{\geqslant x})$ (*resp.* $f^{-1}(Q_{\leqslant x})$) *is contractible as a topological space. Then* f *induces a homotopy equivalence between the order complexes* $\Delta(P)$ *and* $\Delta(Q)$.

A conjunctive (resp. subjunctive) element of a poset P is, by definition, an element x of P such that for all $y \in P$ a least upper bound $x \vee y$ exists in P (resp. a greatest lower bound $x \wedge y$ exists in P).

Lemma 2 ([12, (1.5)]). *If a poset P has a conjunctive or a subjunctive element, then the order complex of P is contractible.*

Lemma 3 ([3, Corollary 5]). *Let* $f : P \to Q$ *be an order-preserving map between the two finite posets P and Q. Assume that*

(1) *Q is a meet semi-lattice with unique least element* $\hat{0}$;

(2) *every minimal element in* $Q_{>\hat{0}}$ *is in the image of* f;

(3) *for every* $q \in Q_{>\hat{0}}$, *the complex* $\Delta(f^{-1}(Q_{\leqslant q}))$ *is either contractible or a wedge of* (n_q)*-dimensional spheres, with* $0 \leqslant n_q < n_r$ *if* $q < r$ *in Q.*

Then the order complex $\Delta(P)$ *is homotopy-equivalent to the wedge*

$$(\Delta(f^{-1}\{\hat{0}\}) * \Delta(Q_{>\hat{0}})) \vee \bigvee_{q \in Q_{>\hat{0}}} (\Delta(f^{-1}(Q_{\leqslant q})) * \Delta(Q_{>q})) \tag{1}$$

where for $q \in Q_{>\hat{0}}$ *a fixed point* $c_q \in \Delta(f^{-1}(Q_{\leqslant q}) \subseteq \Delta(f^{-1}(Q_{\leqslant q})) * \Delta(Q_{>q})$ *is identified with* $q \in \Delta(f^{-1}\{\hat{0}\}) * \Delta(Q_{>\hat{0}})$.

Let Q be a subposet of the poset P. We say that P is *an extension of Q by minimal elements* if the following conditions hold:

(1) Q^{op} is an ideal of the opposite poset P^{op};

(2) for every $p \in P$, $Q_{\geqslant p} \neq \varnothing$;

(3) for every $p \in P$, either $p \in Q$ or p is a minimal element of P.

The following is a corollary of [14, Theorem 2.4].

Lemma 4. *Let P be an extension of Q by minimal elements and assume that* $\Delta(Q)$ *is contractible. Then* $\Delta(P)$ *has the same homotopy type as the wedge of suspensions*

$$\bigvee_{m \in P \setminus Q} \mathbf{S}(\Delta(P_{>m})).$$

Proof. In [14, Theorem 2.4] it is proved that the wedge $\bigvee_{m \in P \setminus Q} \mathbf{S}(\Delta(P_{>m}))$ is homotopy-equivalent to $\Delta(P_Q)$ where P_Q is the poset given by P with the adjunction of an extra element 0_Q that lies under all minimal elements of Q. Therefore $\Delta(P_Q) = \Delta(P) \cup \Delta(Q_Q)$, and both $\Delta(Q_Q)$ and $\Delta(P) \cap \Delta(Q_Q) = \Delta(Q)$ are contractible. By the so-called 'Gluing Lemma' ([1, Lemma 10.3]), it follows that $\Delta(P_Q) \simeq \Delta(P)$. □

In the sequel we do not distinguish in our notation between a poset and the associated order complex; we simply write P for both, and the correct interpretation will always be clear from the context.

Proposition 5 ([9, (1.2)]). *The order complexes $\mathcal{N}(G)$, $\mathcal{E}(G)$, $\mathcal{A}(G)$ defined in the Introduction and the commuting complex $\Gamma(G^{\#})$ are all G-homotopy-equivalent.*

Proposition 6 ([9, (2.2)]). *Let G be a finite group. If G has non-trivial center $Z(G)$ then $\mathcal{N}(G)$ is contractible.*

Proof. Since for every nilpotent subgroup X of G the subgroup $Z(G)X$ is again nilpotent, $Z(G)$ is a conjunctive element of $\mathcal{N}(G)$, and hence $\mathcal{N}(G)$ is contractible by Lemma 2. □

