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On the homotopy type of the Quillen complex of finite soluble groups

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Abstract

We prove that the homotopy type of the Quillen complex of a finite soluble group at the prime $p \neq 2$ is that of a wedge of spheres of possible different dimensions.

Keywords: p-Subgroup complex; Homotopy type of posets; Group theory; Homotopy colimits

Introduction

For a poset (= partially ordered set) P the symbol $\Delta(P)$ is used to denote the *order complex* of P; this is the simplicial complex whose k-dimensional simplices are the non-empty chains $x_0 < x_1 < \cdots < x_k$ of P. For a finite group G and a prime number p dividing its order, the *Quillen complex* of G at p is defined as the order complex $\Delta(\mathcal{A}_p(G))$, where $\mathcal{A}_p(G)$ denotes the poset of all non-trivial elementary abelian p-subgroups of G, ordered by set inclusion. Interest in Quillen complexes for finite groups began with the influential paper [8] of D. Quillen. A great amount of attention has received above all his famous conjecture [8, 2.9], which states that $\Delta(\mathcal{A}_p(G))$ is a contractible complex if and only if G has a non-trivial normal p-subgroup (see [3] and the references therein).

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In this paper we treat the, perhaps more general, problem of determining the homotopy type of a Quillen complex. This problem is addressed by J. Pulkus and V. Welker in [7]. Here we make use of the same topological techniques, referring to the theory of diagrams of spaces and homotopy colimits. We examine in greater detail the homotopy type of the "upper intervals" of $\mathcal{A}_p(G)$, which was the hardest obstruction in [7]. By this analysis, we are able to give a positive answer to a question raised by J. Thévenaz (as mentioned in [7]), and we prove, in Theorem 21, that the homotopy type of the Quillen complex of a finite soluble group and for odd prime numbers p, is that of a wedge of spheres of possible different dimensions. According to [9, Corollary 4.17], insoluble groups may have Quillen complexes whose homotopy type is not that of a wedge of spheres.

Notation. Our basic references are [1] for group theory, and [6,10] for topology. The notation of the paper follows these books. In particular, we set some basic facts. For a poset $(P, \geqslant)$ and an element $r \in P$ we denote by $P_{\geqslant r}$ the poset $\{q \in P \mid q \geqslant r\}$. Similarly defined are the posets $P_{>r}$, $P_{\leqslant r}$, $P_{<r}$. A map $f : P \to Q$ between posets is said to be order preserving if $f(x) \leqslant f(y)$ whenever $x \leqslant y$ in P. We reserve the symbol $\approx$ for poset isomorphisms, $\simeq$ for homotopy equivalence between topological spaces, and $\cong$ for group isomorphisms. The topological spaces we are dealing with are simplicial complexes, thus in particular CW-complexes; the basic facts of their theory are assumed as granted. For two simplicial complexes Δ_1, Δ_2, we define the *join*, $\Delta_1 * \Delta_2$, as in [10]; in particular, $\Delta * \emptyset = \emptyset * \Delta = \Delta$ and $\Delta_1 * \Delta_2$ is contractible if and only if at least one of the two complexes is. Similarly we define the *wedge*, $\Delta_1 \vee \Delta_2$, of simplicial complexes (and in general of topological spaces). Note that this is unambiguously defined if and only if the spaces involved are path-connected, otherwise we have to specify the points to which the two complexes are wedged. This will be crucial in our wedge decomposition formulas (Lemma 3 and its applications) where the wedge of the spaces is not formed using a single point, instead for each space in the wedge we have to declare a precise point to where it is wedged to. With S^k is denoted, as usual, the k-dimensional sphere, assuming S^{-1} to be the empty set and S^0 the set constituted by two disjoint points. The space constituted by a unique single point is simply called *one-single point*. In order to simplify formulations in this paper, an empty wedge of spheres is to be considered a one-single point. The suspension of a space Δ is denoted by $\mathbf{S}(\Delta)$ and defined as $S^0 * \Delta$. The symbol $\sqcup$ indicates disjoint unions of objects.

1. Topological tools

In this section we expose the topological methods we use for determining the homotopy type of the Quillen complex. We follow the same track as J. Pulkus and V. Welker in [7], which consists in making use of some techniques of the so-called theory of diagrams of spaces and homotopy colimits. Our basic reference for this theory is the work [12].

Let $(P, \leqslant)$ be a poset. A $(P, \leqslant)$-*diagram of topological spaces* is a functor $\mathcal{D}$ from $(P, \leqslant)$ to the category of topological spaces. Fixing the notation, if $\mathcal{D}$ is a diagram of spaces on $(P, \leqslant)$, the space associated to the element $r \in P$ is denoted by D_r and the morphism corresponding to $q \leqslant r$ is denoted by d_{rq} (thus $d_{rq} : D_r \to D_q$ and if $p \leqslant q \leqslant r$, $d_{qp} \circ d_{rq} = d_{rp}$ and, for every $r \in P$, d_{rr} is the identity map on D_r).

The basic example is the following. Suppose that $\mathcal{U} := \{X_i\}_{i \in I}$ is a covering of a topological space X by a finite number of subspaces X_i. The *intersection poset* $P^{\mathcal{U}}$ is, by definition, the partially ordered set of all the intersections $X_J := \bigcap_{i \in J} X_i$ for $J \subseteq I$, with reversed inclusion as order relation. The *natural diagram of spaces* $\mathcal{D}^{\mathcal{U}}$ associated to the poset $P^{\mathcal{U}}$ is defined as follows: for every $r = X_J \in P^{\mathcal{U}}$ the space D_r is just the subspace X_J, and for $p \leqslant r$ the morphism d_{rp} is the inclusion map.

Let $\mathcal{D}$ be an arbitrary diagram of spaces over a finite poset P. Then to $\mathcal{D}$ there is associated a topological space called the *homotopy colimit* of $\mathcal{D}$ and denoted by $\operatorname{hocolim} \mathcal{D}$ (for the explicit definition of $\operatorname{hocolim} \mathcal{D}$ we refer to [12, 1.3]). Given a finite covering $\mathcal{U}$ of a space X the homotopy direct limit of the diagram $\mathcal{D}^{\mathcal{U}}$ and the space X are homotopy equivalent. This is in fact the content of the following lemma, which for convenience we state for simplicial complexes. The interested reader may find the proofs of this and the next two lemmas in the appendix of [12].

Lemma 1 (Projection lemma [12, 1.6]). *Let $\mathcal{U}$ be a covering of the complex X by a finite number of subcomplexes and let $\mathcal{D}^{\mathcal{U}}$ be the natural diagram of spaces associated. Then* $\operatorname{hocolim} \mathcal{D}^{\mathcal{U}} \simeq X$.

