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J. London Math. Soc. Page 1 of 16
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ON SUBNORMALITY CRITERIA FOR SUBGROUPS IN FINITE GROUPS

FRANCESCO FUMAGALLI

Abstract

Let H be a subgroup of a finite group G and let $S^1_G(H)$ be the set of all elements g of G such that H is subnormal in $\langle H, H^g\rangle$. A result of Wielandt states that H is subnormal in G if and only if $G = S^1_G(H)$. In this paper, we let A be a subgroup of G contained in $S^1_G(H)$ and ask if this implies (and therefore is equivalent to) the subnormality of H in $\langle H, A\rangle$. We show with an example that the answer is no, even for soluble groups with Sylow subgroups of nilpotency class at most 2. However, we prove that the two conditions are equivalent whenever either A is subnormal in G or it has p-power index in G (for p any prime number). Q1

Introduction

Let G be a finite group and H a subgroup of G. Wielandt [**9**] proved the following criteria for H to be subnormal in G.

Theorem A. *The subgroup H is subnormal in G provided that one of the following holds.*
(1) *H is subnormal in $\langle H, g\rangle$ for every $g \in G$.*
(2) *H is subnormal in $\langle H, H^g\rangle$ for every $g \in G$.*

This result suggests a study of the so-called subnormalizers of a subgroup H of G; as introduced in [**7**], these are defined as

$$S_G(H) := \{g \in G \mid H \,\mathrm{sn}\, \langle H, g\rangle\},$$
$$S^1_G(H) := \{g \in G \mid H \,\mathrm{sn}\, \langle H, H^g\rangle\}.$$

In general, neither $S_G(H)$ nor $S^1_G(H)$ needs to be a subgroup of G (examples are in [**7**, 7.7; **10**]). Wielandt's criteria tell us, in particular, that if G is finite and $S_G(H)$ (or $S^1_G(H)$) is a subgroup of G, then this is the maximal subgroup of G in which H is subnormal. In particular, $H \,\mathrm{sn}\, G$ if and only if $G = S_G(H) = S^1_G(H)$.

Wielandt's criteria have been generalized in various directions. In particular, Wielandt himself [**9**] demonstrated the existence of 'test sets' $T \subset G$, with the property that $T \subset S^1_G(H)$ is equivalent to $H \,\mathrm{sn}\, G$.

In this paper, we analyse the special situation in which a proper non-trivial subgroup A of G is entirely contained in $S^1_G(H)$. Using a result of Wielandt [**8**, Hilfssatz 2.2], it is easy to see under the further assumption that H permutes with any conjugates H^a, $a \in A$, that H is subnormal in $\langle H, A\rangle$. However, our first observation (Example 2) shows that the condition $A \subseteq S^1_G(H)$ alone is not enough to guarantee the subnormality of H in $\langle H, A\rangle$, even if G is a soluble group having abelian Sylow p-subgroups for all primes p except one. Thus, our interest is focused in two directions. First, we look for some 'easily definable' classes of finite groups satisfying $\forall\, H, A \leqslant G,\ A \subseteq S^1_G(H) \Rightarrow H \,\mathrm{sn}\, \langle H, A\rangle$.

A satisfactory result, in the light of Example 2, is given by the following.

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Theorem (Theorem 1). *Let G be a finite group which, modulo its Fitting subgroup, has abelian Sylow p-subgroups for every prime p. If H and A are two arbitrary subgroups of G such that $A \subseteq S^1_G(H)$, then H is subnormal in $\langle H, A\rangle$.*

In another direction, we search for some extra assumptions on A, in particular, related to its embedding in G, that, together with the condition $A \subseteq S^1_G(H)$, guarantee $H \operatorname{sn} \langle H, A\rangle$.

We have been able to prove the following.

Theorem (Theorem 2). *Let G be a finite group, and A and H be two subgroups of G such that $A \subseteq S^1_G(H)$. If A is subnormal in G, then $H \operatorname{sn} \langle H, A\rangle$.*

Theorem (Theorem 3). *Let G be a finite group, and A and H be two subgroups of G such that $A \subseteq S^1_G(H)$. If the index of A in G is a prime power, then H is subnormal in $\langle H, A\rangle$.*

Theorem 3 can be considered the main result of this paper. Its proof makes use of a result of Guralnick which relies on the *Classification of finite simple groups*.

We mention that a particular case of our general problem has been dealt with by Ho and Völklein in [**3**, **4**]. They treat the situation in which H is a p-subgroup of G and A is a Sylow p-subgroup contained in $S_G(H)$. With the use of the classification theorem and under the assumption $p \geqslant 5$, they prove that $H \operatorname{sn} \langle H, A\rangle$ (which in that context just means $H \leqslant A$).

1. *Preliminary facts, Example 2 and Theorem 1*

Throughout this section and the rest of the paper, G will always denote a finite group and H one of its subgroups.

We collect some basic facts about the subnormalizers $S_G(H)$ and $S^1_G(H)$, defined in the Introduction (as in [**7**, 7.7]). These are, in general, different subsets of G, the inclusion $S_G(H) \subseteq S^1_G(H)$ being strict. The following example shows that there do exist finite groups admitting a subgroup A contained in $S^1_G(H)$ and intersecting $S_G(H)$ trivially.

Example 1. Let $G = S_7$ be the symmetric group on seven objects, H the subgroup generated by the element $h = (12)(34)$ and A the one generated by the element $a = (235)(467)$. As $hh^a = (1234)(56)$ and $hh^{a^{-1}} = (1725)(34)$, both the subgroups $\langle H, H^a\rangle$ and $\left\langle H, H^{a^{-1}}\right\rangle$ are isomorphic to the dihedral group D_8, and so A lies in $S^1_G(H)$. However,

$$hh^a h^{a^{-1}} = (1723456),$$

so H^A is not a 2-group and H is not subnormal in $\langle H, a\rangle$.

For elements of order 2, being in $S_G(H)$ or in $S^1_G(H)$ are equivalent, as stated in the following lemma. The proof is left to the reader.

Lemma 1. *Let x be an involution in $S^1_G(H)$. Then $x \in S_G(H)$.*

We also omit the easy proof of the following fact.

Lemma 2. *The subnormalizer $S^1_G(H)$ is closed under right and left multiplication by elements of the normalizer of H in G.*

From now on A will always denote a subgroup of G lying in $S_G^1(H)$. As we see in Example 2 this condition is not enough to guarantee the subnormality of H in $\langle H, A\rangle$ (even if G is soluble with all Sylow subgroups of nilpotency class at most 2).

EXAMPLE 2. In $\mathrm{GL}(2,5)$ let T be the subgroup $Q \circ Z \rtimes A_0$, where Q is a Sylow 2-subgroup of $\mathrm{SL}(2,5)$, Z a cyclic subgroup of order 4 generated by a scalar matrix z having non-zero entries equal to a primitive fourth root of unity and A_0 a cyclic subgroup of order 3 of the normalizer of Q, generated by a (the symbol '$\circ$' denotes the central product). Let Q be a quaternion group Q3
of order 8 and its subgroups of order 4, say $\langle x\rangle$, $\langle y\rangle$ and $\langle w\rangle$, are transitively permuted by $\langle a\rangle$ using the rule $x^a = y$, $y^a = w$, $w^a = x$. The subgroup $\langle z^2\rangle$ is the center of Q and acts like the inversion on the natural module M of order 25. Set $G := M \rtimes T$. In G let H be the subgroup generated by the element $h := xz$. This H has order 2, and since $H \operatorname{sn} T$, $A_0 \subseteq S_G^1(H)$. We claim that there exists a conjugate A_1 of A_0 for which $A_1 \subseteq S_G^1(H)$ and H is not subnormal in $\langle H, A_1\rangle$. The centralizer of h in M is 1-dimensional. Since A_0 acts fixed-point-freely on M we can write $C_M(h) = \langle v\rangle$, with $v = [m, a^{-1}]$ for some $m \in M$; in particular, va is equal to a^m. Set $A_1 := A_0^m = \langle va\rangle$. As $va \in C_M(H)A_0$ and $a^{-1}v^{-1} \in A_0 C_M(H)$, by Lemma 2, we have $A_1 \subseteq S_G^1(H)$. If H were subnormal in $\langle H, A_1\rangle$, then $H^{A_1} = \left\langle h, h^{va}, h^{a^{-1}v^{-1}}\right\rangle$ would be a 2-subgroup containing the element $z^2 = (h \cdot h^{va})^2$. Since z^2 acts like the inversion on M, it is easy to see that the subgroup $D := Q \circ Z$ is the only 2-Sylow subgroup of G containing z^2. Therefore $H^{A_1} \leqslant D$ and $H \leqslant D \cap D^{va}$. By Frattini's argument, there exists an element $m_1 \in M$ such that $D^{va} = D^{m_1}$. Thus