Recall that the 'join' $(P_1 * P_2, \leqslant)$ of two posets $(P_1, \leqslant_1)$ and $(P_2, \leqslant_2)$ is a new poset whose underlying set is the disjoint union of the underlying sets P_1 and P_2 and whose order relation is defined as follows: for $x, y \in P_1 * P_2$ we have

$$x \leqslant y \text{ if and only if } \begin{cases} x \leqslant_1 y & \text{if both } x, y \in P_1 \\ x \leqslant_2 y & \text{if both } x, y \in P_2 \\ x \in P_1, \ y \in P_2. \end{cases}$$

Proposition 7. *If G is the direct product $G_1 \times G_2$, then $\mathcal{N}(G)$ is homotopy-equivalent to $\mathcal{N}(G_1) * \mathcal{N}(G_2)$.*

Proof. By [12, Proposition 1.9] the order complex of the join poset $\mathcal{N}(G_1) * \mathcal{N}(G_2)$ is homotopy-equivalent to the order complex of

$$P := \mathbf{C}(\mathcal{N}(G_1)) \times \mathbf{C}(\mathcal{N}(G_2)) \setminus \{(1,1)\}.$$

This latter poset consists of the collection of all non-trivial nilpotent subgroups of the form $K_1 \times K_2$ with $K_i \in \mathcal{N}(G_i) \cup \{1\}$ for $i = 1, 2$. We prove that the inclusion map

$$j : P \to \mathcal{N}(G_1 \times G_2)$$

induces a homotopy equivalence between order complexes. For $i = 1, 2$, let $\pi_i : G_1 \times G_2 \to G_i$ be the projection map onto G_i, and note that if K is a non-trivial nilpotent subgroup of $G_1 \times G_2$ then $\pi_1(K) \times \pi_2(K)$ is again a nilpotent subgroup of $G_1 \times G_2$ containing K. Thus the upper fiber $j^{-1}_{\geqslant K}$ contains $\pi_1(K) \times \pi_2(K)$ as its unique minimal element, and so it is a conically contractible space. By Fiber Lemma 1, the map j induces a homotopy equivalence. □

The problem of establishing which groups G have a connected commuting complex has already been dealt with satisfactory. In [9], S. Lucido proves that $\mathcal{N}(G)$ is connected if and only if the ‘prime graph’ of G is connected (this is the graph having as vertices the prime divisors of $|G|$ and with two vertices p and q joined by an edge if and only if G has an element of order pq). The structure of finite groups whose prime graph is not connected was determined by K. Gruenberg and O. Kegel in unpublished work (see [6], [7], [8], [10] and [16]).

We end this section by recalling basic facts about Carter subgroups (i.e. nilpotent and self-normalizing subgroups of a group).

In finite soluble groups the Carter subgroups are exactly the $\mathcal{N}$-projectors of the group (here $\mathcal{N}$ denotes the Schunck class of nilpotent groups); in other words, C is a Carter subgroup of G if and only if for every endomorphism φ of G, $\varphi(C)$ is a maximal nilpotent subgroup of $\varphi(G)$.

Proposition 8 ([4, (VI 12.1), (VI 12.4)]). *Suppose that G is a finite soluble group. Then*

(1) *G has Carter subgroups,*

(2) *all Carter subgroups are conjugate in G, and*

(3) *if N is a nilpotent normal subgroup of G, D/N a Carter subgroup of G/N and C a maximal nilpotent subgroup of D such that $D = NC$, then C is a Carter subgroup of G.*

2 The homotopy type of the complex $\mathcal{N}(G)$ for metanilpotent groups

Throughout this section (and the rest of the paper) let G be a finite metanilpotent group. It is well known (see for example [4, Theorem 12.4]) that $G = FC$, where $F := F(G)$ is the Fitting subgroup of G and C is any Carter subgroup C of G. We consider the case in which $F \cap C \neq 1$.

Proposition 9. *If $F \cap C \neq 1$, then $\mathcal{N}(G)$ is contractible.*

Proof. The hypotheses imply that every nilpotent subgroup of G has a non-trivial centralizer in F. In fact, assume by contradiction that there exists a non-trivial nilpotent subgroup A of G such that $C_F(A) = 1$. Then A is a maximal nilpotent subgroup of FA and, since $F \cap A = 1$, A is a Carter subgroup of FA. As $F(C \cap (FA)) = FA$, any maximal nilpotent subgroup of FA that contains $C \cap FA$ is a Carter subgroup of FA (for example by Proposition 8(3)); if D is any such subgroup, then D and A are conjugate under some element g that can be taken to be in F. We obtain that

$$F \cap D = F \cap A^g = (F \cap A)^g = 1,$$

which contradicts the fact that $F \cap D$ contains the non-trivial subgroup $F \cap C$.