Sometimes we need to modify the maps d_{rq} and the spaces D_r of a diagram $\mathcal{D}$ in a way that the homotopy type of $\operatorname{hocolim} \mathcal{D}$ remains unchanged. This can be done, with some accuracy, by the use of the following lemma.

Lemma 2 (Homotopy lemma [12, 1.7]). *Let $\mathcal{D} := (D_r, d_{rq})$ and $\mathcal{D}' := (D'_r, d'_{rq})$ be two diagrams of spaces on the same poset P. Assume that for each $r \in P$ there is a map $f_r : D_r \to D'_r$ such that f_r induces a homotopy equivalence between D_r and D'_r and that for $q \leqslant r$ in P, $d'_{rq} \circ f_r = f_q \circ d_{rq}$. Then $\operatorname{hocolim} \mathcal{D} \simeq \operatorname{hocolim} \mathcal{D}'$.*

Finally, if strong assumptions are guaranteed, the homotopy type of $\operatorname{hocolim} \mathcal{D}$ can be explicitly computed by the use of the following lemma.

Lemma 3 (Wedge lemma [12, 1.8]). *Let $\mathcal{D} := (D_r, d_{rq})$ be a diagram of spaces over some poset P, endowed with a unique maximal element $\hat{1}$, such that for every $q \in P$, $q \neq \hat{1}$, there is a point $c_q \in D_q$ such that $d_{rq}(x) = c_q$ for all $r > q$ and $x \in D_r$. Then $\operatorname{hocolim} \mathcal{D}$ is homotopy equivalent to*

$$\big(D_{\hat{1}} * \Delta(P_{<\hat{1}})\big) \vee \bigvee_{r \in P_{<\hat{1}}} \big(D_r * \Delta(P_{<r})\big), \tag{1}$$

where the wedge is formed by identifying for every $r < \hat{1}$ the point $c_r \in D_r * \Delta(P_{<r})$ with $r \in D_{\hat{1}} * \Delta(P_{<\hat{1}})$.

By these three steps, passing through the concept of homotopy colimit, the homotopy type of the given space can be recognized via a formula, whose entries consist, roughly speaking, in the covering subspaces and in the way they intersect each other.

A famous result, the so-called "Nerve Theorem," whose first version seems to date back to 1945 (see [5]), can be seen as a special consequence of the lemmas above. This treats in fact the situation in which the arrangement of subspaces gives rise to contractible or empty intersections.

Theorem 4 (Nerve Theorem). *Let Δ be a simplicial complex and let $\mathcal{F}$ be a finite family of subcomplexes which covers Δ. Suppose that every non-empty intersection of elements of $\mathcal{F}$ is contractible. Then Δ and the intersection poset of the family $\mathcal{F}$ are homotopy equivalent.*

From the three lemmas aforementioned we extrapolate the following corollary. (This is essentially a corollary to [4, Theorem 2.5]; another version of it is [7, 2.4].)

Corollary 5. *Let $f : P \to Q$ be an order preserving map between the two finite posets P and Q. Assume that*

(i) *Q is a meet semi-lattice with a unique least element $\hat{0}$;*
(ii) *for every $q \in Q_{>\hat{0}}$, $\Delta(f^{-1}(Q_{\leqslant q})) \supsetneq \Delta(f^{-1}\{\hat{0}\})$;*
(iii) *for every $q \in Q_{>\hat{0}}$, the complex $\Delta(f^{-1}(Q_{\leqslant q}))$ is either contractible or a wedge of n_q-dimensional spheres, with $0 \leqslant n_q < n_r$ if $q < r$ in Q.*

Then the order complex $\Delta(P)$ is homotopy equivalent to the wedge

$$\big(\Delta\big(f^{-1}\{\hat{0}\}\big) * \Delta(Q_{>\hat{0}})\big) \vee \bigvee_{q \in Q_{>\hat{0}}} \big(\Delta\big(f^{-1}(Q_{\leqslant q})\big) * \Delta(Q_{>q})\big), \tag{2}$$

where for $q \in Q_{>\hat{0}}$ a fixed point $c_q \in \Delta(f^{-1}(Q_{\leqslant q}) \subseteq \Delta(f^{-1}(Q_{\leqslant q})) * \Delta(Q_{>q})$ is identified with $q \in \Delta(f^{-1}\{\hat{0}\}) * \Delta(Q_{>\hat{0}})$.

Lemma 6 (Fiber lemma [8, 1.6]). *Let $f : P \to Q$ be an order preserving map amongst the finite posets P and Q. Suppose that all the upper (lower) fibers $f^{-1}(Q_{\geqslant x})(f^{-1}(Q_{\leqslant x}))$, at $x \in Q$, are contractible as topological spaces. Then f induces a homotopy equivalence between $\Delta(P)$ and $\Delta(Q)$.*

Let P be a poset and x, y be two of its elements. If there exists a least upper bound (greatest lower bound) of x and y, this is denoted by $x \vee y$ (respectively by $x \wedge y$). An element x of P is said *conjunctive* (*subjunctive*) if for all $y \in P$ there exists $x \vee y \in P$ (respectively there exists $x \wedge y \in P$). The following is a useful criterion to prove that a poset is contractible.

Lemma 7 [8, 1.5]. *Suppose that x is a conjunctive (subjunctive) element of the poset P. Then the order complex of P is contractible.*

We set some basic terminology we use throughout. We say that a complex Δ is *spherical* if it has the same homotopy type of a wedge of spheres of an appropriate dimension (if this is, say, n the complex will be called n-spherical). For simplicial complexes of finite dimension $n \geqslant 0$, the property of being n-spherical is equivalent to being $(n-1)$-connected (in the sense of [8]). The complex Δ is called a *wedge of spheres* if it has the same homotopy type as a collection of a finite number of spheres, of possible different dimensions, all joined together to a unique common point. We admit an empty wedge of spheres to be a one-point space.

Lemma 8 [10, I, 6.6]. $S^n * S^m \simeq S^{n+m+1}$.

In the next lemma we state some properties of the operations of join, wedge, and disjoint union between topological spaces.

Lemma 9. *Let A, X, and Y be simplicial complexes. The following holds*:

(i) $(X * Y) \simeq (Y * X)$;
(ii) $A * (X * Y) \simeq (A * X) * Y$;
(iii) $A * (X \vee Y) \simeq (A * X) \vee (A * Y)$;
(iv) $A * (X \sqcup Y) \simeq (A * X) \vee (A * Y) \vee (A * S^0)$;
(v) $\mathbf{S}(X \sqcup Y) \simeq \mathbf{S}(X) \vee \mathbf{S}(Y) \vee S^1$.

