Journal of Algebra 322 (2009) 1919–1945

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A characterization of solvability for finite groups in terms of their frame

Francesco Fumagalli

Dipartimento di Matematica "Ulisse Dini", Università di Firenze, Viale Morgagni, 67/A, 50134 Firenze, Italy

ARTICLE INFO

Article history:
Received 10 April 2008
Available online 22 July 2009
Communicated by Ronald Solomon

Keywords:
Finite groups
Conjugacy classes
Maximal subgroups

ABSTRACT

The frame of a group is the poset of conjugacy classes of all its proper subgroups. In this paper we will prove that a finite group is solvable if and only if every collection of maximal elements of its frame has a well-defined meet and the poset consisting of all such meets (including the meet of the empty set) is a modular lattice.

Introduction

Let G be a finite group. For any subgroup H of G, denote with $[H]_G := \{H^g \mid g \in G\}$ the conjugacy class of H in G. Define $\mathcal{C}(G)$ to be the poset whose elements are the conjugacy classes of proper subgroups of G, ordered in the natural way ($[H]_G \leqslant [K]_G$ if and only if there exists $g \in G$ such that $H^g \leqslant K$). We refer to $\mathcal{C}(G)$ as the *frame* of the group G.

In this paper we will prove the following characterization of solvable groups.

Theorem. *A finite group G is solvable if and only if every collection of coatoms of $\mathcal{C}(G)$ has a well-defined meet and the poset consisting of all such meets* (*including the meet* $[G]$ *of the empty set*) *is a modular lattice.*

This result deals with the structure of maximal subgroups in finite groups, and the proof of one implication makes use of the Classification Theorem of finite simple groups. It may be considered analogous to a well-known result of J. Shareshian's [20], in which the solvability of finite groups is characterized in terms of a combinatorial property (non-pure shellability) of the subgroup lattice. Throughout the paper we denote with $\mathcal{M}(G)$ the set of all classes of proper subgroups of G that

E-mail address: fumagalli@math.unifi.it.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.06.005

contain a representative that is the intersection of some collection of pairwise non-conjugate maximal subgroups of G. Our theorem can also be formulated in this way

Theorem. *A finite group G is solvable if and only if $\mathcal{M}(G)$ is a modular meet semilattice.*

In the course of our analysis of the frame of solvable groups, we also will be able to show that $\mathcal{M}(G)$ admits a recursive coatom ordering (Theorem 4), and therefore it is shellable (whenever G is a solvable group). This fact will furnish an easy proof of an earlier result [24] of V. Welker, which says that the homotopy type of the order complex of the frame of a solvable group is that of a wedge of spheres of fixed dimension.

I thank J. Shareshian, R. Solomon and C. Casolo for their encouragement and their helpful comments, and R. Bryce for having sent to me an unpublished copy of the work [7]. I also want to express my gratitude to Washington University in St. Louis for the financial support and the happy atmosphere I received during the period of my visit. Last but not least, I would like to thank the referee.

Notation

Our conventions for expressing the structure of groups are the following. If X and Y are arbitrary finite groups, with $X.Y$ we denote an extension of X by Y. The expressions $X:Y$ and $X{}^{\cdot}Y$ denote respectively split and non-split extensions. We write $X \circ Y$ for a central product of X and Y, and X^m for a direct product of m copies of X. For a natural integer d, $\frac{1}{d}X$ refers to a subgroup of index d in X. The cyclic group of order m is simply denoted by m, while the symbol $[m]$ denotes an arbitrary group of order m. Other notation for group structure is standard and follows basically that of the Atlas [10].

1. Solvable case

In this section we deal with frames of finite solvable groups. In proving Theorem 1 we make use of some classical results due to Ore. The interested reader can find the proofs for instance in [12, Section A, Chapter 16]. Recall that for a subgroup H of G the symbol H_G denotes the *core of H in G*, namely the intersection of all the conjugates of H in G.

Lemma 1. *Let G be a finite solvable group, L and M two maximal subgroups of G. Then L and M are conjugate in G if and only if $L_G = M_G$. Moreover, if $M_G \nleqslant L_G$ then $L \cap M$ is a maximal subgroup of L.*

As it is stated in the Introduction, we denote with $\mathcal{M}(G)$ the set of all classes of subgroups of G that contain a representative that is the intersection of some nonempty collection of pairwise non-conjugate maximal subgroups of G. In particular $\mathcal{M}(G)$ is the subposet of $\mathcal{C}(G)$ containing all the coatoms of $\mathcal{C}(G)$ and all the possible intersections of arbitrary collections of them. We start with proving that for a solvable group G the poset $\mathcal{M}(G)$ is a meet semilattice.

Theorem 1. *If G is a finite solvable group, then $\mathcal{M}(G) \cup \{[G]\}$ is a lattice.*

Proof. Since a meet semilattice with maximum element is a lattice, we show that $\mathcal{M}(G)$ is a meet semilattice.

Let $\{[M_i]_G\}_{i=1}^n$ be an arbitrary collection of pairwise distinct coatoms of $\mathcal{M}(G)$. We will show that they have a unique meet in $\mathcal{M}(G)$, which can be represented by the subgroup $X_n := M_1 \cap M_2 \cap \cdots \cap M_n$. Namely we will show that for any n-tuple $(g_1, g_2, \dots, g_n)$ of elements of G the subgroup $M_1^{g_1} \cap M_2^{g_2} \cap \cdots \cap M_n^{g_n}$ is a conjugate of X_n. We proceed by induction on n.

The case $n = 2$ is straightforward. Since the maximal subgroups M_1 and M_2 are not conjugate, by Lemma 1 their cores are distinct. This implies in particular that we may always write G as the product of any conjugate of M_1 by any conjugate of M_2. Given two arbitrary elements g_1 and g_2

of G, and write $g_1 = m_1 y_2$ with $m_1 \in M_1$, $y_2 \in M_2^{g_2}$ and $g_2 = m_2 y_1$, with $m_2 \in M_2$ and $y_1 \in M_1$, it follows that

$$M_1^{g_1} \cap M_2^{g_2} = M_1^{y_2} \cap M_2^{g_2} = \left(M_1 \cap M_2^{g_2}\right)^{y_2} = \left(M_1 \cap M_2^{y_1}\right)^{y_2} = (M_1 \cap M_2)^{y_1 y_2},$$

as wished.

Let $n \geqslant 3$ and assume that the meet of m distinct coatoms of $\mathcal{M}(G)$ is well defined, for every $m < n$. Without loss of generality we can therefore assume the collection $\{[M_i]_G\}_{i=1}^n$ is irredundant (in the sense that any meet of $n-1$ of its elements, existing by the inductive assumption, is not contained in the last conjugacy class). We claim that, up to rearrangement, if we set $X_0 := G$, $X_i := M_1 \cap \cdots \cap M_i$ for $i = 1, \ldots, n$, then

$$X_i = X_{i+1}(X_i \cap M_n), \quad \text{for every } i \leqslant n-2. \tag{1}$$

By Lemma 1 the cores in G of the subgroups M_i are pairwise distinct. Thus we may assume that $(M_1)_G$ does not contain any $(M_i)_G$, for all $1 < i \leqslant n$. Moreover, any subgroup $M_1 \cap M_i$, $i \geqslant 2$, is a maximal in M_1 (Lemma 1), and by the irredundancy assumption, these are all pairwise non-conjugate. It follows that, as in the case $n = 2$, we may write $M_1 = (M_1 \cap M_i)(M_1 \cap M_j)$, for every $i \neq j \in \{2, 3, \ldots, n\}$. In particular (1) holds for $i = 1$. By Lemma 1 again, we may assume that the normal core in M_1 of $M_1 \cap M_2$ does not contain any other $(M_1 \cap M_i)_{M_1}$, for $i \geqslant 3$. We can repeat our argument to show that the groups $M_1 \cap M_2 \cap M_i$ $(3 \leqslant i \leqslant n)$ are all in $M_1 \cap M_2$ and no two of these groups are conjugate in $M_1 \cap M_2$, and that $M_1 \cap M_2$ is the product of any two of these groups. Proceeding in this way until we exhaust all the subgroups, we obtain a sequence of subgroups each maximal in the next. We may assume this sequence is

$$X_{n-1} \lessdot X_{n-2} \lessdot \cdots \lessdot X_1 = M_1 \lessdot X_0 = G.$$

Also for every $i \leqslant n-2$, X_{i+1} and $X_i \cap M_n$ are two non-conjugate maximal subgroups of X_i, therefore $X_i = X_{i+1}(X_i \cap M_n)$.

Now

$$G = M_1 M_n = X_2(M_1 \cap M_n) M_n = X_2 M_n = \cdots = X_{n-1} M_n,$$

and since our arguments depend only on the conjugacy classes of subgroups and not on the chosen representatives, we may write G as a product of any conjugate of X_{n-1} by any conjugate of M_n. Finally let $g_1, g_2, \ldots, g_n$ be arbitrary elements of G and write, by the inductive assumption,

$$M_1^{g_1} \cap M_2^{g_2} \cap \cdots \cap M_{n-1}^{g_{n-1}} = X_{n-1}^g,$$

for some $g \in G$. Let also $g = x_1 y$, with $x_1 \in X_{n-1}$, $y \in M_n^{g_n}$ and $g_n = m x_2$, with $m \in M_n$, $x_2 \in X_{n-1}$ then

$$M_1^{g_1} \cap M_2^{g_2} \cap \cdots \cap M_n^{g_n} = X_{n-1}^y \cap M_n^{g_n} = \left(X_{n-1} \cap M_n^{g_n}\right)^y = \left(X_{n-1} \cap M_n^{x_2}\right)^y = (X_n)^{x_2 y},$$

which completes the proof. $\square$

Remark 1. Note that in the course of the proof of Theorem 1 we also show that an arbitrary element $[M_1]_G \wedge \cdots \wedge [M_n]_G$ of $\mathcal{M}(G)$ can be represented by any subgroup of the type $M_1^{g_1} \cap \cdots \cap M_n^{g_n}$, where $g_1, \ldots, g_n \in G$.

In [7] the authors study sublattices in the frame of a finite solvable group. In particular, after fixing a Hall system Σ for a solvable group G, they consider the maximal subgroups M_i of G into which Σ reduces (i.e. the members of Σ intersected with each M_i constitute a Hall system for the group M_i). They define $\mathcal{IMS}_\Sigma(G)$ to be the subposet of the subgroup lattice of G whose elements are all the possible intersections of maximal subgroups of G into with Σ reduces, and prove the following.

Theorem 2. *(See [7, Theorems 5.6 and 5.7].) Let Σ be Hall system of a finite solvable group G. Then the following holds.*

1. *$\mathcal{IMS}_\Sigma(G)$ is a sublattice of the subgroup lattice of G.*
2. *The join of two subgroups in $\mathcal{IMS}_\Sigma(G)$ is their setwise product.*
3. *The map $[\cdot]$ sending any subgroup X of $\mathcal{IMS}_\Sigma(G)$ into $[X]_G \in \mathcal{C}(G)$ is order preserving and injective.*

As a consequence of these results and of Theorem 1 we have the following

Proposition 1. *If G is solvable and Σ is a Hall system for G, then $\mathcal{M}(G)$ is a lattice isomorphic to $\mathcal{IMS}_\Sigma(G)$.*

Proof. By Theorem 2(3) the map $[\cdot]: \mathcal{IMS}_\Sigma(G) \to \mathcal{M}(G)$ is order preserving and injective. This is also surjective since, by the remark after Theorem 1, any element of $\mathcal{M}(G)$ can be represented by any intersection between conjugates of maximal subgroups and, by [7, (4.6)], given any maximal subgroup M of G there is a unique conjugate of M into which Σ reduces. □

We recall that a finite lattice $\mathcal{L}$ is said to be *modular* if it satisfies the modular law on its elements, i.e. if for every $x, y, z \in \mathcal{L}$ such that $x \leqslant z$ then

$$(x \vee y) \wedge z = x \vee (y \wedge z). \tag{2}$$

Equivalently, $\mathcal{L}$ is modular if it does not contain any pentagon with vertices: a, b, c, d, e such that $a < c$, $d = a \wedge b = c \wedge b$ and $e = a \vee b = c \vee b$.

We may now summarize our results for solvable groups in the following

Theorem 3. *If G is a finite solvable group, then $\mathcal{M}(G) \cup \{[G]\}$ is a modular lattice.*

Proof. By Theorem 1 and Proposition 1 $\mathcal{M}(G)$ is a meet semilattice isomorphic to $\mathcal{IMS}_\Sigma(G)$ (Σ any Hall system of G). We prove that $\mathcal{IMS}_\Sigma(G)$ is modular. If not there exist $A, B, C \in \mathcal{IMS}_\Sigma(G)$ such that $A < C$, $A \wedge B = C \wedge B$ and $A \vee B = C \vee B$. But by Theorem 2(2) $CB = C \vee B = A \vee B = AB$. Thus, using the modular law in the subgroup lattice of G, we have

$$C = C \cap AB = A(C \cap B) = A(A \cap B) = A,$$

which is a contradiction. □

The rest of this section is devoted in showing that for a solvable group G the lattice $\mathcal{M}(G) \cup \{[G]\}$ satisfies the property of being *pure shellable*. Recall that a poset is called pure (or graded) if all its maximal chains have the same length. For the definition and the main features of (pure) shellability the interested reader is referred to [3–5] and [6]. Here, we prove the equivalent statement that $\mathcal{M}(G) \cup \{[G]\}$ admits a so-called *recursive coatom ordering* (see [4]). Moreover, in [4] it is also shown that for modular lattices the concept of recursive coatom ordering is equivalent to the one of coatom ordering, whose definition we recall here.

Definition 1. Let $\mathcal{L}$ be a lattice. A *coatom ordering* on $\mathcal{L}$ is a total ordering $\prec$ on the set of coatoms of $\mathcal{L}$ such that the following condition holds.

For every pair of coatoms a and b of $\mathcal{L}$ such that $a \prec b$ there exists a coatom $c \prec b$, such that $a \wedge b \leqslant c \wedge b \lessdot b$.

