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Some structural results on the non-abelian tensor square of groups

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Abstract. We study the non-abelian tensor square $G \otimes G$ for the class of groups G that are finitely generated modulo their derived subgroup. In particular, we find conditions on G/G' so that $G \otimes G$ is isomorphic to the direct product of $\nabla(G)$ and the non-abelian exterior square $G \wedge G$. For any group G, we characterize the non-abelian exterior square $G \wedge G$ in terms of a presentation of G. Finally, we apply our results to some classes of groups, such as the classes of free solvable and free nilpotent groups of finite rank, and some classes of finite p-groups.

Introduction

The non-abelian tensor square $G \otimes G$ of a group G is a special case of the non-abelian tensor product $G \otimes H$ of two arbitrary groups G and H that was introduced by Brown and Loday in [5], [6] and it arises from applications of a generalized Van Kampen theorem in homotopy theory.

For all $g, h \in G$ let ${}^g h = ghg^{-1}$ and $[g, h] = ghg^{-1}h^{-1}$. Then $G \otimes G$ is defined as the group generated by the symbols $g \otimes h$, for $g, h \in G$, subject to the relations

$$gh \otimes k = ({}^g h \otimes {}^g k)(g \otimes k) \quad \text{and} \quad g \otimes hk = (g \otimes h)({}^h g \otimes {}^h k).$$

The definition guarantees the existence of an epimorphism $\kappa : G \otimes G \to G'$, defined on the generators by $\kappa(g \otimes h) = [g, h]$ for all $g, h \in G$. Let $J(G)$ be the kernel of the map κ, and let $\nabla(G)$ be the normal subgroup generated by the elements $g \otimes g$, for all $g \in G$. The group $(G \otimes G)/\nabla(G)$ is called *the non-abelian exterior square* of G, and is denoted by $G \wedge G$. The map κ factorizes modulo $\nabla(G)$, thus inducing an epimorphism $\kappa' : G \wedge G \to G'$. By results in [5], [6], the kernel of the map κ' is isomorphic to the Schur multiplicator $M(G)$ of G. Let $\Gamma(G/G')$ be Whitehead's quadratic

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functor, as defined in [16]. Then results in [5], [6] give a commutative diagram with exact rows and central extensions as columns:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & \Gamma(G/G') & \longrightarrow & J(G) & \longrightarrow & M(G) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \nabla(G) & \longrightarrow & G \otimes G & \longrightarrow & G \wedge G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \kappa'} & & \\
 & & 1 & \longrightarrow & G' & \xrightarrow{\text{id}} & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

We are interested in the case when the middle row of the above diagram splits. Our main result in this context is the following.

Theorem 1. *Let G be a group such that G/G' is finitely generated. If G/G' has no elements of order 2, or if G' has a complement in G, then $G \otimes G \simeq \nabla(G) \times (G \wedge G)$.*

We will see that, under the hypotheses of Theorem 1, the structure of the tensor square $G \otimes G$ is completely determined once the structures of G/G' and $G \wedge G$ are known. In [4] Brown, Johnson and Robertson proved that if $M(G)$ is finitely generated then $G \wedge G$ is isomorphic to the derived subgroup of any covering group $\hat{G}$ of G (the notion of a covering group is well known if G is finite, see [11], and in the general case the authors of [4] adopted a similar definition). Our contribution is the following.

Theorem 2. *Let G be a group and let F be a free group such that $G \simeq F/R$ for some normal subgroup R of F. Then $G \wedge G \simeq F'/[F, R]$.*

As corollaries of Theorems 1 and 2, we deduce the structure of non-abelian tensor squares of finitely generated groups that are free in some variety.

The paper is organized as follows. In the first section we collect some background material and prove Theorem 1, while in Section 2 we prove Theorem 2 and derive several consequences. Section 3 deals with finite p-groups G; in particular some upper bounds on the orders of $G \otimes G$ and $M(G)$ are found.

The notation used in this paper is standard (the reader is referred for example to [11]), with the only exception that conjugation and commutation are as defined in the second paragraph of this Introduction.

1 Structure of the non-abelian tensor square

Let G be an arbitrary group. In order to investigate the structure of $G \otimes G$, it is sometimes more convenient to consider the following construction, which was introduced in [14].