Proof. For the commutative and the associative property of the join operation the reader may consult [6, VIII, 62]. The distributive law between join and wedge and (iv) may be proved by standard topology, as well as via the aforementioned techniques of diagrams of spaces. (v) is a special case of (iv) with $A = S^0$ and with the considerations of Lemma 8 ($S^0 * S^0 \simeq S^1$). □

The previous two results yield the following important fact:

Proposition 10. *The class of wedges of spheres is closed under the operations of wedge, join, and suspension.*

In our forthcoming analysis of the Quillen complex of finite groups, we will make use of following topological result. The proof proposed here is suggested by Sandro Buoncristiano (private communication).

Proposition 11. *Let X and Y be two non-empty CW-complexes, with X simply connected. Assume $X \vee Y$ is homotopy equivalent to a wedge of spheres. Then X too is homotopy equivalent to a wedge of spheres.*

Proof. Let S be the wedge of spheres and $f : X \vee Y \to S$ be the continuous map realizing the homotopy equivalence. We let the spheres be numbered, and we denote with S^n_j the jth n-dimensional sphere of S; we write

$$S = \bigvee_n \bigvee_{j=1}^{t_n} S_j^n$$

meaning that S consists of the one point wedge of exactly t_0 0-spheres, t_1 1-spheres, etc. The map f induces isomorphisms between the integral coefficients homology groups; namely for every natural integer n, we call

$$f_n : H_n(X \vee Y) \longrightarrow H_n(S)$$

the group isomorphism induced by f between the nth homology groups of $X \vee Y$ and S. The reduced homology of a wedge of spaces is the direct sum of the reduced homologies of the spaces involved. In particular, for all $n > 0$,

$$H_n(X \vee Y) = H_n(X) \oplus H_n(Y)$$

and, since f_n is a group isomorphism,

$$H_n(S) = f_n\big(H_n(X)\big) \oplus f_n\big(H_n(Y)\big).$$

Moreover, from $S = \bigvee_n \bigvee_{j=1}^{t_n} S_j^n$ and the homology of the sphere, we have that

$$H_n(S) = \bigoplus_{j=1,\dots,t_n} H_n\big(S_j^n\big) \cong \mathbb{Z}^{t_n}. \tag{3}$$

By Krüll–Schmidt theorem, we can find a group automorphism g_n of $H_n(S)$ such that the subgroup $g_n(f_n(H_n(X)))$ is the one generated by the first k_n generators of $H_n(S)$ in formula (3) (we assume $f_n(H_n(X)) \cong \mathbb{Z}^{k_n}$, for some $0 \leqslant k_n \leqslant t_n$). Repeating this argument for every dimension of the spheres of S, we obtain a set of isomorphisms $(g_n)_n$ with the same property for every n. There exists a continuous map $g : S \to S$ that induces this set of isomorphisms. (Such a map g can be constructed "piece by piece" as an application of the fact that for any h-uple of integers $(l_1, l_2, \dots, l_h)$ and for any dimension $m \geqslant 1$, there is a continuous map from S^m to the wedge $\bigvee_{i=1}^h S_i^m$ of the wished degree l_i on S_i^m, for all $i = 1, \dots, h$; see [10, IV, 8].) Let U be the topological subspace of S defined by

$$U = \bigvee_n \bigvee_{i=1}^{k_n} S_i^n$$

and $\pi : S \to U$ be the projection of S onto U. Then the map

$$\pi \circ g \circ f_{|X} : X \longrightarrow U$$

is continuous and such that it induces an isomorphism in homology. In fact, by construction, for every n we have

$$H_n(X) \cong f_n\big(H_n(X)\big) \cong g_n\big(f_n\big(H_n(X)\big)\big) \cong H_n(U).$$

Moreover, since X is simply connected by assumption, $0 = H_1(X) = H_1(U)$, thus in the wedge decomposition of U there are no circles, which means that U is simply connected too. We can apply a version of Whitehead theorem for CW-complexes [10, VII, Section 6, Theorem 25], and deduce that X and U have the same homotopy type, proving that X is homotopy equivalent to a wedge of spheres. □

2. Applications of the topological tools to the poset $\mathcal{A}_p(G)$

The contents of this section are essentially the same as those in [7, Section 3]; we report them here for reader's convenience.

We recall that for any finite group G and any prime number p, $\mathcal{A}_p(G)$ denotes the poset of all non-trivial elementary abelian p-subgroups of G ordered by inclusion. With $\Omega_1(G)$ we indicate the subgroup of G generated by all the elements of order p. If N is a normal p'-subgroup of G, we use the "bar" notation to denote the quotient subgroups, namely we write $\overline{H}$ for the image HN/N of a subgroup H of G under the epimorphism $\pi : G \to G/N$. Note that if A is an elementary abelian p-subgroup of G, then its image $\overline{A}$ is isomorphic to A. Therefore π induces a map f from $\mathcal{A}_p(G)$ to $\mathcal{A}_p(\overline{G})$ sending any A to $\overline{A}$. We review some simple, but crucial, facts about the poset $\mathcal{A}_p(G)$ and the map f.

Lemma 12. *Let N be a normal p'-subgroup of a finite group G. Then*:

(i) $\mathcal{A}_p(G) \cup \{\hat{0}\}$, *with the unique minimal element $\hat{0}$, is a meet-semilattice.*
(ii) *If A is an elementary abelian p-subgroup, $\mathcal{A}_p(G)_{>A} = \mathcal{A}_p(C_G(A))_{>A}$.*
(iii) *If A is not contained in $\Omega_1(Z(G))$, then the order complex of $\mathcal{A}_p(G)_{>A}$ is contractible.*
(iv) *The map $f : \mathcal{A}_p(G) \to \mathcal{A}_p(\overline{G})$, $A \mapsto \overline{A}$ is surjective and order preserving.*
(v) *If $\overline{A} \in \mathcal{A}_p(\overline{G})$, the lower fiber of f is equal to $f^{-1}(\mathcal{A}_p(\overline{G})_{\leqslant \overline{A}}) = \mathcal{A}_p(NA)$.*

Proof. Except for (iii), all these statements are part of [7, Lemma 3.1]. For (iii), note that if $A \nleqslant \Omega_1(Z(G))$, then $A\Omega_1(Z(G))$ is an elementary abelian p-subgroup strictly containing A, thus an element of $\mathcal{A}_p(G)_{>A}$. Moreover, for any element B of $\mathcal{A}_p(G)_{>A}$,

$$B \cdot A\Omega_1\big(Z(G)\big) = B\Omega_1\big(Z(G)\big)$$

is still an element of $\mathcal{A}_p(G)_{>A}$. This shows that $A\Omega_1(Z(G))$ is a conjunctive element of $\mathcal{A}_p(G)_{>A}$. By Lemma 7, $\Delta(\mathcal{A}_p(G)_{>A})$ is contractible. □

In order to apply Corollary 5 to the map f between the posets $P := \mathcal{A}_p(G)$ and $Q := \mathcal{A}_p(\overline{G}) \cup \{\hat{0}\}$, we need the following result which is a reformulation of a theorem of Quillen [8, 11.2]. We warn the reader that the hypothesis of solubility in this lemma is fundamental. Whether the same holds for insoluble groups is still an open problem (see [8, 2.3] and the weaker conjecture in [2]).