$$H^{m_1} \leqslant D^{m_1} \cap MH = H(D^{m_1} \cap M) = H$$

and so $m_1 \in C_M(H)$. Also, a and $v^a m_1^{-1}$ lie in $N_G(D)$, thus $v^a m_1^{-1} \in N_G(D) = T$, and therefore $v^a m_1^{-1} \in T \cap M = 1$; this forces

$$v^a = m_1 \in C_M(H) \cap C_M(H^a) = C_M(\langle H, H^a\rangle) \leqslant C_M(z^2) = 1,$$

by which we obtain the contradiction $v = 1$.

We now collect some useful results. As introduced by Wielandt [**8**] (see also [**7**, p. 129]), an *operator* on a lattice Σ of subgroups of G is a function $\omega : \Sigma \longrightarrow \Sigma$ such that for every $H, K \in \Sigma$:

(i) $\langle H, K\rangle^\omega = \langle H^\omega, K^\omega\rangle$;
(ii) $H \trianglelefteq K$ implies $H^\omega \trianglelefteq K$.

LEMMA 3. *Let ω be an operator on the lattice Σ with elements that are intersections and* Q3
joins of subgroups of the family $\{H^a | a \in A\}$. Assume that ω commutes with the conjugation action of A. If $A \subseteq S_G^1(H)$, then H normalizes $(H^\omega)^A$.

Proof. For every $a \in A$, we have

$$\langle H, H^a\rangle^\omega = \langle H^\omega, (H^a)^\omega\rangle = \langle H^\omega, (H^\omega)^a\rangle,$$

therefore

$$[H, (H^\omega)^a] \leqslant [H, \langle H, H^a\rangle^\omega] \leqslant \langle H, H^a\rangle^\omega \leqslant (H^\omega)^A. \qquad \square$$

Our basic application of Lemma 3 is as follows.

COROLLARY 1. *If $A \subseteq S_G^1(H)$, then H normalizes the subgroups $(H^{\mathcal{N}})^A$ and $O^p(H)^A$ ($H^{\mathcal{N}}$ denotes the nilpotent residual of H).*

Proof. By [**7**, Theorems 3.3.1. and 4.1.3], both maps $H \mapsto H^{\mathcal{N}}$ and $H \mapsto O^p(H)$ are operators on the lattice of all subnormal subgroups of a group. The statement is then an immediate consequence of Lemma 3. □

Lemma 4. *Let H be a p-perfect subgroup of G. If K is a normal p-subgroup of G, then $S^1_G(H) \cap K = N_K(H)$.*

Proof. Let $k \in S^1_G(H) \cap K$. Set $L := \langle H, H^k \rangle$ and Y the normal closure of H^L. Then, as any H^x $(x \in L)$ is subnormal in L, the map $X \mapsto O^p(X)$ when restricted to the family $\{H^x | x \in L\}$ is an operator; in particular, we have Q4

$$O^p(Y) = \langle O^p(H^x) | x \in L \rangle = \langle H^x | x \in L \rangle = Y.$$

Therefore $Y \leqslant O^p(L)$. However, $L = YH^k$ and L/Y is p-perfect, thus $L = O^p(L)$. Moreover, $L = L \cap KH = H(L \cap K)$, and so

$$O^p(L) = O^p(H)O^p(L \cap K) = O^p(H) = H,$$

that is, $L = H$ and $H^k = H$. □

We say that a group G lies in the class (S_1) if

for every pair of subgroups H, A of G, the condition $A \subseteq S^1_G(H)$ implies H sn$\langle H, A \rangle$.

We also say that G lies in $(S_1)_P$ if

for every pair of subgroups H, A of G, with H a group of prime power order, $A \subseteq S^1_G(H)$ implies H sn$\langle H, A \rangle$.

In searching for the finite groups that lie in (S_1), the key ingredient is the following.

Proposition 1. *Let $\mathfrak{X}$ be a class of groups closed under quotients and subgroups. Then $\mathfrak{X} \subseteq (S_1)$ if and only if $\mathfrak{X} \subseteq (S_1)_P$.*

Proof. Assume that $\mathfrak{X} \subseteq (S_1)_P$. Let $G \in \mathfrak{X}$ and H, A arbitrary subgroups of G, with $A \subseteq S^1_G(H)$; we prove that H sn$\langle H, A \rangle$, by induction on $|G| + |H|$.

By the $\mathfrak{s}$-closure of $\mathfrak{X}$, we may assume that $G = \langle H, A \rangle$.

(1) Suppose first that H has a subnormal subgroup S, which is a simple non-abelian group.

Let D be the subgroup generated by all the subnormal subgroups of H isomorphic to S, namely the S-components of H. Then $1 \neq D \trianglelefteq H$ and D is the direct product of all the S-components of H. Let $a \in A$, then $D \trianglelefteq H$ sn $\left\langle H, H^{a^{-1}} \right\rangle$, and so

$$D^a \trianglelefteq H^a \,\text{sn}\, \langle H, H^a \rangle .$$

If we set D_a the product of the S-components of $\langle H, H^a \rangle$, then $D^a \trianglelefteq D_a$. By [**7**, Theorem 4.6.3], D_a normalizes every subnormal subgroup of $\langle H, H^a \rangle$; in particular, it normalizes H. Thus $D_a \cap H \trianglelefteq D_a$ and as $D_a \cap H$ is a product of S-components, we have $D_a \cap H = D$. Therefore for every $a \in A$,

$$[D^a, H] \leqslant [D_a, H] \leqslant D_a \cap H = D,$$

and so

$$[D^A, H] \leqslant D \leqslant D^A \cap H. \tag{1}$$

This shows that D^A is normalized by H and so $D^A \trianglelefteq \langle H, A \rangle = G$. Since $D^A \neq 1$, by the inductive hypothesis, $HD^A/D^A \,\text{sn}\, G/D^A$, and therefore $HD^A \,\text{sn}\, G$. Since $H \trianglelefteq HD^A$ by (1), we conclude that $H \,\text{sn}\, G$.

(2) Assume now that the minimal subnormal subgroups of H are all abelian.

Let p be a prime divisor of the order of H such that $O_p(H) \neq 1$. Set also $X := O^p(H)$.