Let G^φ be a group isomorphic to G via the isomorphism $\varphi : G \to G^\varphi$, and consider the group

$$\nu(G) := \langle G, G^\varphi \,|\, \mathcal{R}, \mathcal{R}^\varphi, {}^{g_3}[g_1, g_2^\varphi] = [{}^{g_3}g_1, ({}^{g_3}g_2)^\varphi] = {}^{g_3^\varphi}[g_1, g_2^\varphi], \text{ for all } g_1, g_2, g_3 \in G\rangle,$$

where $\mathcal{R}, \mathcal{R}^\varphi$ are the defining relations of G and G^φ respectively.

In [14, Proposition 2.6], the non-abelian tensor square $G \otimes G$ is proved to be isomorphic to the commutator subgroup $[G, G^\varphi]$ inside $\nu(G)$.

From now on we identify $G \otimes G$ with $[G, G^\varphi]$ and, unless differently specified, we write $[g, h^\varphi]$ in place of $g \otimes h$ (for $g, h \in G$). For the reader's convenience we report here some results that we will often use.

Lemma 1.1 ([14, Lemma 2.1], [3, Lemma 2.1], [15, Lemma 3.1]). *Let G be any group. The following relations hold in $\nu(G)$.*

(i) ${}^{[g_3, g_4^\varphi]}[g_1, g_2^\varphi] = {}^{[g_3, g_4]}[g_1, g_2^\varphi] = {}^{[g_3^\varphi, g_4]}[g_1, g_2^\varphi]$, *for all* $g_1, g_2, g_3, g_4 \in G$.
(ii) $[g_1^\varphi, g_2, g_3] = [g_1, g_2^\varphi, g_3] = [g_1, g_2, g_3^\varphi] = [g_1^\varphi, g_2^\varphi, g_3] = [g_1^\varphi, g_2, g_3^\varphi] = [g_1, g_2^\varphi, g_3^\varphi]$, *for all* $g_1, g_2, g_3 \in G$.
(iii) $[g_1, [g_2, g_3]^\varphi] = [g_2, g_3, g_1^\varphi]^{-1}$, *for all* $g_1, g_2, g_3 \in G$.
(iv) $[g, g^\varphi]$ *is central in* $\nu(G)$, *for all* $g \in G$.
(v) $[g, g^\varphi] = 1$, *for all* $g \in G'$.
(vi) *If* $g_1 \in G'$ *or* $g_2 \in G'$, *then* $[g_1, g_2^\varphi]^{-1} = [g_2, g_1^\varphi]$.

For a finitely generated abelian group A, the non-abelian tensor square is simply the ordinary tensor product of two copies of A. In particular, if $\mathcal{A} = \{a_1, \ldots, a_s\}$ is a set of generators of A such that A is the direct product of the cyclic groups $\langle a_1\rangle, \ldots, \langle a_s\rangle$, then we can write

$$A \otimes A = \nabla(A) \times E_{\mathcal{A}}(A),$$

where

$$\nabla(A) = \langle [a_i, a_i^\varphi], [a_i, a_j^\varphi][a_j, a_i^\varphi] \mid 1 \leqslant i < j \leqslant s\rangle$$

and

$$E_{\mathcal{A}}(A) = \langle [a_i, a_j^\varphi] \mid 1 \leqslant i < j \leqslant s\rangle.$$

Observe that $\nabla(A)$ is independent of the set of generators $\mathcal{A}$ of A, since in fact $\nabla(A) = \langle [a, a^\varphi] \mid a \in A\rangle$, while $E_{\mathcal{A}}(A)$ does depend on the choice of $\mathcal{A}$.

It turns out that for any group G such that G^{ab} is finitely generated (in particular, for any finitely generated group G), the structure of $\nabla(G)$ essentially depends on G^{ab}.

The following lemma, which improves the result [15, Proposition 3.3] of Rocco, makes this observation precise.

Lemma 1.2. *Let G be a group such that G^{ab} is finitely generated. Assume that G^{ab} is the direct product of the cyclic groups $\langle x_1 G'\rangle, \dots, \langle x_s G'\rangle$ and set $E(G)$ to be $\langle [x_i, x_j^{\varphi}] \mid i < j\rangle [G', G^{\varphi}]$. Then the following hold*:

(i) $\nabla(G)$ *is generated by the set* $\{[x_i, x_i^{\varphi}], [x_i, x_j^{\varphi}][x_j, x_i^{\varphi}] \mid 1 \leqslant i < j \leqslant s\}$;
(ii) $[G, G^{\varphi}] = \nabla(G)E(G)$.