Lemma 13 [8, 11.2]. *Let p be any prime number and $G = NA$ be a finite p-nilpotent soluble group, with elementary abelian Sylow p-subgroup A. Then the Quillen complex $\Delta(\mathcal{A}_p(G))$ is either contractible (if $C_A(N) \neq 1$) or Cohen–Macaulay (and in particular spherical) of dimension* $\mathrm{rk}(A) - 1$.

We now prove the following homotopy type decomposition formula for the Quillen complex of a group admitting a soluble normal p'-subgroup.

Lemma 14 [7, Theorem 1.1]. *Let G be a finite group and N a soluble normal p'-subgroup of G. Then $\Delta(\mathcal{A}_p(G))$ is homotopy equivalent to the wedge*

$$\Delta\bigl(\mathcal{A}_p(\overline{G})\bigr) \vee \bigvee_{\overline{A} \in \mathcal{A}_p(\overline{G})} \bigl(\Delta\bigl(\mathcal{A}_p(NA)\bigr) * \Delta\bigl(\mathcal{A}_p(\overline{G})_{>\overline{A}}\bigr)\bigr), \tag{4}$$

where for each $\overline{A} \in \mathcal{A}_p(\overline{G})$ an arbitrary chosen point $c_{\overline{A}} \in \Delta(\mathcal{A}_p(NA))$ is identified with $\overline{A} \in \Delta(\mathcal{A}_p(\overline{G}))$.

Proof. For the proof apply Corollary 5 to the mapping $f : \mathcal{A}_p(G) \to \mathcal{A}_p(\overline{G}) \cup \{\hat{0}\}$. By Lemma 12(i) and (iv) and Lemma 13 all the assumptions of Corollary 5 are fulfilled, hence

$$\Delta\bigl(\mathcal{A}_p(G)\bigr) \simeq \bigvee_{\overline{A} \in \mathcal{A}_p(\overline{G}) \cup \{\hat{0}\}} \bigl(\Delta\bigl(f^{-1}\bigl(\mathcal{A}_p(\overline{G})_{\leqslant \overline{A}}\bigr)\bigr) * \Delta\bigl(\mathcal{A}_p(\overline{G})_{>\overline{A}}\bigr)\bigr).$$

Lemma 12(v) and our convention $\Delta * \emptyset = \Delta$ show the claimed formula. $\square$

Formula (4) can be sometimes reduced and simplified (for this see [7, Remark 3.4]); nevertheless its meaning is directly connected with the homotopy type of the order complex of an upper interval $\mathcal{A}_p(G)_{>A}$. This will be analyzed in the next section.

3. The homotopy type of the order complexes of the upper intervals

In this section we abuse the notation by denoting the order complex of a poset with the same letter of the poset itself. This is done in order to make more readable our formulas. In the sequel p will always denote an *odd* prime number (unless differently specified).

Let A be an elementary abelian p-subgroup of the finite group G. In order to analyze the homotopy type of $\mathbf{S}(\mathcal{A}_p(G)_{>A})$, by virtue of Lemma 12(ii), we can assume A central in G.

The next lemma treats the case in which G is a p-group.

Lemma 15. *Let P be a finite p-group of exponent p with derived subgroup P' cyclic of order p. Then $\mathcal{A}_p(P)_{>Z(P)}$ is* $(\mathrm{rk}(P) - \mathrm{rk}(Z(P)) - 1)$*-spherical.*

Proof. Note that $P' = \Phi(P) \leqslant Z(P)$. If $P' = Z(P)$, the group P is extraspecial and the statement is the content of [8, Example 10.4]. Assume that $P < Z(P)$. Let E/P' be a complement of $Z(P)/P'$ in the elementary abelian group P/P'. Then P is the central product $P = Z(P) \circ E$. Note that E is an extraspecial p-group, and that the mapping

$$\mathcal{A}_p(E)_{>Z(E)} \longrightarrow \mathcal{A}_p(P)_{>Z(P)}, \quad A \longmapsto Z(P)A$$

realizes an isomorphism between the posets $\mathcal{A}_p(E)_{>Z(E)}$ and $\mathcal{A}_p(P)_{>Z(P)}$. Thus, by the result for the extraspecial case, $\mathcal{A}_p(P)_{>Z(P)}$ is $(\mathrm{rk}(E)-2)$-spherical. Finally, $\mathrm{rk}(P) = \mathrm{rk}(E) + \mathrm{rk}(Z(P)) - 1$, and this completes the proof. $\square$

For arbitrary groups G and for any $A \in \mathcal{A}_p(G) \cup \{1\}$ we set

$$\mathcal{M}_A(G) := \left\{X \in \mathcal{S}_p(G) \;\middle|\; A < X,\ X = \Omega_1(X),\ \Phi(X) \leqslant A \leqslant Z(X)\right\}$$

(where $\mathcal{S}_p(G)$ denotes the set of all non-trivial p-subgroups of G).

Any subgroup X belonging to $\mathcal{M}_A(G)$ has nilpotency class at most 2. Since p is odd and X is generated by elements of order p, X has exponent p [1, 23.11]. In particular, this fact implies that $\mathcal{M}_A(G)$ is an order ideal, in the sense that if $A < X \leqslant Y$ and $Y \in \mathcal{M}_A(G)$, then X too lies in $\mathcal{M}_A(G)$ (this property fails if $p = 2$ as Remark 18 shows).

Observe that $\mathcal{M}_{\{1\}}(G) = \mathcal{A}_p(G)$.

Generally we write simply $\mathcal{M}_A$ for $\mathcal{M}_A(G)$.