(Here $c \wedge b \lessdot b$ means that b covers $c \wedge b$, i.e. there does not exist an element $x \in \mathcal{L}$ such that $c \wedge b < x < b$.)

Theorem 4. *If G is a finite solvable group, then $\mathcal{M}(G)$ admits a recursive coatom ordering. In particular $\mathcal{M}(G)$ is pure shellable.*

Proof. By Theorem 3, $\mathcal{M}(G) \cup \{[G]\}$ is a modular lattice, thus in particular it is pure, and we may limit our consideration in proving that it admits a coatom ordering. To the set $\mathcal{M}^*$ of the coatoms we give the following partial ordering:

$$[L]_G \prec [M]_G \quad \text{iff} \quad M_G < L_G.$$

We let $[M_1]_G, [M_2]_G, \dots, [M_r]_G$ be a linear extension of this ordering. It is immediate to show that this satisfies the condition of Definition 1. In fact, let $1 \leqslant i < k \leqslant r$, then of course $(M_i)_G \nleqslant (M_k)_G$, and, by Lemma 1, $M_i \cap M_k$ is a maximal subgroup of M_k. Thus we simply take $c = a = [M_i]_G$ to show the condition of Definition 1. □

We may now describe the homotopy type of the frame of a finite solvable group. The complete result that follows was obtained by V. Welker using a different approach.

Corollary 1. *(See V. Welker [24, 4.12].) For a finite solvable group G of chief length s, the order complex of the frame of G is either contractible or spherical of dimension $s - 2$.*

Proof. If P is any finite poset and if we denote with $I(P)$ the subposet of P consisting of all the possible intersections between arbitrary collections of coatoms of P, then a well-known application of the Nerve theorem says that the order complexes $\Delta(P)$ and $\Delta(I(P))$ are homotopy equivalent (see for instance [23]). Another well-known fact is that if P is pure shellable, then $\Delta(P)$ is homotopy equivalent to a wedge of spheres of fixed dimension. Finally in [24] it is shown that $\Delta(\mathcal{C}(G))$ is contractible if and only if it the poset $\mathcal{C}(G)$ is not complemented, otherwise the dimension of the spheres is exactly $s - 2$. □

2. Insolvable case

In this section we prove the converse of Theorem 3. We begin with some preliminary lemmas that are useful to reduce the problem to one about finite simple groups.

In the course of our analysis we have to treat the following situation. A finite group G has a unique minimal normal subgroup K which is non-abelian, and so a direct product of isomorphic simple groups S_i, for $i = 1, 2, \dots, r$ say. Since G acts transitively on the set $\{S_i\}_{i=1}^r$, for every $i \leqslant r$ we set $S_i = S_1^{g_i}$, for some $g_i \in G$, and we chose $g_1 = 1$. Given any subgroup L_1 of S_1 we also let

$$L^* := L_1 \times L_2 \times \cdots \times L_r$$

the subgroup of K such that for every $i = 1, 2, \dots, r$, $L_i := L_1^{g_i}$.

Moreover, for every $i \leqslant r$ we denote with π_i the projection map from K onto S_i, and for an arbitrary subgroup X of K we let $X_i := X \cap S_i$ and $X^i := \pi_i(X)$. Then for every $i \leqslant r$, $X_i \trianglelefteq X^i \leqslant S_i$. Finally, we put $X^- := \prod_i X_i$ and $X^+ := \prod_i X^i$, so that $X^- \leqslant X \leqslant X^+$.

Lemma 2. *Let G be a finite group. Assume that K is the unique minimal normal subgroup of G and that K is non-abelian. With the above notation, if the proper S_1-class $[L_1]_{S_1}$ is (maximal with respect to being) $N_G(S_1)$-invariant, then the K-class $[L^*]_K$ is (maximal with respect to being) G-invariant. Conversely, if $[L]_K$ is* *a proper K-class, such that $L_1 \neq 1$ and such that it is (maximal with respect to being) G-invariant, then $[L_1]_{S_1}$ is (maximal with respect to being) $N_G(S_1)$-invariant.*

Proof. Let $[L_1]_{S_1}$ be $N_G(S_1)$-invariant. We will prove that $[L^*]_K$ is G-invariant.

For every $x \in G$ denote by $\sigma_x \in \mathrm{Sym}(r)$ the permutation induced by x on the set $\{S_i \mid i \leqslant r\}$, so that, in our notation, $\forall i \leqslant r$:

$$S_1^{g_i x} = S_i^x = S_{\sigma_x(i)} = S_1^{g_{\sigma_x(i)}}.$$

Now for every i, the component S_i^x contains both the subgroups L_i^x and $L_{\sigma_x(i)}$, and we claim that these are conjugate subgroups in S_i^x. Note that $g_i x g_{\sigma_x(i)}^{-1} \in N_G(S_1)$ and since $[L_1]_{S_1}$ is $N_G(S_1)$-invariant, there exists an element $s_i \in S_1$ such that

$$L_1^{g_i x g_{\sigma_x(i)}^{-1}} = L_1^{s_i},$$

equivalently

$$L_i^x = L_i^{g_i x} = L_1^{s_i g_{\sigma_x(i)}} = (L_{\sigma_x(i)})^{g_{\sigma_x(i)}^{-1} s_i g_{\sigma_x(i)}},$$

which proves our claim since $g_{\sigma_x(i)}^{-1} s_i g_{\sigma_x(i)} \in S_i^x$. If we set

$$k := \prod_{i=1}^{r} \left(g_{\sigma_x(i)}^{-1} s_i g_{\sigma_x(i)}\right) \in K,$$

then $(L^*)^x = (L^*)^k$, and so $[L^*]_K$ is G-invariant.

Assume now that $[L]_K$ is a proper G-invariant K-class such that $L_1 \neq 1$. Let $g \in N_G(S_1)$ and let $k \in K$ such that $L^g = L^k$. Then

$$L_1^g = (L \cap S_1)^g = L^g \cap S_1^g = L^k \cap S_1 = (L \cap S_1)^k = (L_1)^{\pi_1(k)},$$

which shows that $[L_1]_{S_1}$ is $N_G(S_1)$-invariant. A similar argument shows also that $[L_1]_G = \{(L_j)^{s_j} \mid s_j \in S_j,\ j \leqslant r\}$. Let now $g \in G$ such that $S_1^g = S_i$, and let $k \in K$ such that $L^g = L^k$. Then $(\pi_1(L))^g = \pi_i(L^g)$, equivalently

$$\left(L^1\right)^g = \left(L^g\right)^i = \left(L^k\right)^i = \left(L^i\right)^k = \left(L^i\right)^{\pi_i(k)}.$$

From this it follows that $[L^+]_K$ is a proper G-invariant K-class, as, for every $g \in G$,

$$\left(L^+\right)^g = \left(\prod_i L^i\right)^g = \prod_i \left(L^i\right)^g = \prod_j \left(L^j\right)^k = \left(L^+\right)^k,$$

if as before we assume $L^g = L^k$. Note that if $L^+ = K$, then, as $L_1 \neq 1$, also $L^- = K$, forcing the contradiction $L = K$. Now if $[L]_K$ is a maximal G-invariant K-class, with $L_1 \neq 1$, then $L = L^+$ and so also $L = L^-$. In particular, $L^1 = L_1 = L \cap S_1$ and clearly $[L_1]_{S_1}$ is a proper maximal $N_G(S_1)$-invariant class. On the other hand, if $[L_1]_{S_1}$ is a class maximal with respect to being $N_G(S_1)$-invariant, then it is now immediate to see $[L^*]_K$ is a maximal G-invariant class. □

Lemma 3. *Let K be a normal subgroup of G and let $[R]_K$ be a proper K-conjugacy class maximal with respect to being G-invariant. Then $G = K N_G(R)$ and $N_G(R)$ is a maximal subgroup of G such that $N_G(R) \cap K = R$.*

Proof. Since $G = KN_G(R)$ follows immediately from a Frattini argument, we need only to show that if M is a proper subgroup of G containing the normalizer of R, then $M = N_G(R)$ and $M \cap K = R$. By the modular law, we have $M = N_G(R)(M \cap K)$. Since $G = KM$ and $M \cap K$ is normal in M, the K-class $[M \cap K]_K$ is G-invariant, thus, by the maximality of the class $[R]_K$, we have $M \cap K = R$, and so also $M = N_G(R)$. □

In the literature if S is a finite non-abelian simple group and G a group such that $S < G \leqslant \mathrm{Aut}(S)$, the non-maximal subgroups of S whose normalizers in G are maximal subgroups of G are sometimes called *novelties*.

Definition 2. We say that a finite lattice $\mathcal{L}$ satisfies the property (max) if for every pair x, y of coatoms of $\mathcal{L}$, their meet $x \wedge y$ is covered by both x and y (i.e. there does not exist any $z \in \mathcal{L}$ such that $x \wedge y \lneqq z \lneqq x$ or $x \wedge y \lneqq z \lneqq y$).

Note that any modular lattice satisfies (max), this property being just a reformulation of the non-existence of a pentagon with vertices a, b, c, d, e ($a < c$, $d = a \wedge b = c \wedge b$, $e = a \vee b = c \vee b$) in which b and c are coatoms.

Lemma 4. *Assume that G is a finite group such that $\mathcal{M}(G)$ is a modular meet semilattice. Let K be a normal subgroup of G and denote with $\mathcal{M}(K)^G$ the subposet of $\mathcal{C}(K)$ consisting of all the possible meets between arbitrary collections of proper K-classes which are maximal with respect to being G-invariant. Then $\mathcal{M}(K)^G$ is a meet semilattice satisfying* (max).

Proof. We first note that, since K is normal in G, the meet between the class $[K]_G$ with an arbitrary class $[X]_G$ is always well defined, being equal to $[X \cap K]_G$. Therefore in order to prove that $\mathcal{M}(K)^G$ is a meet semilattice, we show that $\mathcal{M}(K)^G$ coincides with the subposet

$$\mathcal{A} := \bigl\{[X]_G \wedge [K]_G \bigm| [X]_G \in \mathcal{M}(G)\bigr\} \setminus [K]_G.$$

We have to prove that the elements of $\mathcal{A}$ are G-invariant K-classes, and, since K is normal in G, this is equivalent to say that

$$[X \cap K]_K = [X \cap K]_G, \quad \forall [X]_G \in \mathcal{M}(G),$$

in other terms that

$$G = KN_G(X \cap K), \quad \forall [X]_G \in \mathcal{M}(G). \tag{3}$$

Note that by Lemma 3 this is true whenever $[X]_G$ is a coatom of $\mathcal{M}(G)$. We proceed by showing (3) by induction downwards on the level l of $[X]_G$ in $\mathcal{M}(G)$. Let $l \geqslant 2$, and let

$$[X]_G = \bigwedge_{i=1}^{l} [X_i]_G \tag{4}$$

be an irredundant writing of $[X]_G$ as intersection of coatoms $[X_i]_G$ of $\mathcal{M}(G)$ (note that such a writing can always be found since by assumption $\mathcal{M}(G)$ is modular and so pure). Assume that T is a maximal subgroup of G containing $KN_G(X \cap K)$. We can of course assume that T is not a G-conjugate to any of the X_i's ($i = 1, \dots, l$), otherwise $X \cap K = Y \cap K$, where $[Y]_G$ is an intersection of $l - 1$ elements, and so an element of level $\leqslant l - 1$ in $\mathcal{M}(G)$, thus by the inductive assumption we would have that $[X \cap K]_K$ is G-invariant. Set for all $i = 1, \dots, l$, $[Y_i]_G := \bigwedge_{j=1}^{i} [X_j]_G$. Since (4) is an irredundant writing of the element $[X]_G$, every $[Y_i]_G$ has level exactly i, in particular the following is a maximal chain in $\mathcal{M}(G)$:

$$[X]_G = [Y_l]_G \lessdot [Y_{l-1}]_G \lessdot \cdots \lessdot [Y_2]_G \lessdot [Y_1]_G = [X_1]_G. \tag{5}$$

Intersecting this chain with the coatom $[T]_G$ we obtain the following strictly increasing chain

$$[X]_G < [Y_{l-1}]_G \wedge [T]_G < \cdots < [Y_2]_G \wedge [T]_G < [X_1]_G \wedge [T]_G < [X_1]_G. \tag{6}$$

Note in fact that if for some j, $[Y_{j+1}]_G \wedge [T]_G = [Y_j]_G \wedge [T]_G$, then

$$[X \cap K]_G = [X]_G \wedge [K]_G = \bigwedge_{i \neq j+1} [X_i]_G \wedge [K]_G,$$

and therefore in this case the result would follow by the inductive assumption. Thus (6) is strictly increasing of length l, but then we have reached a contradiction to the fact that all the maximal chains of the interval $[[X]_G, [X_1]_G]$ in $\mathcal{M}(G)$ have the same length $l-1$, $\mathcal{M}(G)$ being a pure lattice. Thus (3) holds and $\mathcal{A} = \mathcal{M}(K)^G$ is a meet semilattice.

We prove now that $\mathcal{M}(K)^G$ satisfies (max).