Proof. (i) Let $\mathrm{Y} = \{y_\alpha\}_{\alpha \in I}$ be a set of generators for G' and let $\mathrm{X} = \{x_i\}_{i=1}^{s}$. Then $\mathscr{G} = \mathrm{X} \cup \mathrm{Y}$ generates G. By [2, Lemma 17] (or [15, Proposition 3.3]), $\nabla(G)$ is generated by $\{[a, a^{\varphi}], [a, b^{\varphi}][b, a^{\varphi}] \mid a, b \in \mathscr{G}\}$.

Note that $[a, a^{\varphi}] = 1$ if $a \in \mathrm{Y}$ (by Lemma 1.1 (v)) and similarly $[a, b^{\varphi}][b, a^{\varphi}] = 1$ if at least one of a and b lies in Y (Lemma 1.1 (vi)).

(ii) Consider the map $f : [G, G^{\varphi}] \to [G^{\mathrm{ab}}, (G^{\mathrm{ab}})^{\varphi}]$ induced by the projection onto G^{ab}. Then $\mathrm{Im} f = f(\nabla(G)\langle [x_i, x_j^{\varphi}] \mid i < j\rangle)$ and $\mathrm{Ker} f = [G', G^{\varphi}] = [G, (G')^{\varphi}]$ (see [14, Remark 3]), so $[G, G^{\varphi}] = \nabla(G)E(G)$. □

We are now able to describe the structure of the non-abelian tensor square $G \otimes G$ in terms of $\nabla(G)$ and the non-abelian exterior square $G \wedge G$. Our result generalizes [4, Proposition 8] and [3, Proposition 3.1].

Theorem 1.3. *Assume that G^{ab} is finitely generated. Then, with the notation of Lemma 1.2, the following hold.*

(i) *The map f_1 defined to be the restriction $f|_{\nabla(G)} : \nabla(G) \to \nabla(G^{\mathrm{ab}})$ of the projection $f : G \to G^{\mathrm{ab}}$ onto G^{ab} has kernel $N = E(G) \cap \nabla(G)$. Moreover, N is a central elementary abelian 2-subgroup of $[G, G^{\varphi}]$ of rank at most the 2-rank $\mathrm{rk}_2(G^{\mathrm{ab}})$ of G^{ab}.*
(ii) $[G, G^{\varphi}]/N \simeq \nabla(G^{\mathrm{ab}}) \times (G \wedge G)$.
(iii) *Suppose either that G^{ab} has no elements of order 2 or that G' has a complement in G. Then $\nabla(G) \simeq \nabla(G^{\mathrm{ab}})$ and $G \otimes G \simeq \nabla(G) \times (G \wedge G)$.*

Proof. (i) Let $w \in \nabla(G) \cap E(G)$. Then

$$f_1(w) = f(w) \in \nabla(G^{\mathrm{ab}}) \cap E(G^{\mathrm{ab}}) = 1,$$

and so $N \leqslant \mathrm{Ker}(f_1)$. Conversely,

$$\mathrm{Ker}(f_1) = \mathrm{Ker}(f) \cap \nabla(G) = [G', G^{\varphi}] \cap \nabla(G) \leqslant N.$$

Moreover, N is a central subgroup of $[G, G^{\varphi}]$, as $N \leqslant \nabla(G)$ (see Lemma 1.1 (iv)). We now recall that there is a sequence of epimorphisms

$$\Gamma(G^{\mathrm{ab}}) \xrightarrow{\psi} \nabla(G) \xrightarrow{f_1} \nabla(G^{\mathrm{ab}}),$$

where $\Gamma(G^{ab})$ is the Whitehead functor on G (see [4]). In particular, if $N_2 = \mathrm{Ker}(\psi f_1)$ and $N_1 = \mathrm{Ker}(\psi)$, then $N \simeq N_2/N_1$. Now [14, Remark 6] proves that N_2 is an elementary abelian 2-group of rank $r = \mathrm{rk}_2(G^{ab})$. Since $N \simeq N_2/N_1$, the result follows.

(ii) By Lemma 1.2 and (i), we have

$$[G, G^{\varphi}]/N \simeq \nabla(G)/N \times E(G)/N.$$

Note that $\nabla(G)/N \simeq \nabla(G^{ab})$ and $E(G)/N \simeq [G, G^{\varphi}]/\nabla(G) = G \wedge G$.