If $X = 1$, then H is a p-subgroup and so it is subnormal in G by the assumption $\mathfrak{X} \subseteq (S_1)_P$. Thus X is not trivial. Assume now that $X = H$. As $H \operatorname{sn}\langle H, H^a\rangle$ for every $a \in A$, we have that $1 \neq O_p(\langle H, H^a\rangle)$ and this normalizes H by Lemma 4. In particular for every $a \in A$,

$$[O_p(H)^a, H] \leqslant H \cap O_p(\langle H, H^a\rangle) \leqslant O_p(H).$$

Then $[O_p(H)^A, H] \leqslant O_p(H)$, that is, $O_p(H)^A$ is normal in G and $H \trianglelefteq HO_p(H)^A$. Working modulo $O_p(H)^A$ we obtain that $HO_p(H)^A \trianglelefteq G$, and so $H \operatorname{sn} G$, as required. Assume therefore that X is a proper non-trivial subgroup of H. We claim that $X \operatorname{sn} G$. In fact, $A \subseteq S^1_G(X)$ and, by induction on $|H|$, we have $X \operatorname{sn} X^A$. Moreover by Corollary 1, X^A is normalized by H, thus $X^A \trianglelefteq G$ and so $X \operatorname{sn} G$. Since H has only abelian components, the same occurs to X, in particular, there exists a prime number q such that $O_q(X) \neq 1$. As $X \operatorname{sn} G$, we have $O_q(G) \neq 1$. If $q = p$, then since $G = S^1_G(X)$, by Lemma 4, $O_p(G)$ normalizes X. Thus X is normal in $HO_p(G)$ and since $HO_p(G)/X$ is a p-group, we have $H/X \operatorname{sn} HO_p(G)/X$, by which $H \operatorname{sn} HO_p(G)$. Moreover, as $O_p(G) \neq 1$, working modulo $O_p(G)$, we deduce that $HO_p(G) \operatorname{sn} G$, and $H \operatorname{sn} G$. Let therefore q not equal p. If $O^q(H) \neq H$, arguing as before we have that $O^q(H)$ is subnormal in G, and so $H = \langle O^p(H), O^q(H)\rangle$ is subnormal in G. Thus we assume that $O^q(H)$ equals H. Then $1 \neq O_q(H) \leqslant O_q(\langle H, H^a\rangle)$, and this latter normalizes H. Therefore, for every $a \in A$,

$$[O_q(H)^a, H] \leqslant H \cap O_q(\langle H, H^a\rangle) \leqslant O_q(H),$$

by which we deduce that $O_q(H)^A \trianglelefteq \langle H, A\rangle = G$ and $H \trianglelefteq HO_q(H)^A$. Considerations modulo $O_q(H)^A$ bring to $HO_q(H)^A \operatorname{sn} G$ and thus $H \operatorname{sn} G$. □ Q5

A group is said to be an *A-group* if all its Sylow subgroups are abelian. As a corollary of the previous result we have the following.

THEOREM 1. *Let G be a finite group such that $G/\mathrm{Fit}\,(G)$ is an A-group. If H and A are two arbitrary subgroups of G with $A \subseteq S^1_G(H)$, then H is subnormal in $\langle H, A\rangle$.*

Proof. The class of finite groups T such that $T/\mathrm{Fit}\,(T)$ is an A-group is closed under subgroups and quotients. Therefore, by Proposition 1 we reduce to proving the statement in the case in which H is a p-subgroup for some prime number p. In this situation for every $a_1, a_2 \in A$, $\langle H^{a_1}, H^{a_2}\rangle$ is a p-group, and the assumption on G implies that

$$[H^{a_1}, H^{a_2}] \leqslant O_p(G).$$

Therefore we have proved that

$$(H^A)' \leqslant O_p(G).$$

As $H^A/(H^A)'$ is generated by p-groups, we deduce that H^A is a p-group. In particular, $H \operatorname{sn} H^A \trianglelefteq \langle H, A\rangle$, and so $H \operatorname{sn} \langle H, A\rangle$, as required. □

REMARK 1. This result does not furnish a complete characterization of the finite groups in (S_1) (for instance, it can easily be checked that this class contains the symmetric group S_5).

2. *Theorems* 2 *and* 3

We have already introduced the term component in the course of the proof of Proposition 1. We recall the precise definition.

DEFINITION 1. A subgroup S of G is a *component* of G if S is subnormal and quasisimple (this means that S is a perfect group and $S/Z(S)$ is simple non-abelian).

LEMMA 5. *Let G be a finite group, S a component of G and H a subgroup of G. If $S \subseteq S_G(H)$, then either $S \leqslant H$ or $[H,S] = 1$.*

Proof. We may assume that $G = \langle H, S\rangle = S^H H$. It is enough to prove that H is subnormal in G; for then the result will follow from a well-known fact about components [**6**, 6.5.2]. We Q6 proceed by induction on $|G| + |H|$. Let T be equal to $S^H \cap H$. If $T = H$, then S is normal in G and $G = SH$. By Lemma 2 and the Wielandt criterion, H is subnormal in G. Therefore assume that T is properly contained in H. By induction on the order of H, T is normalized by S. Thus $T \trianglelefteq G$. If T is not trivial then, by the minimality of G, $H \operatorname{sn} G$. Then assume that $T = 1$ and $G = S^H \rtimes H$. Let H_0 be a proper subgroup of H and s be an arbitrary element of S. As H sn$\langle H, s\rangle$, there exists an integer m such that

$$[\langle s\rangle^{H_0}, {}_m H_0] \leqslant H \cap S^H = 1,$$

which means that H_0 is subnormal in $\langle H_0, s\rangle$, that is, $S \subseteq S_G(H_0)$ for every subgroup H_0 of H. By induction on $|H|$, S normalizes every proper subgroup of H. Therefore H must contain a unique maximal subgroup; in other words, H is a cyclic p-group, for some prime p. Moreover, since the maximal subgroup of H is normal in G, we can reduce to the case $|H| = p$. Let H be equal to $\langle h\rangle$. Now if S is normal in G by the Wielandt criterion, then we immediately have $H \operatorname{sn} G$. Therefore assume that S^H is the direct product of p copies of S. Let y be an element of p'-order of S. For some integer m,

$$[\langle y\rangle^H, {}_m H] \leqslant S^H \cap H = 1$$

and since the action of H on $\langle y\rangle^H$ is coprime, $[\langle y\rangle^H, H] = [\langle y\rangle^H, {}_m H]$, so every p'-element of S centralizes H. However, then $[S,H] = 1$ and H is normal in G. □

THEOREM 2. *Let $A \subseteq S^1_G(H)$. If A is subnormal in G, then $H \operatorname{sn} \langle H, A\rangle$.*

Proof. We proceed by induction on $|G| + |G:H|$. We assume that G is a minimal counterexample and H is maximal in G for which the statement is not true. In particular, we have $G = \langle H, A\rangle$.

We claim that $\operatorname{Fit}(G)$ is a p-group, for some prime p.

Assume that p and q are two distinct primes and that M and N are minimal normal subgroups of G with M of p-power order and N of q-power order. By the minimality of G, we have that neither of them is contained in H. By the inductive hypothesis, HM and HN are subnormal in G and so also $HN \cap HM$ is such. Let $H_0 := H \cap MN$ and let π_M and π_N be, respectively, the projection maps from H_0 to M and to N. We have that $\operatorname{Ker}(\pi_M) = H_0 \cap N$, $\operatorname{Im}(\pi_M) = H_0N \cap M$, and similar statements for the map π_N. By the theorems of isomorphisms, we have

$$\frac{H_0N \cap M}{H_0 \cap M} \simeq \frac{H_0}{(H_0 \cap N)(H_0 \cap M)} \simeq \frac{H_0M \cap N}{H_0 \cap N}.$$

Therefore, since $p \neq q$, we must have $H_0 = (H_0 \cap M)(H_0 \cap N)$, that is, $H \cap MN = (H \cap M)(H \cap N)$. However then, by the modular law, $HM \cap HN = H(M \cap HN) = H(H_0 \cap M) = H$, and thus H is subnormal in G. Therefore we can assume that $\operatorname{Fit}(G) = O_p(G)$, for some prime p.