By contradiction, for $i = 1, 2, 3$, let $[R_i]_K$ be three distinct maximal elements of $\mathcal{M}(K)^G$ such that

$$[R_{13}]_K := [R_1]_K \wedge [R_3]_K \lneqq [R_1]_K \wedge [R_2]_K =: [R_{12}]_K.$$

Let M_i be the normalizer in G of R_i. Using (3), up to conjugation we may assume that $R_1 \cap R_2 = M_{12} \cap K$ and $R_1 \cap R_3 = M_{13} \cap K$ (where $M_{12} := M_1 \cap M_2$ and $M_{13} := M_1 \cap M_3$ are such that $[M_{12}]_G := [M_1]_G \wedge [M_2]_G$ and $[M_{13}]_G := [M_1]_G \wedge [M_3]_G$). In particular, $[M_{12}]_G$ and $[M_{13}]_G$ are distinct, forcing that $[M_{123}]_G := [M_1]_G \wedge [M_2]_G \wedge [M_3]_G$ lies strictly below $[M_{12}]_G$. Consider the subgroup KM_{123}. If $G = KM_{123}$, then, using the modular law we would have

$$M_{13} = (M_{13} \cap K)M_{123} = R_{13}M_{123} \lneqq (R_{12})M_{123} = (M_{12} \cap K)M_{123} = M_{12},$$

and this contradicts the fact that $\mathcal{M}(G)$ satisfies (max), being a modular lattice. Therefore KM_{123} is a proper subgroup of G. Let T be a maximal subgroup of G containing it. Then $T \cap M_1$ contains $(K \cap M_1)M_{123} = R_1M_{123}$ and $T \cap M_{12}$ contains $(R_{12})M_{123}$ and does not contain R_1. This yields the following

$$[M_{123}]_G \lneqq [M_{12}]_G \wedge [T]_G \lneqq [M_1]_G \wedge [T]_G \lneqq [M_1]_G.$$

But this is in contradiction to the fact that the closed interval $[[M_{123}]_G, [M_1]_G]$ of $\mathcal{M}(G)$ has rank 2, being $\mathcal{M}(G)$ a pure meet semilattice. $\square$

Definition 3. Given two distinct conjugacy classes, $[A]_G$ and $[B]_G$, of a group G, we say that $A^x \cap B^y$ is a *maximal intersection of type* $(A|B)$ if $A^x \cap B^y$ is not strictly contained in any subgroup of G of the form $A^h \cap B^k$, with $h, k \in G$.

The following lemma shows that, for most of the simple Lie type groups G, $\mathcal{M}(G) \cup \{[G]\}$ fails to be a lattice.

Lemma 5. *Let G be a finite simple Lie type group. Assume that the Lie rank l of G is greater or equal to 2 and that G is not of type $B_2(q)$, $G_2(q)$ or $D_4(q)$. Then $\mathcal{M}(G) \cup \{[G]\}$ is not a lattice.*

Proof. In the course of this proof we mostly follow the notation of Carter's book [8]. In particular we denote with I the set of all the simple roots of G; these, unlike Carter's, are simply denoted using natural numbers from 1 to l (= the Lie rank of G). An arbitrary Borel subgroup is denoted by B. B is the semidirect product $U \rtimes H$, where H is a maximal torus of G, whose normalizer N in G is such that $N/H \simeq W$, the Weyl group of the root system, and U is a Sylow p-subgroup of G (p being the characteristic of G). U is generated by the root subgroups X_i, for $i = 1, 2, \dots, l$. An arbitrary parabolic subgroup associated to a subset J of I is denoted by P_J, and the symbol $\hat{}$ is used with an exclusive meaning, so that for every $i = 1, 2, \dots, l$ we have

$$P_{\hat{i}} := P_{I\setminus\{i\}} = \langle B, n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_l\rangle = \langle B, X_{-1}, \dots, X_{-(i-1)}, X_{-(i+1)}, \dots, X_{-l}\rangle,$$

where for every $j \in I$, n_j is an element of N which projects onto the simple reflection $w_j \in W$, via the epimorphism $N \to W$ with kernel H. The elements n_j act on the root subgroups in this way: $(X_i)^{n_j} = X_{w_j(i)}$, for all $i, j \in I$. We do not follow [8] in denoting the conjugation action on subgroups, thus for us Y^x means, as in the rest of this paper, the subgroup $x^{-1}Yx$.

By Theorems 8.3.2 and 8.3.3 in [8], $[P_{\hat{1}}]_G$, $[P_{\hat{2}}]_G$ and $[P_{\hat{3}}]_G$ (if $l \geqslant 3$) are distinct coatoms of $\mathcal{M}(G)$. Moreover, if J_1 and J_2 are two distinct nonempty proper subsets of I, then the parabolic subgroup $P_{J_1\cap J_2} = P_{J_1} \cap P_{J_2}$ represents a maximal intersection of type $(P_{J_1}|P_{J_2})$ (by [8, 8.3.4]). In particular, if we assume that $\mathcal{M}(G)$ is a meet semilattice, each of $[P_J]_G$, for J proper subset of I, lies in $\mathcal{M}(G)$, and each of $P_{\hat{1},\hat{2}}$, $P_{\hat{2},\hat{3}}$ and $P_{\hat{1},\hat{2},\hat{3}}$ is a unique (up to conjugation) maximal intersection of type respectively $(P_{\hat{1}}|P_{\hat{2}})$, $(P_{\hat{2}}|P_{\hat{3}})$ and $(P_{\hat{1},\hat{2}}|P_{\hat{2},\hat{3}})$. Now let $g = n_1n_2 \in N$ and consider the subgroup $(P_{\hat{1}})^g$. Since N normalizes H, $(P_{\hat{1}})^g$ contains H. Moreover, it contains also the subgroups $X_{\pm i}$, for every $i \geqslant 4$, as these are normalized by n_1n_2 (note that here we used the fact that $G \neq D_4(q)$ in order to include $X_{\pm 4}$). Also $(P_{\hat{1}})^g$ contains the root subgroups $(X_{\pm 2})^{n_1n_2} = X_{\pm w_1w_2(2)}$, and, since $w_1w_2(2) = 1$, we have that $(P_{\hat{1}})^g$ contains a Levi subgroup of $P_{\hat{2},\hat{3}}$, namely

$$L_{\hat{2},\hat{3}} := \langle H, X_{\pm 1}, X_{\pm 4}, \dots, X_{\pm l}\rangle.$$

Of course, $L_{\hat{2},\hat{3}}$ is also contained in $P_{\hat{2}}$ and $P_{\hat{3}}$, thus

$$L_{\hat{2},\hat{3}} \leqslant (P_{\hat{1}})^g \cap P_{\hat{2}} \cap P_{\hat{3}}.$$

Assuming therefore that $\mathcal{M}(G)$ is a meet semilattice, we have $(P_{\hat{1}})^g \cap P_{\hat{2}} \cap P_{\hat{3}} \leqslant (P_{\hat{1},\hat{2},\hat{3}})^{x^{-1}}$, for some element $x \in G$. It follows that $(L_{\hat{2},\hat{3}})^x \leqslant P_{\hat{1},\hat{2}} \cap P_{\hat{2},\hat{3}}$. The subgroup $(L_{\hat{2},\hat{3}})^x$ has trivial intersection with $O_p(P_{\hat{2},\hat{3}})$, in fact this intersection is a normal p-subgroup of $(L_{\hat{2},\hat{3}})^x$ and $O_p((P_{\hat{2},\hat{3}})^x)$ avoids $(L_{\hat{2},\hat{3}})^x$. From this it follows that $(L_{\hat{2},\hat{3}})^x$ is a complement of $O_p(P_{\hat{2},\hat{3}})$ in $P_{\hat{2},\hat{3}}$, and thus it is a Levi subgroup of $P_{\hat{2},\hat{3}}$. But then $P_{\hat{2},\hat{3}} = \langle B, (L_{\hat{2},\hat{3}})^x\rangle$, forcing $P_{\hat{2},\hat{3}} \leqslant P_{\hat{1},\hat{2}}$, and this contradicts the fact that the parabolic subgroups containing a fixed Borel subgroup form a lattice isomorphic to the lattice of subsets of I [8, 8.3.4]. □

Remark 2. Since the proof of Lemma 5 depends uniquely on the axioms of (B, N)-pair and on the action of the Weyl group on root subgroups, the previous lemma extends easily to any simple twisted Lie group with Weyl group W^1 not of the following types: $W(A_1)$, $W(B_2)$, $W(G_2)$, $W(D_4)$ and, eventually, D_{16}.

A crucial point in the proof of our Main Theorem (5) is the following result which makes use of the classification of finite non-abelian simple groups.

Lemma 6. *Let S be a finite non-abelian simple group and let G be any subgroup such that $S \leqslant G \leqslant \mathrm{Aut}(S)$. Denote with $\mathcal{M}(S)^G$ the subposet of the frame of S whose elements are meets of maximal G-invariant S-classes. Then one of the following holds.*

(α) *$\mathcal{M}(S)^G$ is not a lattice,*
(β) *$\mathcal{M}(S)^G$ is a lattice that does not satisfy the property* (max).

Remark 3. Before giving the proof of the lemma we need to make some comments.

(1) Note that in order to show condition (α) it is enough to find a pair of elements $[A]_S$ and $[B]_S$ of $\mathcal{M}(S)^G$ for which their meet is not well defined. This means exactly that, assuming $A \cap B$ is a maximal intersection of type $(A|B)$, there exists some $x \in S$ such that

$$A^x \cap B \nleqslant (A \cap B)^y,$$

for all $y \in S$.

(2) Sometimes in the course of the proof of Lemma 6, depending on the various cases, condition (α) could be difficult to prove, instead it can be much easier to assume that $\mathcal{M}(S)^G$ is a lattice and show that it admits three distinct coatoms, say $[M_1]_S$, $[M_2]_S$ and $[M_3]_S$, such that

$$[M_1]_S \wedge [M_2]_S \lneqq [M_1]_S \wedge [M_3]_S.$$

When we decide to adopt this strategy we simply say that prove condition (β), and tacitly we assume that all the meets involved are well defined.

(3) Note that, under the assumption that $\mathcal{M}(S)^G$ is a lattice, in proving condition of type $[A]_S \leqslant [B]_S$ we may use arguments similar to the following. If for instance B contains a Sylow p-subgroup of S for some prime p dividing the order of A, then a p-Sylow of A lies completely in a conjugate of B. In particular, all the p-part of $|A|$ divides the order of a representative subgroup of meet $[A]_S \wedge [B]_S$. Of course, if this holds for every prime divisor of $|A|$, we have that $[A]_S = [A]_S \wedge [B]_S \leqslant [B]_S$.

We proceed now with the proof of Lemma 6.

Proof of Lemma 6.
Alternating groups Assume $S = \mathrm{Alt}(n)$ is an alternating group of degree $n \geqslant 5$, and $n \neq 6$, so that in particular $\mathrm{Aut}(S) = \mathrm{Sym}(n)$ (the case $\mathrm{Alt}(6) \simeq L_2(9)$ will be treated as a linear group). If $n = 5$ we take M_1 and M_2 respectively the stabilizer of one point and the stabilizer of a set of cardinality two. Of course $M_1 \simeq \mathrm{Alt}(4)$ and $M_2 \simeq \mathrm{Sym}(3)$ are maximal subgroups of Alt(5), and their classes are Sym(5)-invariant. Finally note that in the frame of Alt(5), these classes admit two different intersections, one is the class of subgroups of order two, the other being the ones of order three. Thus condition (α) holds. The same choice for the subgroups M_1 and M_2 also works well when $n \geqslant 7$. It is easy to show that these stabilizers represent two non-conjugate maximal subgroups of $\mathrm{Alt}(n)$ (see for instance exercises 5.2.8 and 5.2.9 in [11]). Moreover both conjugacy classes $[M_1]_S$ and $[M_2]_S$ are invariant by the action of $\mathrm{Sym}(n)$. We show that the meet $[M_1]_S \wedge [M_2]_S$ is not well defined. We say that two representative subgroups M_1 and M_2 are incident when the point stabilized by M_1 lies in the 2-subset stabilized by M_2, note that if so, then $M_1 \cap M_2$ consists in the stabilizer of two distinct points, which is maximal in M_1 (for the same reason as before) and therefore it is a maximal intersection of type $(M_1|M_2)$. On the other hand if M_1 and M_2 are not incident, then $M_1 \cap M_2$ is never contained in the stabilizer of two distinct points (whenever $n \geqslant 5$). As $\mathrm{Alt}(n)$ is transitive on the set of n objects, we may always choose two representative M_1 and M_2 and an element $g \in \mathrm{Alt}(n)$, such that M_1 and M_2 are incident and M_1^g and M_2 are not. This completes the proof of (α).

Untwisted Lie type groups To treat the classical groups we use here the same notations as [17], unless differently specified. In particular if the classical group S has Lie rank l, with P_i $(i = 1, \dots, l)$ we denote the arbitrary maximal parabolic subgroup correlated with the node i of the Dynkin diagram (thus P_i denotes here what in Lemma 5 was $P_{\hat{i}}$). The subgroups P_i are the stabilizers of a totally singular (t.s.) subspace of dimension i of the underlying space. These are almost-always maximal subgroups of S (the only exception is when $S = P\Omega_{2l}^{+}(q)$ and $i = l-1$, see f.i. [13, Theorems 4.1 and 4.2]). For the other simple Lie type groups, case by case, we adopt the same notation as the papers to which the reader is referred. In general, we say that two parabolic subgroups of S are *incident* if and only if they contain the same Borel subgroup B of S. In particular note that if P_i and P_j are two distinct incident maximal parabolic subgroups of S, then $P_i \cap P_j$ is a maximal intersection of type $(P_i|P_j)$ [8, Theorem 8.3.4].

$A_l(q)$.

Whenever l is greater than 1 the full automorphism group of $A_l(q)$ admits a duality automorphism, denoted by ι, which acts as the inverse-transpose map on each matrix. In particular ι fuses the maximal parabolic P_j and P_{l+1-j}, for all $j \leqslant [l+1/2]$.

We examine separately the three different cases:

(1) $l \geqslant 3$, (2) $l = 2$, (3) $l = 1$.

(1) Let $l \geqslant 3$.