Lemma 16. *Let P be a finite p-group and A a central elementary abelian subgroup of P. If $X_1, X_2, \ldots, X_n$ are $n > 1$ maximal elements of $\mathcal{M}_A(P)$, then $Z(X_1 \cap X_2 \cap \cdots \cap X_n)$ strictly contains A.*

Proof. Assume P is a counterexample of minimal order; let $X_1, X_2, \ldots, X_n$ be $n > 1$ maximal elements of $\mathcal{M}_A(P)$, let Y be their intersection, and assume $Z(Y) = A$. We may suppose that P is generated by the union of the subgroups X_i, for $i = 1, 2, \ldots, n$. Set Z_2/A the center of P/A. Note that, as X_i/A is abelian for every i, Y is contained in Z_2. Since every group in $\mathcal{M}_A$ has exponent p, the same holds for Y. Therefore $Y \leqslant \Omega_1(Z_2)$. Conversely, note that for every i, the subgroup $\Omega_1(Z_2)X_i$ lies in $\mathcal{M}_A$. The maximality of the X_i implies that $\Omega_1(Z_2) \leqslant X_i$, and since this holds for all i, $Y = \Omega_1(Z_2)$. Since $[X_i, Y] \leqslant X_i' \leqslant A$ and P is generated by the subgroups X_i, $[P, Y] \leqslant A$. Therefore, as A is central in P, $[P, Y, P] = 1$. By the 3-subgroups lemma [1, 8.7], it follows $[P', Y] = 1$. But $P' = \Phi(P)$, and so $\Phi(P) \cap Y \leqslant Z(Y) = A$. Now set $S := (X_1)_P$ the normal core of X_1 in P (i.e., the largest normal subgroup of P contained in X_1), and $R := (S \cap \Phi(P))A$. Assume that R strictly contains A. Being R a normal subgroup of P, the group $B/A := R/A \cap Z(P/A)$ is not trivial. Note that B is normal in P and, as $B \leqslant S \leqslant X_1$, it is of exponent p. Moreover, it is easy to check that for every $i = 1, \ldots, n$, BX_i lies in $\mathcal{M}_A$. The maximality of the X_i implies that $B \leqslant X_i$, for every $i = 1, \ldots, n$. Therefore $B \leqslant Y \cap \Phi(P)A = A$. Thus $R = A$, which means $S \cap \Phi(P) \leqslant A$. Then S/A is central in P/A. Moreover, as S is of exponent p, $S \leqslant \Omega_1(Z_2) = Y$, and so $S = Y$. Being $n > 1$, it is $Y \neq X_1$; thus, by the definition of normal core, Y coincides with the intersection

$X_1 \cap X_1^{g_1} \cap \cdots \cap X_s^{g_s}$, for some elements $g_1, \ldots, g_s$ of P ($s \geqslant 1$). Obviously, the subgroups $X_1^{g_i}$ are maximal elements of $\mathcal{M}_A$. But the subgroup $\langle X_1, X_1^{g_1}, \ldots, X_1^{g_s} \rangle$ is contained in the normal closure of X_1 in P (i.e., the smallest normal subgroup of P containing X_1), which, being in a p-group, is a proper subgroup of P. By the inductive hypothesis it follows that A is strictly contained in $Z(Y)$, that is the desired contradiction.

Corollary 17. *If P is a p-group and A a central elementary abelian subgroup of P, then $\mathcal{M}_A(P)$ is contractible.*

Proof. We show that $\mathcal{M}_A(P)$ conically contracts. This is obvious if it has just a unique maximal element. Otherwise, by the previous lemma, all the maximal elements of $\mathcal{M}_A(P)$ do intersect in a common element which is easily seen to be conjunctive in $\mathcal{M}_A(P)$. Lemma 7 completes the proof. $\square$

Remark 18. If $p = 2$, the previous result fails. For instance, in the dihedral group of order 16 the maximal elements of $\mathcal{M}_{Z(P)}(P)$ do intersect in a subgroup strictly containing $Z(P)$, but this is no more generated by prime order elements, and so it does not lie in $\mathcal{M}_{Z(P)}(P)$. In this situation $\mathcal{M}_{Z(P)}(P)$ is homotopy equivalent to a 0-sphere.

We say that a finite group G satisfies ($\star$) if

> for every $A \in \mathcal{A}_p(G)$ the suspension $\mathbf{S}(\mathcal{A}_p(G)_{>A})$ has the homotopy type of a wedge of spheres.

In Proposition 20, we will prove that all finite soluble groups have this property for odd prime numbers. We need the following lemma.

Lemma 19. *Let G be a finite group such that every proper subgroup of G satisfies ($\star$). Let A be a central elementary abelian p-subgroup of G, and $1 \leqslant X \leqslant Y \leqslant A$. Then the suspension $\mathbf{S}((\mathcal{M}_X)_{>A})$ is homotopy equivalent to a wedge of spheres if and only if this holds for $\mathbf{S}((\mathcal{M}_Y)_{>A})$. In particular, $\mathbf{S}(\mathcal{A}_p(G)_{>A})$ is homotopy equivalent to a wedge of spheres if and only if $\mathbf{S}(\mathcal{M}_A(G))$ is so.*

Proof. Clearly it suffices to prove the lemma for X of index p in Y.

Let $|Y : X| = p$ and choose a supplement R of Y in A, such that $Y \cap R = X$. Since A is a central subgroup of G, any element of $(\mathcal{M}_X)_{>A}$ lies also in $(\mathcal{M}_Y)_{>A}$, thus $(\mathcal{M}_X)_{>A}$ injects into $(\mathcal{M}_Y)_{>A}$. Call i this injection. The lower fiber of an element U of $(\mathcal{M}_Y)_{>A}$ is

$$i^{-1}_{\leqslant U} = \{T \in (\mathcal{M}_X)_{>A} \mid T \leqslant U\} = (\mathcal{M}_X(U))_{>A}.$$

We claim that $(\mathcal{M}_X(U))_{>A}$ is a poset isomorphic to $(\mathcal{A}_p(U/R))_{>A/R}$. In fact, the mapping

$$(\mathcal{M}_X(U))_{>A} \longrightarrow (\mathcal{A}_p(U/R))_{>A/R}, \quad T \longmapsto T/R$$

is trivially order preserving and injective. To prove it is also surjective, note that if T/R is an elementary abelian subgroup of U/R over A/R, then $T = \Omega_1(T)$, as T is contained in U whose exponent is p. Moreover, being $\Phi(U) \leqslant Y$, $\Phi(T) \leqslant Y \cap R = X$, and so T lies in $(\mathcal{M}_X(U))_{>A}$. Thus $(\mathcal{M}_X(U))_{>A} \approx (\mathcal{A}_p(U/R))_{>A/R}$. In particular, since U has exponent p, $(\mathcal{M}_X(U))_{>A}$ is never empty, and it is either contractible, if $Z(U/R) > A/R$ (by Lemma 12(iii)), or it is $(\mathrm{rk}(U/R) - \mathrm{rk}(A/R) - 1)$-spherical (by Lemma 15). We apply Corollary 5 to the mapping $i : (\mathcal{M}_X)_{>A} \to (\mathcal{M}_Y)_{\geqslant A}$. It yields the formula

$$(\mathcal{M}_X)_{>A} \simeq (\mathcal{M}_Y)_{>A} \vee \bigvee_{U \in (\mathcal{M}_Y)_{>A}} \big(\mathcal{A}_p(U/R)_{>A/R} * (\mathcal{M}_Y)_{>U}\big), \tag{5}$$

where the wedge is made by identifying any element U of $(\mathcal{M}_Y)_{>A}$ with a specific point of $\mathcal{A}_p(U/R)_{>A/R} * (\mathcal{M}_Y)_{>U}$.