(iii) If G^{ab} has no elements of order 2, then 2 does not divide the order of the torsion part of $\Gamma(G^{ab})$, and so $\Gamma(G^{ab}) \simeq \nabla(G) \simeq \nabla(G^{ab})$, forcing the result.

Assume now that G' has a complement A in G. If we write $g \in G$ as $g = xa$, with $x \in G'$ and $a \in A$, by [15, Lemma 3.1 (iv)] we have $[g, g^{\varphi}] = [a, a^{\varphi}]$, forcing

$$\nabla(G) = \langle [g, g^{\varphi}] \mid g \in G \rangle = \langle [a, a^{\varphi}] \mid a \in A \rangle \simeq \nabla(G^{ab}),$$

and $N = 1$. $\square$

Observation. In [14, Remark 6] it is proved that in the proof of Theorem 1.3 (i) if we have $|x_i| = |x_i G'|$, for $i = 1, \dots, r$, then N_1 has rank r, so $N \simeq N_2/N_1 = 1$, $\nabla(G) \simeq \nabla(G^{ab})$ and $G \otimes G \simeq \nabla(G) \times (G \wedge G)$.

As a consequence of Theorem 1.3 and the fact that the Schur multiplicator $M(G)$ is isomorphic to the quotient $J(G)/\nabla(G)$ (by [15, Proposition 2.8]), we have the following:

Corollary 1.4. *Let G be a group such that G^{ab} is a finitely generated abelian group with no elements of order 2. Then $J(G) \simeq \Gamma(G^{ab}) \times M(G)$.*

We recall the notions of *non-abelian tensor center $Z^{\otimes}(G)$* and *non-abelian exterior center $Z^{\wedge}(G)$* of a group G. These groups are defined in [7] as

$$\begin{aligned} Z^{\otimes}(G) &= \{g \in G \mid [g, x^{\varphi}] = 1, \text{ for all } x \in G\}, \\ Z^{\wedge}(G) &= \{g \in G \mid [g, x^{\varphi}] \in \nabla(G), \text{ for all } x \in G\}. \end{aligned}$$

As Ellis showed in [7] and [8], $Z^{\otimes}(G)$ is a characteristic central subgroup of G and is the largest normal subgroup L of G such that $G \otimes G \simeq G/L \otimes G/L$. The non-abelian exterior center $Z^{\wedge}(G)$ is a central subgroup of G and is equal to the epicenter $Z^{*}(G)$ of G.

Corollary 1.5. *Let G be any group such that G^{ab} is finitely generated. With the notation of Theorem 1.3, if $N = 1$ then $Z^{\otimes}(G) = Z^{\wedge}(G) \cap G'$. In particular, the conclusion holds if G is a finite group of odd order.*

Proof. By the definition of exterior center we have that $[Z^{\wedge}(G) \cap G', G^{\varphi}] \leqslant N = 1$. Therefore $Z^{\wedge}(G) \cap G' \leqslant Z^{\otimes}(G)$. Conversely, we trivially have $Z^{\otimes}(G) \leqslant Z^{\wedge}(G)$.

Let $x \in Z^{\otimes}(G)$. Under the natural map from $G \otimes G$ to $G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}$, the element $1 = [x, x^{\varphi}]$ is mapped to $[xG', (xG')^{\varphi}]$, which is thus the trivial element of the tensor product $G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}$. Hence xG' is the identity element of G^{ab}, so $x \in G'$. We conclude that $Z^{\otimes}(G) \leqslant G'$. □

Question. With the notation of Theorem 1.3, is it always true that

$$N = [Z^{\wedge}(G) \cap G', G^{\varphi}]?$$

Note that a positive answer to this question will imply, by [4, Proposition 9], that $G \otimes G/N$ is isomorphic to the tensor square of G/H, where H is defined to be $Z^{\wedge}(G) \cap G'$.

2 Structure of the non-abelian exterior square

We will now describe the structure of the non-abelian exterior square $G \wedge G$ of a group G. Throughout this section we view the non-abelian tensor square $G \otimes G$ as defined at the beginning of the paper, with generators $g_1 \otimes g_2$, rather than via the isomorphic subgroup $[G, G^{\varphi}]$ of $\nu(G)$. We denote by $g_1 \wedge g_2$ the coset of $G \wedge G$ containing $g_1 \otimes g_2$.