We prove condition (α). If G lies inside the group $P\Gamma$ of inner, diagonal and field automorphisms of S, then G does not induce a graph automorphism on S, and the result is an immediate consequence of Lemmas 4 and 5. Assume therefore that G contains an element not in $P\Gamma$, say $\phi = \gamma\iota$ (with $\gamma \in P\Gamma$). Since ϕ acts on the S-classes of parabolic subgroups $[P_j]_S$ in the same way as ι does, in what follows without loss of generality we assume that G contains ι. In this situation the S-classes $[P_1]_S$ and $[P_l]_S$ are fused together (by the action of ι) and therefore they are not elements of $\mathcal{M}(S)^G$. The same happens for the classes $[P_2]_S$ and $[P_{l-1}]_S$. The maximal G-invariant classes that we may consider are therefore represented by the two subgroups $M_1 := P_1 \cap P_1^{\iota}$ and $M_2 := P_2 \cap P_2^{\iota}$. In terms of (projective) matrices, the elements of M_1 and M_2 are of block-diagonal shape, with block degrees respectively: $1, l$ (for M_1) and $2, l-1$ (for M_2). These are invariant by the inverse-transpose automorphism, and so are maximal G-invariant classes (see also [17, Table 3.5.A]). We show that $[M_1]_S$ and $[M_2]_S$ do not admit a unique meet. Assume first $l > 3$. If P_1 and P_2 are incident, then $M_1 \cap M_2$ is a maximal intersection of type $(M_1|M_2)$. This consists of elements that, up to a suitable basis, have (projective) matrix shape of diagonal blocks of degrees: $1, 1, l-1$. Now there exists some $g \in S$ such that the 1-subspace stabilized by M_1^g lies in the $(l-1)$-subspace stabilized by M_2, and therefore the elements of $M_1^g \cap M_2$, up to a suitable basis, have diagonal block shape of degrees: $2, 1, l-2$. In particular, as soon as $l-2 \geqslant 2$, $M_1^g \cap M_2$ cannot be contained in any conjugate of $M_1 \cap M_2$. If $l = 3$, note that now $[P_2]_S$ is fixed by the action of ι. As P_1 and P_2 are incident, $M_1 \cap P_2$ is contained in the stabilizer of two distinct points of the underlying space. As before, let $g \in S$ such that the 1-subspace stabilized by M_1^g lies outside the line stabilized by P_2; then $M_1^g \cap P_2$ does not stabilize two distinct 1-subspaces and so it cannot be contained in any conjugate of $M_1 \cap P_2$.

(2) Let $l = 2$.

The group $A_2(2) \simeq A_1(7)$ will be treated in (3), thus now assume $q > 2$.

If G lies inside $P\Gamma$, then G does not induce a graph automorphism we reach a contradiction by Lemmas 4 and 5. Let G be outside $P\Gamma$ and, as before assume that G contains the element ι. We prove condition (β). Let B be the generic Borel subgroup of S and $C := P_1 \cap P_1^{\iota}$, the subgroup whose elements have a (projective) matrix shape with diagonal blocks of degrees 2 and 1. $[B]_S$ and $[C]_S$ are coatoms of $\mathcal{M}(S)^G$. As a third coatom, we take the one represented by the normalizer N of a maximal torus H. By [13, Theorem 4.5], except in the case $q = 4$, N is a maximal subgroup of S. By [17, Table 3.5.A], $[N]_S$ is $\mathrm{Aut}(S)$-invariant, and so it is a coatom of $\mathcal{M}(S)^G$. Assuming $q \neq 2, 4$, we note that

$$B = U \rtimes H \simeq q^3 : (q-1)^2/\mu,$$

$$N := N_S(H) \simeq (q-1)^2/\mu : S_3,$$

where $\mu = (q-1, 3)$ and U is a Sylow p-subgroup of S (p being the characteristic of S and $q = p^f$). In particular $H \leqslant B \cap N$. As we are implicitly assuming that $\mathcal{M}(S)^G$ is a lattice, necessarily p is different from 2 and 3. Otherwise if $[X]_S := [B]_S \wedge [N]_S$, then since B contains a p-Sylow subgroup of S, $|X| = p \cdot (q-1)^2/\mu$, and so X contains a copy of H as a subgroup of index respectively 3 or 2.

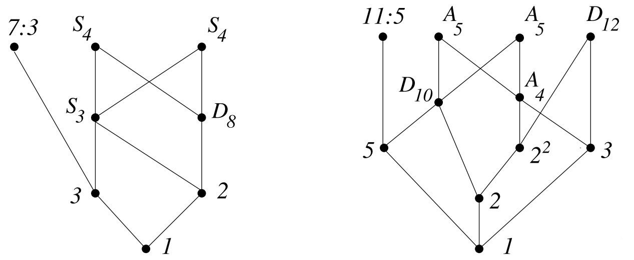


Fig. 1. $\mathcal{M}(L_2(7))$ and $\mathcal{M}(L_2(11))$.

But since X normalizes both its p-Sylow and its p'-part, we reach a contradiction with the fact that H is a maximal abelian subgroup of S. Thus $p \neq 2, 3$ and $[B]_S \wedge [N]_S = [H]_S$. Finally note that the subgroup $B \cap C$ strictly contains H, and therefore

$$[B]_S \wedge [N]_S \lneq [B]_S \wedge [C]_S.$$

In the case $q = 4$ we may replace the class $[N]_S$ with the unique class of maximal subgroups isomorphic to $3^2 : Q_8$. It is immediate to note that a unique meet between this class and the class of Borel subgroups is not defined.

(3) Let $l = 1$.

Assume first that q is either 8 or $\geqslant 13$, and prove (β). Under this assumption, the normalizer N of a maximal torus H is a maximal subgroup of S (see for instance [13, Corollary 2.2]). Moreover $[N]_S$ is $\mathrm{Aut}(S)$-invariant [17, Table 3.5.A]. Therefore $[N]_S$ is a coatom of $\mathcal{M}(S)^G$, as is the class $[B]_S$ of Borel subgroups. We necessarily have that q is odd. Otherwise, since $H \leqslant B \cap N$ and B contains a Sylow 2-subgroup of S, the class $[X]_S := [B]_S \wedge [N]_S$ would be represented by a subgroup of order $2(q-1) = |N|$, forcing a contradiction. Thus we can assume q odd $\geqslant 13$. As a third coatom we take the one consisting of the normalizers D of Singer cycles. D is a maximal subgroup of S, if $q \geqslant 13$ is odd [13, Corollary 2.2]. Moreover $[D]_S$ is $\mathrm{Aut}(S)$-invariant [17, Table 3.5.A]. We have the following isomorphisms

$$B \simeq q : (q-1)/2,$$

$$N \simeq (q-1)/2 : 2,$$

$$D \simeq (q+1)/2 : 2.$$

Also $[B]_S \wedge [N]_S = [H]_S$ (with $H \simeq (q-1)/2$). Now by a matter of orders, any intersection of type $B^x \cap D^y$ $(x, y \in S)$ is either trivial or a 2-group strictly contained in a conjugate of H. Therefore $[B]_S \wedge [D]_S \lneq [B]_S \wedge [N]_S$.

The cases $A_1(7), A_1(9)$ and $A_1(11)$ are drawn in Figs. 1 and 2. In particular note that in these cases the subgroups N and D are not maximal. However, in any case, the S-classes strictly containing one of them fuse pairwise together in some overgroup $G \leqslant \mathrm{Aut}(S)$. Therefore if $G = S$ (or Sym(6) in the case $S = A_1(9)$), then Figs. 1 and 2 show that $\mathcal{M}(S)^G$ is not a lattice. If otherwise $G > S$ (and $G \neq \mathrm{Sym}(6)$ in the case $A_1(9)$) we always have that $[N]_S$ and $[D]_S$ are maximal G-invariant classes and we can complete the proof as in the case $q \geqslant 13$.

The groups $A_1(4)$ and $A_1(5)$ are both isomorphic to A_5, while $A_1(2)$ and $A_1(3)$ are not simple groups.

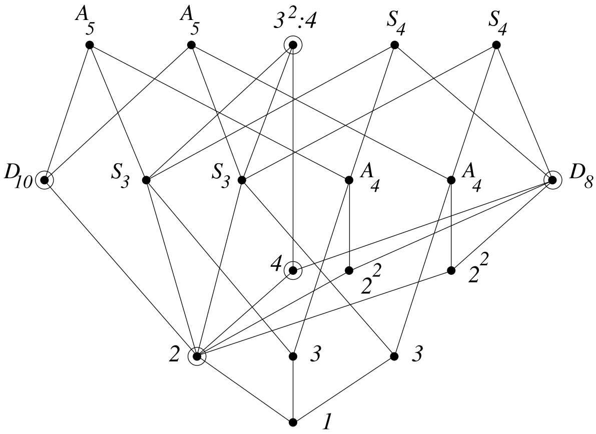


Fig. 2. $\mathcal{M}(L_2(9))$, the elements marked by $\odot$ constitute $\mathcal{M}(S)^G$, in the cases $S < G \neq \mathrm{Sym}(6)$.

$B_l(q)$.

We distinguish the two cases:
(1) $l \geqslant 3$, (2) $l = 2$.

(1) Let $l \geqslant 3$.
Lemma 5 yields that the conjugacy classes of maximal parabolics P_1 and P_2 have no unique meet in $\mathcal{M}(S)^G$.

(2) Let $l = 2$.
(2.1) Assume first q is odd ($q \neq 3$).
As $B_2(3) \simeq {}^2A_2(2)$, we treat this group later as a unitary group. In the following we show condition (β). The maximal classes $[P_1]_S$ and $[P_2]_S$ are $\mathrm{Aut}(S)$-invariant (Table 3.5.C in [17]). As a third class we take the one whose members are the stabilizers of a decomposition in t.s. 2-dimensional subspaces. Call R an arbitrary representative subgroup of this class, then R is a maximal subgroup of S [13, Theorem 4.6], and $[R]_S$ is $\mathrm{Aut}(S)$-invariant [17, Table 3.5.C]. By 4.1.9 and 4.2.5 in [17] we have

$$P_1 \simeq \left[q^3\right].(q-1).L_2(q),$$
$$P_2 \simeq \left[q^3\right].(q-1)/2.PGL_2(q),$$
$$R \simeq \frac{(q-1)}{2}.PGL_2(q).2.$$

We extend our terminology by saying that a parabolic subgroup P_i is incident to a subgroup R of type $\mathcal{C}_2$ if the t.s. subspace stabilized by P_i lies completely in a member of the t.s. factorization stabilized by R. It is then easy to see that when P_i and R are incident we have that $P_i \cap R$ is a maximal intersection of type $(P_i|R)$, for $i = 1, 2$. Assuming that P_1, P_2 and R are all pairwise incident, we have that $P_2 \cap R$ has index two in R and it consists of elements whose preimages in the full symplectic group $Sp_4(q)$ are matrices of the block-diagonal shape $\mathrm{diag}(A, A^*)$, where A lies in the group $GL_2(q)$ and A^* denotes the inverse-transpose matrix of A. On the other hand note that the elements of $P_1 \cap R$ have preimages (in $Sp_4(q)$) of shape $\mathrm{diag}(C, C^*)$, where C is a Borel subgroup of A, and so we have $[P_1]_S \wedge [R]_S \lneq [P_2]_S \wedge [R]_S$.
(2.2) Let $q = 2^f > 4$.
(Note that $B_2(2) \simeq \mathrm{Sym}(6)$.)
The group S admits a graph automorphism of order two, and $\mathrm{Out}(S)$ is a cyclic group of order $2f$ (see for instance [8, Proposition 12.3.3], or [17, p. 25]). We distinguish the two cases:

(2.2.1) G does not induce a graph automorphism on S,
(2.2.2) G induces a non-trivial graph automorphism on S.

(2.2.1) Then $G \leqslant P\Gamma$ and it is easy to see that $[P_2]_S$ and $[R]_S$ do not admit a unique meet.
(2.2.2) We show condition (β). According to [1, Section 14], the group G fuses some members of $\mathcal{C}$ in this way:

1. G fuses the maximal parabolic subgroups P_1 and P_2,
2. G fuses the members of $\mathcal{C}_2$ with the ones in $\mathcal{C}_8$ preserving a quadratic form of sign $+1$,
3. G fuses the members of $\mathcal{C}_3$ with the ones in $\mathcal{C}_8$ preserving a quadratic form of sign -1.

As coatoms of $\mathcal{M}(S)^G$ we can therefore consider the classes represented by the following novelties:

$$B = \text{Borel subgroup},$$
$$C = N_S\big(q^2+1\big),\ \text{the normalizer of a Singer cycle},$$
$$D = N_S\big((q+1)^2\big),\ \text{the normalizer of a maximal torus of type } (q+1)^2.$$

Note that

$$|B| = q^4(q-1)^2, \qquad |C| = 4\big(q^2+1\big) \quad \text{and} \quad |D| = 8(q+1)^2.$$

As the numbers $q-1$, $q+1$ and q^2+1 are pairwise coprime we immediately have

$$[4]_S = [C]_S \wedge [D]_S \lneq [D]_S \wedge [B]_S = [D_8]_S.$$

$C_l(q)$.

As $C_2 = B_2$ and $C_l(2^f) = B_l(2^f)$, we assume here that $l \geqslant 3$ and q is odd. Lemma 5 yields that the conjugacy classes of maximal parabolics P_1 and P_2 have no meet in $\mathcal{M}(S)^G$.

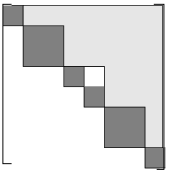
$D_l(q)$.

We distinguish the two cases:
(1) $l > 4$, (2) $l = 4$.

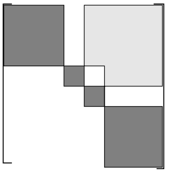
(1) Let $l > 4$.
Lemma 5 yields that the conjugacy classes of the maximal parabolic subgroups P_1 and P_2 have not a unique meet in $\mathcal{M}(S)^G$.

(2) Let $l = 4$.
We refer the reader to [14] for a complete classification of the maximal subgroups of S and of any group G such that $S \leqslant G \leqslant \mathrm{Aut}(S)$. Here, we also change our notation and adopt the same as [14]; in particular we use the symbol R_{si} to denote the stabilizer of a t.s. i-dimensional subspace of the underlying space V.
We distinguish the two situations:
(2.1) G does not induce on S a 'triality' graph automorphism,
(2.2) G induces a triality on S.
In both situations we prove condition (α).