Suppose now that $A \cap O_p(G) \neq 1$. If A is a p-subgroup, then $A \leqslant O_p(G)$ and $G = HO_p(G)$. Call $R := O^p(H)$, then $A \leqslant S^1_G(R) \cap O_p(G)$ and by Lemma 4, A normalizes R. Thus $R \trianglelefteq \langle H, A\rangle = G$. By minimality of G, we deduce that $R = 1$, forcing H to be a p-group and the same for G, which is a contradiction. Therefore A is not a p-subgroup. By Lemma 4

$$[O_p(G), O^p(A)] \leqslant O_p(G) \cap O^p(A) \leqslant O_p(A). \tag{2}$$

Moreover $O^p(A)^H = O^p(A^H)$, which is normal in G. The subgroup $T := [O_p(G), O^p(A)^H]$ is then normal in G and contained in $O_p(A)^H$. As $O_p(A) \operatorname{sn} G$ and $O_p(A) \subseteq S^1_G(H)$, by the previous case we deduce $H \operatorname{sn} O_p(A)^H H$, so also $H \operatorname{sn} HT$. As $T \trianglelefteq G$, $A \subseteq S^1_G(HT)$, thus if $T \nleqslant H$, by induction on $|G : H|$, we have $HT \operatorname{sn} G$, and H subnormal in G. Otherwise, if $T \leqslant H$ and $T \neq 1$, by the minimality of G, we have $H \operatorname{sn} G$. Thus T is equal to 1, in particular, $O^p(A) \leqslant C_G(O_p(G))$. As $A \operatorname{sn} G$, if K is any component of G, by [**6**, 6.5.2], either $K \leqslant A$ or $[K, A] = 1$. Since A is not a p-group, $O^p(A)$ cannot centralize every component of G, otherwise it centralizes the generalized Fitting subgroup of G and so by [**6**, 6.5.8] $O^p(A) \leqslant \operatorname{Fit}(G) = O_p(G)$. Let therefore K be a component of G contained in A and let $Y := K^G \cap A$. As $[K^G, A] \leqslant Y$, Y^H is normalized by both H and A. Thus Y^H is equal to K^G. By induction on the index of H in G, $HK^G \operatorname{sn} G$. Moreover by Lemma 5, Y normalizes H, so also does K^G. Thus $H \trianglelefteq HK^G$, which is itself subnormal in G if HK^G properly contains H. Hence $K^G \leqslant H$, but then by minimality of G we again reach a contradiction.

Then we reduced to the case $A \cap \operatorname{Fit}(G) = 1$. In particular, any minimal subnormal subgroup of G contained in A is necessarily a non-abelian simple group. Let S be one of these. Let $Y := S^G \cap A$, then $[S^G, A] \leqslant Y$. Arguing as in the last part of the previous case, we have that $Y^H = S^G$. By induction on the index of H in G, we can assume that $HS^G \operatorname{sn} G$. Now $Y \leqslant A$ and Y normalizes H, by Lemma 5. Therefore $[S^G, H] = [Y^H, H] \leqslant H$ and $H \trianglelefteq HS^G \operatorname{sn} G$. □

The following four lemmas are easy facts that will be needed in the proof of our main result (Theorem 3).

Lemma 6. *Let H be a p-subgroup of G and A a subgroup of G contained in $S^1_G(H)$. Assume that G has abelian Sylow p-subgroups. Then $H \operatorname{sn} \langle H, A\rangle$.*

Proof. For every $a \in A$, H and H^a commute pairwise. H is then a central subgroup of H^A and so $H \operatorname{sn} \langle H, A\rangle$. □

Lemma 7. *Let A and K be two subgroups of G. Assume that K is subnormal in G. Then $|K : K \cap A|$ divides $|G : A|$.*

Proof. We use induction on the defect d of K in G. The result is clear if K is normal in G, so assume that $d > 1$. By the inductive step, $|K : K \cap A|$ divides $\left|K^G : K^G \cap A\right| = \left|K^G A : A\right|$, so it also divides $|G : A|$. □

Lemma 8. *Let P be a p-subgroup of G. If the index in G of $N_G(P)$ is a power of p, then $P \leqslant O_p(G)$.*

Proof. We prove that P is contained in any p-Sylow of G. Let S be one of those and let P be contained in S^g, for some $g \in G$. By assumption $G = SN_G(P)$, so we can write $g = sn$ with $s \in S$ and $n \in N_G(P)$. Then $P \leqslant S^g = S^n$ and so $P = P^{n^{-1}} \leqslant S$. □

We prove the following lemma under the strong assumption of p-solubility. It would be interesting to know if it works without this assumption. For the analogous problem with the 'zero'-subnormalizer $S_G(H)$ we refer the interested reader to the works of Ho and Völklein [**3**, **4**]. (For a different proof of the following result see [**1**, Lemma 2.17].)

Lemma 9. *Assume that G is p-soluble, H a p-subgroup of G and P a Sylow p-subgroup of G. If $P \subseteq S^1_G(H)$, then $H \leqslant P$.*

Proof. Let G be a minimal counterexample. If $O_p(G) \neq 1$, by induction on $|G|$ we have that $HO_p(G)/O_p(G) \leqslant P/O_p(G)$, and so $H \leqslant P$. Thus $O_p(G)$ equals 1. Let S be a non-trivial normal p'-subgroup of G. Working modulo S, we obtain $H \leqslant PS$. By the Schur–Zassenhaus theorem there exists an element $s \in S$ such that $H^s \leqslant P$. For an arbitrary h in H, then

$$[h, h^{-s}] \in S \cap \langle H, H^{h^{-s}} \rangle$$

since $[h, h^{-s}]$ is equal both to $[h, s][h^{-1}, s] \in S$ and to $h^{-1} \cdot h^{h^{-s}}$. Since $h^{-s} \in P \subseteq S^1_G(H)$, $\langle H, H^{h^{-s}} \rangle$ is a p-group, and as $|S|$ is coprime with p, we have $[h, h^{-s}] = 1$. Then $[h^{-1}, s] = [h, s]^{-1} = [s, h]$, and so

$$\begin{aligned} [s, h, h] = [h^{-1}, s, h] &= [s, h^{-1}]h^{-1}[h^{-1}, s]h \\ &= [s, h^{-1}][s, h] = [s, h^{-1}][h^{-1}, s] = 1, \end{aligned}$$

which means that $[s, h]$ commutes with h and thus its order is a p-power. As $[s, h]$ lies also in S, $[s, h] = 1$. Since this happens for all $h \in H$, $H = H^s \leqslant P$. □

We are now ready to state and prove our main result.

THEOREM 3. *Let G be a finite group, and A and H two subgroups of G such that $A \subseteq S^1_G(H)$. Assume that the index of A in G is a power of some prime number p. Then H is subnormal in $\langle H, A \rangle$.*

Proof. We prove the theorem by induction on $|G| + |H|$. We let G be a minimal counterexample; in particular, $G = \langle A, H \rangle$.

We discuss separately the cases: G is soluble or not.

Assume first that G is a soluble group.