Applying the suspension operator to formula (5), together with Lemma 9(iii), we obtain

$$\mathbf{S}(\mathcal{M}_X)_{>A} \simeq \mathbf{S}(\mathcal{M}_Y)_{>A} \vee \bigvee_{U \in (\mathcal{M}_Y)_{>A}} \mathbf{S}\big(\mathcal{A}_p(U/R)_{>A/R} * (\mathcal{M}_Y)_{>U}\big). \tag{6}$$

We first prove the implication "from bottom to top," thus we assume that $\mathbf{S}((\mathcal{M}_X)_{>A})$ is homotopy equivalent to a wedge of spheres and we show the same for $\mathbf{S}((\mathcal{M}_Y)_{>A})$.

If $\mathbf{S}((\mathcal{M}_X)_{>A})$ is contractible, by formula (6), it is immediate that $\mathbf{S}(\mathcal{M}_Y)_{>A}$ is so. Assume therefore that $\mathbf{S}((\mathcal{M}_X)_{>A})$ is a non-empty wedge of spheres. If $C_1, C_2, \ldots, C_k$ are the connected components of $(\mathcal{M}_Y)_{>A}$, by Lemma 9(v) iterated, the suspension $\mathbf{S}((\mathcal{M}_Y)_{>A})$ is homotopy equivalent to

$$\mathbf{S}(C_1) \vee \mathbf{S}(C_2) \vee \cdots \vee \mathbf{S}(C_k) \vee \bigvee_{k-1} S^1.$$

In particular, if all the connected components of $(\mathcal{M}_Y)_{>A}$ are contractible, then $\mathbf{S}((\mathcal{M}_Y)_{>A})$ is a wedge of 1-dimensional spheres and we are done. Thus, assume that C is a non-contractible connected component of $(\mathcal{M}_Y)_{>A}$, and let $C \vee D$ be the corresponding connected component of $(\mathcal{M}_X)_{>A}$ given by formula (5). By (6) we obtain that $\mathbf{S}(C \vee D) \vee Z \simeq \mathbf{S}((\mathcal{M}_X)_{>A})$ for some topological space Z. Since $C \vee D$ is connected, $\mathbf{S}(C \vee D)$ is simply connected [10, VIII, 5, Corollary 3]. We can therefore apply Lemma 11, and deduce that $\mathbf{S}(C \vee D)$ is homotopy equivalent to a wedge of spheres. By Lemma 9(iii), $\mathbf{S}(C \vee D) \simeq \mathbf{S}(C) \vee \mathbf{S}(D)$ and, finally, by Lemma 11 again, $\mathbf{S}(C)$ is a wedge of spheres, proving our claim and the first part of the lemma.

To prove the opposite implication, let us assume that $\mathbf{S}(\mathcal{M}_Y)_{>A}$ is a wedge of spheres. By formula (6) it is enough to show that for $U \in (\mathcal{M}_Y)_{>A}$ the spaces

$$\mathbf{S}\big(\mathcal{A}_p(U/R)_{>A/R} * \ (\mathcal{M}_Y)_{>U}\big)$$

are wedges of spheres. Set for simplicity $D_U := \mathcal{A}_p(U/R)_{>A/R} * (\mathcal{M}_Y)_{>U}$. Note that

$$\mathbf{S}(D_U) \simeq \mathcal{A}_p(U/R)_{>A/R} * \mathbf{S}(\mathcal{M}_Y)_{>U}$$

and that $\mathcal{A}_p(U/R)_{>A/R}$ is either contractible or $(\mathrm{rk}(U) - \mathrm{rk}(A) - 1)$-spherical by Lemma 15. We can therefore limit our analysis to the subcomplex $(\mathcal{M}_Y)_{>U}$. Moreover, we restrict to the subgroups U for which $Z(U) = A$ (otherwise $\mathcal{A}_p(U/R)_{>A/R}$ conically contracts by Lemma 12(iii) and so D_U too), and such that $(\mathcal{M}_Y)_{>U}$ is not empty (otherwise D_U reduces to $\mathcal{A}_p(U/R)_{>A/R}$). Let U be such a subgroup. We set $H := \langle V \mid V \in (\mathcal{M}_Y)_{>U}\rangle$. U is normal in H and $U/Y \leqslant Z(H/Y)$. Let W be the subgroup $H'H^p$. We claim that $[W, U] = 1$. From $[H, U] \leqslant Y \leqslant Z(H)$, it follows that $[H, U, H] = 1$. Therefore, by the 3-subgroups lemma, $[H', U] = 1$. Moreover, for $h \in H$ and $u \in U$, $[h, u]$ commutes both with h and u, and, using the fact that U has exponent p,

$$\left[h^p, u\right] = [h, u]^p = \left[h, u^p\right] = [h, 1] = 1$$

by which $[H^p, U] = 1$ and so our claim follows. In particular, $W \cap U \leqslant Z(U) = A$, and so, by the modular law, $WA \cap U = A$. The group H/WA is an elementary abelian p-group, we choose in it a complement K/WA of WU/WA. The posets $(\mathcal{M}_Y)_{>U}$ and $(\mathcal{M}_Y(K))_{>A}$ are isomorphic via the map

$$\phi : (\mathcal{M}_Y)_{>U} \longrightarrow \left(\mathcal{M}_Y(K)\right)_{>A}, \quad T \quad \longmapsto \quad T \cap K.$$

In fact, as H/A is the direct product $U/A \times K/A$, every element $T \in (\mathcal{M}_Y)_{>U}$ can be written as $T = U(T \cap K)$. By this, one easily sees that the map ϕ is well-defined, order preserving, and injective. To prove it is surjective too, let R be any element of $(\mathcal{M}_Y(K))_{>A}$, then, since U centralizes R modulo Y, UR lies in $(\mathcal{M}_Y)_{>U}$, moreover $\phi(UR) = UR \cap K = R(U \cap K) = RA = R$. Thus $(\mathcal{M}_Y)_{>U} \approx (\mathcal{M}_Y(K))_{>A}$. Finally note that, being K a proper subgroup of G, by assumption $\mathbf{S}(\mathcal{A}_p(K)_{>A})$ is homotopy equivalent to a wedge of spheres. Since K satisfies the hypothesis of the lemma, we can use the implication "from bottom to top" previously proved to deduce that $\mathbf{S}(\mathcal{M}_Y(K))_{>A}$ is a wedge of spheres too. This completes the proof of the lemma. □

Proposition 20. *Finite soluble groups satisfy the property* $(\star)$.