(2.1) Under the action of S the 8-dimensional space V admits just two orbits of t.s. 4-dimensional spaces (or *solids*). Two t.s. solids lie in the same orbit if and only if their intersection has even dimension; therefore each t.s. 3-dimensional space (*plane*) lies in exactly two t.s. solids (one in each orbit). By this fact the stabilizer R_{s3} of a t.s. plane is the intersection of the stabilizers, R^1_{s4} and R^2_{s4}, of the two t.s. solids containing it. Even if R_{s3} is never a maximal subgroup of S, the class $[R_{s3}]_S$ is always G-invariant (under the assumption (2.1)). Moreover, it is a member of $\mathcal{M}(S)^G$, in fact either G does not fuse $[R^1_{s4}]_S$ and $[R^2_{s4}]_S$ and therefore $[R_{s3}]_S = [R^1_{s4}]_S \wedge [R^2_{s4}]_S$, or G fuses the two classes of stabilizers of t.s. solids, but then $[R_{s3}]_S$ itself is a maximal G-invariant class. We conclude with this case by showing that the meet between $[R_{s1}]_S$ and $[R_{s3}]_S$ is not well defined. Assume that the parabolics R_{s1} and R_{s3} are incident, so that $R_{s1} \cap R_{s3}$ is a maximal intersection of type $(R_{s1}|R_{s3})$. Up to a suitable orthogonal basis $\mathcal{B} = \{e_1, e_2, e_3, e_4, f_4, f_3, f_2, f_1\}$, the elements of $R_{s1} \cap R_{s3}$ have the following projective matrix shape:



We may choose an element g in S that interchanges $\langle e_1\rangle$ with $\langle e_4\rangle$ (and consequently $\langle f_1\rangle$ with $\langle f_4\rangle$); then $R^g_{s1} \cap R_{s3}$ consists of elements of the form:



In particular $R^g_{s1} \cap R_{s3}$ contains a linear subgroup that acts irreducibly on a t.s. plane, and since this fact does not happen in $R_{s1} \cap R_{s3}$, we have that

$$R^g_{s1} \cap R_{s3} \nleqslant (R_{s1} \cap R_{s3})^x \quad \text{for every } x \in S,$$

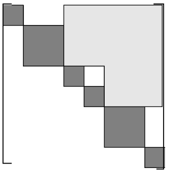
forcing condition (α).

(2.2) Let G induce a triality automorphism τ on the Dynkin diagram of S. The parabolic subgroups R_{s1}, R^1_{s4} and R^2_{s4} are all fused together by τ. In particular the class denoted by $[P_2]_S := [R_{s1} \cap (R_{s1})^\tau \cap (R_{s1})^{\tau^2}]_S$ is a coatom of $\mathcal{M}(S)^G$ [14, Table I]. The parabolic subgroup R_{s2} correlated to the node 2 is always maximal in S and with $\mathrm{Aut}(S)$-invariant class. We conclude by showing that $[P_2]_S$ and $[R_{s2}]_S$ have no a unique meet in $\mathcal{M}(S)^G$. Assume that R_{s2} is the stabilizer of the subspace $\langle e_1, e_2\rangle$ and P_2 the stabilizer of the chains

$$\langle e_1\rangle < \langle e_1, e_2, e_3\rangle < \langle e_1, e_2, e_3, e'_4\rangle,$$

$$\langle e_1\rangle < \langle e_1, e_2, e_3\rangle < \langle e_1, e_2, e_3, e''_4\rangle.$$

In particular R_{s2} and P_2 are incident, and so $R_{s2} \cap P_2$ is a Borel subgroup of S (of course a maximal intersection of type $(R_{s2}|P_2)$). Take $g \in S$ such that $(R_{s2})^g$ is the stabilizer of $\langle e_2, e_3\rangle$, then $(R_{s2})^g \cap P_2$ consists of elements of the following matrix shape:



This contains a copy of $L_2(q)$, which is impossible for any Borel subgroup of $D_4(q)$. Thus, for every $x \in S$, $(R_{s2})^g \cap P_2 \nleqslant (R_{s2} \cap P_2)^x$, which shows (α).

$E_l(q)$, $l \in \{6, 7, 8\}$.

Lemma 5 proves condition (α) in all the cases except when $S = E_6$ and G induces a non-trivial graph automorphism on the Dynkin diagram. In this latter case it is not difficult to show (α) using an argument similar to that of Lemma 5. We leave the details to the reader.

$G_2(q)$.

We distinguish the two distinct cases:
(1) G does not induce a non-trivial graph automorphism on S,
(2) G induces a non-trivial graph automorphism on S.

(1.1) Let q be odd and prove (β).

We refer to Theorem A in [16] for the structure of maximal subgroups of S. With the notation of [16], we consider the following pairwise non-conjugate maximal subgroups of S:

$$P_a \simeq [q^5] : GL_2(q),$$
$$P_b \simeq [q^5] : GL_2(q),$$
$$K_+ \simeq SL_3(q) : 2.$$

Set also $[X_a]_S := [P_a]_S \wedge [K_+]_S$ and $[X_b]_S := [P_b]_S \wedge [K_+]_S$ and show $[X_b]_S \lneqq [X_a]_S$. Note that $|P_a| = q^6(q-1)^2(q+1)$, $|P_b| = q^6(q-1)^2(q+1)$ and $|K_+| = 2q^3(q-1)^2(q+1)(q^2+q+1)$. Since both P_a and P_b contain a Sylow p-subgroup of S and since S admits a unique class of maximal tori T_+ isomorphic to $(q-1)^2$ [16, Table I], both X_a and X_b have orders divisible by $q^3(q-1)^2$. Moreover by [2, (2.15)], the Levi complement of P_a lies completely in a conjugate of K_+. It follows that $|X_a| = q^3(q-1)^2(q+1)$ and X_a is a maximal parabolic subgroup of $K \simeq SL_3(q)$. As $q^3(q-1)^2$ divides $|X_b|$, X_b contains a Borel subgroup of K. We distinguish the two cases: (1.1.1) $X_b \leqslant K$, (1.1.2) $X_b \nleqslant K$.

(1.1.1) If $X_b = B_0$ a Borel subgroup of K, then we immediately have $[X_b]_S \lneqq [X_a]_S$. Assume therefore that X_b is a maximal parabolic subgroup of K. Then since any involution of $K_+ \setminus K$ acts like the inverse-transpose map on K [16, step 5 in the proof of Proposition 2.2], we have that X_b and X_a are conjugate in K_+, and so $[X_a]_S = [X_b]_S$. But then the cyclic subgroup $\simeq q+1$ of X_b lies in a conjugate of P_a, and so $[P_a]_S \wedge [P_b]_S$ cannot be the class of Borel subgroups of S, contradiction.

(1.1.2) $X_b \nleqslant K$. Then $X_b = B_0 : 2$ (otherwise, arguing as before, we reach the contradiction $[X_a]_S < [X_b]_S$). In particular any involution of $X_b \setminus B_0$ acts as the inversion on a copy of $T_+ \simeq (q-1)^2$. But inside the parabolic subgroup $P_b \simeq [q^5] : GL_2(q)$ there does not exist any involution which inverts T_+, and this completes the proof.

(1.2) Let q be even, $S = G_2(2^n)$, $n \geqslant 2$.

The reader is referred to [9]. By Theorems 2.3 and 2.4. in there we see that we may take the same subgroups as before, P_a, P_b and K_+, as representatives for coatoms of $\mathcal{M}(S)^G$. We can repeat similar arguments of the previous case to reach the same conclusion.

(2) We necessarily have $q = 3^{2m+1}$. We refer to [16, Theorem B] for the structure of the maximal G-invariant classes of S.

(2.1) Let $q > 3$ and prove (β).

Among the maximal G-invariant classes in the list of Theorem B in [16], we consider the following so represented:

$$B \simeq \left[q^6\right] : (q-1)^2, \text{ the Borel subgroup,}$$
$$C_S(s_2) \simeq \left(SL_2(q) \circ SL_2(q)\right) \cdot 2, \text{ the involution centralizer,}$$
$$N_S(T_5) \simeq \left(q^2 - q + 1\right) : 6, \text{ the normalizer of the maximal torus } T_5 \simeq q^2 - q + 1.$$

Set $[X]_S := [N_S(T_5)]_S \wedge [B]_S$ and $[Y]_S := [C_S(s_2)]_S \wedge [B]_S$. Since $(q-1, q^2-q+1) = 1$ and B contains a Sylow 3-subgroup of S, we necessarily have $|X| = 3$ or 6; in any case X centralizes an involution and therefore, since S admits a unique class of involution centralizers, we have $[X]_S \leqslant [C_S(s_2)]_S$, and $[X]_S \lneqq [Y]_S$, being, for instance, $|Y|$ divisible by 3^2.

(2.2) Let $q = 3$ and prove (β).

Consider the maximal G-invariant classes represented by the following subgroups: B, $C_S(s_2)$ and $L_2(13)$ [16, Theorem B]. Set

$$[X]_S := [B]_S \wedge \left[C_S(s_2)\right]_S,$$
$$[Y]_S := \left[L_2(13)\right]_S \wedge \left[C_S(s_2)\right]_S,$$
$$[Z]_S := [B]_S \wedge \left[L_2(13)\right]_S.$$

We claim that either $[Z]_S \lneqq [Y]_S$ or $[Z]_S \lneqq [X]_S$. Note that B, $C_S(s_2)$ and $L_2(13)$ have orders respectively $2^2 \cdot 3^6$, $2^6 \cdot 3^2$ and $2^2 \cdot 3 \cdot 7 \cdot 13$. Since B contains a Sylow 3-subgroup of S and $C_S(s_2)$ a Sylow 2-subgroup, we necessarily have $|X| = 2^2 \cdot 3^2$. Similarly we have $2^2 \mid |Y|$. Moreover, any 3-Sylow subgroup of $L_2(13)$ centralizes an involution and therefore lies in a suitable conjugate of $C_S(s_2)$, thus $3 \mid |Y|$, and it follows that $|Y| = 2^2 \cdot 3$ (Y being the normalizer of a maximal torus $\simeq 6$ in $L_2(13)$). The order of Z may be either 3 or $2 \cdot 3$ or $2^2 \cdot 3$, in any case Z normalizes its 3-Sylow subgroup, thus Z lies in a conjugate of Y. If $|Z|$ were 3 or 6, then $[Z]_S \lneqq [Y]_S$. Otherwise $|Z| = 2^2 \cdot 3$, and $[Z]_S = [Y]_S \lneqq [C_S(s_2)]_S$, forcing $[Z]_S \lneqq [X]_S$.

$F_4(q)$.

If G does not induce on S a non-trivial graph automorphism, then the maximal parabolic subgroups P_i, $i = 1, 2, 3, 4$, represent four distinct coatoms of $\mathcal{M}(S)^G$, and Lemma 5 shows that condition (α) is satisfied.

Assume therefore that G induces a non-trivial graph automorphism on S. In particular q is even. We refer the reader to [18] in which subgroups of maximal rank in finite exceptional Lie type groups are classified.

(1) Let $q \geqslant 8$, and prove (α).

By Table 5.2 in [18], the normalizer N of the maximal torus $H \simeq (q-1)^4$ is a novelty of $F_4(q)$, thus $[N]_S$ is a coatom of $\mathcal{M}(S)^G$. Note that $N \simeq (q-1)^4.W(F_4)$ has order $2^7 \cdot 3^6 \cdot (q-1)^4$. Another element of $\mathcal{M}(S)^G$ is of course the class of Borel subgroups B, whose order is $2^{24} \cdot (q-1)^4$. Since S contains a unique class $[H]$ of maximal tori and since B contains a Sylow 2-subgroup of S, then $|X| = 2^7 \cdot (q-1)^4$, where X is a subgroup representing a maximal intersection of type $(B|N)$. Moreover the structures of B and N imply that $X \simeq [2^7] \times (q-1)^4$, but this is a contradiction, H being self-centralizing in S.

(2) Let $q = 4$, and prove (β).

By [18, Table 5.2], the metacyclic subgroup $M \simeq 241.12$ represents a coatom of $\mathcal{M}(S)^G$. Since any semisimple element of S lies in a maximal torus and since there is a unique maximal torus of type $H \simeq 3^4$ [18], we have that the subgroup of order 12 of M lies in some Borel subgroup. In particular, if for $i=0,1$ we set $[P_{i,4-i}]_S$ to be the coatom of $\mathcal{M}(S)^G$ that in $\mathcal{M}(S)$ is covered only by the coatoms $[P_i]_S$ and $[P_{4-i}]_S$, we have that

$$[12]_S = [M]_S \wedge [P_{1,4}]_S \lneqq [P_{1,4}]_S \wedge [P_{2,3}]_S = [B]_S.$$

(3) Let $q=2$, and prove (α).

The Atlas [10] shows that S admits the following two maximal subgroups whose classes are $\mathrm{Aut}(S)$-invariant:

$$X := {}^2F_4(2) \quad \text{and} \quad Y := L_4(3):2.$$

We have that $|X| = 2^{12} \cdot 3^3 \cdot 5^2 \cdot 13$ and $|Y| = 2^8 \cdot 3^6 \cdot 5 \cdot 13$. In particular X contains a 5-Sylow and a 13-Sylow subgroup, and Y contains a 3-Sylow and a 13-Sylow subgroup of S. Set $[Z]_S := [X]_S \wedge [Y]_S$. A counting argument based on Sylow's theorems yields that $|Z| = 2^a \cdot 3^3 \cdot 5 \cdot 13$, for some $a \leqslant 8$. But this is in contradiction to the fact that $L_4(3)$ has no proper subgroups of orders divisible by 65 [10, p. 68].

Twisted Lie type groups

${}^2A_l(q)$.

We treat separately the different cases:

(1) $l \geqslant 5$, (2) $l=4$, (3) $l=3$, (4) $l=2$.

(1) Let $l \geqslant 5$.

Condition (α) follows immediately from the remark after Lemma 5 and the fact that the Weyl group W^1 is either $W(B_l)$ or $W(C_l)$, with $l \geqslant 3$ [8, 13.3].

(2) Let $l=4$.

We prove condition (α). As maximal S-classes that are $\mathrm{Aut}(S)$-invariant we take the classes associated to the stabilizers Q_1 and Q_2 respectively of a non-singular point and of a non-degenerate line. By [13, Theorem 4.3], Q_1 and Q_2 are maximal subgroups of S. By [17, Table 3.5.B], their classes are $\mathrm{Aut}(S)$-invariant, and therefore coatoms of $\mathcal{M}(S)^G$, for every $S \leqslant G \leqslant \mathrm{Aut}(S)$. Note that the elements of Q_i, up to a suitable basis, have projective matrix form in two diagonal blocks each of unitary type and whose degrees are respectively i and $5-i$. Now if the n.s. point stabilized by Q_1 lies in the n.d. line stabilized by Q_2, we have that $Q_1 \cap Q_2$ consists of unitary matrices having a block-diagonal shape with blocks of degrees: $1,1,3$. Now if we take $g \in S$ such that Q_1^g is the stabilizer of a point lying in the 3-dimensional n.d. subspace stabilized by Q_2, then $Q_1^g \cap Q_2$ consists of block-diagonal unitary matrices with blocks of degrees: $2,1,2$. Note that both of $Q_1 \cap Q_2$ and $Q_1^g \cap Q_2$ are maximal intersections of type $(Q_1|Q_2)$, and, since none of the two is contained in a conjugate of the other, we have that there does not exist a unique meet between $[Q_1]_S$ and $[Q_2]_S$.

