Mathematical Models for problems with free boundaries

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Overview

Some phenomena are modeled by differential equations defined in domains in which the boundary is partially or entirely unknown. This kind of problems are termed *Free Boundary Problems* (FBPs) and require a further condition to exclude indeterminacy.

FBPs raise interesting mathematical issues such as *existence of solution in function spaces, uniqueness, regularity properties, stability and numerical approximation procedures*.

Examples of FBPs occurs in phase transition, filtration through porous media, ferromagnetism, reaction-diffusion, fluid dynamics, biomathematics and so on.

Aim of the course

The main scope of the course is to provide some example of processes that can be modeled by means of FBPs. In particular we will focus on the mathematical formulation of the problems and on the derivation of the free boundary conditions. We will also be dealing with analytical issues such as well posedness and regularity.

We will study the following problems:

1. Phase transition models (Stefan)
2. Viscoplastic models (Bingham)
3. Reaction-diffusion models (R-D with dead cores, Oxygen consumption)

Mathematical Prerequisites

1. basic knowledge on continuum mechanics (kinematics, general balance laws)
2. basic knowledge on parabolic equations (representation formulas, maximum principles)
3. some basic tool from funcional analysis (fixed point theorems)
4. basic knowledge on function spaces and weak formulation for parabolic pde (test functions)

The Stefan Problem

Determine the thermal evolution of a homogeneous medium undergoing a phase change (e.g. ice passing to water) and describe the evolving boundary separating the two phases.

Mathematical formulation: FBP

Phase Transition (solid-liquid): T temperature, c heat capacity, k heat conductivity

$$\varrho c \frac{\partial T}{\partial t} - \nabla \cdot (\kappa \nabla T) = 0, \qquad \in \Omega_t$$

Find T in the liquid and in the solid domain separated by Γ (the free boundary $S(\vec{x}, t) = 0$). Interface conditions are

$$T = T_o, \qquad [\![k \frac{\partial T}{\partial \vec{n}}]\!] \cdot \nabla S + \mathcal{L} \frac{\partial S}{\partial t} = 0 \qquad \text{on} \quad \Gamma$$

Mathematical analysis

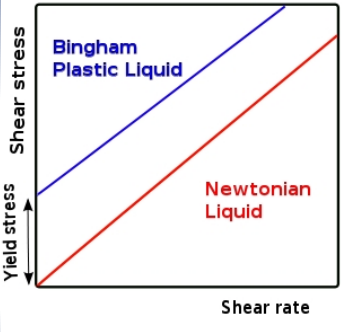
1. Existence of self-similar solutions for the one-phase problem
2. Classical solution $T(\vec{x}, t) \in C^{2,1}(\Omega_t)$ for one-phase and two-phase
3. Stability results (continuous dependence, continuation, etc)
4. Weak formulation and well posedness in Sobolev space $W^{2,1}(\Omega_t)$

REFERENCES

(1) A. FASANO, Mathematical models of some diffusive processes with free boundaries, e-Lecture Notes SIMAI, 1 (2008)

(2) L.I. RUBINSTEIN, The Stefan Problem, Translations of Mathematical Monographs 27, American Mathematical Society (1971)

The Bingham model



Describe the motion of a fluid that behaves like a rigid body for low stresses and as a viscous fluid for high stresses (Bingham).

Mathematical formulation: FBP

Constitutive equation

$$\mathbf{T} = \begin{cases} -P\mathbf{I} + \left[2\eta + \dfrac{\tau_o}{II_D}\right]\mathbf{D}, & II_T > \tau_o \\ \mathbf{D} = 0, & II_T \leq \tau_o \end{cases}$$

Solid-Liquid interface conditions

$$[\![\vec{v}]\!] \cdot \vec{t} = 0, \qquad II_T = \tau_o$$

$$[\![\mathbf{T}\vec{n}]\!] \cdot \vec{t} = 0, \qquad [\![\mathbf{T}\vec{n}]\!] \cdot \vec{n} = 0$$

Mathematical analysis

1. One dimensional case: reduction to a Stefan Problem with Cauchy data
2. Extension to elastic and visco elastic cores (well posedness in the weak sense)
3. Two dimensional case: the lubrication approach
4. Approximating models

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(1) L.I. RUBINSTEIN, The Stefan Problem, Translations of Mathematical Monographs 27, American Mathematical Society (1971)

(2) E. COMPARINI, A one dimensional Bingham flow, J. Math. Anal. App., 169, 127-139 (1992)

(3) L. FUSI, A. FARINA, F. ROSSO, Flow of a Bingham-like fluid in a finite channel of varying width: A two-scale approach, Journal of Non-Newtonian Fluid Mechanics 76–88 (2012)

Reaction-Diffusion Model

Determine the concentration of one or more substances under the influence of chemical reactions and diffusion.

Mathematical formulation: FBP

$$
\begin{cases}
\dfrac{\partial c}{\partial t} - \nabla \cdot (D(c)\nabla c) = \mathcal{F}(c, T) \\[2ex]
\varrho c \dfrac{\partial T}{\partial t} - \nabla \cdot (k(T)\nabla T) = \mathcal{G}(c, T)
\end{cases}
$$

Dead core (regions where $c = 0$) interface conditions

$$
c = 0, \qquad \frac{\partial c}{\partial \vec{n}} = 0
$$

Mathematical analysis

1. One dimensional case with $\mathcal{F}$, $\mathcal{G}$ given by the "Arrhenius factor"
2. Constraints on the existence of dead cores
3. The problem for oxygen consumption $m = 0$
4. Analytical issues (existence, uniqueness, stability, etc)

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(1) A. FASANO, Mathematical models of some diffusive processes with free boundaries, e-Lecture Notes SIMAI, 1 (2008)

(2) I. STAKGOLD, Reaction-diffusion problems in chemical engineering, "Non-linear diffusion problems" (CIME), Lecture Notes in Mathematics, Springer-Verlag (1986)

(3) A. FASANO, M. PRIMICERIO, New results on some classical parabolic free boundary problems, Quart. App. Math. 38, 439-460 (1981)