We claim that the normal core A_G of A is trivial. Otherwise let $\overline{G}$ be the group G/A_G and use the 'bar' notation to denote its subgroups. By the minimality of G the subgroup $\overline{HA_G}$ is subnormal in $\overline{G}$, and hence $HA_G \operatorname{sn} G$. By Lemma 2, $HA_G \subseteq S^1_G(H)$, and so by the Wielandt criterion $H \operatorname{sn} HA_G$ and $H \operatorname{sn} G$. Thus, assume that A_G equals 1 and let M be a minimal normal subgroup of G. Then M is an elementary abelian p-group, and $\mathrm{Fit}\,(G) = O_p(G)$. Let q be a prime divisor of $|H|$ such that $O^q(H) \lneq H$. By induction on the order of H, we have $O^q(H) \operatorname{sn} \langle O^q(H), A \rangle$. By Corollary 1, H normalizes $O^q(H)^A$; then $O^q(H)^A \trianglelefteq G = \langle A, H \rangle$ and so $O^q(H) \operatorname{sn} G$. Consider first the case where H is a q-group, that is, $O^q(H) = 1$. If $q = p$, then H is subnormal in the p-group $HO_p(G)$. Working modulo $O_p(G)$, $HO_p(G) \operatorname{sn} G$, and then $H \operatorname{sn} G$. Let therefore $q \neq p$ and let Q be a Sylow q-subgroup of G contained in A. By Lemma 9, $H \leqslant Q$, but then $H \leqslant A$ and so $A = G$ and $H \operatorname{sn} G$ by the Wielandt criterion. Therefore $O^q(H)$ is a non-trivial subnormal subgroup of G properly contained in H. Since $\mathrm{Fit}\,(G) = O_p(G)$, we have $O_p(O^q(H)) \neq 1$, and so also $O_p(H) \neq 1$. Consider the subgroup $O^p(H)$, that we can assume not trivial. If $O^p(H)$ is a proper subgroup of H, arguing as before we have $O^p(H) \operatorname{sn} G$. In particular, by Lemma 4, $O_p(G)$ normalizes $O^p(H)$, and then the subgroup $HO_p(G)$ normalizes $O^p(H)$. Since $HO_p(G)/O^p(H)$ is a p-group, H is subnormal in $HO_p(G)$, which is itself subnormal in G, and we conclude that $H \operatorname{sn} G$. Finally we are reduced to consider the case $O^p(H) = H$. Then by Lemma 4, for every $a \in A$, $O_p(\langle H, H^a \rangle)$ normalizes H; in particular

$$[O_p(H)^a, H] \leqslant O_p(\langle H, H^a \rangle) \cap H \leqslant O_p(H).$$

Therefore $O_p(H)^A$ is a non-trivial normal subgroup of G that normalizes H. Working modulo $O_p(H)^A$, we obtain that $HO_p(H)^A$ is subnormal in G, but then $H \operatorname{sn} G$, and this completes the proof in the case where G is soluble.

Assume that G is not soluble. We prove a series of reductions on the structures of the group G and of the subgroup H.

2.1. *H is a nilpotent subgroup*

By contradiction, assume that the nilpotent residual $H^{\mathcal{N}}$ of H is not trivial. By [**7**, Lemma 7.6.6(a)], $H^{\mathcal{N}}$ is subnormal in H^A, and so in G too. Set $N := (H^{\mathcal{N}})^A = \langle (H^{\mathcal{N}})^a | a \in A \rangle$. By Corollary 1, N is normalized by H, and so by the whole group G. By induction on $|G|$, we can assume that $HN \operatorname{sn} G$. By [**7**, Lemma 7.6.6(b)], H is then subnormal in HN, and so in G. Therefore $H^{\mathcal{N}} = 1$, and H is nilpotent.

2.2. *H is a t-group, for some prime number t*

By contradiction, let t and r be two different prime divisors of $|H|$ and let T and R be, respectively, the non-trivial t- and r-Sylow subgroups of H. By induction on $|H|$, $O^r(H)$ is subnormal in $\langle O^r(H), A\rangle$. In particular, using Corollary 1,

$$O^r(H) \operatorname{sn} O^r(H)^A = O^r(H)^G \trianglelefteq G.$$

As $1 \neq T \trianglelefteq O^r(H) \operatorname{sn} G$, we get $O_t(G) \neq 1$. Arguing in a similar way $O^t(H)$ is subnormal in G. By Lemma 4, $O_t(G)$ normalizes $O^t(H)$, thus $O^t(H) \trianglelefteq HO_t(G) \operatorname{sn} G$, where the last is by induction on $|G|$. As $HO_t(G)/O^t(H)$ is a t-group, we have $H \operatorname{sn} HO_t(G) \operatorname{sn} G$, which contradicts our assumption.

2.3. Fit $(G) = 1$

Let M be an abelian minimal normal subgroup of G. As we can assume that $A_G = 1$, M is an elementary abelian p-subgroup of G. By induction on $|G|$, we have $HM/M \leqslant O_t(G/M) =: X/M$. Let T be a Sylow t-subgroup of A such that $T \cap X$ is a Sylow t-subgroup of X and $X = M \rtimes (T \cap X)$. Then $T \cap X \leqslant A \subseteq S^1_T(H)$, and so, by the soluble case treated before, H is subnormal in $\langle H, T \cap X\rangle$. However, we then have $\langle H, T \cap X \rangle = T \cap X$, which forces $H \leqslant A$, $G = A$ and H subnormal in G by the Wielandt criterion.

2.4. *$G = MH$, where M is a minimal normal subgroup of G*

By contradiction, assume that MH is properly contained in G. Then, working modulo M, $\overline{MH} \operatorname{sn} \overline{G}$, forcing $MH \operatorname{sn} G$. By Lemma 7, the index of $MH \cap A$ in MH is a power of p. Thus by induction on $|G|$, H is subnormal in the subgroup $W := \langle H, MH \cap A \rangle$ and so $H \leqslant O_t(W)$. In particular, $t = p$, otherwise $O_t(W) \leqslant MH \cap A$, which implies $H \leqslant A$, leading immediately to a contradiction. Since W has p-power index in MH, by Lemma 8, $O_p(W) \leqslant O_p(MH)$. Then $H \operatorname{sn} O_p(MH)$, and we conclude in this case that $H \operatorname{sn} G$.

2.5. *M is a non-abelian simple group*

Assume that M is the direct product of, say, $k > 1$ isomorphic copies, $\{S_i\}_{i=1,\dots,k}$, of a non-abelian simple group S. As the index of $M \cap A$ in M divides $|G : A|$, for every $i = 1, 2, \dots, k$, $|S_i : S_i \cap A|$ is a p-power. Let a be an arbitrary element of $S_1 \cap A$ and let $h \in H$. If h does not normalize S_1, then $a^h \in S_j$, for some $j \neq 1$, thus the element $a^{-1}a^h = [a, h]$ has order $|a|$. However, $[a, h] = (h^{-1})^a h$ also lies in $\langle H, H^a\rangle$ and so it must be a t-element. This shows that $S_1 \cap A$ is a t-group, which is impossible as S_1 is simple non-abelian.

2.6. *H is cyclic, moreover if $t = p$, $|H| = p$*

Let K be a maximal subgroup of H, with MK a normal subgroup of G of index t. By applying the inductive hypothesis on it, K is subnormal in the subgroup $W := \langle K, MK \cap A\rangle$. If $t = p$, then K lies in $O_p(W)$. Since W has p power index in G, by Lemma 8, we have $O_p(W) \leqslant O_p(G) = 1$, in particular, $K = 1$ and $|H| = p$. Assume that $t \neq p$, then $K \leqslant O_t(W) \leqslant MK \cap A$, as this has index in W coprime with t. In particular $K \leqslant A$ and H is cyclic having $H \cap A$ as its unique maximal subgroup.

Summarizing Sections 2.1–2.6, our minimal counterexample G is an insoluble group $G = \langle A, H \rangle = MH$, where M is a finite non-abelian simple group and H is a cyclic t-group, for some prime t. In particular, the condition $A \subseteq S^1_G(H)$ simply means that every subgroup $\langle H, H^a \rangle$, $a \in A$, is a t-group.

From now on set $H = \langle h \rangle$ and assume that it acts on M non-trivially; also set $A^* := M \cap A$.

Guralnick [**2**] gives a complete classification of all finite non-abelian simple groups admitting a subgroup of prime power index. With our notation these are precisely the ones listed here.

(1) M is the alternating group A_n and $A^* \simeq A_{n-1}$, with $n = p^a$.

(2) $M = \mathrm{PSL}(n, q)$ and A^* is the stabilizer of a projective point or a hyperplane such that $|M : A^*| = (q^n - 1)/(q - 1) = p^a$.

(3) $M = \mathrm{PSL}(2, 11)$ and $A^* \simeq A_5$.

(4) M is the Mathieu group M_{23} and $A^* \simeq M_{22}$, or $M = M_{11}$ and $A^* \simeq M_{10}$.