(3) Let $l=3$.

(3.1) Let $q \neq 2$ and $q \neq 3$.

We prove condition (β). As pairwise non-conjugate maximal subgroups we take the two non-conjugate parabolics, P_1 and P_2, and the stabilizer of a decomposition in t.s. 2-dimensional subspaces, which we call N_2. This last subgroup, whenever q is not 2 or 3, is a maximal subgroup of S [13, Theorem 4.6]. Moreover, the three classes, $[P_1]_S$, $[P_2]_S$ and $[N_2]_S$, are $\mathrm{Aut}(S)$-invariant [17, Table 3.5.B]. Let P_1 and N_2 be such that $[P_1 \cap N_2]_S = [P_1]_S \wedge [N_2]_S$, and assume that $P_1 := \mathrm{Stab}_S(\langle e_1 \rangle)$ and $N_2 := \mathrm{Stab}_S(U_1 \oplus U_2)$, with e_1 isotropic vector and U_i t.s. subspace of dimension 2 ($i=1,2$). Now if $e_1 \in U_i$, for some i, then of course we have that $P_1 \cap N_2$ is contained in $\mathrm{Stab}_S(U_1)$ which is a conjugate to P_2. Therefore we have $[P_1]_S \wedge [N_2]_S \lneqq [P_1]_S \wedge [P_2]_S$, since N_2 does not contain any Borel subgroup of S. Thus assume that $e_1 = u_1 + u_2$ for some $u_i \in U_i \setminus \{0\}$, $i=1,2$. Now the elements of $P_1 \cap N_2$ either stabilize both the 1-dimensional subspaces $\langle u_1 \rangle$ and $\langle u_2 \rangle$, or interchange them. In any case $P_1 \cap N_2$ stabilizes the line $W := \langle u_1, u_2 \rangle$. Since e_1, u_1 and u_2 are isotropic vectors we have that

$0 = (e_1, e_1) = 2(u_1, u_2)$. Therefore if p is odd W is a t.s. line, thus we have that, up to conjugation, $P_1 \cap N_2 \leqslant P_2$ and so condition (β). Assume therefore that $p = 2$ and also that W is non-degenerate. Note that the stabilizer in S of a n.d. line W is never a maximal subgroup of S, being contained in the stabilizer of the decomposition $W \perp W^{\perp}$. We call $[D_2]_S$ this class of maximal subgroups, and we refer the reader to [13, Theorem 4.7] and to [17, Table 3.5.B] for the properties of maximality and Aut(S)-invariance of this class. In this situation we have that $[P_1]_S \wedge [N_2]_S \leqslant [P_1]_S \wedge [D_2]_S$. Finally we note that this inclusion is strict. Let $P_1 = \mathrm{Stab}_S(\langle e_1\rangle)$ and $D_2 := \mathrm{Stab}_S(\langle e_1, f_1\rangle \perp \langle e_2, f_2\rangle)$ (being $\langle e_1, e_2, f_2, f_1\rangle$ a unitary basis for the underlying space), then $P_1 \cap D_2$ acts on $\langle e_2, f_2\rangle$ in a unitary way, and any maximal intersection of type $(P_1|D_2)$ must contain a copy of the unitary group $U_2(q)$. But N_2 does not (see [17, Proposition 4.2.4]).

(3.2) Let $q = 3$.

We prove condition (β). A look at the Atlas [10] shows that, with the same notation as the previous case, the classes $[P_1]_S$, $[P_2]_S$ and $[D_2]_S$ are maximal and Aut(S)-invariant. We claim that $[P_2]_S \wedge [D_2]_S \lneqq [P_1]_S \wedge [P_2]_S$. This follows from the fact that $D_2 \simeq 2(A_4 \times A_4).4$ has order $2^7 \cdot 3^2$, and it contains a Sylow 2-subgroup of S. Since also P_2 contains a Sylow 3-subgroup, if we set $[X]_S := [P_2]_S \wedge [D_2]_S$, then $|X| = 2^3 \cdot 3^2$. Moreover, the 2-Sylow of X being, up to conjugation, also a 2-Sylow of P_2, lies in the diagonal subgroup H, thus in particular in a Borel subgroup B, and so $[X]_S \lneqq [B]_S = [P_1]_S \wedge [P_2]_S$.

(3.3) Let $q = 2$.

We prove condition (β). From [10] we know that S admits five distinct conjugacy classes of maximal subgroups and any of these is also Aut(S)-invariant. We consider the ones represented by the following subgroups:

$$Q_1 \simeq 3^{1+2}_+ : 2A_4,$$
$$P_1 \simeq 2^{\cdot}(A_4 \times A_4).2,$$
$$P_2 \simeq 2^4 : A_5.$$

Q_1 is the stabilizer of a point in S, viewed as the symplectic group $PSp_4(3)$, P_1 and P_2 are the stabilizers respectively of a singular point and an isotropic line in S viewed as $PSU_4(2)$. Note that $|Q_1| = 3^4 \cdot 2^3$, $|P_1| = 2^6 \cdot 3^2$ and $|P_2| = 2^6 \cdot 3 \cdot 5$. In particular these latter both contain a Sylow 2-subgroup of S. Set $[B]_S := [P_1]_S \wedge [P_2]_S$, the class of Borel subgroups of S, so that $|B| = 2^6 \cdot 3$. In particular, up to conjugation, P_1 contains a Sylow 3-subgroup of P_2. An argument that makes use of Sylow's Theorem easily shows that, if we set $[Y_1]_S := [Q_1]_S \wedge [P_1]_S$ and $[Y_2]_S := [Q_1]_S \wedge [P_2]_S$, then $|Y_1| = 2^3 \cdot 3^2$ and $|Y_2| = 2^3 \cdot 3$. But then we have $[Y_2]_S \lneqq [Y_1]_S$, since this latter class contains both a Sylow 2- and a Sylow 3-subgroup of the former.

(4) Let $l = 2$.

(4.1) Let $q \neq 3$ and $q \neq 5$, and prove condition (β).

The conjugacy classes $[P_1]_S$ and $[Q_1]_S$, associated respectively to the stabilizers of an isotropic point and of a non-isotropic one, are two distinct coatoms of $\mathcal{M}(S)^G$ (the maximality of these subgroups can be checked in [13, Theorem 4.3] and in the [10] for the case $U_3(4)$, the Aut(S)-invariance in [17, Table 3.5.B]). The maximal torus H, contained in some P_1, stabilizes also a non-isotropic point. Therefore H lies in a conjugate of Q_1. If we set $[X]_S := [P_1]_S \wedge [Q_1]_S$ and assume $X = P_1 \cap Q_1$, we have that $H \leqslant X$. Now the order of Q_1 is divisible by q and since P_1 contains a Sylow p-subgroup of S, thus q divides the order of any maximal intersection of type $(P_1|Q_1)$. In particular we have that X contains a (conjugate of) $q : H$.

Let now C be a maximal subgroup in the Aschbacher's class $\mathcal{C}_3$, induced by the field extension $\mathbb{F}_q < \mathbb{F}_{q^3}$. The conjugacy class $[C]_S$ is always Aut(S)-invariant [17, Table 3.5.B]. Moreover C is a maximal subgroup of S, except in the cases $U_3(3)$ and $U_3(5)$ (see [13, Theorems 2.6 and 2.7]). We will examine these cases later, thus now $[C]_S$ is a third coatom of $\mathcal{M}(S)^G$. Note that

$$C \simeq \frac{q^2 - q + 1}{\mu} : 3,$$

where $\mu = (q+1, 3)$ [17, Proposition 4.3.6]. Also,

$$|P_1| = q^3(q+1)(q-1)/\mu,$$
$$|Q_1| = q(q+1)^2(q-1)/\mu.$$

Since $q^2 - q + 1$ is coprime with q, we have that if $p \neq 3$, $|C|$ is coprime with q, and therefore $[P_1]_S \wedge [C]_S \leqslant [H]_S$, forcing

$$[P_1]_S \wedge [C]_S \lneqq [Q_1]_S \wedge [P_1]_S.$$

Assume therefore $p = 3$. Since 3 is the only possible prime dividing both $r^2 - r + 1$ and $(r+1)^2(r-1)$ (whenever r is a prime power), we have that $|C|$ is coprime with $(q+1)^2(q-1)$. In particular any maximal intersection of type $(Q_1|C)$ is a 3-group of order at most q. But then

$$[Q_1]_S \wedge [C]_S \lneqq [Q_1]_S \wedge [P_1]_S.$$

(4.2) Let $q = 5$, and prove (α).

The reader is referred to [10] for the structure and the fusion of the conjugacy classes of maximal subgroups of S. The only maximal subgroups of S that are not novelties are P_1 and Q_1, respectively the stabilizer of an isotropic point and of a non-isotropic one. We have that

$$P_1 \simeq 5_+^{1+2} : 8 \quad \text{and} \quad Q_1 \simeq 2S_5.$$

In particular, as $|S| = 2^4 \cdot 3^2 \cdot 5^3 \cdot 7$, P_1 is a Borel subgroup of S and Q_1 contains a 2-Sylow of S. Set $[X]_S := [P_1]_S \wedge [Q_1]_S$, an argument based on Sylow's Theorem shows that $X \simeq 5:8$. This in particular forces that the normalizer in $2S_5 (\simeq Q_1)$ of a 5-Sylow contains a cyclic group of order 8, and this is a contradiction since $N_{2S_5}(5) \simeq 5 : (4 \times 2)$, $2S_5$ being a subgroup of $GL_2(5)$.

(4.3) Let $q = 3$, and prove (α).

We refer the reader to [10] for the structure and the fusion of the conjugacy classes of maximal subgroups of S. Every conjugacy class of maximal subgroups is invariant in $\mathrm{Aut}(S)$. We consider the following

$$P_1 \simeq 3_+^{1+2} : 8, \qquad Q_1 \simeq 4^{\cdot} S_4, \qquad D \simeq 4^2 : S_3.$$

Since both Q_1 and D contain a Sylow 2-subgroup of S as a subgroup of index 3, if $\mathcal{M}(S)^G$ is a lattice, then

$$[Q_1]_S \wedge [D]_S = \{\text{2-Sylow subgroups}\}. \tag{7}$$

Now as P_1 contains a Sylow 3-subgroup of S, if $[X_1]_S := [P_1]_S \wedge [Q_1]_S$ and $[X_2]_S := [P_1]_S \wedge [D]_S$, we have $X_1 \simeq X_2 \simeq 3:8$, and $[X_1]_S \neq [X_2]_S$, since by (7) the Sylow 3-subgroups of Q_1 and D lie in different classes. But [10] shows that S contains unique classes of cyclic subgroups isomorphic respectively to 4, 8 and 12. We conclude that there does not exist a unique $[X_1]_S \wedge [X_2]_S$, since otherwise this class would contain both $[8]_S$ and $[12]_S$, and thus we should have $[X_1]_S = [X_2]_S$.

${}^2B_2(2^{2m+1})$.

For the Suzuki groups we refer the reader to [21, Theorem 9] where these groups are denoted by $G(q)$, $q = 2^{2m+1}$. Condition (α) is easily verified by taking the maximal classes $[B]_S$, of Borel subgroups, $B = UH$, and $[N]_S$, $N := N_S(H) \simeq D_{2(q-1)}$.

$^2D_l(q)$.

As $^2D_2(q) \simeq A_1(q^2)$ and $^2D_3(q) \simeq {}^2A_3(q)$, we assume $l \geqslant 4$. This case follows immediately by the remark after Lemma 5 and the fact that the Weyl groups W^1 are isomorphic to $W(B_{l-1})$.

$^2E_6(q)$.

Condition (α) follows by the remark after Lemma 5 and the fact that the Weyl groups of $^2E_6$ are isomorphic to $W^1 \simeq W(F_4)$.

$^2F_4(2^{2m+1})$ and $^2F_4(2)'$.

The Ree groups $^2F_4(2^{2m+1})$ are simple whenever $m \geqslant 1$. Assume first that $m \geqslant 1$ and treat the case of the Tits group $^2F_4(2)'$ later.

Let $m \geqslant 1$ and prove (β).

For the structure of the maximal subgroups we refer to [19], Main Theorem (but note that here we use q where q^2 is used in [19]). The three coatoms of $\mathcal{M}(S)^G$ that we consider are respectively represented by the following subgroups:

$$P_a = \left[q^{11}\right] : \left(L_2(q) \times (q-1)\right),$$
$$P_b = \left[q^{10}\right] : \left({}^2B_2(q) \times (q-1)\right),$$
$$N_S(T_8) = \left((q+1) \times (q+1)\right) : GL_2(3).$$

P_a and P_b are maximal parabolic subgroups of S, T_8 denotes the, unique up to conjugation, maximal torus isomorphic to $(q+1)^2$. We have $[P_a]_S \wedge [P_b]_S = [B]_S$, the class of Borel subgroups. Note that $B \simeq [q^{12}] : (q-1)^2$, $|P_b| = q^{12}(q-1)^2(q^2+1)$ and $|N_S(T_8)| = 2^4 \cdot 3 \cdot (q+1)^2$. Since $q = 2^{2m+1} \equiv -1 \pmod 3$, 3 divides $q+1$ but not $|P_b|$; it follows that 2 is the only prime dividing both $|P_b|$ and $|N_S(T_8)|$. As P_b contains a Sylow 2-subgroup of S, $[P_b]_S \wedge [N_S(T_8)]_S = [2^4]_S$, forcing

$$[P_b]_S \wedge \left[N_S(T_8)\right]_S \lneqq [P_b]_S \wedge [P_a]_S.$$

Let $S = {}^2F_4(2)'$, and prove (β).

