Mathematical modeling of flows through inelastic porous solids

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Abstract

We propose a framework, based on classical mixture theory, to describe the isothermal flow of an incompressible fluid through a deformable inelastic porous solid. The modeling of the behavior of the inelastic solid takes into account changes in the elastic response due to evolution in the microstructure of the material. We apply the model to a compression layer problem. The mathematical problem generated by the model is a free boundary problem.

Keywords: Mixtures theory; Deformable inelastic porous solids; Free boundary problems

1. Introduction

The research on flow through deformable porous solid media traces its roots to the seminal work by Darcy [19] and further developed by research workers interested in the fields of soil mechanics, soil consolidation under loading, ground water hydrology, petroleum engineering and extraction. We refer to [4,3,29,30,32] where the early papers in the field are cited and to the papers by Terzaghi [47] and Biot [6–9]. The article by Scheidegger [44] also provides a thorough list of references concerning flow through porous media that is pertinent to the above areas.

When a fluid flows through a deformable porous medium the forces associated with the flow deform the porous solid. In turn, the deformation of the porous medium influences the flow. The competition among the stresses in the solid and the stresses in the fluid determines the evolution of the system, which is very different from that observed when the interaction between the flow and the deformation of the solid porous medium is absent.

This paper is concerned with the modeling of the mechanical behavior of a deformable porous solid when a liquid flows through it. The main goal of the paper is to present a rational methodology for formulating constitutive models for solid–fluid mixtures within the context of the recent theory of “natural configurations” developed by Rajagopal et al. [36,38–40,42]. We consider, in fact, the case in which the solid component undergoes microstructural changes as it deforms which in turn alters its response characteristics. We thus develop a technique for describing the constitutive response for certain classes of mixtures which takes into account such microstructural changes.

The general form of constitutive equations for elastic solids is based on the assumption that the stress arises due to a single micromechanism and that the micromechanism does not change as the material deforms. Therefore, it is assumed that the stress depends only on the gradient of mapping of the current configuration with respect to a reference
configuration. Such an approach is inadequate if the solid component of the mixture undergoes microstructural changes leading, for instance, to new physical phenomena such as permanent set upon release of the surface traction (the latter is a rather common phenomenon in both soil mechanics and paper technology where the solid skeleton deforms permanently). To deal with these phenomena, which are usually associated with solids that are classified as “plastic”, one needs to take into account the evolution of the underlying stress-free state (“natural configuration”) as the body deforms.

We will find it convenient to use a theory that has been recently developed to describe bodies that possess more than one stress-free state. The central feature of the theory is that some body possesses numerous stress-free configurations (i.e. natural configurations [36]) whose evolution in time is determined by the rate of dissipation, the response of the body being elastic from each natural configurations.

Although no specific applications are considered here, we analyze an illustrative example concerning the compression of a homogeneous layer of a porous solid saturated with a liquid (water for instance). The mathematical problem turns out to be a very peculiar free boundary problem.

Problems similar to the one considered here arise quite often in paper technology and soil mechanics.

The paper is divided into four sections. Section 2 deals with the mathematical tools which are necessary for modeling mixtures. The main features of mixture theory are briefly recalled and a constitutive model for binary mixtures in which the solid constituent undergo plastic deformation is developed.

In Section 3, we make specific constitutive assumptions and, operating within the framework defined above, we develop the general equations.

Finally, in Section 4, we focus our attention on an application of the theory to model a layer pressing process. We show how the model allows one to incorporate the stresses and strains in the problem of the flow of the fluid through the porous media. The latter is an important feature of the model since in several important technological problems the porous solid is far from being rigid and the stresses can generate significant permanent deformations.

2. Modeling flow through porous inelastic solids

We now recall the main features of mixtures theory (Sections 2.1 and 2.2) and use the notion of “natural configurations” to a mixture composed by an inelastic solid and a fluid (Sections 2.3 and 2.4).

2.1. Basic assumptions

Modeling the behavior of a porous solid and of the fluid (or fluids) that permeates it is not an easy task. It is clear that a complete knowledge of the behavior of the system is achieved when the flow of the fluid in the pores and the displacement of the solid is known. This is, however, an indomitable task, both from the viewpoint of theoretical as well as practical considerations. Rarely, one is interested in information at the microscopic level in most technological considerations. Rather, one is more interested in obtaining a description of some “gross features” of the motion, which then have to be predictable, repeatable and insensitive to small changes in the initial and boundary conditions.

The definition of “gross features” depends on the technological application one is interested in. For instance, one might be interested in measuring macroscopic quantities such as the swelling of the solid or the flow rate of the fluid filtering through the sample and so on.

One should therefore aim at deducing a sufficiently accurate model able to describe in a satisfactory way the important phenomena involved and which is at the same time simple to apply.

There is basically one method for achieving this purpose. Such a method is usually called mixture theory and is one of the fundamental tools to describe the macroscopic behavior of complex systems in which different continua interact.

The basic premise of the theory is that the space occupied by a mixture can be considered to be occupied co-jointly by the various constituents of the mixture, each considered as a continuum in its own right. Thus, at each point in the space occupied by the mixture, there will be a particle belonging to each of the constituent. This approach (also called effective media approach or continuum mechanics approach) to study the evolution of deformable porous media presupposes that the pore sizes are much smaller than any other characteristic length involved in the problem. In this case actual material points are hardly identifiable from a macroscopic point of view and this justifies the assumption of co-occupying continua.
The components of the mixture are capable of moving relative to each other, on the application of forces. For each constituent we can define its motion, mass density, partial stress tensor, internal energy, temperature, entropy and other relevant physical quantities as we do for a single continuum.

Great strides have been made in developing a rational theoretical basis for studying the mechanics of mixtures in the context of such an effective media approach. Starting from the pioneering works of Truesdell [48,49], these papers have been followed by papers on the subject, too numerous to document. The interested reader can find references to them in the articles by Atkin and Craine [1,2], Bowen [11–14], Bedford and Drumheller [5], de Boer [10], the numerous appendices in Rational Thermodynamics by Truesdell [50] and the books by Samohyl [43] and Rajagopal and Tao [37].

2.2. Kinematics and field equations

This section contains an introduction to the kinematics and balance equations for the constituents of a mixture. Here we follow the notation used by Rajagopal and Tao [41].

Let us consider a mixture of two constituents: a solid and a fluid, labeled by the index \( x = s, f \), respectively. The motion of the \( x \)th constituent is described by the position occupied, at time \( t \), by the particle labeled \( \tilde{X}^x \)

\[
\tilde{x} = \chi^x(\tilde{X}^x, t),
\]

where \( \tilde{X}^x \) refers to the position of the particle in the reference configuration \( \kappa^x_o \) relative to the \( x \)th component. The function \( \chi^x \) in (1), usually referred to as deformation function,\(^1\) represents the mapping between \( \kappa^x_o \) and the configuration at time \( t \), the configuration denoted by \( \kappa^x_t \). Its inverse function is assumed to exist and is denoted by \( \chi^x \). Therefore

\[
\tilde{X}^x = \chi^x(\tilde{x}, t).
\]

The spatial description of the velocity and of the acceleration fields are\(^2\)

\[
\begin{align*}
\dot{\tilde{v}}^x(\tilde{x}, t) &= \frac{\partial \chi^x(\tilde{X}^x, t)}{\partial t} \bigg|_{\tilde{X}^x = \chi^x(\tilde{x}, t)} , \\
\ddot{a}^x(\tilde{x}, t) &= \frac{\partial^2 \chi^x(\tilde{X}^x, t)}{\partial t^2} \bigg|_{\tilde{X}^x = \chi^x(\tilde{x}, t)} ,
\end{align*}
\]

respectively. Obviously

\[
\ddot{\tilde{a}}^x(\tilde{x}, t) = \frac{\text{D}^2 \tilde{v}^x(\tilde{x}, t)}{\text{Dt}},
\]

where \( \text{D}^2/\text{Dt} \) denotes the material derivative following the \( x \)th component of the mixture, i.e.

\[
\frac{\text{D}^2 f(\tilde{x}, t)}{\text{Dt}} = \frac{\partial f(\tilde{x}, t)}{\partial t} + \tilde{v}^x(\tilde{x}, t) \cdot \nabla f(\tilde{x}, t),
\]

where \( \nabla \) is the gradient operator with respect to the spatial position \( \tilde{x} \), i.e. with respect to the Eulerian spatial variable.