(5) $M = \mathrm{PSU}(4, 2) \simeq \mathrm{PS}p(4, 3)$ and A^* is a parabolic subgroup of index 27.

We examine separately the different cases and show how to reach a contradiction in any of these.

2.6.1. *Alternating and symmetric groups.* Let M be the alternating group A_n of degree $n = p^a \geqslant 5$. The group $G = M\langle h \rangle$ is either A_n or S_n, according to whether h lies in M or not. In any case, the subgroup A of p-power index in G is the stabilizer of some point and it is isomorphic either to A_{n-1} or to S_{n-1}.

Consider first the case $G = M = A_n$. Let h_1 be the element of prime order t in H. We claim that $h_1 \notin A$. Otherwise, $A \subseteq S^1_G(\langle h_1 \rangle)$, and by the Wielandt criterion $\langle h_1 \rangle$ is subnormal in A, contradicting the simplicity of A, if $n > 5$. Note that if $n = 5$, then it must be that $t = 2$, but then, as the Sylow 2-subgroups of G are elementary abelian of order 4, $h = h_1$ and so $h_1 \in A$ would imply $A = G$, which is a contradiction. Therefore $h_1 \notin A$, and thus $h = h_1$. Write h as the product of, say, $k \geqslant 1$ t-cycles σ_i $(i = 1, 2, \ldots, k)$. Without loss of generality, we can assume that A is the stabilizer of the point 1 and that $\sigma_1 = (12 \ldots t)$. The element $a_1 := (234)$ belongs to A and

$$h^{-1}h^{a_1} = (235),$$

forcing $t = 3$. If $h = \sigma_1 = (123)$, then $\langle h, h^{a_1} \rangle \simeq A_4$, and so it is not a 3-subgroup. Thus there are at least two t-cycles in the factorization of h. Again there is no loss in assuming $\sigma_2 = (456)$. Take $a_2 := (24)(35)$, then

$$h^{-1}h^{a_2} = (16)(24)$$

which, being not a 3-element, leads to a contradiction.

Assume now that $h \notin M = A_n$ so that $G = S_n$. The subgroup $\langle h \rangle$ is then a cyclic 2-group. Without loss of generality, we assume again that the stabilizer of 1 in A_n, namely $A_n(1)$, is contained in A. Since h is an odd permutation not fixing 1, we can write

$$h = \sigma_1\sigma_2 \ldots \sigma_t$$

as a product of an odd number t of disjoint cycles, each of order a power of 2. Assume that the point 1 lies in the orbit of σ_1. If $t = 1$, then we can assume that $h = \sigma_2 = (12 \ldots 2^m)$. Take the element $a_1 := (234)$ of A. A computation shows that $h^{-1}h^{a_1}$ has order 3, forcing $\langle h, h^{a_1} \rangle$ to be not a 2-subgroup, again a contradiction. Thus $t > 1$. We can suppose that 2, 3 and 4 are points, respectively, in the orbits of σ_1, σ_2 and σ_3. Again the element $a_1 = (234)$ of A is such that $h^{-1}h^{a_1}$ has order 3, producing the same contradiction.

2.6.2. *Projective groups.* Let M be the projective special linear group $\mathrm{PSL}(n, q)$ and A^* the stabilizer in M of a projective point or of a hyperplane. Subgroups of these two types are fused in Aut (M), therefore without loss of generality we can always assume A^* to be the stabilizer of some projective point. Note that $|M : A^*| = (q^n - 1)/(q - 1) = p^a$, and, since p is the unique primitive divisor of $q^n - 1$, A^* is a p'-Hall subgroup of M.

The arguments we use to reach a contradiction require the following lemma more than once. We prefer to state and prove it now separately.

LEMMA 10. *Let $M = \mathrm{PSL}(n, q)$, $q = r^f$, r being the characteristic of the field, $G = M\langle h\rangle$ and $h \notin M$ acting on M as an outer automorphism of order a power of r. Then there does not exist any Borel subgroup of M that lies in $S^1_G(H)$. (In particular, $A^* \nsubseteq S^1_G(\langle h\rangle)$.)*

Proof. By contradiction, let B be a Borel subgroup of M in $S^1_G(H)$. Write $B = U \rtimes C$, with U the unipotent radical and C a Cartan complement; set also $N := N_M(C)$. Then M is equal to BNB. Let U_1 be an r-Sylow subgroup normalized by H, and $B_1 := N_M(U_1)$. Let $g \in M$ be such that $B_1 = B^g$; if we write $g = b_1 n b_2$, with $b_i \in B$ and $n \in N$, then

$$B \cap B_1 \geqslant C^{b_2} =: C_2.$$

Since for all $x \in C_2$, $[h, x]$ is an r-element of B_1, we have that $[H, C_2] \leqslant U_1$. A look at the structure of outer automorphisms of M shows the following dichotomy.

(a) either $G = M\langle\mu\rangle$ for some r-element μ of G that acts on V like a field automorphism or

(b) $r = 2$ and $G = M\langle\mu i\rangle$ for some field automorphism μ and some graph automorphism i of M.

Case 1: Up to conjugation we can assume that μ normalizes U_1. Thus μ also normalizes B_1, and $B_1\langle h\rangle = B_1\langle\mu\rangle$ (otherwise $N_M(B_1) > B_1$ which is a contradiction, as B_1 contains the normalizer in M of an r-Sylow of M). Therefore we can write $h = y\mu^s$, for some r-element $y \in U_1$ and some $s \geqslant 1$. Since for all $x \in C_2$,

$$[h, x] = [y\mu^s, x] = [y, x]^{\mu^s}[\mu^s, x]$$

lies in U_1, we deduce that $[\mu^s, x] \in U_1$. However, μ normalizes B_1, thus in particular, with respect to a basis for V under which the elements of B_1 have upper unitriangular shape, μ acts on the entries of these matrices as a field automorphism, and therefore it normalizes C_2. Then

$$[\mu^s, x] \in C_2 \cap U_1 = 1$$

and $\mathbb{F}_q \subseteq \mathrm{Fix}(\mu^s)$, which means that $\mu^s = 1$ and $h \in M$, which is a contradiction.

Case 2: If h is not associated to any field automorphism of M and $h \notin M$, then G/M is isomorphic to a cyclic subgroup of the abelian group

$$\frac{A(n, q)}{\mathrm{PGL}(n, q)} \simeq \langle\nu\rangle \times \langle i\rangle$$

(where $\langle\nu\rangle$ is the full group of field automorphisms and $\langle i\rangle$ is the group of graph automorphisms of order 2) containing an element not in $\langle\nu\rangle$. Therefore $Mh = M\mu i$, for some field automorphism μ. Moreover, with the same notation as before, we can think that both μ and i are defined on the same base $\mathcal{B}$ under which the elements of U_1 have unitriangular shape and the ones of C_2 have diagonal shape. This means that μ acts on the elements of U_1 as a field automorphism on every entry of such matrices, and i as the inverse transpose; in particular for every $x \in C_2$

$$[i, x] = x^\tau x = x^2.$$

By Sylow's theorem, there exists some element $m \in M$ such that $U_1\langle h\rangle = U_1\langle\mu i\rangle^m$. Let $h = u_1(\mu i)^m$ for some $u_1 \in U_1$; for all $x \in C_2$ we have that

$$[h, x] = [u_1, x]^{\mu i^m}[(\mu i)^m, x] \in U_1$$

and so $[(\mu i)^m, x] \in U_1$. Then U_1C_2 equals $U_1C_2^{(\mu i)^m}$. By the Schur–Zassenhaus theorem there exists some $u_2 \in U_1$ such that $(\mu i)^m u_2 \in N_G(C_2)$. Then

$$[(\mu i)^m u_2, x] \leqslant U_1 \cap C_2 = 1.$$

Now

$$N_G(C_2) = M\,\langle \mu i \rangle \cap N_G(C_2) = N_M(C_2)\,\langle \mu i \rangle$$

so we can write

$$(\mu i)^m u_2 = \mu i n$$

for some element $n \in N_M(C_2)$. Therefore for all $x \in C_2$

$$1 = [\mu i n, x] = [\mu i, x]^n [n, x] = (x^\mu x)^n [n, x]$$

forcing

$$n x n^{-1} = x^{-\mu}.$$

This can happen only if $n \in C_2$ and μ inverts the elements of C_2. However, then μi acts like the transpose on the matrices representing the elements of M in the base $\mathcal{B}$, and so μi is not an automorphism of M, which is the required contradiction. □

We subdivide our analysis into two cases, according to the dimension n being 2 or greater.