We refer the reader to [10] or [22].

The group S admits a unique class of involutions, the centralizer, $C_S(2)$, of one of these is a maximal subgroup of S isomorphic to $2.[2^8] : 5 : 4$, and of course its conjugacy class is $\mathrm{Aut}(S)$-invariant. Another maximal class, which is also $\mathrm{Aut}(S)$-invariant, is represented by the normalizer of a four-group $N_S(2^2) \simeq 2^2.[2^8] : S_3$. A simple question of orders yields $[C_S(2)]_S \wedge [N_S(2^2)]_S = [2^{11}]$ the class of Sylow 2-subgroups. Moreover, S admits two distinct classes of maximal subgroups isomorphic to $L_3(3) : 2$, which are fused in $\mathrm{Aut}(S)$. Now as $\mathrm{Out}(S) = 2$, we let $[A]_S$ be the class represented either by a copy of $L_3(3) : 2$ if $G = S$, or by the subgroup $13 : 6$ if $G > S$. Then $[A]_S$ is a coatom of $\mathcal{M}(S)^G$, and we have that $[2]_S = [A]_S \wedge [C_S(2)]_S \lneqq [C_S(2)]_S \wedge [N_S(2^2)]_S$.

$^2G_2(3^{2m+1})$, $m \geqslant 1$.

For the structure of maximal subgroups we refer the reader to [16, Theorem C].

We prove condition (β).

Using the same notation as [16], we consider the following three coatoms of $\mathcal{M}(S)^G$, represented respectively by

$$P \simeq \left[q^5\right] : (q-1), \text{ the Borel subgroup},$$
$$N_S(i) \simeq 2 \times L_2(q), \text{ an involution centralizer},$$
$$N_S(\langle i, j\rangle) \simeq \left(2^2 \times D_{(q+1)/2}\right) : 3, \text{ a four-group normalizer}.$$

Also recall that S admits a unique class of involutions and a unique class of four-groups. Set $[X]_S := [N_S(\langle i, j\rangle)]_S \wedge [P]_S$ and $[Y]_S := [C_S(i)]_S \wedge [P]_S$. Let $2s = (q+1)/2$ and note that s is odd. Moreover $(q(q-1), s) = 1$ and, since P and $N_S(\langle i, j\rangle)$ do contain respectively a 3-Sylow and a 2-Sylow of S, we necessarily have $|X| = 6$. As X is contained in a conjugate of $N_S(\langle i, j\rangle)$, X is a cyclic group. Thus X centralizes an involution, and so $[X]_S \leqslant [Y]_S$. Finally note that $q > 3$ and $q||Y|$, yields $[X]_S \lneqq [Y]_S$.

$^3D_4(q)$.

For the structure of the maximal subgroups of S we refer the reader to [15], whose notation here we adopt. We prove condition (β).

(1) Let q be odd.

We consider the three pairwise distinct coatoms of $\mathcal{M}(S)^G$, represented by

$$\begin{aligned} N_S(T_5) &\simeq \left(q^4 - q^2 + 1\right) : 4, \\ C_S(g_1) &\simeq G_2(q), \\ C_S(s_2) &\simeq \left(SL_2\left(q^3\right) \circ SL_2(q)\right) \cdot 2, \end{aligned}$$

where T_5 denotes a maximal torus of type $q^4 - q^2 + 1$, g_1 an element of order 3 in $\mathrm{Aut}(S) \setminus S$, and s_2 is an involution of S. Both $C_S(g_1)$ and $C_S(s_2)$ contain a Sylow 2-subgroup of S. Moreover, under the assumption q odd, in S there is a unique class of involutions [15, Lemma 2.3]. Set $[X]_S := [N_S(T_5)]_S \wedge [C_S(g_1)]_S$ and $[Y]_S := [C_S(s_2)]_S \wedge [C_S(g_1)]_S$. As $|G_2(q)| = q^6(q^6-1)(q^2-1)$, we have that $q^4 - q^2 + 1$ is coprime with $|C_S(g_1)|$. Therefore, since a Sylow 2-subgroup of S contained in $C_S(g_1)$, we have $|X| = 4$. Since also $C_S(s_2)$ contains a Sylow 2-subgroup of S, Y contains a Sylow 2-subgroup, and as its order is bigger than 4, we have that $[X]_S \lneqq [Y]_S$.

(2) Let q be even.

We argue as before using the maximal parabolic subgroups P_a and P_b in place of $C_S(g_1)$ and $C_S(s_2)$. Note that

$$\begin{aligned} P_a &\simeq \left[q^9\right] : \left(SL_2\left(q^3\right) \circ (q-1)\right), \\ P_b &\simeq \left[q^{11}\right] : \left(\left(q^3 - 1\right) \circ SL_2(q)\right), \end{aligned}$$

in particular they have orders respectively $q^{12}(q^6-1)(q-1)$ and $q^{12}(q^3-1)(q^2-1)$. Now $[P_a]_S \wedge [P_b]_S$ is the class of Borel subgroups, which strictly contains $[4]_S = [P_a]_S \wedge [N_S(T_5)]_S$.

Sporadic groups Table 1 summarizes the proof of the lemma in the case in which S is one of the 26 sporadic groups. The first column of the table denotes the sporadic group S, the second column the subgroups, representing pairwise distinct coatoms of $\mathcal{M}(S)^G$ that we have chosen to prove the conditions (α) and (β). In all the cases, except the one of the Held group $S = He$, each conjugacy class is $\mathrm{Aut}(S)$-invariant, so we do not have to worry about the overgroup G. The table is to be understood in this way. When there are just two subgroups, A and B, the meaning is that there does not exist a unique meet between the classes $[A]_S$ and $[B]_S$, if otherwise there are three subgroups, A, B and C, this means that $[A]_S \wedge [B]_S \lneqq [B]_S \wedge [C]_S$ (whenever these meets have a meaning). The basic reference is [10].

We proceed by showing that Table 1 holds. Assuming that the classes $[D]_S := [A]_S \wedge [B]_S$, $[E]_S := [B]_S \wedge [C]_S$ and eventually $[D]_S \wedge [E]_S$ are always well defined, we either reach a contradiction or show that $[D]_S \lneqq [E]_S$. Without loss of generality, we implicitly assume that $D = A \cap B$ and $E = B \cap C$. We examine separately the various cases.

$S = M_{11}$.

$|S| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$, $|A| = 2^4 \cdot 3^2 \cdot 5$, $|B| = 2^4 \cdot 3$.

Therefore $|D| = 2^4 \cdot 3$ and so $D = B$, contradiction.

Table 1
Sporadic groups.

S	Representative subgroups	S	Representative subgroups
M_{11}	$A = M_{10}$ $B = M_8 : S_3$	Fi_{22}	$A = 2^{5+8} : (S_3 \times A_6)$ $B = (2 \times 2_+^{1+8} : U_4(2)) : 2$
M_{12}	$A = 4^2 : D_{12}$ $B = L_2(11)$ $C = 2 \times S_5$	Fi_{23}	$A = L_2(17)$ $B = 2^{11 \cdot} M_{23}$ $C = 2^{2 \cdot} U_6(2).2$
M_{22}	$A = 2^4 : A_6$ $B = 2^4 : S_5$	Fi'_{24}	$A = 29 : 14$ $B = 3^3.[3^{10}].GL_3(3)$ $C = 2^{6+8}.(S_3 \times A_8)$
M_{23}	$A = L_3(4) : 2$ $B = 2^4 : A_7$	HN	$A = 5_+^{1+4} : 2_-^{1+4}.5.4$ $B = 3^4 : 2(A_4 \times A_4).4$ $C = 2^3.2^2.2^6.(3 \times L_3(2))$
M_{24}	$A = 2^6 : (L_3(2) \times S_3)$ $B = L_2(7)$	Th	$A = 31 : 15$ $B = 3^2.[3^7].2S_4$ $C = 3^5 : 2S_6$
J_2	$A = L_3(2) : 2$ $B = 2^{1+4} : A_5$ $C = 2^{2+4} : (3 \times S_3)$	B	$A = 47 : 23$ $B = Th$ $C = 2_+^{1+22} : Co_2$
Suz	$A = 3^{2+4} : 2(A_4 \times 2^2).2$ $B = L_2(25)$ $C = 2_-^{1+6 \cdot} U_4(2)$	M	$A = 41 : 40$ $B = 13^{1+2} : (3 \times 4S_4)$ $C = 2^5.2^{10}.2^{20}.(S_3 \times L_5(2))$
HS	$A = 2^4.S_6$ $B = 4^3 : L_3(2)$ $C = 4^\cdot 2^4 : S_5$	J_1	$A = 2^3 : 7 : 3$ $B = 7 : 6$
M^cL	$A = U_4(3)$ $B = U_3(5)$	$O'N$	$A = 3^4 : 2_-^{1+4} D_{10}$ $B = 4^{3 \cdot} L_3(2)$ $C = 4^\cdot L_3(4) : 2$
Co_3	$A = HS$ $B = M^cL : 2$	J_3	$A = 3^2.(3 \times 3^2) : 8$ $B = 2_-^{1+4} : A_5$ $C = 2^{2+4} : (3 \times S_3)$
Co_2	$A = U_6(2)$ $B = 2^{10} : M_{22} : 2$	Ly	$A = 37 : 18$ $B = 67 : 22$ $C = 2^\cdot A_{11}$
Co_1	$A = Co_2$ $B = 2^{11} : M_{24}$	Ru	$A = L_2(13) : 2$ $B = 5_+^{1+2} : S_5$ $C = 2^\cdot 2^{4+6} : S_5$
He	$A = 2_+^{1+6}.L_3(2)$ $B = 5^2 : 4.A_4$ $C_1 = 2^6 : 3^\cdot S_6$ $C_2 = 2^{4+4}.(S_3 \times S_3)$	J_4	$A = 37 : 12$ $B = 43 : 14$ $C = 29 : 28$

$S = M_{12}$.
$|S| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$, $|A| = 2^6 \cdot 3$, $|B| = 2^2 \cdot 3 \cdot 5 \cdot 11$, $|C| = 2^4 \cdot 3 \cdot 5$.
Thus $|D| = 2^2 \cdot 3^a$, with $a = 0, 1$. Moreover, both B and C contains an element of the class $3B$, therefore $|E| = 2^2 \cdot 3 \cdot 5$, since $L_2(11)$ has no subgroups of index 22 or 44. It follows $[D]_S \lneq [E]_S$.

$S = M_{22}$.
$|S| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, $|A| = 2^7 \cdot 3^2 \cdot 5$, $|B| = 2^7 \cdot 3 \cdot 5$.
Therefore $|D| = 2^7 \cdot 3 \cdot 5$, forcing $[D]_S = [B]_S$, contradiction.

$S = M_{23}$.
$|S| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, $|A| = 2^7 \cdot 3^2 \cdot 5 \cdot 7$, $|B| = 2^7 \cdot 3^2 \cdot 5 \cdot 7$.
Thus $[D]_S = [A]_S = [B]_S$, contradiction.

$S = M_{24}$.
$|S| = 2^{10} \cdot 3^3 \cdot 5 \cdot 11 \cdot 23$, $|A| = 2^{10} \cdot 3^2 \cdot 7$, $|B| = 2^3 \cdot 3 \cdot 7$.
Thus D is a subgroup of index $\leqslant 3$ in $L_2(7)$, contradiction.

$S = J_2$.
$|S| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 11$, $|A| = 2^4 \cdot 3 \cdot 7$, $|B| = 2^7 \cdot 3 \cdot 5$, $|C| = 2^7 \cdot 3^2$.
Moreover $C = N_S(2A^2)$ and $B = N_S(2A)$. In particular a subgroup of order three of C centralizes an element of the class $2A$, forcing $|E| = 2^7 \cdot 3$, and $[D]_S \lneqq [E]_S$.

$S = Suz$.
$|S| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, $|A| = 2^7 \cdot 3^7$, $|B| = 2^3 \cdot 3 \cdot 5^2 \cdot 13$, $|C| = 2^{13} \cdot 3^4 \cdot 5$.
Since $L_2(25)$ has no subgroups of index 195, then $|E| = 2^3 \cdot 3 \cdot 5$. As $|D| = 2^a \cdot 3$, for $a \leqslant 3$, we have $[D]_S \lneqq [E]_S$.

$S = HS$.
$|S| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$, $|A| = 2^8 \cdot 3^2 \cdot 5$, $|B| = 2^9 \cdot 3 \cdot 7$, $|C| = 2^9 \cdot 3 \cdot 5$.
Moreover S admits a unique class of elements of order three, therefore $|E| = 2^9 \cdot 3$, $|D| = 2^8 \cdot 3$ and $[D]_S \lneqq [E]_S$.

$S = M^cL$.
$|S| = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$, $|A| = 2^7 \cdot 3^6 \cdot 5 \cdot 7$, $|B| = 2^4 \cdot 3^2 \cdot 5^3 \cdot 7$.
We have $|D| = 2^4 \cdot 3^2 \cdot 5 \cdot 7$, but this is in contradiction to the fact that $U_3(5)$ has no subgroup of index 25 (see [10, p. 34]).

$S = Co_3$.
$|S| = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$, $|A| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$, $|B| = 2^8 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$.
We have that the order of D is divisible by $2^7 \cdot 5^3 \cdot 7 \cdot 11$ and divides $2^8 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$. Thus the index of $D \cap M^cL$ in M^cL is in particular coprime with 5 and not divisible by 8. But M^cL has no such subgroup (see [10, p. 100]).

$S = Co_2$.
$|S| = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$, $|A| = 2^{16} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$, $|B| = 2^{18} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$.
We have that the order of D is divisible by $2^{16} \cdot 3^2 \cdot 7 \cdot 11$ and divides $2^{16} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$. In particular the index of $D \cap U_6(2)$ in $U_6(2)$ is odd and coprime with 11. But $U_6(2)$ has not such a subgroup (see [10, p. 115]).