We introduce the deformation gradient as the linear transformation defined by\(^3\)

\[
\mathbf{F}^x(\tilde{X}^x, t) = \text{Grad}^2[\chi^x(\tilde{X}^x, t)],
\]

where \( \text{Grad}^2 \) denotes the differentiation with respect to \( \tilde{X}^x \). The inverse of \( \mathbf{F}^x \) is

\[
\mathbf{F}^{x^{-1}}(\tilde{x}, t) = \nabla \chi^x(\tilde{x}, t).
\]

---

1 We shall suppose that \( \chi^x \) is \( C^2 \) diffeomorphism.

2 The usual Einstein summation convention does not apply. Summation is carried out on repeated indices only if the summation symbol is expressed explicitly.

3 Denoting by \( \mathcal{F}_{\chi^x}(\kappa^x_o) \) and \( \mathcal{F}_{\chi^x}(\kappa^x_t) \) the tangent spaces of \( \kappa^x_o \) at \( \tilde{X}^x \) and of \( \kappa^x_t \) at \( \tilde{x} \), the second-order tensor \( \mathbf{F}^x \) can be regarded as a linear mapping from \( \mathcal{F}_{\chi^x}(\kappa^x_o) \times \mathbb{R} \) to \( \mathcal{F}_{\chi^x}(\kappa^x_t) \).
The velocity gradient is defined as
\[ \mathbf{L}^x = \nabla \bar{v}^x(\bar{x}, t). \] (9)

In particular, using the chain rule, it can be shown that
\[ \mathbf{L}^x(\bar{x}, t) = \frac{D^2 \mathbf{F}^x}{Dt} \mathbf{F}^{x^{-1}}, \] (10)

where the time derivative of \( \mathbf{F}^x \) is performed keeping fixed \( \bar{X}^x \), i.e.
\[ \frac{D^2 \mathbf{F}^x}{Dt} = \left. \frac{\partial \mathbf{F}^x(\bar{X}^x, t)}{\partial t} \right|_{\bar{X}^x = \bar{X}^x(\bar{x}, t)}. \] (11)

If \( \rho^x \) denotes the density of the \( x \)th constituent, then the density of the mixture is defined through
\[ \rho = \sum_{x=s, f} \rho^x(\bar{x}, t). \] (12)

Physically \( \rho^x \) represents the mass of the \( x \)th constituent per unit volume of the mixture and it is sometimes called bulk density. The “true” density for the \( x \)th constituent is denoted by \( \rho^x_R \) and represents the mass of the \( x \)th constituent per unit volume of the \( x \)th constituent.

In mixture theory great care has to be exercised in interpreting what one means by incompressibility. In fact, if the \( x \)th constituent is incompressible in its pure state, it cannot be then made to occupy a domain that is larger than that occupied by it in the first place. Thus, the domain of the mixture has to be the same as the domain of each individual pure constituent if each of them is incompressible. But this is not what one means by co-occupancy. In the homogenized co-occupied state, i.e. the mixed state, the volume of the mixture is larger than that of the original pure constituent. It is in the mixture that the bulk density \( \rho^x \) is defined and this need not be a constant. If in the pure state the constituents are incompressible we expect the volume of the mixture to be the sum of the volumes of the individual constituents and we homogenize each constituent over the volume of the mixture. Thus, we can impose the constraint that the volumes of the constituents be additive. It is, of course, possible to enforce that the volume of two constituents be additive and neither of the constituents be incompressible in their pure state. We consider a mixture whose volumes are additive and whose constituents in the pure state are incompressible. Thus
\[ \rho^x_R = \text{const.} \quad x = s, f, \] (13)

and
\[ \phi^s + \phi^f = 1, \] (14)

where
\[ \phi^x = \frac{\rho^x}{\rho^x_R}, \quad x = s, f, \] (15)

is the so-called volume fraction of the \( x \)th constituent.

The velocity of the mixture at \((\bar{x}, t)\) is the vector field
\[ \bar{v}(\bar{x}, t) = \frac{1}{\rho} \sum_{x=s, f} \rho^x \bar{v}^x. \] (16)

The diffusion velocity of the \( x \)th component with respect to the mixture is
\[ \bar{u}^x = \bar{v}^x - \bar{v}. \] (17)

It is useful to define the material time derivative following the mixture, that is
\[ \frac{Df(\bar{x}, t)}{Dt} = \frac{\partial f(\bar{x}, t)}{\partial t} + \bar{v}(\bar{x}, t) \cdot \nabla f(\bar{x}, t). \] (18)
According to (6)
\[
\frac{D^2 f}{Dt^2} - \frac{D f}{Dt} = \vec{u} \cdot \nabla \phi.
\]  

(19)

Following the Eulerian formalism the equations for the balance of mass and linear momentum are
\[
\frac{\partial \rho^x}{\partial t} + \text{div} (\rho^x \vec{v}^x) = 0,
\]

(20)  
\[
\frac{\partial}{\partial t} (\rho^x \vec{v}^x) + \text{div} (\rho^x \vec{v}^x \otimes \vec{v}^x) = \text{div} \mathbf{T}^x + \vec{p}^x,
\]

(21)  
\[
\sum_{x=s,f} \vec{p}^x = 0,
\]

(22)

where

- \( \mathbf{T}^x \) is the partial Cauchy stress tensor, i.e. the stress tensor of the \( x \)th component when the other component is co-occupying the same point in space.
- \( \vec{p}^x \) is the momentum supply (also referred to as interaction force) due to local interactions between constituents.

The balance of angular momentum for the mixture states that, in absence of angular momentum supply

\[
\sum_{x=s,f} \mathbf{T}^x = \left( \sum_{x=s,f} \mathbf{T}^x \right)^T.
\]

(23)

Thus, the individual partial stress tensors for the individual constituents need not to be symmetric. In what follow, we shall assume that the partial stress tensors are symmetric.

The balance of energy for the \( x \)th constituents takes the form
\[
\frac{\partial}{\partial t} \left[ \rho^x \left( \varepsilon^x + \frac{|\vec{v}^x|^2}{2} \right) \right] + \text{div} \left[ \rho^x \left( \varepsilon^x + \frac{|\vec{v}^x|^2}{2} \right) \vec{v}^x \right] = \text{div} (\mathbf{T}^x \vec{v}^x - \vec{q}^x) + \vec{p}^x \cdot \vec{v}^x + \mathbf{s}^x,
\]

(24)  
\[
\sum_{x=s,f} (\vec{p}^x \cdot \vec{v}^x + \mathbf{s}^x) = 0,
\]

(25)

where

- \( \varepsilon^x \) is the specific internal energy of the \( x \)th component.
- \( \vec{q}^x \) is the partial heat influx vector associated with the \( x \)th component.
- \( \mathbf{s}^x \) is the energy supply due to energy exchanges between constituents.

In documenting (20), (21) and (24) we have neglected the mass conversion between constituents, body forces and the external energy supply, respectively.

We remark that the mass balance equation can also be written in the following ways:
\[
\frac{D \rho^x}{Dt} + \vec{u} \cdot \nabla \rho^x = -\rho^x \text{tr} \mathbf{L}^x,
\]

(26)  
\[
\frac{D^2 (\rho^x \det \mathbf{F}^x)}{Dt^2} = 0.
\]

(27)

In particular, recalling (15), we have
\[
\rho^x = \frac{\rho_0^x}{\det \mathbf{F}^x} \leftrightarrow \phi^x = \frac{\phi_0^x}{\det \mathbf{F}^x},
\]

(28)
where \( \rho^s_0, \phi^s_0 \) are the mass density and the volume fraction of the \( s \)th constituent in the reference configuration \( \kappa^s_0 \), respectively.

We also notice that, by virtue of (21), Eq. (24) can be rewritten as

\[
\rho^s \frac{D^s \dot{e}^s}{Dt} = \text{tr}(T^s L^s) - \text{div} \tilde{q}^s + s^s.
\]  

(29)

If a tensor \( \hat{T} \) is defined for the mixture as a whole through (see [48,49])

\[
\hat{T} = \sum_{s,f} (T^s - \rho^s \bar{u}^s \otimes \bar{u}^s),
\]  

(30)

Eqs. (20)–(25) lead to the following balance equations for the mixture as a whole:

\[
\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0,
\]  

(31)

\[
\rho \frac{D \mathbf{v}}{Dt} = \text{div} \hat{T},
\]  

(32)

\[
\rho \frac{D e}{Dt} = \sum_{s,f} \left[ \text{tr}(T^s L^s) - \text{div} (\tilde{q}^s + \rho^s \bar{u}^s \cdot \bar{\mathbf{v}}^s) - \mathbf{v}^s \cdot \bar{p}^s \right].
\]  

(33)

However, we shall find it convenient to define the total stress of the mixture as (see [27,28])

\[
T^t = \sum_{s,f} T^s,
\]  

(34)

and consequently Eq. (32) will be modified.