Therefore the map

$$f : \mathcal{N}(G) \to \mathcal{N}(F); \quad A \mapsto C_F(A)$$

is well defined. This map f is an order-reversing map between posets, and we claim that its upper fibers are contractible. Given any non-trivial subgroup L of F, the upper fiber over L is

$$f^{-1}_{\geqslant L} := \{X \in \mathcal{N}(G) \mid C_F(X) \geqslant L\}.$$

Since L is nilpotent, $Z(L)$ is non-trivial, and thus it is an element of $f^{-1}_{\geqslant L}$. Now for $X \in f^{-1}_{\geqslant L}$ the subgroup $XZ(L)$ is again nilpotent, as L centralizes X, and we have

$$C_F(XZ(L)) \geqslant C_F(X) \cap C_F(Z(L)) \geqslant L,$$

so that $XZ(L) \in f^{-1}_{\geqslant L}$. This shows that $Z(L)$ is a conjunctive element of $f^{-1}_{\geqslant L}$, and so by Corollary 2 this fiber is contractible. Hence, by Lemma 1, f induces a homotopy equivalence between the complexes $\mathcal{N}(G)$ and $\mathcal{N}(F)$; since the latter is a cone we conclude that $\mathcal{N}(G) \simeq 1$. □

Corollary 10. *If $C_F(C) \neq 1$, then $\mathcal{N}(G)$ is contractible.*

Proof. Since C equals its normalizer, $C_F(C) \subseteq F \cap C$, and thus $F \cap C \neq 1$ so that we can apply Proposition 9. □

Remark 11. Proposition 9 implies in particular that there exist finite groups with trivial center and contractible complex $\mathcal{N}(G)$. For instance, an extension of the extraspecial group of order 27 and exponent 3 by the group generated by an involution fixing one of the two standard generators and inverting the other is of this type. This negatively answers [11, Question 2.9].

We treat now the case in which $G = F \rtimes C$ is a split extension of its Fitting subgroup F by a Carter subgroup C. Note that this assumption implies that G has trivial center.

Our starting point is the following

Lemma 12. *If X is a nilpotent subgroup of $G = F \rtimes C$ such that $C_F(X)$ is trivial, then X is contained in one and only one Carter subgroup of G.*

Proof. Let X be as in the statement and let D be a maximal nilpotent subgroup of G containing X. Then $D \cap F = 1$, since otherwise $D \cap F$ is a non-trivial normal subgroup of D and therefore, since D is nilpotent, $1 \neq Z(D) \cap F$, contradicting the fact that $Z(D) \cap F \leqslant C_F(X) = 1$. In particular $N_F(D) = C_F(D) \leqslant C_F(X) = 1$, i.e. $N_G(D)$ has trivial intersection with F. Therefore $N_G(D)$, being isomorphic to a subgroup of G/F, is nilpotent. The maximality of D implies $N_G(D) = D$, and this is a Carter subgroup containing X.

Suppose now that X is contained in two Carter subgroups of G, say D_1 and D_2. As these subgroups are conjugate (by Proposition 8(2)), we may write $D_1 = D_2^g$ for some g that can be taken to be in F. But then X and X^g both lie in $FX \cap D_1$, and, by repeated use of Dedekind's modular law ([4, (12.12)]), we have

$$X^g = X^g(F \cap D_1) = FX^g \cap D_1 = FX \cap D_1 = X(F \cap D_1) = X.$$

Therefore g is an element of F normalizing X. But $C_F(X) = 1$, and so g is trivial and $D_1 = D_2$. □

The previous observation permits us to deform the complex $\mathcal{N}(G)$ by, roughly speaking, contracting any subgroup X with trivial centralizer in F into the single Carter subgroup containing it. This operation does not affect the homotopy type of the complex $\mathcal{N}(G)$:

Lemma 13. *If $G = F \rtimes C$, then $\mathcal{N}(G)$ is homotopy-equivalent to $\mathcal{B} \cup \mathcal{C}$, where $\mathcal{B} := \{X \in \mathcal{N}(G) \mid C_F(X) \neq 1\}$ and $\mathcal{C}$ is the set of Carter subgroups of G.*

Proof. Consider the inclusion map $i : \mathcal{B} \cup \mathcal{C} \to \mathcal{N}(G)$ and choose an arbitrary element $Y \in \mathcal{N}(G)$. We claim that the upper fiber $i^{-1}_{\geqslant Y}$ has a unique minimal element. If $C_F(Y) = 1$, then this happens also for every subgroup containing Y, and thus by Lemma 12, the fiber $i^{-1}_{\geqslant Y}$ consists of a unique element of G (which is a Carter subgroup of G). Let $C_F(Y) \neq 1$, so that Y lies in $\mathcal{B}$ and is the unique minimal element of $i^{-1}_{\geqslant Y}$. Therefore every upper fiber is contractible. Fiber Lemma 1 completes the proof. □

For every Carter subgroup C of G let

$$C_{(F)} := \{X \leqslant C \mid X \neq 1, C_F(X) \neq 1\}.$$

The homotopy type of $\mathcal{N}(G)$ can be completely described in terms of that of $C_{(F)}$.