$S = Co_1$.
$|S| = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$, $|A| = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$, $|B| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$.
The order of D is divisible by $2^{18} \cdot 11 \cdot 23$ and divides $2^{18} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. In particular its index in Co_2 is coprime with $2 \cdot 11 \cdot 23$. But Co_2 has not such a subgroup (see [10, p. 154]).

$S = He$.
$|S| = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$ and $\mathrm{Out}(S) = 2$.
If G does not induce an outer automorphism, then choose the following three representative subgroups:

$$A = 5^2 : 4.A_4, \qquad B = 2_+^{1+6}.L_3(2), \qquad C_1 = 2^6 : 3^{\cdot}S_6.$$

We have that $|A| = 2^4 \cdot 3 \cdot 5^2$, $|B| = 2^{10} \cdot 3 \cdot 7$ and $|C_1| = 2^{10} \cdot 3^3 \cdot 5$. Thus $|B \cap C_1| = 2^{10} \cdot 3$ and $|A \cap B| \in \{2^4, 2^4 \cdot 3\}$, so, in any case, $[A \cap B]_S \lneqq [B \cap C_1]_S$.

If G does induce a non-trivial outer automorphism, then take the same A and B but, instead of C_1, choose as representative of a maximal element of $\mathcal{M}(S)^G$ the subgroup $C_2 = 2^{4+4}.(S_3 \times S_3)$. Since $|C_2| = 2^{10} \cdot 3^2$, we have two possibilities, either $|B \cap C_2| = 2^{10} \cdot 3$, or $|B \cap C_2| = 2^{10}$. In the first case we conclude immediately that $[A \cap B] \lneqq [B \cap C_2]$. In the second case, if $|A \cap B| = 2^4$, then $[A \cap B] \lneqq [B \cap C_2]$, otherwise $|A \cap B| = 2^4 \cdot 3$, but then $|A \cap C_2| = 2^4$ (as in this situation the 3-Sylow of A is the same of that of B and thus cannot be in C_2), forcing $[A \cap C_2] \lneqq [B \cap C_2]$.

$S = Fi_{22}$.
$|S| = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, $|A| = 2^{17} \cdot 3^3 \cdot 5$, $|B| = 2^{17} \cdot 3^4 \cdot 5$.
Since S admits a unique conjugacy class of subgroups of order 5, $|D| = 2^{17} \cdot 3^a \cdot 5$, with $a \leqslant 2$. Moreover, as $U_4(2)$ has no subgroups of index 3^2 [10, p. 26] and A_6 has no subgroups of index 3 or 9, we have $a = 0$, i.e. $|D| = 2^{17} \cdot 5$, but this is in contradiction to fact that A_6 has no subgroups of index 9.

$S = Fi_{23}$.
$|S| = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$, $|A| = 2^5 \cdot 3^2 \cdot 17$, $|B| = 2^{18} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, $|C| = 2^{18} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$.
Since $L_2(17)$ has no subgroups of index 17 or 51, $|D| = 2^5$, forcing $[D]_S \lneqq [E]_S$.

$S = Fi'_{24}$.
$|S| = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$, $|A| = 2 \cdot 7 \cdot 29$, $|B| = 2^5 \cdot 3^{16} \cdot 13$, $|C| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7$.
Trivially, $[2]_S = [D]_S \lneqq [E]_S$.

$S = HN$.
$|S| = 2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$, $|A| = 2^7 \cdot 5^6$, $|B| = 2^7 \cdot 3^6$, $|C| = 2^{14} \cdot 3^2 \cdot 7$.
D is a 2-group and thus contained in a conjugate of E.

$S = Th$.
$|S| = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$, $|A| = 3 \cdot 5 \cdot 31$, $|B| = 2^4 \cdot 3^{10}$, $|C| = 2^5 \cdot 3^7 \cdot 5$.
We have $|D| = 3$, therefore $[D]_S \lneqq [E]_S$.

$S = B$.
$|S| = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$, $|A| = 23 \cdot 47$, $|B| = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$, $|C| = 2^{41} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$.
Trivial since $D = 1$.

$S = M$.
$|S| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$, $|A| = 2^3 \cdot 5 \cdot 41$, $|B| = 2^5 \cdot 3^2 \cdot 13^3$, $|C| = 2^{46} \cdot 3^3 \cdot 5 \cdot 7 \cdot 31$.
D is strictly contained in a 2-Sylow of B which lies completely in C, thus $[D]_S \lneqq [E]_S$.

$S = J_1$.
$|S| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, $|A| = 2^3 \cdot 3 \cdot 7$, $|B| = 2 \cdot 3 \cdot 7$.
We should have $|D| = |B|$, and so the contradiction $[D]_S = [B]_S$.

$S = O'N$.
$|S| = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$, $|A| = 2^5 \cdot 3^4 \cdot 5$, $|B| = 2^9 \cdot 3 \cdot 7$, $|C| = 2^9 \cdot 3^2 \cdot 5 \cdot 7$.
We have $|D| = 2^5 \cdot 3$ and, since S admits a unique conjugacy class of elements of order 3, E contains both a 2-Sylow and a 3-Sylow of B. Thus $[D]_S \lneqq [E]_S$.

$S = J_3$.
$|S| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$, $|A| = 2^3 \cdot 3^5$, $|B| = 2^7 \cdot 3 \cdot 5$, $|C| = 2^7 \cdot 3^2$.
We have $|D| = 2^3 \cdot 3$. Moreover, S contains a unique class of involutions, $2A$, $B = N_S(2A)$, $C = N_S(2A^2)$ is the normalizer of a Klein 4-group and the Sylow 3-subgroups of C are elementary abelian of order 9. In particular a subgroup of order 3 of C centralizes $2A$, forcing $|E| = 2^7 \cdot 3$, and $[D]_S \lneqq [E]_S$.

$S = Ly$.
$|S| = 2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$, $|A| = 2 \cdot 3^2 \cdot 37$, $|B| = 2 \cdot 11 \cdot 67$, $|C| = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$.
There exists a unique class of involutions in S, and $[2]_S = [D]_S \lneqq [E]_S$.

$S = Ru$.
$|S| = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$, $|A| = 2^3 \cdot 3 \cdot 7 \cdot 13$, $|B| = 2^5 \cdot 5^3$, $|C| = 2^{14} \cdot 3 \cdot 5$.
D is a 2-subgroup strictly contained in a 2-Sylow of B, thus $[D]_S \lneqq [E]_S$.

$S = J_4$.
$|S| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$, $|A| = 2^2 \cdot 3 \cdot 37$, $|B| = 2 \cdot 7 \cdot 43$, $|C| = 2^2 \cdot 7 \cdot 29$.
Since S admits a unique conjugacy class of cyclic subgroups of order 14, we have $[D]_{\leqslant}[2]_S \lneqq [14]_S = [E]_S$. $\square$

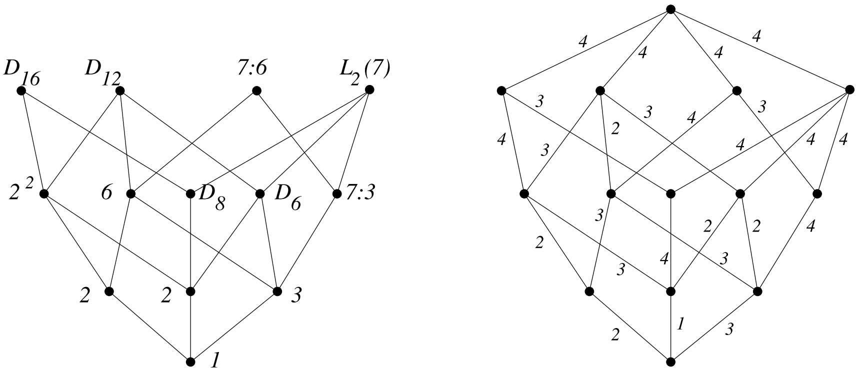


Fig. 3. $\mathcal{M}(PGL_2(7))$ and one of its shellings.

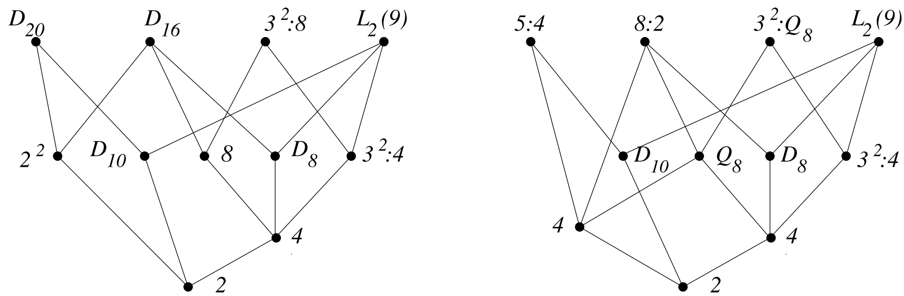


Fig. 4. $\mathcal{M}(PGL_2(9))$ and $\mathcal{M}(M_{10})$.

Theorem 5. *A finite group G is solvable if and only if $\mathcal{M}(G) \cup \{[G]\}$ is a modular lattice.*

Proof. Theorem 3 shows that if G is solvable $\mathcal{M}(G)$ is a modular semilattice. We prove now the opposite implication. Assume that G is a minimal counterexample. Let K be a minimal normal subgroup of G. If we denote with $[X]$ the meet of all coatoms in $\mathcal{C}(G)$ that contain $[K]$, then $\mathcal{M}(G/K)$ is isomorphic to the interval $[[X], [G]]$ in $\mathcal{M}(G) \cup \{[G]\}$. In particular G/K satisfies the assumptions of the theorem, and therefore, by the minimality of G, G/K is solvable. Moreover, K is the unique minimal normal subgroup of G. If H were another one, then G, being embedded into $G/K \times G/H$, would be solvable, which is not the case. K is the direct product of some copies of isomorphic non-abelian simple groups S_i. By Lemma 4, $\mathcal{M}(K)^G$ is a lattice satisfying the property (max). By Lemma 2 the poset $\mathcal{M}(S_1)^{N_G(S_1)}$ is a lattice that satisfies the property (max). But this is a contradiction with Lemma 6. □

Remark 4. In the following are drawn the poset $\mathcal{M}(G)$ for the simple groups $A_1(7)$, $A_1(9)$ and $A_1(11)$, and for some of their extensions, namely: $PGL_2(7)$, $PGL_2(9)$ and M_{10}. See Figs. 3 and 4. In the figures the brackets representing the conjugacy classes have been voluntarily omitted. Note that $\mathcal{M}(PGL_2(9))$ and $\mathcal{M}(M_{10})$ are lattices, and $\mathcal{M}(PGL_2(7))$ is a pure shellable lattice (we explicitly exhibit a shelling).

References

[1] M. Aschbacher, On the maximal subgroup of the finite classical groups, Invent. Math. 76 (1984) 469–515.
[2] M. Aschbacher, Chevalley groups of type G_2 as the group of a trilinear form, J. Algebra 109 (1987) 193–259.
[3] A. Björner, Shellable and Cohen–Macaulay partially ordered sets, Trans. Amer. Math. Soc. 260 (1980) 159–183.
[4] A. Björner, M. Wachs, On lexicographic shellable posets, Trans. Amer. Math. Soc. 277 (1983) 323–341.
[5] A. Björner, M. Wachs, Shellable nonpure complexes and posets, I, Trans. Amer. Math. Soc. 348 (1996) 1299–1327.

[6] A. Björner, M. Wachs, Shellable nonpure complexes and posets, II, Trans. Amer. Math. Soc. 349 (10) (1997) 3945–3975.
[7] R.A. Bryce, T.O. Hawkes, Lattices in the frame of a finite soluble group, preprint.
[8] R.W. Carter, Simple Groups of Lie Type, John Wiley & Sons, London, 1972.
[9] B.N. Cooperstein, Maximal subgroups of $G_2(2^n)$, J. Algebra 70 (1981) 23–36.
[10] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, An ATLAS of Finite Groups, Oxford Univ. Press, 1985.
[11] J.D. Dixon, B. Mortimer, Permutation Groups, Springer-Verlag, 1991.
[12] K. Doerk, T. Hawkes, Finite Soluble Groups, de Gruyter Exp. Math., vol. 4, Walter de Gruyter, Berlin, 1992.
[13] O.H. King, The subgroup structure of finite classical groups in terms of geometric configurations, in: Surveys in Combinatorics, in: London Math. Soc. Lecture Note Ser., vol. 327, Cambridge Univ. Press, 2005, pp. 29–56.
[14] P. Kleidman, The maximal subgroups of the finite 8-dimensional orthogonal groups $P\Omega_8^+(q)$ and of their automorphism groups, J. Algebra 110 (1987) 182–199.
[15] P. Kleidman, The maximal subgroups of the Steinberg triality groups ${}^3D_4(q)$ and of their automorphism groups, J. Algebra 115 (1) (1988) 297–325.
[16] P. Kleidman, The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, the Ree groups ${}^2G_2(q)$, and their automorphism groups, J. Algebra 117 (1988) 30–71.
[17] P. Kleidman, M. Liebeck, The Subgroup Structure of the Finite Classical Groups, Cambridge Univ. Press, London, 1990.
[18] M. Liebeck, J. Saxl, G.M. Seitz, Subgroups of maximal rank in finite exceptional groups of Lie type, J. Algebra 115 (1988) 182–199.
[19] G. Malle, The maximal subgroups of ${}^2F_4(q^2)$, J. Algebra 139 (1991) 52–69.
[20] J. Shareshian, On the shellability of the order complex of the subgroup lattice of a finite group, Trans. Amer. Math. Soc. 353 (7) (2001) 2689–2703.
[21] M. Suzuki, On a class of doubly transitive groups, Ann. of Math. 75 (1) (1962) 105–145.
[22] K.B. Tchakerian, The maximal subgroups of the Tits simple group, C. R. Acad. Bulgare Sci. 34 (1981) 1637.
[23] J. Thévenaz, P.J. Webb, Homotopy equivalence of posets with a group action, J. Combin. Theory Ser. A 56 (1991) 173–181.
[24] V. Welker, Equivariant homotopy of posets and some applications to subgroup lattices, J. Combin. Theory Ser. A 69 (1995) 61–86.