Proof. Assume G is a counterexample of minimum order and let A be an elementary abelian p-subgroup of G such that $\mathbf{S}(\mathcal{A}_p(G)_{>A})$ is not a wedge of spheres. Since the suspension operator preserves these classes of spaces (Proposition 10), $\mathcal{A}_p(G)_{>A}$ is not a wedge of spheres. By Lemma 12(iii), A is a central subgroup of G. Set $N = O_{p'}(G)$. If N is not trivial set $\overline{G} = G/N$. We apply Corollary 5 to the mapping $f : R \mapsto \overline{R}$ from $\mathcal{A}_p(G)_{>A}$ to $\mathcal{A}_p(\overline{G})_{\geqslant \overline{A}}$. To show that all the assumptions are satisfied, choose $\overline{R} \in \mathcal{A}_p(\overline{G})_{>\overline{A}}$ and consider the lower fiber $f^{-1}_{\leqslant \overline{R}}$. This is

$$\left\{S \in \mathcal{A}_p(G)_{>A} \;\middle|\; \overline{S} \leqslant \overline{R}\right\} = \left\{S \in \mathcal{A}_p(G)_{>A} \;\middle|\; NS \leqslant NR\right\} = \mathcal{A}_p(NR)_{>A},$$

where R is a Sylow p-subgroup in the preimage of $\overline{R}$. The poset $\mathcal{A}_p(NR)_{>A}$ is easily seen to be isomorphic to $\mathcal{A}_p(NR/A)$, via the quotient map sending every element $T \in \mathcal{A}_p(NR)_{>A}$ into $T/A \in \mathcal{A}_p(NR/A)$. Therefore, by Lemma 13, the fiber $f^{-1}_{\leqslant \overline{R}}$ is

$(\mathrm{rk}(\overline{R})-1)$-spherical. All the hypotheses of Corollary 5 are satisfied, according with it, we obtain the following homotopy equivalence:

$$\mathcal{A}_p(G)_{>A} \simeq \mathcal{A}_p(\overline{G})_{>\overline{A}} \vee \bigvee_{\overline{R}\in\mathcal{A}_p(\overline{G})_{>\overline{A}}} \big(\mathcal{A}_p(NR/A) * \mathcal{A}_p(\overline{G})_{>\overline{R}}\big), \tag{7}$$

where R is a Sylow p-subgroup in the preimage of $\overline{R}$ (this formula expresses the meaning of [7, Remark 3.4]). Passing to the suspensions (together with Lemma 9(ii) and (iii))

$$\mathbf{S}\big(\mathcal{A}_p(G)_{>A}\big) \simeq \mathbf{S}\big(\mathcal{A}_p(\overline{G})_{>\overline{A}}\big) \vee \bigvee_{\overline{R}\in\mathcal{A}_p(\overline{G})_{>\overline{A}}} \big(\mathcal{A}_p(NR/A) * \mathbf{S}\big(\mathcal{A}_p(\overline{G})_{>\overline{R}}\big)\big). \tag{8}$$

By our choice of G and by Lemma 13, every term in the right-hand side of (8) is a wedge of spheres, thus also $\mathbf{S}(\mathcal{A}_p(G)_{>A})$ is.

Therefore $N=1$. As G is a soluble group $O_p(G)\neq 1$. If G is a p-group, by Corollary 17, $\mathcal{M}_A(G)$ is contractible, and so its suspension too. By Lemma 19, the space $\mathbf{S}(\mathcal{A}_p(G)_{>A})$ must then be a wedge of spheres. Thus we assume G is not a p-group. We claim that $\mathcal{M}_A(G)$ is contractible in this case too. By our reduction, the Fitting subgroup of G consists just in $O_p(G)$, and so $C_G(O_p(G)) \leqslant O_p(G)$ [1, 31.10]. Since A is a central elementary abelian p-subgroup of G and G is not a p-group, the previous inequality yields that A is strictly contained in $O_p(G)$. Moreover, since p is odd, any p'-element of G acts faithfully by conjugation on $\Omega_1(O_p(G))$ [1, 24.8], thus A is strictly contained $\Omega_1(O_p(G))$. In particular, we have that $\mathcal{M}_A(O_p(G))$ is not empty. If $\mathcal{M}_A(O_p(G))$ has a unique maximal element, call it B, otherwise let B be the center of the intersection of all the maximal elements of $\mathcal{M}_A(O_p(G))$, and note that it strictly contains A in virtue of Lemma 16. In any case, B is a normal p-subgroup of G lying in $\mathcal{M}_A(G)$. For every subgroup C of B strictly containing A, set, for simplicity, $\mathcal{L}_C := \mathcal{M}_A(C_G(C/A))$. Let V be an arbitrary element of $\mathcal{M}_A(G)$, then, as V is a p-group acting on the p-group B/A, if we set $C_V/A := C_{B/A}(V)$, this is not trivial, by which we have that V lies in $\mathcal{L}_{C_V}$. This proves that $\mathcal{M}_A(G)$ is covered by the subcomplexes $\mathcal{L}_C$ (when C varies among the subgroups of B strictly containing A). Now note that, by the definition of $\mathcal{L}_C$, C is a conjunctive element of $\mathcal{L}_C$. Lemma 7 therefore yields that the complexes $\mathcal{L}_C$ are all contractible. If $C_1, C_2, \ldots, C_n$ are n subgroups of B strictly containing A, then

$$\mathcal{L}_{C_1} \cap \mathcal{L}_{C_2} \cap \cdots \cap \mathcal{L}_{C_n} = \mathcal{L}_{\langle C_1, C_2, \ldots, C_n\rangle},$$

which is still contractible since $\langle C_1, C_2, \ldots, C_n\rangle \leqslant B$. Applying the Nerve Theorem 4, we deduce that $\mathcal{M}_A(G)$ is homotopy equivalent to the intersection poset of the family $\{\mathcal{L}_C\}_{A<C\leqslant B}$. This family has a minimum element, which is $\mathcal{L}_B$, and so the intersection poset is contractible, proving our claim. Since $\mathcal{M}_A(G)$ is contractible, so is its suspension, and finally, by Lemma 19, $\mathbf{S}(\mathcal{A}_p(G)_{>A})$ is homotopy equivalent to a wedge of spheres. This contradicts our assumption on G and completes the proof. $\square$

4. The Quillen complex for soluble groups

Theorem 21. *Let G be a finite soluble group and p an odd prime number dividing the order of G. The Quillen complex of G at p is homotopy equivalent to a wedge of spheres.*

Proof. We can assume $O_p(G) = 1$, otherwise $\Omega_1(Z(O_p(G)))$ is a conjunctive element of the Quillen complex, which is contractible by Lemma 7.