Theorem 14. *Let G be any finite metanilpotent group, F its Fitting subgroup and C a Carter subgroup of G. Then the complex $\mathcal{N}(G)$ has the same homotopy type as the wedge of $|F|$ copies of the suspension of $C_{(F)}$, namely*

$$\mathcal{N}(G) \simeq \bigvee_{|F|} \mathbf{S}(C_{(F)}). \tag{2}$$

Proof. Assume first that $F \cap C \neq 1$. Then $1 \neq Z(C) \cap F \leqslant C_F(C)$, and C is the unique maximal element of $C_{(F)}$, so that $C_{(F)}$ is contractible. Since wedges and suspensions of contractible spaces are contractible,

$$\bigvee_{|F|} \mathbf{S}(C_{(F)}) \simeq 1 \simeq \mathcal{N}(G),$$

where the last equivalence comes from Proposition 9.

Let $F \cap C = 1$. We set $P := \mathcal{B} \cup \mathcal{C}$ and $\mathcal{D} := \bigcup_{C \in \mathcal{C}} P_{\leqslant C}$. Then $\mathcal{B}$ and $\mathcal{D}$ are subposets of P such that $P = \mathcal{B} \cup \mathcal{D}$. Note that $\mathcal{B}$ is an ideal of P. Moreover we claim that $\mathcal{B}$, considered as a complex, is contractible. Let ϕ be the map from $\mathcal{B}$ to $\mathcal{N}(F)$

sending any element X to its centralizer in F. This is a well-defined order-reversing map. We show that its upper fibers are all contractible. For any non-trivial subgroup Y of F, we have $1 \neq Z(Y) \leqslant C_F(Y)$, and thus $Z(Y)$ is an element of the upper fiber $\phi^{-1}(\mathcal{N}(F)_{\geqslant Y}) := \{X \in \mathcal{B} \mid C_F(X) \geqslant Y\}$. Let X be an arbitrary element of this fiber. Since Y centralizes X the subgroup $XZ(Y)$ is nilpotent and satisfies

$$C_F(XZ(Y)) = C_F(X) \cap C_F(Z(Y)) \geqslant Y.$$

This means that $Z(Y)$ is a conjunctive element of the fiber, and therefore the fiber is contractible by Lemma 2. Fiber Lemma 1 implies then that $\mathcal{B}$ and $\mathcal{N}(F)$ are homotopy-equivalent. Since $\mathcal{N}(F)$ is a cone we conclude that $\mathcal{B}$ is contractible. Now if $C \in \mathcal{C}$ then $P_{\leqslant C}$, regarded as a set, is the union $\{C\} \cup C_{(F)}$. Since all Carter subgroups are isomorphic (by Proposition 8) the posets $P_{\leqslant C}$ are all isomorphic. In particular, if for some Carter subgroup C the poset $C_{(F)}$ is empty, then all are empty and thus $\mathcal{D} = \mathcal{C}$. In this case, $\mathcal{B} \cap \mathcal{D} = \varnothing$ and P is therefore homotopy-equivalent to an antichain of $1 + |\mathcal{C}|$ points. Since $|\mathcal{C}| = |F|$, $\mathcal{N}(G)$ is homotopy-equivalent to a wedge of $|F|$ 0-spheres, and so to

$$\bigvee_{|F|} \mathbf{S}(C_{(F)}),$$

since $C_{(F)}$ is empty and $\mathbf{S}(\varnothing) \simeq S^0$. Therefore we assume now that $C_{(F)} \neq \varnothing$ for all $C \in \mathcal{C}$. We can apply Lemma 4 to the opposite poset $P' := P^{\mathrm{op}}$ and with $Q := \mathcal{B}^{\mathrm{op}}$. In fact, we have already seen that Q is an ideal of P and that $\Delta(Q)$ is contractible. Moreover the set $P \backslash Q$ is exactly $\mathcal{C}$, and so it consists of minimal elements of P'. Finally, if $p \in P'$, then either $\mathcal{B}_{\leqslant p}$ contains p, or $p = C \in \mathcal{C}$ and $\mathcal{B}_{\leqslant p} = C_{(F)} \neq \varnothing$. All the hypotheses of Lemma 4 are thus satisfied. This yields the following homotopy equivalences:

$$\mathcal{N}(G) \simeq \mathcal{B} \cup \mathcal{C} \simeq \bigvee_{x \in F} \mathbf{S}((\mathcal{B} \cup \mathcal{C})_{<C^x}) \simeq \bigvee_{x \in F} \mathbf{S}(C^x_{(F)}) \simeq \bigvee_{|F|} \mathbf{S}(C_{(F)}),$$

and the result follows. $\square$

In the case when the Carter subgroups are elementary abelian p-subgroups, the structure of $C_{(F)}$ is rather simple. The next result on the complex of nilpotent subgroups is similar to what happens for the Quillen complex (see [12, Theorem 11.2]).

Proposition 15. *Let G be a finite metanilpotent group and assume that G has an elementary abelian p-subgroup A as a Carter subgroup. Then the complex $\mathcal{N}(G)$ is either contractible or spherical of dimension* $\mathrm{rk}(A) - 1$, *and it is contractible if and only if $C_F(A) \neq 1$.*

Proof. According to Theorem 14, we have to examine the homotopy type of $A_{(F)}$.

Assume first that $C_A(F) \neq 1$. Since $C_A(F) \leqslant F$, we have $A \cap F \neq 1$. Proposition 9 implies $\mathcal{N}(G)$ is contractible. The same conclusion holds, by Corollary 10, if $C_F(A) \neq 1$.

Consider therefore the case in which A acts faithfully and fixed-point freely on $F \backslash \{1\}$. In particular p does not divide $|F|$, since otherwise $P := O_p(F)A$ will be a Sylow p-subgroup of G, and $Z(P) \cap O_p(F)$ will be non-trivial; but then $C_F(A) \neq 1$. Thus p does not divide $|F|$, and $F \cap A = 1$. We prove by induction on $\mathrm{rk}(A)$ that $\mathcal{N}(G)$ is $(\mathrm{rk}(A) - 1)$-spherical (which means that it is homotopy-equivalent to a wedge of spheres each of dimension $\mathrm{rk}(A) - 1$). By formula (2) of Theorem 14 and the fact that in making the suspension the dimensions of the spheres increase by 1 (see [12, Example 8.1]), it is enough to show that the subposet $A_{(F)}$ is contractible or $(\mathrm{rk}(A) - 2)$-spherical.

If $\mathrm{rk}(A) = 1$ then, as $C_F(A) = 1$, $A_{(F)}$ is empty and so (-1)-spherical, and the statement is true. Suppose that $\mathrm{rk}(A) \geqslant 2$. By the 'Generation Lemma' ([5, (1.9)]) we know that F can be generated by the centralizers in F of the maximal subgroups of A. In particular, there exists a maximal subgroup, say M, of A such that $C_F(M) \neq 1$. Set $P := A_{(F)}$ and $Q := \{X \in P \mid X \cap M \neq 1\}$. Notice that $Q = \{X \lneqq A \mid X \cap M \neq 1\}$ and that Q^{op} is an ideal of P^{op}. Moreover each element Y of $P \backslash Q$ is a complement to M in A, and so a minimal element of P. The map sending each $X \in Q$ to $X \cap M$ establishes a homotopy equivalence between the complex Q and the complex of all non-trivial subgroups of M; the latter, having a unique maximal element, is contractible, and so Q is contractible. Let $Y \in P$. If $Q_{\geqslant Y} = \varnothing$ this forces that Y is a complement of M and M has order p. Thus, in this case, $A_{(F)}$ is an antichain and Theorem 14 implies that $\mathcal{N}(G)$ is homotopy-equivalent to a wedge of spheres of dimension $1 = \mathrm{rk}(A) - 1$. We can therefore suppose that for all $Y \in P$ we have $Q_{\geqslant Y} \neq \varnothing$. All hypotheses of Lemma 4 are therefore satisfied; by applying it we obtain

$$A_{(F)} \simeq \bigvee \mathbf{S}((A_{(F)})_{>X}), \tag{3}$$

where the wedge is taken over all the complements X of M in P. Now note that $(A_{(F)})_{>X} = \{Y \in A_{(F)} \mid X < Y\}$ is isomorphic to the poset

$$M_{(C_F(X))} := \{R \leqslant M \mid 1 \neq R,\ C_{C_F(X)}(R) \neq 1\}.$$

In fact, by the modular law each $Y \in (A_{(F)})_{>X}$ can be written in the form $Y = (Y \cap M)X$, and thus