(1) Let $n = 2$.

According to [**2**], the condition $q + 1 = p^a$ occurs exactly when:

(i) $q = r$ is a Mersenne prime of the form $2^a - 1$, $p = 2$;

(ii) $q = 2^f$, p is a Fermat prime and $a = 1$;

(iii) $q = 8$ and $p^a = 9$.

(i) Let $M = \mathrm{PSL}(2, r)$, where $r = 2^a - 1$ is a Mersenne prime, and $a \geqslant 3$. As $|\mathrm{Out}\,(M)| = 2$, either $G = M = \mathrm{PSL}(2, r)$ or $G = \mathrm{PGL}(2, r)$. In both situations, for $t \neq 2$ the Sylow t-subgroups of G are cyclic [**5**, II.8.10]. Thus by Lemma 6 we reach a contradiction with the fact that $\mathrm{Fit}\,(G) = 1$. Therefore t equals 2. Note that $t = p$, and so by Subsection 2.6 in the reductive sections, we can assume that h is an involution of G. Let $\langle v_1 \rangle$ be the projective point, in the natural module V, stabilized by A. Since $\langle v_1 \rangle$ is not $\langle h \rangle$-invariant, we fix $\mathcal{B} := \{v_1, v_1^h\}$ as a basis for V. Let α be an element of the ground field $\mathbb{F}_r$ of multiplicative odd order and let a be the element of A represented by the diagonal matrix $\mathrm{diag}(\alpha, \alpha^{-1})$, with respect to $\mathcal{B}$. Then

$$[h, a] = \mathrm{diag}(\alpha^2, \alpha^{-2}),$$

which is an element of odd order, in contradiction to the fact that it must lie in the 2-subgroup $\langle h, h^a \rangle$.

(ii) Let $p = 2^f + 1$ be a Fermat prime and $M = \mathrm{PSL}(2, 2^f)$. The group M has abelian Sylow subgroups [**5**, II.8.27]. Therefore if $G = M$ we reach a contradiction by Lemma 6 and the simplicity of G. Assume that $h \notin M$. The order the outer automorphism group of M is f, which is a power of 2, p being a Fermat prime. Therefore $t = 2 = r$. We apply Lemma 10 to obtain the required contradiction.

(iii) Let $M = \mathrm{PSL}(2, 8)$. Suppose that M has abelian Sylow subgroups, thus by Lemma 6 we can assume that M is strictly contained in G. Therefore $\langle h \rangle$ has order 3 and $G = M\,\langle h \rangle = P\Gamma L(2, 8)$. Note that A^* is a Hall $3'$-subgroup of G and is the normalizer in M of a Sylow 2-subgroup of G. By order arguments, we have that the intersection of any two conjugates of A^* contains a Sylow 7-subgroup of G. Let $\langle x \rangle$ be a subgroup of order 7 in $A \cap A^{h^{-1}}$, then Q7

$$[x, h] = (h^x)^{-1} h = x^{-1} x^h$$

lies both in $\langle h, h^x\rangle$, which is a 3-group and in A^*, which is a $3'$-subgroup, therefore $[x,h]=1$, and the subgroup H centralizes a 7-Sylow of G. This is impossible, since the normalizers in G of the 7-Sylow subgroups are Frobenius groups of order 42.

(2) Now let $n \geqslant 3$. The condition $(q^n-1)/(q-1)=p^a$ implies that p is the unique primitive divisor of q^n-1. In particular n is a prime number and $p^a \equiv 1 \pmod n$.

Lemma 11. *$t=r$, the characteristic of the field.*

Proof. Proceed by contradiction. Assume first that $t=p$. As p is the unique primitive divisor of $r^{fn}-1$, it is easy to see that $p \nmid f$. Moreover $p \neq 2$ and $p \neq n$ (as $p^a \equiv 1 \pmod n$). Therefore $p \nmid 2df = |\mathrm{Out}\,(M)|$ (where $d=(n,q-1)$), and so, in this situation, $\langle h\rangle$ lies in M. As the Sylow p-subgroups of M are cyclic [**5**, II.7.3], we reach a contradiction by Lemma 6. Assume that $t \neq p$. Since A has index p^a in G, $G=MA$ and $\langle h\rangle$ is contained in a Sylow t-subgroup of some conjugate of A, say $H \leqslant A^m$ (for $m \in M$). Under our assumptions, $(A^*)^m=(A\cap M)^m$ is the stabilizer in M of some projective point, say $\langle v_1\rangle$. In particular, $O_r(A\cap M)\neq 1$. Moreover we can assume that $O_r(A\cap M)=O_r(A)$, otherwise we would have $G=MO_r(A)$, and thus $t=r$. As $h \notin A$, A^* is the stabilizer in M of some $\langle v_2\rangle \neq \langle v_1\rangle$. Set $X := O_r(A^m)\cap A$. Then $X \leqslant M$ and for all $x \in X$, the element

$$[h,x] \in \langle h, h^x\rangle \cap O_r(A^m)$$

is both a t-element and an r-element. If it were $t \neq r$, then we conclude that $[H,X]=1$. Take any $a \in A\cap A^m\cap M$ and b any element of X, then

$$[a,b,h] \in [O_r(A^m)\cap A, \langle h\rangle] = [X,\langle h\rangle] = 1$$

and

$$[b,h,a] \in [[X,\langle h\rangle],A] = 1.$$

By the three-subgroup lemma, $[h,a] \in C_{A^m\cap M}(O_r(A^m)\cap A)$, which is an r-subgroup of $\mathrm{PSL}(n,q)$. Therefore if $t \neq r$, we must have

$$[\langle h\rangle, A\cap A^m\cap M]=1.$$

Let now $Y := O_r(A)\cap A^m$. Then $Y \leqslant A\cap A^m\cap M$ and $[O_r(A),Y]=1$, since $O_r(A)$ is abelian. By the three-subgroup lemma again, we conclude that

$$[H,O_r(A)] \leqslant C_M(Y).$$

Again a matrix computation shows that $C_M(Y)$ is an r-group, and therefore under our contradictory assumption,

$$[\langle h\rangle, O_r(A)]=1.$$

However, then $O_r(A)$ is a non-trivial normal subgroup of G, and this is impossible. □

By Lemmas 10 and 11, we are reduced to consider only the case when $G=M=\mathrm{PSL}(n,q)$ and $\langle h\rangle$ is an r-subgroup, r being the characteristic of the field. We show now how to reach the last contradiction.