Denote with N the subgroup $O_{p'}(G)$ and call $\overline{G}$ the factor group G/N. According to Lemma 14,

$$\Delta\big(\mathcal{A}_p(G)\big) \simeq \bigvee_{\overline{A}\in\mathcal{A}_p(\overline{G})} \big(\Delta\big(\mathcal{A}_p(NA)\big) * \Delta\big(\mathcal{A}_p(\overline{G})_{>\overline{A}}\big)\big), \tag{9}$$

where we used the fact that $\Delta(\mathcal{A}_p(\overline{G}))$ is homotopy equivalent to a point, having $\overline{G}$ a non-trivial normal p-subgroup. The significant contributions to formula (9) are given by the non-contractible terms $\Delta(\mathcal{A}_p(NA)) * \Delta(\mathcal{A}_p(\overline{G})_{>\overline{A}})$, for which in particular $\Delta(\mathcal{A}_p(NA))$ is, by virtue of Lemma 13, a non-empty wedge of some spheres of dimension $\mathrm{rk}(A) - 1 \geqslant 0$. In considering the homotopy type of the order complex of $\mathcal{A}_p(NA) * \mathcal{A}_p(\overline{G})_{>\overline{A}}$ in this case, we apply Lemmas 8 and 9(ii) and (iii); it follows that

$$\begin{aligned}\Delta\big(\mathcal{A}_p(NA)\big) * \Delta\big(\mathcal{A}_p(\overline{G})_{>\overline{A}}\big) &\simeq \Big(\bigvee \big(S^{\mathrm{rk}(A)-2} * S^0\big)\Big) * \Delta\big(\mathcal{A}_p(\overline{G})_{>\overline{A}}\big) \\ &\simeq \bigvee S^{\mathrm{rk}(A)-2} * \mathbf{S}\big(\Delta\big(\mathcal{A}_p(\overline{G})_{>\overline{A}}\big)\big),\end{aligned}$$

where, in this situation, all the spaces on the right-hand side are wedged to a unique common point. By Propositions 20 and 10, the space $\Delta(\mathcal{A}_p(NA)) * \Delta(\mathcal{A}_p(\overline{G})_{>\overline{A}})$ is therefore homotopy equivalent to a wedge of spheres. Thus is every non-contractible term in (9), and the proof of the theorem is completed.

Remarks.

(1) As stated in [9, Corollary 4.17], software computation shows that $\widetilde{H}_2(\mathcal{A}_3(S_{13}))$ is not torsion free. This fact implies that the Quillen complex at 3 of the symmetric group S_{13} is not homotopy equivalent to a wedge of spheres.

(2) In [7, Section 5], two examples of soluble groups whose Quillen complex consists of a wedge of spheres of *different* dimensions are described.

(3) Our definition of the class $(\star)$ of groups, based on the suspensions of the upper intervals and not directly on them, is motivated exclusively by the use of Whitehead theorem in proving Proposition 11. Whitehead's result does not hold without the assumption of simply connectivity on the CW-complexes (see [11, IV, 7, Example 3]). Of course, passing to the suspensions this hypothesis is immediately guaranteed. We do not know if Proposition 11 holds without the assumption of simply connectivity. If so, we could have defined $(\star)$ as the class of groups for which, for all $A \in \mathcal{A}_p(G)$ the interval $\mathcal{A}_p(G)_{>A}$ is homotopy equivalent to a wedge of spheres. In this way, with the same arguments used, we could have proved an analogous of Proposition 20, that

is the homotopy type of any upper intervals $\mathcal{A}_p(G)_{>A}$, for soluble groups and odd primes, is the one of a wedge of spheres (of possible different dimensions). We have not been able to prove it, and at the moment the homotopy structure of these intervals remains, even for soluble groups, unclear. As a consequence of Proposition 20, and the fact that for any space X and any integer i, $\widetilde{H}_i(X) \cong \widetilde{H}_{i+1}(\mathbf{S}(X))$, we know of course that $\mathcal{A}_p(G)_{>A}$ have homology free groups. Moreover, in the case G is a p-group ($p \neq 2$) these upper intervals are indeed wedges of spheres. This is motivated by the following lemma.

Lemma. *Let p be an odd prime, P a p-group and $A \leqslant \Omega_1(Z(P))$. Let $X_1, X_2, \ldots, X_m$ be the maximal elements of $\mathcal{M}_A(P)$, then*

$$\Delta\big(\mathcal{A}_p(P)_{>A}\big) \simeq \bigvee_{i=1,\ldots,m} \Delta\big(\mathcal{A}_p(X_i)_{>A}\big).$$

Proof. We apply the Nerve Theorem 4 to the covering $\{\Delta(\mathcal{A}_p(X_i)_{>A})\}_{i=1}^m$ of $\Delta(\mathcal{A}_p(P)_{>A})$. Note that any intersection of at least two different elements of this family is contractible since, by Lemma 16, it possesses a conjunctive element. □

This lemma reduces the analysis to the p-groups X_i of exponent p such that modulo their central subgroup A are elementary abelian. With the same techniques exposed in this paper, it can be proved that the homotopy type of the upper intervals, $\mathcal{A}_p(X_i)_{>A}$, of such a p-group (and therefore of every p-group), is that of a wedge of spheres.

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References

[1] M. Aschbacher, Finite Group Theory, Cambridge Univ. Press, Cambridge, UK, 1986.
[2] M. Aschbacher, Simple connectivity of p-group complexes, Israel J. Math. 82 (1993) 1–44.
[3] M. Aschbacher, S. Smith, On Quillen's conjecture for the p-subgroups complex, Ann. of Math. 137 (1993) 473–529.
[4] A. Björner, M.L. Wachs, V. Welker, A fiber poset theorem, preprint.
[5] J. Leray, Sur la position d'un ensemble fermé de points d'un espace topologique, J. Math. Pures Appl. 24 (1945) 169–199.
[6] J.R. Munkres, Elements of algebraic topology, Addison–Wesley, Menlo Park, CA, 1984.
[7] J. Pulkus, V. Welker, On the homotopy type of the p-subgroup complex for finite solvable groups, J. Austral. Math. Soc. Ser. A 69 (2000) 212–228.
[8] D. Quillen, Homotopy properties of the poset of nontrivial p-subgroups of a group, Adv. Math. 28 (1978) 101–128.
[9] J. Shareshian, Hypergraph matching complexes and Quillen complexes of symmetric groups, J. Combin. Theory Ser. A 106 (2004) 299–314.

[10] E.H. Spanier, Algebraic Topology, McGraw–Hill, New York, 1966.
[11] G.W. Whitehead, Elements of Homotopy Theory, Grad. Texts in Math., vol. 61, Springer-Verlag, Berlin, 1978.
[12] G.M. Ziegler, R.T. Živaljeviź, Homotopy type of subspaces arrangements via diagrams of spaces, Math. Ann. 295 (1993) 527–548.