Also, in Eq. (33) \( e \) is the internal energy density for the mixture

\[
e = \frac{1}{\rho} \sum_{s,f} \rho^s \dot{e}^s.
\]  

(35)

Following Bowen [12] and Rajagopal and Tao [41], we introduce the specific entropy \( \eta^s \) for the \( s \)th component (i.e. entropy per unit mass) so that

\[
\mathcal{H}^s(\mathcal{U}) = \int_{\mathcal{U}} \rho^s \eta^s \, d^3 \mathbf{x}
\]  

(36)

is the entropy of the portion of the \( s \)th constituent that at time \( t \) occupies the region \( \mathcal{U} \).

We choose to express the Clausius–Duhem inequality for the mixture as a whole in the following way (see [12,41]):

\[
\frac{\partial}{\partial t} \left( \int_{\mathcal{U}} \sum_{s,f} \rho^s \eta^s \, d^3 \mathbf{x} \right) + \int_{\partial \mathcal{U}} \left( \sum_{s,f} \rho^s \eta^s \bar{u}^s \cdot \bar{n} \, ds \right) \geq \int_{\partial \mathcal{U}} \left( \sum_{s,f} \frac{\mathbf{\tilde{h}}^s}{\theta} \cdot \bar{n} \, ds \right),
\]  

(37)

where

- \( \bar{n} \) is the outward normal to \( \partial \mathcal{U} \),
- \( \mathbf{\tilde{h}}^s \) is the entropy flux (not yet related to \( \tilde{q}^s \)),
- \( \theta \) is the absolute temperature. It is possible to associate different temperatures with different constituents. Here, we shall assume that all the constituents have the same temperature associated with them.
If we define the specific entropy for the mixture through

\[ \eta = \frac{1}{\rho} \sum_{x=s,f} \rho^x \eta^x, \]  

(38)

then inequality (37) can be rewritten as

\[ \rho \frac{D\eta}{Dt} \geq \sigma_c, \]  

(39)

where

\[ \sigma_c = -\text{div} \sum_{x=s,f} \left( \frac{\tilde{h}^x + \rho^x \eta^x \theta \tilde{u}^x}{\theta} \right). \]  

(40)

Here, \( \sigma_c \) represents the entropy production rate due to interactions with surroundings. We can rewrite (39) as an equality

\[ \rho \frac{D\eta}{Dt} = \sigma_c + \sigma_i, \]  

(41)

where \( \sigma_i \geq 0 \) represents the internal entropy production rate per unit volume. Inequality (39) assumes the form

\[ \rho \theta \frac{D\eta}{Dt} = -\theta \text{div} \left( \sum_{x=s,f} \frac{\tilde{q}^x + \rho^x \eta^x \theta \tilde{u}^x}{\theta} \right) + \theta \sigma_i. \]  

(42)

If now we define the rate of dissipation for the mixture as

\[ \zeta = \sigma_i \theta \geq 0, \]  

(43)

from (42) we obtain

\[ \zeta = \rho \theta \frac{D\eta}{Dt} + \text{div} \sum_{x=s,f} (\tilde{h}^x + \rho^x \eta^x \theta \tilde{u}^x) + \theta \nabla \left( \frac{1}{\theta} \right) \cdot \sum_{x=s,f} (\tilde{h}^x + \rho^x \eta^x \theta \tilde{u}^x). \]  

(44)

On introducing then the specific Helmholtz free energy of the \( x \)th constituent through

\[ \psi^x = \psi^x - \theta \eta^x, \]  

(45)

and of the mixture through

\[ \psi = \frac{1}{\rho} \sum_{x=s,f} \rho^x \psi^x, \]  

(46)

we have

\[ \psi = \varepsilon - \theta \eta, \]  

(47)

and

\[ \frac{D\psi}{Dt} = \frac{De}{Dt} - \theta \frac{D\eta}{Dt} - \eta \frac{D\theta}{Dt}. \]  

(48)
So, under isothermal conditions (i.e. $\theta$ constant in space and time) (44) acquires the form
\[
\ddot{\varphi} = D_e \frac{\partial \varphi}{\partial t} + \text{div} \left( \sum_{x=s,f} [\tilde{h}^2 + \rho^x (\tilde{v}^x - \psi^x)] \right) - \rho \frac{D\psi^x}{D t}
\]
\[
= \sum_{x=s,f} [\text{tr}(T^x L^x) - \tilde{p}^x \cdot \tilde{v}^x] - \text{div} \left( \sum_{x=s,f} (\rho^x \psi^x \tilde{u}^x + \tilde{d}^x - \tilde{h}^x) \right) - \rho \frac{D\psi^x}{D t},
\]
(49)
where the energy balance (33) has been taken into account.

Eq. (49) is the starting point in the determination of the constitutive equations. We will use (49) to place some restrictions on the structure of $T^x$, $s$, $f$, and of $\tilde{p}^f$ since by (22)
\[
\tilde{p}^s = -\tilde{p}^f.
\]
(50)

### 2.3. Thermodynamical restrictions on the constitutive equations

As mentioned in the Introduction, we are interested in modeling a solid–fluid mixture whose solid component undergoes microstructural changes. Considering the solid, the reference configuration $\kappa_s^0$ might be identified as stress free. If changes in the internal structure of the solid occur then the stress free configuration is no longer $\kappa_s^0$. In other words, the fraction of the solid undergoing microstructural changes will have a different reference configuration. Let $\kappa_s^p(t)$ denote the stress free configuration of the mass fraction of the solid that has undergone a microstructural change at time $t$ (see, for instance, [36]). Obviously $\kappa_s^p(t)$, often referred to as natural configuration, may evolve due to the continuous modifications which take place during motion.

We measure the deformation from the natural configuration $\kappa_s^0$ through the tensor $F^s_{p(t)}$, while the path from $\kappa_s^0$ to $\kappa_s^p(t)$ is described by the tensor $G^s$, so that the following decomposition holds:
\[
F^s_{p(t)} = F^s G^s^{-1}.
\]
(51)

We remark that in general the tensors $F^s_{p(t)}$ and $G^s$ need not to be the gradient of any particular functions, as $F^s$ does. In other words, there does not exist, in general, a global mapping $\zeta^s$ from $\kappa_0^s$ to $\kappa_s^p(t)$ such that
\[
G^s = \text{Grad} \zeta^s (\tilde{X}^s, t).
\]
(52)

We can, however, define appropriate linear transformations from the tangent spaces as points belonging to $\kappa_0^s$ and $\kappa_s^p(t)$, and from $\kappa_0^s$ to $\kappa_s^p(t)$, respectively. In analogy with formula (10) we define
\[
L^s_G = \frac{D^s G^s}{D t} G^s^{-1},
\]
(53)
\[
L^s_{p(t)} = \frac{D^s F^s_{p(t)}}{D t} F^s_{p(t)} F^s_{p(t)}^{-1}.
\]
(54)

We remark, however, that (53) and (54) are definitions, while (10) comes from calculations since $L^s$ is defined by (9), and does not require an unloading to a natural configuration.

Henceforth, in order to have a simple and compact notation, tensors $G^s$, $F^s_{p(t)}$, $L^s_G$, $L^s_{p(t)}$, and $F^s$ will be denoted by $G$, $F_p$, $L_G$, $L_p$, and $F$, respectively, and $\tilde{v}_{s,f}$ will denote the velocity of the solid with respect to the fluid, i.e.
\[
\tilde{v}_{s,f} = \tilde{v}^s - \tilde{v}^f,
\]
(55)
while $\tilde{v}_{f,s}$ the velocity of the fluid with respect to the solid
\[
\tilde{v}_{f,s} = \tilde{v}^f - \tilde{v}^s,
\]
(56)
which is also known as seepage or superficial velocity.
Starting from Eq. (49), we look for some thermodynamical restrictions on $\vec{p}$ and for $T^z$, $z = s, f$, assuming that the functions

$$\psi = \hat{\psi}(F_p, G),$$

$$\zeta = \hat{\zeta}(L_G, F_p, G, \vec{v}_s, f)$$

are given.