Let G be a group and let $R \xrightarrow{i} F \xrightarrow{\pi} G$ be a presentation for G, where F is a free group. The following theorem is the main result of this section. As the proof uses an argument similar to that of [12, Theorem 2], we will just sketch it.

Theorem 2.1. *Let G be a group and let F be a free group such that $G \simeq F/R$ for some normal subgroup R of F. Then $G \wedge G \simeq F'/[F, R]$.*

Proof. Set F° to be the quotient $F/[F, R]$ and set R° to be $R/[F, R]$, so that

$$1 \to R^{\circ} \xrightarrow{i} F^{\circ} \xrightarrow{\eta} G \to 1 \tag{1}$$

is a central exact sequence. From the sequence (1) and by [4, Proposition 7], there exists a homomorphism $\xi : G \otimes G \to (F^{\circ})'$ such that $\eta\xi$ is the commutator map $\kappa : G \otimes G \to G'$. In particular, ξ operates as follows on the generators $g_1 \otimes g_2$ of $G \otimes G$: $\xi(g_1 \otimes g_2) = [f_1, f_2][F, R]$, where f_1 and f_2 are any two preimages of g_1 and g_2 in F, respectively. Of course, ξ is trivial on the central subgroup $\nabla(G)$, and so it induces a homomorphism

$$\bar{\xi} : G \wedge G \to (F^{\circ})' \simeq F'/[F, R]. \tag{2}$$

It turns out that the map $\bar{\xi}$ is an isomorphism.

The surjectivity of $\bar{\xi}$ follows immediately from its definition. To prove that $\bar{\xi}$ is injective one can consider its restriction $\phi : M(G) \to (F^{\circ})'$ to $M(G)$ and using the same argument as in the proof of [12, Theorem 2] one concludes that ϕ is injective. Then one applies the short five lemma ([1, Proposition 2.10]) to the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M(G) & \xrightarrow{i} & G \wedge G & \xrightarrow{\kappa'} & G' & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \bar{\xi}} & & \Big\downarrow{\scriptstyle 1_{G'}} & & \\ 1 & \longrightarrow & (R \cap F')/[F,R] & \xrightarrow{\tilde{i}} & F'/[F,R] & \xrightarrow{\tilde{\eta}} & G' & \longrightarrow & 1, \end{array}$$

where $\tilde{i}$ is the restriction of the map i to $R^\circ \cap (F^\circ)'$ and $\tilde{\eta}$ is the restriction of η to $(F^\circ)'$. □

As consequences of the results above we can describe the structures of the non-abelian tensor squares of some groups that are ‘universal’ in the sense that they are free in suitable varieties. The next two results are already known, but one may now give immediate proofs for each using Theorems 1.3 and 2.1.

Corollary 2.2 ([4, Proposition 6]). *Let F_n be a free group of rank n. Then*

$$F_n \otimes F_n \simeq \mathbb{Z}^{n(n+1)/2} \times (F_n)'.$$

Corollary 2.3 ([3, Corollary 1.7]). *Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of rank $n > 1$ and class $c \geqslant 1$. Then*

$$G \otimes G \simeq \mathbb{Z}^{n(n+1)/2} \times (\mathcal{N}_{n,c+1})'.$$

Corollary 2.4. *Let F be the free group of finite rank $n > 1$, let d be a natural number, and let $G = F/F^{(d)}$ be the free solvable group $\mathcal{S}_{n,d}$ of derived length d and rank $n > 1$. Then*

$$G \otimes G \simeq \mathbb{Z}^{n(n+1)/2} \times F'/[F, F^{(d)}]$$

is an extension of a nilpotent group of class at most 3 by a free solvable group of derived length $d - 2$ and infinite rank. In particular, if $d = 2$, then $G \otimes G$ is nilpotent.

Proof. Theorems 1.3 and 2.1 imply that $G \otimes G$ has the described factorization. Note that $F^{(d-1)}/[F, F^{(d)}]$ is a normal subgroup of the group $F'/[F, F^{(d)}]$ and that $F^{(d-1)}/[F, F^{(d)}]$ is nilpotent of class at most 3, as it is a quotient of $F^{(d-1)}/\gamma_3(F^{(d-1)})$. Thus $M = \mathbb{Z}^{n(n+1)/2} \times F^{(d-1)}/[F, F^{(d)}]$ is also nilpotent of class at most 3 and $G \otimes G/M$ is isomorphic to $F'/F^{(d-1)}$, so it is free solvable of derived length $d - 2$. The fact that $F'/F^{(d-1)}$ is of infinite rank follows from the well-known fact that F' is not finitely generated. □

We end this section by applying our results to a particular finite p-group.