$$1 \neq C_F(Y) = C_F(Y \cap M) \cap C_F(X) = C_{C_F(X)}(Y \cap M).$$

A routine argument shows that the map sending $Y \in (A_{(F)})_{>X}$ to $Y \cap M$ establishes an isomorphism between the posets $(A_{(F)})_{>X}$ and $M_{(C_F(X))}$. By the inductive hypothesis we may assume that $(A_{(F)})_{>X}$ is, if not contractible, spherical of dimension $\mathrm{rk}(M) - 2$ i.e. of dimension $\mathrm{rk}(A) - 3$. Thus the suspension $\mathbf{S}((A_{(F)})_{>X})$ is either contractible (if $(A_{(F)})_{>X}$ is contractible) or $(\mathrm{rk}(A) - 2)$-spherical. Formula (3) yields that $A_{(F)}$ is either contractible or a spherical complex of dimension $\text{rk}(A) - 2$ as claimed. $\square$

Remark 16. The above result is no longer true if the Carter subgroups of G are not elementary abelian p-groups; in fact even if they are products of two elementary abelian groups of coprime orders, spheres of different dimensions can appear in the complex $\mathcal{N}(G)$. For example, let α be a cube root of unity in the field $\mathbb{F}_7$ of 7 elements. Take C to be the abelian group $\langle a, b, c, d \rangle \simeq C_3 \times C_3 \times C_2 \times C_2$ and let C act on a 5-dimensional vector space V over $\mathbb{F}_7$ as follows:

$$a = \text{diag}(\alpha, \alpha, 1, 1, \alpha), \qquad c = \text{diag}(1, -1, 1, -1, -1),$$
$$b = \text{diag}(1, 1, \alpha, \alpha, \alpha^2), \quad d = \text{diag}(-1, 1, -1, 1, -1).$$

Set $G := V \rtimes C$. Drawing the poset $C_{(V)}$ of all non-trivial subgroups of C with non-trivial centralizer in V, one sees that $C_{(V)}$ has the same homotopy type as the wedge of a 1-dimensional sphere and a 0-sphere, i.e. as $S^1 \vee S^0$. Therefore by formula (2) of Theorem 14, $\mathcal{N}(G)$ is homotopy-equivalent to the wedge of $|F| = 35$ copies of the space $\mathbf{S}(S^1 \vee S^0) = S^2 \vee S^1$.

We make the following

Conjecture. The homotopy type of the complex of non-trivial nilpotent subgroups of a finite metanilpotent group is that of a wedge of spheres, possibly of different dimensions.

Some motivation is given by the following extension of Proposition 15.

Proposition 17. *Let G be a finite metanilpotent group and assume that the Carter subgroups of G are Sylow p-subgroups, for some prime $p \neq 2$ dividing the order of G. Then $\mathcal{N}(G)$ is either contractible or a wedge of spheres (possibly of different dimensions).*

Proof. By [4, Theorem 12.4] we have $G = FP$, where $P \in \text{Syl}_p(G)$ and $F = F(G)$ is the Fitting subgroup. Hence since P is a Carter subgroup $C_F(P) = 1$ and since $F = F(G)$ we have $C_P(F) = 1$. Moreover, by Proposition 15 we may assume that P is not elementary abelian. Under these assumptions we work with the complex $\mathcal{E}(G)$ of non-trivial direct products of elementary abelian subgroups of G, which is homotopy-equivalent to $\mathcal{N}(G)$, by Lemma 5.

Consider the map

$$f : \mathcal{E}(G) \to \widehat{\mathcal{A}_p}(\bar{G}) := \mathcal{A}_p(G/F) \cup \{1_{G/F}\}; \quad A \mapsto \bar{A}$$

induced by the projection to the quotient group. Note that the lower fiber of some $\overline{Y} \in \widehat{\mathcal{A}_p}(\bar{G})$ is

$$f_{\leqslant Y}^{-1} = \mathscr{E}(FY),$$

which by Proposition 15 is either contractible (if $C_F(Y) \neq 1$), or spherical of dimension $\mathrm{rk}(Y) - 1$. Lemma 3 yields the formula

$$\mathscr{N}(G) \simeq \bigvee_{X \in \mathscr{A}_p(P)} (\mathscr{E}(FX) * \mathscr{A}_p(P)_{>X}). \tag{4}$$