Notice that, because of (13)–(15) and (27)

$$\rho^s = \rho^s_R \frac{\phi^s_0}{\det F} = \rho^s_R \frac{\phi^s_0}{\det F_p \det G},$$

$$\rho^f = \rho^f_R \left( 1 - \frac{\phi^s_0}{\det F} \right),$$

where

$$\gamma = \frac{\rho^f_R}{\rho^s_R}.$$  

So $\hat{\psi}$ and $\hat{\zeta}$ depend on $\rho^s$ and $\rho^f$ through $F_p$ and $G$. It is customary to write inequality (49) in a form such that the thermodynamic restrictions on constitutive equations can be easily obtained. To do that one has to explicitly evaluate $D\hat{\psi}/Dt$. It follows from (57) that

$$\frac{D\hat{\psi}}{Dt} = \frac{\partial \hat{\psi}}{\partial F_p} \cdot \frac{DF_p}{Dt} + \frac{\partial \hat{\psi}}{\partial G} \cdot \frac{DG}{Dt}. \tag{62}$$

The explicit evaluation of $(\partial \hat{\psi}/\partial F_p) \cdot (DF_p/Dt)$ is laborious. On using (19) we obtain

$$\frac{\partial \hat{\psi}}{\partial F_p} \cdot \frac{DF_p}{Dt} = \text{tr} \left( \frac{\partial \hat{\psi}}{\partial F_p^T} \right) \frac{DF_p}{Dt} \left[ \frac{D^4(F_p)_{ij}}{Dt} - (L^4)_{ij} \right]. \tag{63}$$

It follows from (10) and (51) that:

$$\frac{D^4(F_p)_{ij}}{Dt} = \left( \frac{D^4F}{Dt} G^{-1} \right)_{ij} + \left( F \frac{D^4G^{-1}}{Dt} \right)_{ij}$$

$$= (L^4F_{ij})_{ij} - \left( FG^{-1} \frac{D^4G}{Dt} G^{-1} \right)_{ij}$$

$$= (L^4F_p)_{ij} - (F_p L)_{ij}, \tag{64}$$
where definition (53) has been exploited. Substituting (64) into (63) yields
\[
\text{tr} \left( \frac{\partial \hat{\psi}}{\partial F_p} \frac{DF_p}{Dt} \right) = \text{tr} \left( F_p \frac{\partial \hat{\psi}}{\partial F_p} L^T - \text{tr} \left[ \frac{\partial \hat{\psi}}{\partial F_p} (\nabla F_p \hat{\mu}^s) \right] \right) - \text{tr} \left( F_p^T \frac{\partial \hat{\psi}}{\partial F_p} L^T_G \right). \tag{65}
\]

Operating in a similar manner, the following expression for \((\partial \hat{\psi}/\partial G) \cdot (DG/\text{Dt})\) can be obtained
\[
\frac{\partial \hat{\psi}}{\partial G} \frac{DG}{\text{Dt}} = \text{tr} \left( \frac{\partial \hat{\psi}}{\partial G} DG^T \right) = \text{tr} \left( \frac{\partial \hat{\psi}}{\partial G} G^T L^T_G \right) - \text{tr} \left[ \frac{\partial \hat{\psi}}{\partial G^T} (\nabla G \hat{\mu}^s) \right]. \tag{66}
\]

It follows from (49), (62), (65) and (66) that:
\[
\dot{\xi} = - \nabla \cdot \left( \rho^2 \psi \hat{\mu}^s + \hat{q}^s - \hat{h}^s \right) - \sum_{x=s,f} \hat{v}^x \cdot \hat{m}^x + \rho \text{tr} \left[ \frac{\partial \hat{\psi}}{\partial G^T} (\nabla G \hat{\mu}^s) \right] + \rho \text{tr} \left[ \frac{\partial \hat{\psi}}{\partial F_p} (\nabla F_p \hat{\mu}^s) \right] + \text{tr} \left( T^s - \rho F_p \frac{\partial \hat{\psi}}{\partial F_p} \right) L^s + T^f L^f + \rho \left( F_p \frac{\partial \hat{\psi}}{\partial F_p} - \frac{\partial \hat{\psi}}{\partial G} G^T \right) L^T_G \geq 0. \tag{67}
\]

Now, because of (57) we have
\[
\hat{u}^s \cdot \nabla \hat{\psi} = \text{tr} \left[ \frac{\partial \hat{\psi}}{\partial G^T} (\nabla G \hat{\mu}^s) \right] + \text{tr} \left[ \frac{\partial \hat{\psi}}{\partial F_p} (\nabla F_p \hat{\mu}^s) \right], \tag{68}
\]
and so, taking (50) and (55) into account, (67) can be rewritten as
\[
\dot{\xi} = - \nabla \cdot \left( \rho^2 \psi \hat{\mu}^s + \hat{q}^s - \hat{h}^s \right) + \text{tr} \left[ \left( T^s - \rho F_p \frac{\partial \hat{\psi}}{\partial F_p} \right) L^s + T^f L^f \right] + \rho \left( F_p \frac{\partial \hat{\psi}}{\partial F_p} - \frac{\partial \hat{\psi}}{\partial G} G^T \right) L^T_G + \rho \tilde{\epsilon} + \rho^f \nabla \hat{\psi} \cdot \hat{v}_{s,f}, \tag{69}
\]
where we have exploited the following relation:
\[
\rho \hat{u}^s = \rho^f (\hat{u}^s - \hat{u}^f). \tag{70}
\]

The assumption that the volumes of the constituents of the mixture are additive, implies the volume additivity constraint (14). There is, as remarked in Section 2.2, a philosophical problem associated with co-occupancy of the constituents if they are incompressible and their volumes add up. This volume additivity applies to the pure constituents, the homogenized constituents co-occupying the total volume. Such a constraint leads to a Lagrange multiplier and we shall assume that the constraint does no work. It is important to recognize that this is an assumption. In general, when constraints arise in dissipative systems, we cannot assume that they do not perform work (see [40]).
Adding the mass balance equations (26) leads to
\[ \phi^s \text{tr} \mathbf{L}^s + \phi^f \text{tr} \mathbf{L}^f + \nabla \phi^s \cdot \mathbf{\tilde{v}}_{s,f} = 0, \]  
(71)
or equivalently
\[ \text{div} \, \mathbf{\tilde{v}}_c = 0 \]  
(72)
with
\[ \mathbf{\tilde{v}}_c = \phi^s \mathbf{\tilde{v}}^s + \phi^f \mathbf{\tilde{v}}^f. \]  
(73)

Since Eq. (69) is expected to hold, under the above constraint, a Lagrange multiplier \( \lambda \) comes into play, namely
\[
\hat{\zeta} = \sum_{\alpha=s,f} \text{div} [\mathbf{\tilde{h}}^\alpha - (\mathbf{\tilde{q}}^\alpha + \rho^\alpha \psi^2 \mathbf{\hat{u}}^2)] + \text{tr} \left[ \left( \mathbf{T}^\alpha - \rho \mathbf{F}_p \frac{\partial \hat{\psi}}{\partial \mathbf{F}_p} + \lambda \phi^\alpha \mathbf{I} \right) \mathbf{L}^\alpha \right] \\
+ \text{tr} \left( \mathbf{T}^f + \lambda \phi^f \mathbf{I} \right) \mathbf{L}^f \\
+ \text{tr} \left[ \rho \left( \mathbf{F}_p^T \frac{\partial \hat{\psi}}{\partial \mathbf{F}_p} - \frac{\partial \hat{\psi}}{\partial \mathbf{G}} \mathbf{G}^T \right) \mathbf{L}_G^T \right] \\
+ \left[ \mathbf{\tilde{p}}^f + \rho^f \nabla \hat{\psi} + \lambda \nabla \phi^f \right] \cdot \mathbf{\tilde{v}}_{s,f}. \]  
(74)

In particular, it will be shown that \( \lambda \) is related to the isotropic part of \( \mathbf{T} \) and therefore, physically interpreted as a hydrostatic pressure acting on the mixture.

Thus, (74) becomes
\[
\hat{\zeta}(\mathbf{L}_G, \mathbf{F}_p, \mathbf{G}, \mathbf{\tilde{v}}_{s,f}, \rho^f) = \sum_{\alpha=s,f} \text{div} [\mathbf{\tilde{h}}^\alpha - (\mathbf{\tilde{q}}^\alpha + \rho^\alpha \psi^2 \mathbf{\hat{u}}^2)] \\
+ \text{tr} \left[ \left( \mathbf{T}^\alpha + \lambda \phi^\alpha \mathbf{I} - \rho \mathbf{F}_p \frac{\partial \hat{\psi}}{\partial \mathbf{F}_p} \right) \mathbf{L}^\alpha \right] \\
+ \text{tr} \left( \mathbf{T}^f + \lambda \phi^f \mathbf{I} \right) \mathbf{L}^f \\
+ \text{tr} \left[ \rho \left( \mathbf{F}_p^T \frac{\partial \hat{\psi}}{\partial \mathbf{F}_p} - \frac{\partial \hat{\psi}}{\partial \mathbf{G}} \mathbf{G}^T \right) \mathbf{L}_G^T \right] \\
+ \left[ \mathbf{\tilde{p}}^f - \lambda \nabla \phi^f + \rho^f \nabla \hat{\psi} \right] \cdot \mathbf{\tilde{v}}_{s,f}. \]  
(75)

At this point we make the usual assumption that
\[ \mathbf{\tilde{h}}^\alpha = -\mathbf{\tilde{q}}^\alpha - \rho^\alpha \psi^2 \mathbf{\hat{u}}^2, \]  
(76)
which means that the entropy flux depends on the diffusive velocity, i.e., on the relative “mixing” of the components.