Let d be an integer and, as before, denote by F_d the free group on d generators. We recall that for every integer i the group $\gamma_i(F_d)/\gamma_{i+1}(F_d)$ is free abelian of rank

$$m_d(i) := \frac{1}{i}\sum_{t|i} \mu(t) d^{i/t},$$

where μ is the Möbius function (see [11, Chapter 3.2]).

We also recall for a fixed prime number p the notion of the *lower central p-series* of a group G. The terms of this series are $\{\lambda_i(G)\}_{i\geqslant 1}$, where $\lambda_1(G) = G$ and $\lambda_{k+1}(G) = [\lambda_k(G), G]\lambda_k(G)^p$, for $k \geqslant 1$. This series is the most rapidly descending central series of G whose factors have exponent p (see [11, Chapter 3]).

For every pair of positive integers d and c, define $G_{d,c}$ to be the quotient $F_d/\lambda_{c+1}(F_d)$. According to [11, Theorem 3.2.10], $G_{d,c}$ is a finite p-group of class c and order p^m, where $m = \sum_{j=1}^{c}(c+1-j)m_d(j)$.

Corollary 2.5. *With the above notation, we have* $G_{d,c} \wedge G_{d,c} \simeq G'_{d,c+1}$ *and*

$$G_{d,c} \otimes G_{d,c} \simeq (\mathbb{Z}_{p^c})^{d(d+1)/2} \times G'_{d,c+1}.$$

Proof. Let $G = G_{d,c}$. We first prove that $G \otimes G \simeq \nabla(G) \times (G \wedge G)$.

For p odd this follows from Theorem 1.3, while for the case $p = 2$ a little more care is needed. More precisely, we observe that if $F_d = \langle f_1, \dots, f_d\rangle$, then the image x_i in $G = F_d/\lambda_{c+1}(F_d)$ of the generator f_i of F_d has order p^c for each i. Moreover, by [11, Theorem 3.2.10], G^{ab} is isomorphic to a direct product of $d = m_d(1)$ cyclic groups $\mathbb{Z}_{p^c}$ of order p^c. So now our claim follows from the observation following Theorem 1.3.

We have $\nabla(G) \simeq (\mathbb{Z}_{p^c})^{d(d+1)/2}$. We show that the derived subgroup of a covering group for G is isomorphic to $G'_{d,c+1}$. In the following, let L_i denote $\lambda_i(F_d)$, $i \geqslant 1$. We note that the group $G_{d,c+1} = F_d/L_{c+2}$ has L_{c+1}/L_{c+2} as a central elementary abelian subgroup. Moreover,

$$M/L_{c+2} = \frac{\gamma_2(F_d)L_{c+2} \cap L_{c+1}}{L_{c+2}} \simeq \frac{L_{c+1} \cap \gamma_2(F_d)}{L_{c+2} \cap \gamma_2(F_d)},$$

which in turn is isomorphic to $M(G)$, by [11, Theorem 3.2.10]. Now let H/L_{c+2} be a complement of M/L_{c+2} in L_{c+1}/L_{c+2} and consider the factor group

$$\overline{G}_{d,c+1} = \frac{G_{d,c+1}}{H/L_{c+2}}.$$

If $N \leqslant G_{d,c+1}$ we denote by $\overline{N}$ the image of N in $\overline{G}_{d,c+1}$ under the canonical projection. It follows that $M(G) \simeq \overline{M} \leqslant Z(\overline{G}_{d,c+1}) \cap \overline{G}'_{d,c+1}$. Moreover,

$$\overline{G}_{d,c+1}/\overline{M} \simeq F_d/L_{c+1} = G,$$

so $\overline{G}_{d,c+1}$ is a covering group for G. Finally, note that $\overline{G}'_{d,c+1} \simeq G'_{d,c+1}$. $\square$

3 Non-abelian tensor squares of finite p-groups

Throughout this section G is a finite p-group, for some prime p. We start with a lemma concerning the lower central p-series of G. We again identify the group $G \otimes G$ with its isomorphic image $[G, G^\varphi]$ in the group $\nu(G)$ defined in Section 2.