Since $r\neq p$, $\langle h\rangle$ lies in a Sylow r-subgroup of some conjugate A^g of A. Assume that A^g and A are, respectively, the stabilizers of the projective points $\langle v_1\rangle$ and $\langle v_2\rangle$. Set W the $\langle h\rangle$-invariant subspace of V generated by $\langle v_1\rangle$ and $\langle v_2\rangle$. Suppose first that $\dim(W)=2$. We can choose an appropriate basis $\mathcal{B}$ for V with respect to which the restriction of h to W can be represented by the following projective matrix

$$\begin{bmatrix} 1 & b \\ 0 & \lambda \end{bmatrix}$$

for some $b, \lambda \in \mathbb{F}_q$, $\lambda \neq 0$. Moreover, as $h \notin A$, $b \neq 0$. Computation then shows that

$$h^r_{|W} = \begin{bmatrix} 1 & b\Phi_r(\lambda) \\ 0 & \lambda^r \end{bmatrix},$$

where $\Phi_r(X)$ denotes the cyclotomic polynomial associated to the prime r. As $h^r \in A$, λ is an rth-root of unity. But $r = \operatorname{char} \mathbb{F}_q$, therefore $\lambda = 1$, that is, with respect to $\mathcal{B}$

$$h_{|W} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}.$$

Let now a be any element of A such that

$$a_{|W} = \begin{bmatrix} 1 & 0 \\ b^{-1} & 1 \end{bmatrix}.$$

Then

$$[h, a]_{|W} = \begin{bmatrix} 3 & b \\ -b^{-1} & 0 \end{bmatrix}.$$

In particular $[h, a] \neq 1$ and $r \neq 3$, otherwise the element $[h, a]_{|W}$ has order 2, which is not a power of r, contrary to the fact that $[h, a]$ lies in $\langle h, h^a \rangle$. However, then a matter of computation shows that the element $h_{|W} \cdot (h^a)_{|W}$ has order 3, contrary to the fact that it must be a power of r.

Assume therefore that $\dim(W) \geqslant 3$. Set $v_3 := {v_2}^h$. If $r \neq 2$, then we choose an involution $a \in A$ such that $a(v_1) = -v_1$, $a(v_2) = -v_2$, $a(v_3) = v_3$. Then $[h, a]$ fixes v_1 and sends v_3 to $-v_3$, its order therefore must be even, contrary to the fact that we are assuming $r \neq 2$. Thus r is equal to 2. Since $h^2 \in A$ we have that $v_3^h \in \langle v_2 \rangle$. Take $a \in A$ such that it interchanges $\langle v_1 \rangle$ with $\langle v_3 \rangle$. Then

$$\begin{aligned} [h, a] : \langle v_1 \rangle &\longmapsto \langle v_2 \rangle \\ \langle v_2 \rangle &\longmapsto \langle v_3 \rangle \\ \langle v_3 \rangle &\longmapsto \langle v_1 \rangle \end{aligned}$$

forcing the order of $[h, a]$ to be a power of 3, in contradiction to the fact that $r = 2$. □ Q8

2.6.3. $M = \mathrm{PSL}(2, 11)$. The subgroups of $\mathrm{PSL}(2, 11)$ of prime power index are isomorphic to A_5 and have index 11. These lie in two conjugacy classes of $\mathrm{PSL}(2, 11)$, which are fused in $\mathrm{PGL}(2, 11)$. In particular, $\mathrm{PGL}(2, 11)$ has no subgroups of index 11. Thus, in our notation, we can exclude the case $h \notin M$. Assume therefore that $G = M$. Since $|\mathrm{PSL}(2, 11)| = 2^2 \cdot 3 \cdot 5 \cdot 11$, G is an A-group. The subnormality of $\langle h \rangle$ in G is guaranteed by Corollary 1, but this contradicts the simplicity of G.

2.6.4. *Mathieu groups.* Let M be either M_{11} or M_{23}. These groups have no outer automorphisms, therefore $h \in M$ and $G = M$. In both cases for a prime $t \neq 2$, the Sylow t-subgroups of G are abelian; Lemma 6 leads therefore to a contradiction if H is not a 2-group. Let $\langle h \rangle$ be a 2-subgroup. Then $\langle h \rangle$, being contained in a conjugate of A, stabilizes some point in the natural permutation action of M, say the point marked by 1. Since M is 2-transitive, we can also assume that A is the stabilizer of 2. Let h_1 be the involution of $\langle h \rangle$, and let 3 be such that $3^h \neq 3$. There exists an element a of A that interchanges 1 and 3 and fixes the element 3^h. In particular $[h, a]$ contains the 3-cycle $(1, 3^h, 3)$ and so it cannot be a 2-element, in contradiction to the fact that $\langle h, h^a \rangle$ must be a 2-subgroup.

2.6.5. $M = \mathrm{PSU}(4, 2)$. Let M be the simple group $\mathrm{PSU}(4, 2)$ having A^* as a maximal parabolic subgroup of index 27. The order of M is $2^6 \cdot 3^4 \cdot 5$, and $\mathrm{Out}\,(M) = C_2$, therefore we limit our considerations to the cases in which $\langle h \rangle$ is either a 2-group or a 3-group. Assume first that $|h|$ is a power of 2. The subgroup A^* is the stabilizer of a unitary projective line, in particular it contains some involutions that are regular unipotent elements of M. Each of these elements, according to [**4**], lies in a unique Sylow 2-subgroup of M. Let $a \in A$ be any of these regular unipotent involutions. As $A \subseteq S^1_G(\langle h \rangle)$ and $|a| = 2$, by Lemma 1, $\langle h, a \rangle$ is a 2-group. Let S be a Sylow 2-subgroup of G containing $\langle h, a \rangle$. Then $S \cap M =: P$ is the unique Sylow 2-subgroup of M that contains the element a. Thus either $h \in P$ or $S = P \cdot \langle h \rangle$; in both cases $\langle h \rangle$ normalizes P. Since we can repeat this argument for every Sylow 2-subgroup of A, and since A is generated by two distinct of these, we have that $\langle h \rangle$ normalizes A, and this is a contradiction.

Consider now the case that $\langle h \rangle$ is a 3-subgroup. Then $G = M$ and, by Section 2.6, we can assume that $|h| = 3$. Let $a \in A$ and let P be a Sylow 3-subgroup of G containing $\langle h, h^a \rangle$. Now P contains a characteristic subgroup X of index 3, which is elementary abelian of order 3^3. Let $N := N_G(X)$. Then N is a maximal subgroup of G of index 40, and by order reasons we have that $|N \cap A| = 2^3 \cdot 3$. The inductive hypothesis shows that $\langle h \rangle$ sn $\langle h, N \cap A \rangle =: W$. Therefore $N = P \cdot W$ and $\langle h \rangle$ is subnormal in both P and W. By [**7**, Theorem 7.7.1] $\langle h \rangle$ is subnormal in N. In a similar way, $\langle h \rangle^a$ sn N. However then $\langle h, h^a \rangle$ sn N, that is, $\langle h, h^a \rangle \leqslant O_3(N) = X$, which is elementary abelian. We conclude that $[h, h^a] = 1$ for all $a \in A$. Therefore $\langle h \rangle$ is a central subgroup of its normal closure $\langle h \rangle^A$, so $\langle h \rangle$ sn G, in contradiction with the simplicity of G. □

3. *Further comments*

(1) Theorems 1–3 of course do hold if we substitute $A \subseteq S^1_G(H)$ with the stronger condition $A \subseteq S_G(H)$. Even the analogs to our initial question for the 'zero'-subnormalizer $S_G(H)$ (replacing $S^1_G(H)$) has a negative answer. In fact the symmetric group S_8 can be generated by the elements

$$h := (12)(34)(56)(78), \quad a_1 := (23)(45)(67), \quad a_2 := (24)(35)(67)$$

and a matter of calculation shows that $A = \langle a_1, a_2 \rangle$ lies in $S_G(\langle h \rangle)$, but of course $\langle h \rangle$ is not subnormal in S_8. However, it would be interesting to find, if it exists, a *soluble* counterexample of this case.

(2) A more general and difficult question, as it generalizes the problem studied in [**4**], is the following.

QUESTION. If H and A are two subgroups of G such that $(|H|, |G : A|) = 1$ and $A \subseteq S^1_G(H)$, is then H subnormal in $\langle H, A \rangle$?

(Note that in our counterexamples both $|H|$ and $|G : A|$ are even.)

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