Next, since the second and the third terms on the r.h.s. of (75) are the only ones that depend on \( \mathbf{L}^s \) and \( \mathbf{L}^f \), we assume that
\[ \mathbf{T}^s = -\phi^s \lambda \mathbf{I} + \rho \mathbf{F}_p \frac{\partial \hat{\psi}}{\partial \mathbf{F}_p}, \]  
(77)
\[ \mathbf{T}^f = -\lambda \phi^f \mathbf{I}. \]  
(78)

We remark that assumption (78) does not mean that there are no frictional effects due to the fluid. It merely implies that frictional effects at the pores (which are reflected in the constitutive assumption for the interactive force) are much more significant than the frictional losses in the fluid.
Then, we stipulate that
\[
\hat{\gamma}_{s}(L_{G}, F_{p}, G, \vec{v}_{s,f}) = \hat{\gamma}_{s}(L_{G}, F_{p}, G) + \hat{\gamma}_{s,f}(F_{p}, G, \vec{v}_{s,f}),
\]
where
- \( \hat{\gamma}_{s} \) represents the rate of dissipation due to changes in the internal solid microstructure.
- \( \hat{\gamma}_{s,f} \) represents the rate of dissipation due to solid–liquid interactions.

In particular, we consider the following form for \( \hat{\gamma}_{s,f} \):
\[
\hat{\gamma}_{s,f}(F_{p}, G, \vec{v}_{s,f}) = \vec{v}_{s,f} \cdot H(F_{p}, G) \vec{v}_{s,f},
\]
with \( H \) a second-order positive definite tensor. This automatically ensures that \( \hat{\gamma}_{s,f} \geq 0 \).

Having settled this choice for \( \hat{\gamma}_{s,f} \), we can proceed to find the form for \( \hat{\gamma}_{s} \) by requiring that the choice maximize the rate of dissipation. Of course, one should ask why the choice of \( \hat{\gamma}_{s} \) be determined by a maximization procedure. The point is that we have some guidance concerning the rate of dissipation \( \hat{\gamma}_{s,f} \) due to drag which we can use to simplify the problem. The maximization of \( \hat{\gamma}_{s} \) would then provide us with the evolution of the natural configuration for the porous solid and hence its inelastic response.

Assumption (80) implies
\[
\vec{p}^{f} = H\vec{v}_{s,f} - \rho^{f} \nabla \hat{\psi} + \lambda \nabla \phi^{f},
\]
with \( \lambda \) the Lagrange multiplier to be determined by (82). Solving (83) we obtain
\[
L_{G} = L_{G}(F_{p}, G).
\]
We now make use of the criterion of the maximization of the rate of dissipation for \( \hat{\gamma}_{s} \) (see [38–40]). We look for \( L_{G} \) that maximizes \( \hat{\gamma}_{s} \) under constraint (82), keeping \( F_{p}, G, \vec{v}_{s,f} \) fixed in (82). We then get the following equation:
\[
\frac{\partial \hat{\gamma}_{s}}{\partial L_{G}} + \lambda_{1} \left[ \frac{\partial \hat{\gamma}_{s}}{\partial L_{G}} - \rho \left( F_{p}^{T} \frac{\partial \hat{\psi}}{\partial F_{p}} - \frac{\partial \hat{\psi}}{\partial G} G^{T} \right) \right] = 0,
\]
where \( \lambda_{1} \) is the Lagrange multiplier to be determined by (82). Solving (83) we obtain
\[
L_{G} = L_{G}(F_{p}, G).
\]
We remark that in general \( \lambda_{1} \) is such that
\[
\lambda_{1} = \lambda_{1}(F_{p}, G),
\]
so that when we select \( \hat{\gamma}_{s} \) and \( \hat{\psi} \) we have to verify that they guarantee the existence of a unique \( \lambda_{1} \) once we substitute (84) into (82). Thus the evolution equation for \( G \) is
\[
\frac{D^{\gamma} G}{D\tau} = L_{G}(F_{p}, G)G,
\]
whose initial condition is
\[
G(0) = I.
\]
2.4. Further restrictions on the constitutive equations imposed by frame invariance

In this section, we discuss restrictions due to assumption of frame indifference on the constitutive assumptions (57), (58). Given two frames \((\vec{x}, t)\) and \((\vec{x}^*, t^*)\) related through

\[
\vec{x}^* = Q(t)\vec{x} + \vec{c}(t), \quad t^* = t - a,
\]

where \(Q(t) \in \text{Orth}^+\), i.e., \(Q\) is a proper rotation, and \(a\) is some constant, the frame indifference requires that

\[
\hat{\psi}(F_p, G) = \hat{\psi}(F^*_p, G^*),
\]

(89)

\[
\hat{\xi}(L_G, F_p, G, \vec{v}_{s,f}) = \hat{\xi}(L^*_G, F^*_p, G^*, \vec{v}^*_{s,f})
\]

(90)

that is \(\hat{\psi}\) and \(\hat{\xi}\) are invariant under transformation (88).

Now recalling (1) and (7) we have

\[
\vec{x}^* = Q(t)\chi^2(\vec{X}^2, t) + \vec{c}(t),
\]

(91)

and therefore

\[
F^* = \text{Grad}^\chi[Q(t)\chi^2(\vec{X}^2, t)] = QF.
\]

(92)

We assume

\[
G^* = G,
\]

(93)

since the natural configuration is defined up to a rigid rotation. Thus, from (51), we obtain

\[
F^*_p = QF_p.
\]

(94)

Substituting (93) and (94) into (89) yields

\[
\hat{\psi}(F_p, G) = \hat{\psi}(QF_p, G).
\]

(95)

In particular, recalling the polar decomposition \(F_p = R_p U_p\), (92) entails

\[
F^*_p = R^*_p U^*_p = QR_p U_p,
\]

(96)

where \(QR_p \in \text{Orth}^+\). Thus, in view of the uniqueness of the polar decomposition

\[
R^*_p = QR_p, \quad U^*_p = U_p.
\]

(97)

We can therefore conclude that, if \(\hat{\psi}\) depends on \(F_p\) through combinations of \(U_p\), i.e., through the right Cauchy–Green tensor

\[
C_p = U^2_p = F^T_p F_p,
\]

(98)

then (95) is satisfied. We also notice that if \(\hat{\psi}\) depends on \(C_p\) any rotation of \(\kappa^2_{p(t)}\) leaves the strain energy unchanged.

As far as the dissipation \(\hat{\xi}\) is concerned it is useful to recall that

\[
\vec{v}^* = QQ^{-1}(\vec{x} - \vec{c}(t)) + Q\vec{v} + \frac{d\vec{c}}{dt},
\]

(99)

and so

\[
\vec{v}^*_{s,f} = Q\vec{v}_{s,f}.
\]

(100)
Thus, from (80)
\[ \vec{v}^s_f \cdot H^* \vec{v}^s_f = \vec{v}^s_f \cdot H \vec{v}^s_f, \] (101)
if
\[ Q^T H^* Q = H. \] (102)

We, therefore, conclude that (90) is verified if \( \hat{\xi} \) depends on \( F_p \), through combinations of \( C_p \) and (101) holds true.

3. A model for the flow of a fluid through an inelastic porous media

Given the restrictions of Sections 2.3 and 2.4, we now illustrate a specific model for the flow of a fluid through an inelastic porous media. A particular application of the model will be analyzed in Section 4.

3.1. Physical assumptions

We restrict ourselves to a mixture of an inelastic solid and a fluid, whose volumes as a mixture are additive. Even in this case, the problem of finding reliable constitutive equations for the solid and the fluid is extremely difficult and cannot be fully developed within this context. For more details we refer the reader to the works by Bowen [13,14], Rajagopal et al. [18,33,45,25,26,37,46], Ehlers [20–22], Crochet and Naghdi [17], Coussy [16] and Prévost [34].

We introduce some physical assumptions which will guide us in our choice and we list these below.

A1. Negligible inertial effects compared with the stress.