Lemma 3.1. *Let G be a finite p-group. Then for every $k \geqslant 1$,*

$$[\lambda_k(G), G^\varphi] = [G, (\lambda_k(G))^\varphi].$$

Proof. We prove the result by induction on k. Since the result is trivial for $k = 1$, we now assume that $[\lambda_k(G), G^\varphi] = [G, (\lambda_k(G))^\varphi]$.

Since $[\lambda_k(G), G, G^\varphi]$ and $[\lambda_k(G)^p, G^\varphi]$ are both normal in $\nu(G)$, we have

$$[\lambda_{k+1}(G), G^\varphi] = [[\lambda_k(G)\lambda_k(G)^p, G], G^\varphi][\lambda_k(G), G, G^\varphi][\lambda_k(G)^p, G^\varphi].$$

Using Lemma 1.1 (ii), we have

$$[\lambda_k(G), G, G^\varphi] = [\lambda_k(G)^\varphi, G^\varphi, G] = [G, [\lambda_k(G), G]^\varphi] \leqslant [G, \lambda_{k+1}(G)^\varphi].$$

Thus our proof will be complete if we show that $[\lambda_k(G)^p, G^\varphi] \leqslant [G, \lambda_{k+1}(G)^\varphi]$.
Define R to be $[\lambda_k(G), G, G^\varphi]$ $(= [G, [\lambda_k(G), G]^\varphi])$.

Note that R contains the derived subgroup of $[\lambda_k(G), G^\varphi]$. To see this, we observe that $[\lambda_k(G), G^\varphi]'$ is generated by the elements

$$[[x, a^\varphi], [y, b^\varphi]], \quad \text{where } x, y \in \lambda_k(G) \text{ and } a^\varphi, b^\varphi \in G^\varphi,$$

and, by Lemma 1.1 (i) and the defining properties of $\nu(G)$, we have

$$[[x, a^\varphi], [y, b^\varphi]] = [[x, a], [y, b]^\varphi] \in R.$$

We claim that the following hold:

$$[x^m, a^\varphi] \in [x, a^\varphi]^m R \quad \text{for all } x \in \lambda_k(G), a^\varphi \in G^\varphi, m \in \mathbb{N}, \tag{3}$$

$$[y, (b^m)^\varphi] \in [y, b^\varphi]^m R \quad \text{for all } y \in G, b^\varphi \in (\lambda_k(G))^\varphi, m \in \mathbb{N}. \tag{4}$$

We prove (3) by induction on m. The proof of (4) is similar.
If $m = 1$ then (3) is trivially true. Let $m \geqslant 2$. Then

$$[x^m, a^\varphi] = [x \cdot x^{m-1}, a^\varphi] = {}^x[x^{m-1}, a^\varphi][x, a^\varphi] = [x^{m-1}, ({}^x a)^\varphi][x, a^\varphi].$$

Now the claim is proved since, by induction on m, the term $[x^{m-1}, ({}^x a)^\varphi]$ lies in the coset

$$\begin{aligned}[x, ({}^x a)^\varphi]^{m-1} R &= [x, [x, a]^\varphi a^\varphi]^{m-1} R = ([x, [x, a]^\varphi] \cdot {}^{[x,a]^\varphi}[x, a^\varphi])^{m-1} R \\ &= ([x, [x, a]^\varphi] \cdot [x, a^\varphi])^{m-1} R = ([x, a^\varphi])^{m-1} R,\end{aligned}$$

by repeated use of Lemma 1.1, and the fact that $[x,[x,a]^{\varphi}] = [x,a,x^{\varphi}]^{-1} \in R$. Therefore claims (3) and (4) are true, and we now complete the proof of the lemma. The group $[\lambda_k(G)^p, G^{\varphi}]$ is generated by elements of the form $[x^p, a^{\varphi}]$ with $x \in \lambda_k(G)$ and $a^{\varphi} \in G^{\varphi}$. By (3) we have $[x^p, a^{\varphi}] \in ([x, a^{\varphi}])^p R$. Now

$$[x, a^{\varphi}] \in [\lambda_k(G), G^{\varphi}] = [G, (\lambda_k(G))^{\varphi}]$$

by the inductive hypothesis, so we may write

$$[x, a^{\varphi}] = w_1 \dots w_l,$$

with $w_i = [y_i, b_i^{\varphi}]$, $y_i \in G$ and $b_i^{\varphi} \in \lambda_k(G)^{\varphi}$ for $i = 1, \dots, l$. In particular, since $[\lambda_k(G), G^{\varphi}]/R$ is abelian we have $[x, a^{\varphi}]^p R = w_1^p \dots w_l^p R$. Finally, by (4) we have $w_i^p R = [y_i, (b_i^p)^{\varphi}]R$ for $i = 1, \dots, l$, forcing

$$[x^p, a^{\varphi}] \in R[G, (\lambda_k(G)^p)^{\varphi}] = [G, (\lambda_{k+1}(G))^{\varphi}]. \quad \square$$

The following result is an improvement of [14, Corollary 3.12]. In [10] it is proved using arguments different from ours.