Note that the first term of formula (1) in Lemma 3 has disappeared here; this is because this term is equal to the join of two spaces one of which is $\mathscr{E}(F)$, and so contractible. Since the join of a contractible space and any other (non-empty) space is contractible, this first term is $\simeq 1$ and so it can be omitted. We analyze the homotopy type of the non-contractible terms in formula (4). Note that if $\mathscr{E}(FX) * \mathscr{A}_p(P)_{>X}$ is any of these, using Proposition 15, it is homotopy-equivalent to

$$\begin{aligned}(\bigvee S^{\mathrm{rk}(X)-1}) * \mathscr{A}_p(P)_{>X} &\simeq (\bigvee (S^{\mathrm{rk}(X)-2} * S^0)) * \mathscr{A}_p(P)_{>X} \\ &\simeq \bigvee S^{\mathrm{rk}(X)-2} * \mathbf{S}(\mathscr{A}_p(P)_{>X}).\end{aligned}$$

Finally, in [3, Proposition 20] the homotopy type of the upper intervals $\mathscr{A}_p(G)_{>X}$ (when G is a finite soluble group and $p \neq 2$) is described. In particular, it is proved that the suspensions of these spaces are wedges of spheres (possibly of different dimensions). Therefore, by the previous equivalence, every non-contractible term in formula (4) is a wedge of spheres (possibly of different dimensions), and hence by (4) the whole space $\mathscr{N}(G)$ is a wedge of spheres. □

3 Contractibility of $\mathscr{N}(G)$

It is interesting to investigate whether the contractibility of the complex $\mathscr{N}(G)$ is equivalent to some specific group-theoretic property of G (as for instance happens for the p-subgroups complex, where it is conjectured by Quillen that its contractibility is equivalent to the fact that the group has a non-trivial normal p-subgroup; see [12]).

In Remark 16 we saw that the non-triviality of the center of the group is not a necessary condition for $\mathscr{N}(G)$ to be contractible, and indeed we have not identified any group-theoretic property that seems likely to be equivalent to the contractibility of $\mathscr{N}(G)$.

For metanilpotent groups we can give another sufficient condition. It is contained in the following result.

Proposition 18. *Let $G = FC$, with F the Fitting subgroup and C a Carter subgroup of G. If every direct product of elementary abelian subgroups of C has a non-trivial centralizer in F then $\mathscr{N}(G)$ is contractible. Moreover if C is a Sylow p-subgroup of G then the reverse implication holds.*

Proof. By Theorem 14 it is enough to show that for every Carter subgroup C of G the subcomplex $C_{(F)}$ is contractible.

Note that $C_{(F)}$ is homotopy-equivalent to the subcomplex $C^0_{(F)}$ consisting of all elements of $C_{(F)}$ that are non-trivial direct products of elementary abelian subgroups. In fact, consider the inclusion map $i : C^0_{(F)} \to C_{(F)}$. We claim that this has contractible fibers. Given any $X \in C_{(F)}$, since X is nilpotent it has non-trivial center $Z(X)$. In particular the subgroup $Z := \Omega(Z(X))$ generated by the elements of $Z(X)$ of prime order is non-trivial. Thus Z lies in the lower fiber $i^{-1}_{\leqslant X} = \{T \in C^0_{(F)} \mid T \leqslant X\}$. Now let T be an arbitrary element of $i^{-1}_{\leqslant X}$; then TZ is a product of elementary abelian groups, and $TZ \leqslant X$. Moreover

$$C_F(TZ) = C_F(T) \cap C_F(Z) \leqslant C_F(X) \neq 1,$$

which means that TZ lies itself in the fiber $i^{-1}_{\leqslant X}$. Thus Z is a conjunctive element of the fiber, which is contractible, by Proposition 2. Lemma 1 guarantees that there is a homotopy equivalence between the order complexes of $C^0_{(F)}$ and $C_{(F)}$. Since C is nilpotent the subgroup $\Omega(Z(C))$ is non-trivial. We claim this is a conjunctive element of the poset $C^0_{(F)}$. By our assumptions, it is trivial that $\Omega(Z(C)) \in C^0_{(F)}$. Moreover, given any $X \in C^0_{(F)}$, the subgroup $X\Omega(Z(C))$ is again a direct product of elementary abelian subgroups of C, and thus, since its centralizer in F non-trivial, it is an element of $C^0_{(F)}$. Therefore $\Omega(Z(C))$ is a conjunctive element of $C^0_{(F)}$, as claimed. By Lemma 2 we have that $C^0_{(F)}$ is contractible. By Theorem 14, $\mathcal{N}(G)$ is a wedge of suspensions of $C^0_{(F)}$, and so it is contractible.