A2. \( \psi^s, \psi^f \) depend on their own constituent variables. In particular, we assume
\[ \psi^f = \psi_0 = \text{const.}, \] (103)
\[ \psi^s = \psi^s(I_{C_p}), \] (104)
with \( \psi^s \) a given function and
\[ I_{C_p} = \text{tr} C_p = \text{tr}(F_p^T F_p). \] (105)

In particular, \( \psi^s(3) = 0 \).

A3. The dissipation \( \hat{\xi}^s \) has the following form:
\[ \hat{\xi}^s = \beta H(I_C^s - I_C)L_G \cdot L_G, \] (106)
where

- \( \beta \) is a positive constant.
- \( I_C = \text{tr} C = \text{tr}(F^T F) \).
- \( H(I_C^s - I_C) \) is the Heaviside function
\[ H(I_C^s - I_C) = \begin{cases} 1 & \text{if } I_C^s \geq I_C, \\ 0 & \text{if } I_C^s < I_C, \end{cases} \] (107)

with \( I_C^s \) having a given prescribed value.

Assumption A1 implies that
\[ \hat{T} = T' = T^f + T^f. \] (108)
The inertial effect, represented in (30) by $\sum_{x=s,f} \rho^x u^x \otimes u^x$, can be neglected. The momentum equations for the single constituent (21) and for the mixture (32) take the form
\[
\begin{align*}
\text{div } T^s - \bar{p}^f &= 0, \\
\text{div } T^f + \bar{p}^f &= 0, \\
\text{div } \hat{T} &= 0,
\end{align*}
\]
respectively.

Assumption A2 and (46) yield the following explicit expression for:
\[
\hat{\psi}(F_p, G) = \frac{\rho^f \psi_0 + \rho^s \hat{\psi}(I_{C_p})}{\rho^f + \rho^s},
\]
where, by virtue of (51), (59) and (60) $\rho^s$ and $\rho^f$ depend on $F_p$ and $G$.

3.2. General 3D model

In this section, we derive explicitly the general 3D model that follows from the hypotheses A1–A3 and constitutive assumptions (112) and (106).

3.2.1. Evolution equation for $\phi^s$

From (77) and (78) we have
\[
\hat{T} = -\lambda I + \tilde{T},
\]
where
\[
\tilde{T} = \rho F_p \frac{\partial \hat{\psi}}{\partial F_p},
\]
is the extra stress while the term $-\lambda I$ represents, as mentioned before, the isotropic part of $T$, i.e., an hydrostatic pressure acting on the mixture. In particular, the stress equilibrium equation (111) reduces to
\[
-\nabla \lambda + \text{div } \tilde{T} = 0.
\]
Using (81) and (78) for $\bar{p}^f$ and $T^f$, respectively, from (110) we obtain
\[
-\phi^f \nabla \lambda - \phi^f \nabla (\rho^f \hat{\psi}) + H \tilde{v}_{s,f} = 0,
\]
and so
\[
\phi^f \tilde{v}_{f,s} = -K \nabla (\lambda + \rho^f \hat{\psi})
\]
with
\[
\frac{K}{\phi^f} = \phi^f H^{-1}.
\]
We remark that Eq. (117) is the usual Darcy’s law.

Recalling now the balance of mass for the solid (i.e., Eq. (20) with $x = s$) and (72) we can write
\[
\phi^s \text{div}(\phi^f \tilde{v}_{f,s}) - \frac{D^s \phi^s}{Dt} = 0.
\]
Thus, using (117) and (115) we end up with the following evolution equation for $\phi^s$:
\[
\frac{D^s \phi^s}{Dt} + \phi^s \text{div}[K(\text{div}(\tilde{T}) + \rho^f \nabla \hat{\psi})] = 0,
\]
with $\tilde{T}$ given by (114).
3.2.2. Constitutive equation for $\tilde{T}$ and evolution equation for $G$

We analyze separately the case $I_C^* \geq I_C$ and the case $I_C^* < I_C$.

**First case:** $I_C^* \geq I_C$. Let us first consider the stress tensor $\tilde{T}$. Starting from (114) we have to evaluate explicitly $\partial \hat{\psi} / \partial F_p^T$ recalling that $\rho^s$ and $\rho^f$ are functions of $F_p$ and $G$. We obtain

\[
\frac{\partial \hat{\psi}}{\partial F_p^T} = \left( \frac{\partial \hat{\psi}}{\partial \rho^s} + \frac{\partial \hat{\psi}}{\partial \rho^f} \frac{\partial \rho^s}{\partial \rho^f} \right) \frac{\partial \rho^s}{\partial F_p^T} + \frac{\partial \hat{\psi}}{\partial \rho^s} \frac{\partial \hat{\psi}}{\partial F_p^T} 
= - \frac{\rho^s \rho^f}{\rho^2} (\hat{\psi} - \psi_0) F_p^{-1} + 2 \frac{\rho^s}{\rho} W_p(I_{C_p}) F_p^T, \tag{121}
\]

where

\[
W_p(I_{C_p}) = \frac{\partial \hat{\psi}^s(I_{C_p})}{\partial I_{C_p}}, \tag{122}
\]

and where the following results have been exploited:

\[
\frac{\partial \rho^s}{\partial F_p^T} = \frac{\partial}{\partial F_p^T} \left[ \frac{\rho^s}{\rho^2} \partial \phi_0 \right] = - \rho^s F_p^{-1}, \tag{123}
\]

\[
\frac{\partial}{\partial F_p^T} \text{tr}(F_p^T F_p) = 2 F_p^T. \tag{124}
\]

Substituting the above into (114) yields

\[
\tilde{T} = \frac{\rho^s \rho^f}{\rho} (\psi_0 - \hat{\psi}^s) I + 2 \rho^s W_p(I_{C_p}) F_p F_p^T. \tag{125}
\]

Let us now consider the evolution equation for $G$. Since

\[
\frac{\partial \hat{\psi}}{\partial G} = - \frac{\rho^s \rho^f}{\rho^2} (\hat{\psi} - \psi_0) G, \tag{126}
\]

from (83) we have

\[
L_G = \frac{\lambda_1}{1 + \lambda_1} \frac{\rho^s W_p(I_{C_p})}{\beta} F_p^T F_p. \tag{127}
\]

To determine $\lambda_1$ it is enough to substitute the above expression into equation (82). We obtain $\lambda_1 = -2$ and so

\[
L_G = 2 \frac{\rho^s W_p(I_{C_p})}{\beta} F_p^T F_p. \tag{128}
\]

Thus, recalling (86) and (51), the evolution equation for $G$ is

\[
\frac{D^t G}{Dt} = \frac{2 \rho^s W_p(I_{C_p})}{\beta} F_p^T F_p G. \tag{129}
\]

**Second case:** $I_C^* < I_C$. In such a case $\hat{\zeta}_s = 0$ and thus from (82) we have

\[
L_G = 0, \tag{130}
\]

which, in view of (86) and (87) leads to

\[
G = I \quad \text{and} \quad F_p = F. \tag{131}
\]
Eq. (114) yields
\[ \tilde{T} = \frac{\rho R^s}{\rho} (\psi_0 - \hat{\psi}^s) I + 2\rho^s W(I_C) FF^T, \]  
(132)
where now
\[ W(I_c) = \frac{\partial \hat{\psi}^s(I_c)}{\partial I_c}. \]  
(133)

Summarizing the results obtained thus far, we have
\[ \tilde{T} = \begin{cases} \frac{\rho R^s}{\rho} (\psi_0 - \hat{\psi}^s) I + 2\rho^s W(I_{C_p}) F_p F_p^T, & I_C^s \geq I_c, \\ \frac{\rho R^s}{\rho} (\psi_0 - \hat{\psi}^s) I + 2\rho^s W(I_C) FF^T, & I_C^s < I_c, \end{cases} \]  
(134)
\[ \frac{D^s G}{Dt} = \begin{cases} 2\rho^s W_p(I_{C_0}) F_{p}^T F_p G, & I_C^s \geq I_c, \\ \beta \frac{\partial G}{\partial I_c}, & I_C^s < I_c, \end{cases} \]  
(135)

4. Application to a compression layer problem

In this section we describe an application of the model that has been developed to a problem in which a layer is compressed under a uniform load. Problems similar to the one we are going to illustrate arise, for instance, in soil consolidation [31], and in composite material manufacturing [23, 24, 35].