Proposition 3.2. *Let G be a finite group of order p^n (with p prime) and let $d = d(G)$ be the minimum number of generators of G. Then $p^{d^2} \leqslant |[G, G^{\varphi}]| \leqslant p^{nd}$.*

Proof. Of course $|[G, G^{\varphi}]| \geqslant p^{d^2}$, as $G \otimes G$ admits $G/\Phi(G) \otimes G/\Phi(G)$ as a quotient, and $G/\Phi(G) \otimes G/\Phi(G)$ is elementary abelian of order p^{d^2}, since it is an ordinary tensor product.

Let $\lambda_k(G)$ be the last non-trivial term of the series $\{\lambda_i(G)\}_i$, and let $\pi : G \to \overline{G} = G/\lambda_k(G)$ be the quotient map. The map π induces a natural epimorphism $\tilde{\pi} : [G, G^{\varphi}] \to [\overline{G}, \overline{G}^{\varphi}]$. According to [14, Remark 3] and using Lemma 3.1, we have

$$\mathrm{Ker}(\tilde{\pi}) = [\lambda_k(G), G^{\varphi}][G, \lambda_k(G)^{\varphi}] = [\lambda_k(G), G^{\varphi}].$$

Since $\lambda_k(G)$ is a central elementary abelian subgroup of G, by Lemma 1.1 (ii), we have that $\mathrm{Ker}(\tilde{\pi})$ is an elementary abelian p-subgroup lying in the center of $\nu(G)$. Thus the map

$$\theta : \lambda_k(G) \times G \to [\lambda_k(G), G^{\varphi}]; \quad (a, g) \longmapsto [a, g^{\varphi}]$$

is bilinear. Let $\lambda_k(G)$ be generated by the set $\{a_i \mid i = 1, \dots, d_k\}$ and let G be generated by $\{g_i \mid i = 1, \dots, d\}$. Therefore $\mathrm{Ker}(\tilde{\pi})$ is generated by the set

$$\{[a_i, g_j^{\varphi}] \mid i = 1, \dots, d_k, j = 1, \dots, d\},$$

forcing $|\mathrm{Ker}(\tilde{\pi})| \leqslant p^{d \cdot d_k}$, and $|[G, G^{\varphi}]| \leqslant p^{d \cdot d_k} |[\overline{G}, \overline{G}^{\varphi}]|$. By induction we obtain that

$$|[G, G^{\varphi}]| \leqslant p^{d \cdot d_k} \dots p^{d^2} = p^{d \sum_{i=1}^{k} d_i} = p^{nd}. \quad \square$$

Remark. Homocyclic abelian groups show that the upper bound in Proposition 3.2 is best possible. Another example for which the upper bound is reached is when G is the group $G_{2,2} = F_2/\lambda_3(F_2)$.

As a consequence of our results we have the following bound for the order of the Schur multiplicator of finite p-groups.

Corollary 3.3. *Let G be a finite p-group of order p^n with $d = d(G)$ generators. If p is odd, the order of the Schur multiplicator $M(G)$ of G is at most $p^{d(n-(d+1)/2)}$. If $p = 2$, then $|M(G)| \leqslant 2^{d(n-(d+3)/2)}$.*

Proof. By Theorem 2.1 and the definition of the exterior square,

$$|M(G)|\,|G'| = |G \wedge G| = \frac{|G \otimes G|}{|\nabla(G)|}.$$

If p is odd, by Theorem 1.3, $\nabla(G) \simeq \nabla(G^{\mathrm{ab}})$, and so $|\nabla(G)| \geqslant p^{d(d+1)/2}$. If $p = 2$, then $|\nabla(G)| \geqslant p^{d(d+3)/2}$. The proof is now completed by using the bounds given in Proposition 3.2. $\square$

For other results on the non-abelian tensor squares of finite p-groups see [13], where powerful p-groups are considered and another bound on order of the Schur multiplicator is given, using different invariants.

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