Assume now that the Carter subgroups of G are Sylow p-subgroups. Then if $\mathcal{N}(G)$ is contractible, in formula (4) of Proposition 17, every term of the form $\mathscr{E}(FX) * \mathscr{A}_p(P)_{>X}$ with $X \in \mathscr{A}_p(P)$ is contractible. In particular, if X is a maximal elementary abelian subgroup of P, then $\mathscr{A}_p(P)_{>X}$ is empty and $\mathscr{E}(FX)$ must be contractible. Proposition 15 implies that $C_F(X) \neq 1$, which completes the proof. $\square$

Remark 19. We note that, in the setting of the previous proposition, if $|\pi(C)| \geqslant 2$ then the condition that all direct products of elementary abelian subgroups of C have non-trivial centralizer in F is not necessary to guarantee $\mathcal{N}(G) \simeq 1$. Let α be a cube root of unity in the field $\mathbb{F}_7$ of 7 elements and C the group $\langle a, b\rangle \simeq C_3 \times C_2$ acting on a 2-dimensional space V over $\mathbb{F}_7$ in the following way: $a = \mathrm{diag}(1, \alpha)$, $b = \mathrm{diag}(-1, -1)$. Let G be the semidirect product $V \rtimes C$. Then V and C are respectively the Fitting subgroup and a Carter subgroup of G; moreover $C_V(C) \leqslant C_V(\langle b\rangle) = 1$. By Theorem 14, $\mathcal{N}(G)$ is homotopy-equivalent to a wedge of suspensions of the subcomplex C_V; this consists of a single element $\langle a\rangle$, and therefore it is contractible, and hence $\mathcal{N}(G)$ is contractible.

Remark 20. The condition that the (direct products of) elementary abelian subgroups of a Carter subgroup C of G have non-trivial centralizers in F does not imply that $C_F(C) \neq 1$. Indeed, let p and q be odd primes such that p divides $q-1$, and let P be the extraspecial group of order p^3 and exponent p. Let F be the direct product $A \times B$, where A and B are elementary abelian groups of orders respectively q^2 and q^{p+1}. We let P act on A faithfully and irreducibly, and on B in such a way that the

$p+1$ direct factors of B are exactly the centralizers of the different maximal elementary abelian subgroups of P. Then the group $G = FP$ is such that $C_F(P) = 1$ but $C_F(X) \neq 1$ for every elementary abelian p-subgroup X of P.

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References

[1] A. Björner. Topological methods. In *Handbook of combinatorics*, vol. 2 (Elsevier, 1995), pp. 1819–1872.

[2] S. Bouc. Homologie de certains ensembles de 2-sous-groupes des groupes symetriques. *J. Algebra* **150** (1992), 158–186.

[3] F. Fumagalli. On the homotopy type of the Quillen complex of finite soluble groups. *J. Algebra* **283** (2005), 639–654.

[4] B. Huppert. *Endliche Gruppen* (Springer-Verlag, 1967).

[5] B. Huppert. *Finite groups*, vol. 3 (Springer-Verlag, 1982).

[6] N. Iiyori and H. Yamaki. Prime graph components of the simple groups of Lie type over the field of even characteristic. *J. Algebra* **155** (1993), 335–343.

[7] A. S. Kondratev. On prime graph components of finite simple groups. *Mat. Sb.* **180** (1989), 787–797.

[8] A. S. Kondratev and V. D. Mazurov. Recognition of alternating groups of prime degree from the orders of their elements. *Sibirsk. Mat. Zh.* **41** (2000), 359–369.

[9] S. Lucido. On the partially ordered set of nilpotent subgroups of finite groups. *Comm. Algebra* **23** (1995), 1825–1836.

[10] S. Lucido. Prime graph components of finite almost simple groups. *Rend. Sem. Mat. Univ. Padova* **102** (1999), 1–22.

[11] J. Pakianathan and E. Yalçin. On commuting and noncommuting complexes. *J. Algebra* **236** (2001), 396–418.

[12] D. Quillen. Homotopy properties of the poset of nontrivial p-subgroups of a group. *Adv. in Math.* **28** (1978), 101–128.

[13] A. S. Rapinchuk, Y. Segev and G. M. Seitz. Finite quotients of the multiplicative group of a finite dimensional division algebra are solvable. *J. Amer. Math. Soc.* **15** (2004), 929–978.

[14] Y. Segev and P. Webb. Extensions of G-posets and Quillen's complex. *J. Austral. Math. Soc. Ser. A* **57** (1994), 60–75.

[15] E. H. Spanier. *Algebraic topology* (McGraw-Hill, 1966).

[16] J. S. Williams. Prime graph components of finite groups. *J. Algebra* **69** (1981), 487–513.

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