4.1. Definition of the problem

Let us consider a layer of an inelastic deformable porous solid fully saturated by a liquid (water, for instance). The top surface of the layer is uniformly loaded while the bottom surface is constrained by a net which allows the liquid to pass with no resistance. The upper boundary of the layer is impervious and so represents a material surface for both the solid and liquid. We further assume that the material is homogeneous in the plane and that the thickness of the layer is much smaller than the other dimensions. We therefore consider a pure compression
\[ \begin{cases} x = f(X,t), & X \in \Omega, \\ y = Y, \\ z = Z, \end{cases} \]  
(136)
where

- \( x \) is the Eulerian coordinate along the vertical direction.
- \( X \) is the Lagrangian vertical coordinate of the solid skeleton.
- \( \Omega \) is the reference configuration, i.e., the domain
\[ \Omega = \{0 \leq X \leq L_0\}, \]  
(137)
where \( L_0 \) is the initial thickness of the layer.
- \( f(X,t) \) is the vertical position (unknown) for the solid particle \( X \) at time \( t \).

The spatial domain occupied by the porous medium is
\[ \omega(t) = \{0 \leq x \leq L(t)\}, \]  
(138)
where \( L(t) \) is the thickness of the layer at time \( t \) (recall that \( L(0) = L_0 \)).
We have
\[ F = \begin{pmatrix} f_X & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} f_X^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \] (139)
So, taking (28) into account, we have
\[ \det F = f_X \iff f_X = \frac{\phi_0}{\phi}, \] (140)
where \( \phi, (1 - \phi) \) denote \( \phi^s, \phi^f \), respectively. In particular,
\[ I_C = \left( \frac{\phi_0}{\phi} \right)^2 + 2. \] (141)
We set
\[ I_C^* = \left( \frac{\phi_0}{\phi^*} \right)^2 + 2, \] (142)
so that
\[ I_C > I_C^* \implies \phi < \phi^*, \]
\[ I_C \leq I_C^* \implies \phi \geq \phi^*. \] (143)
We assume that
\[ G = \begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \implies G^{-1} = G^{-T} = \begin{pmatrix} 1 & 0 & 0 \\ g & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \] (144)
with
\[ g = g(X, t) \quad \text{and} \quad g(X, 0) = 1. \] (145)
From (51) we have
\[ F_p = \begin{pmatrix} \frac{f_X}{g} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \phi_0 & 0 & 0 \\ \phi g & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \] (146)
and so
\[ I_{C_p} = \frac{\phi_0^2}{\phi^2 g^2} + 2. \] (147)
The free energy \( \psi \) and \( \tilde{T}_{11} \) are given by
\[ \psi = \rho R f(X)(1 - \phi)\psi_0 + \rho R \phi \psi_0^s, \] (148)
and
\[ \tilde{T}_{11} = \begin{cases} \frac{\phi_0 f R(X)(\psi_0 - \psi^s)}{\rho} + 2 \rho R W \left( \frac{\phi_0^2}{g^2 \phi^2} \right) \frac{\phi_0^2}{\phi} & \text{if } \phi \geq \phi^*, \\ \frac{\phi_0 f R(X)(\psi_0 - \psi^s)}{\rho} + 2 \rho R W \left( \frac{\phi_0^2}{\phi^2} \right) \frac{\phi_0^2}{g} & \text{if } \phi < \phi^*, \end{cases} \] (149)
respectively.

\[ \text{6 Here and in the sequel } f_X = \frac{\partial f}{\partial X}, f_{XX} = \frac{\partial^2 f}{\partial X^2}, f_t = \frac{\partial f}{\partial t}. \]
The dynamics of $g$ is governed by

$$\frac{D^2 g}{D\tau} = \frac{2 \rho_p^2}{\beta} W_p \left( \frac{\phi_0^2}{g^2 \phi^2} \right) \frac{\phi_0^2}{g\phi} \text{ if } \phi \geq \phi^\ast.$$ \hfill (150)

We now discuss on the evolution equation for $\phi$. To write it explicitly we use the material coordinate $X$. In such a framework the domain where the problem is defined does not vary in time (the moving boundary $x = L(t)$ disappears being replaced by $X = L_0$). Thus, recalling that

$$\frac{\partial}{\partial x} \left[ \frac{\partial}{\partial x} \right] = \frac{1}{f(X, \tau)} \frac{\partial}{\partial X} \frac{\partial}{\partial x},$$ \hfill (151)

Eq. (120) can be rewritten in the following manner:

- $\phi \geq \phi^\ast$

$$\frac{D^2 \phi}{D\tau} = - \frac{\phi_0^2}{\phi_0} \frac{\partial}{\partial X} \left\{ (K)_{11} \frac{\phi_0}{\phi_0} \frac{\partial}{\partial X} [\tilde{T}_{11} + \rho_R^f \tilde{\psi}] \right\}$$

$$\begin{aligned} &= - \frac{\phi_0^2}{\phi_0} \frac{\partial}{\partial X} \left\{ \frac{\phi}{\phi_0} K \frac{\partial}{\partial X} \left[ \rho_R^f \tilde{\psi}_0 + 2 \rho_R^f W_p \left( \frac{\phi_0^2}{g^2 \phi^2} \frac{\phi_0^2}{g\phi} \right) \right] \right\} \\
&= - 2 \rho_R^f \frac{\phi_0^2}{\phi_0} \frac{\partial}{\partial X} \left\{ \frac{\phi}{\phi_0} K \frac{\partial}{\partial X} \left[ W_p \left( \frac{\phi_0^2}{g^2 \phi^2} \frac{\phi_0^2}{g\phi} \right) \right] \right\}. \end{aligned}$$ \hfill (152)

where $(K)_{11} = K$.

- $\phi < \phi^\ast$

$$\frac{D^2 \phi}{D\tau} = - 2 \rho_R^f \frac{\phi_0^2}{\phi_0} \frac{\partial}{\partial X} \left\{ \frac{\phi}{\phi_0} K \frac{\partial}{\partial X} \left[ W \left( \frac{\phi_0^2}{\phi^2} \frac{\phi_0^2}{\phi} \right) \right] \right\}. \hfill (153)$$

We recall that $W_p$ and $W$ in (152) and (153) are given by (122) and (133), respectively.

4.2. Boundary conditions

The top surface, i.e., $x = L(t)$, is a material boundary for both the constituents. Applying (117) yields

$$\tilde{v}_{s.f}|_{x=L(t)} = 0, \quad \Rightarrow \nabla \cdot (\tilde{T} + \rho_R^f \tilde{\psi} I)|_{x=L(t)} = 0.$$ \hfill (154)

Thus we have

$$\begin{cases} \frac{\partial}{\partial X} \left[ W_p \left( \frac{\phi_0^2}{g^2 \phi^2} \frac{\phi_0^2}{g\phi} \right) \right] = 0 \text{ if } \phi \geq \phi^\ast, \\
\frac{\partial}{\partial X} \left[ W \left( \frac{\phi_0^2}{\phi^2} \frac{\phi_0^2}{\phi} \right) \right] = 0 \text{ if } \phi < \phi^\ast. \end{cases} \hfill (155)$$

The boundary condition on the bottom surface, i.e. $x = 0$, is more involved. If $\hat{n}$ denotes the outward unit normal to $x = 0$ and $Q^x$ any quantity relative to the $x$th component which evolves according to the equation

$$\frac{\partial (\rho^x Q^x)}{\partial t} + \nabla \cdot (\rho^x Q^x \tilde{v}^x) = \nabla \cdot \hat{n}^x + g^x.$$

\hfill (156)
we have (see [15, Chapter 8])
\[
\left[ \sum_{\alpha=s,f} \rho^\alpha Q^\alpha (\vec{v}^\alpha - \vec{w}) - \vec{n}^\alpha \right] \cdot \vec{n} = 0, \tag{157}
\]
where
- \( \vec{n} = -\vec{e}_x \).
- \( \vec{w} \) is the surface velocity (\( \vec{w} = 0 \) in this case).
- \( [Q] = Q^+ - Q^- \), with \( Q^+ \), \( Q^- \) denoting the limits of \( Q \) evaluated in the regions that contain \( \vec{n} \) and \( -\vec{n} \), respectively.

In the sequel we write (157) for the cases in which \( Q^\alpha \) represents mass density, momentum and energy.

**Mass density**: In this case \( Q^\alpha = 1 \), \( \alpha = s, f \) and \( \pi^\alpha = 0 \). We have also
\[
[Q^\alpha]^- = 0, \quad [\vec{v}^\alpha]^- = 0, \quad [\rho^\alpha f]^- = \phi^{\text{net}} \rho^\alpha_R,
\]
where \( \phi^{\text{net}} \) is the porosity of the net. Eq. (157) can be rewritten as
\[
[(1 - \phi) v^f]^- = \phi^{\text{net}} [v^f]^+,
\]
where we have assumed that \( [\rho_R^f]^- = [\rho_R^f]^+ \) and where \( v^f = \vec{v}^f \cdot \vec{e}_x \).

**Linear momentum along x**: In this case we assume
\[
[T^f \vec{e}_x]^+ = -\phi^{\text{net}} \rho^{\text{ext}} \vec{e}_x, \quad [T^x \vec{e}_x]^+ = T^{\text{net}} \vec{e}_x,
\]
where \( \rho^{\text{ext}} \) and \( T^{\text{net}} \) are the external liquid pressure (atmospheric pressure for instance) and the stress tensor of the net, respectively. Taking (158) and (159) into account Eq. (157) takes the form
\[
[\rho^f_R]^- ([(v^f)^- - (v^f)^+]) = \phi^{\text{net}} \rho^{\text{ext}} - (T^{\text{net}})_{11} + [(T^f)_{11} + (T^x)_{11}]^-,
\]
which, recalling assumption A1 and using (108) and (113), yields
\[
\phi^{\text{net}} \rho^{\text{ext}} - (T^{\text{net}})_{11} + [-\lambda + (\tilde{T})_{11}]^- = 0.
\]

**Energy**: We now have \( Q^\alpha = \vec{e}^\alpha + [\vec{v}^\alpha]^2/2, \pi^\alpha = T^\alpha \vec{v}^\alpha - \vec{q}^\alpha, \alpha = s, f \). Further we make the following assumptions:
\[
[\vec{e}^f]^+ = [\vec{e}^f]^-, \quad [\vec{q}^\alpha]^+ = 0 = [\vec{q}^\alpha]^-, \quad \alpha = s, f,
\]
justified by the fact that we have assumed an isothermal process. We thus obtain
\[
\rho^f_R [(1 - \phi) v^f]^- \left( \left[ \frac{(v^f)^2}{2} \right]^- - \left[ \frac{(v^f)^2}{2} \right]^+ \right) = [(T^f)_{11}]^- [v^f]^+ + \phi^{\text{net}} \rho^{\text{ext}} [v^f]^+.
\]
If the inertial terms are neglected, with the aid of (159), we obtain
\[
\left[ \frac{(T^f)_{11}}{1 - \phi} \right]^+ + \rho^{\text{ext}} = 0,
\]
which, recalling (78), implies the continuity of the pressure, namely
\[
\rho^{\text{ext}} = [\lambda]^-.
\]
Thus, for a sufficiently coarse net (i.e. for \( \phi^{\text{net}} \approx 1 \)) (162) entails
\[
(T^{\text{net}})_{11} = [\tilde{T}]_{11}^-,
\]
i.e., the load applied on the net is due only to the solid deformation.
In writing explicitly the boundary condition on \( x = 0 \) we have to consider two cases:

1. The compression is controlled by prescribing the velocity of the top surface, \( v_{\text{top}}(t) \) (velocity driven compression).
2. The compression is controlled by prescribing the load applied on \( x = L(t) \), \( P_{\text{top}}(t) \) (pressure driven compression).

We now consider the thorniest issue in developing well-posed problems within the context of the theory of mixtures, namely that of prescribing boundary conditions. Whether it be displacement, velocity or traction boundary conditions, we are privy to only the total displacement, velocity or traction on the boundary, i.e., quantities associated with the mixture. But this immediately implies that there is a non-uniqueness with regard to the prescription of the same quantities on the boundary. While it may seem perfectly reasonable in a displacement boundary value problem to assume that the displacement of both the constituents are the same and that which is undergone by the boundary, in reality all that we know is the displacement of the mixture. This impasse with regard to boundary conditions can only be overcome on physical grounds. Thermodynamics considerations such as Gibbs’ free energy of dilution being zero have been applied in some problems where such a boundary condition may be relevant (see [45,25,26]), a pure mechanical traction splitting (see [37]), continuity of chemical potential, and other conditions have been used. Here we follow yet another approach. The justification for our choice has to be ultimately borne out by its agreement or otherwise to experimental results.

It follows from (72) that

\[
\frac{\partial v_c}{\partial x} = 0, \tag{168}
\]

with \( v_c \) given by (73). Evaluating \( v_c \) on \( x = L(t) \) and on \( x = 0 \) gives

\[
v_{\text{top}}(t) = [(1 - \phi)v^f]^-_{x=0}. \tag{169}
\]

We now evaluate Eq. (117) on the boundary to obtain a boundary condition

\[
[(1 - \phi)v^f]^-_{x=0} = -\left[ K(\phi) \frac{\partial}{\partial x} ((\hat{T})_{11} + \rho_f^f \phi) \right]^-_{x=0}, \tag{170}
\]

that is

\[
v_{\text{top}}(t) = \begin{cases} -2\rho_R \left[ \phi \frac{\phi}{\phi_0} K(\phi) \frac{\partial}{\partial X} \left( W \left( \frac{\phi^2_0}{\phi} \right) \frac{\phi^2_0}{\phi} \right) \right]_{x=0}, & \phi \geq \phi^*, \\ -2\rho_R^e \left[ \phi \frac{\phi}{\phi_0} K(\phi) \frac{\partial}{\partial X} \left( W \left( \frac{\phi^2_0}{\phi^2} \right) \frac{\phi^2_0}{\phi} \right) \right]_{x=0}, & \phi < \phi^*. \end{cases} \tag{171}
\]

The above approach cannot be justified in general as it would imply that the governing equations themselves lead to the boundary conditions. However, in absence of a clear method to form boundary conditions, we shall resort to such an approach. Eventually, we shall solve a problem within the context of the model that is being developed to assess the reasonableness of the assumptions including the one above.

In the second case the load \( P_{\text{top}}(t) \) applied on \( x = L(t) \) is given. Thus, recalling (111) and (113), we have

\[
-\dot{\lambda}(0, t) + (\hat{T}(0, t))_{11} = -\dot{\lambda}(L(t), t) + (\hat{T}(L(t), t))_{11}, \tag{172}
\]

which, because of (166), yields

\[
(\hat{T}(0, t))_{11} = P_{\text{ext}} + [-\dot{\lambda}(L(t), t) + (\hat{T}(L(t), t))_{11}], \tag{173}
\]

that, once evaluated \([-\dot{\lambda}(L(t), t) + (\hat{T}(L(t), t))_{11}] \) in terms of the applied load \( P_{\text{top}}(t) \), represents the boundary condition we were looking for. To this purpose, we apply to the surface \( x = L(t) \) Eq. (157), where now \( \vec{n} = \vec{e}_x \),...
\[ [\hat{\rho}^s]^+ = 0, \quad \hat{w} = [\hat{v}^s]^- = s, \quad f, \]

\[ [\hat{\pi}^s]^- \cdot \hat{n} = (T)_{11} = -\hat{\lambda}(L(t), t) + (\hat{T}(L(t), t))_{11}, \quad (174) \]

\[ [\hat{\pi}^s]^+ \cdot \hat{n} = -P^{\text{top}}(t). \quad (175) \]

We thus obtain

\[ -P^{\text{top}}(t) = -\hat{\lambda}(L(t), t) + (\hat{T}(L(t), t))_{11}. \quad (176) \]

So, in case of pressure driven compression, we have

\[ P^{\text{ext}} - P^{\text{top}}(t) = \begin{cases} 
\left[ \frac{\phi P R_s^f}{\rho} (\psi_0 - \hat{\psi}_s) 
\right]_{X=0} 
\quad & \text{if } \phi \geq \phi^*, \n
+2 \rho R_s^f W \left( \frac{\phi_0^2}{g^2 \phi} \right) \left[ \frac{\phi_0^2}{g^2 \phi} \right]_{X=0} 
\quad & \text{if } \phi < \phi^*. \end{cases} \quad (177) \]

4.3. The mathematical problem

We now discuss the mathematical problem generated by the model presented in the previous sections. We will consider only the case in which the compression process is controlled prescribing the velocity of the top surface (for instance see [23]). Further we assume that at the beginning of the compression the material domain \( \Omega \) is divided into two regions

\[ \Omega_0^{(1)} = 0 \leq X < S_0, \quad S_0 > 0, \quad (178) \]

\[ \Omega_0^{(2)} = S_0 < X \leq L_0, \quad (179) \]

where the initial solid volume fraction is such that

\[ \phi(X, 0) = \hat{\phi}_1(X) > \phi^* \quad \text{if } X \in \Omega_0^{(1)}, \]

\[ \phi(S_0, 0) = \phi^*, \]

\[ \phi(X, 0) = \hat{\phi}_2(X) < \phi^* \quad \text{if } X \in \Omega_0^{(2)}. \]

Due to the complex dynamics of the system \( \Omega_0^{(1)} \) and \( \Omega_0^{(2)} \) will evolve in time so that, at least for a short time after the process have been initiated, we have

\[ \Omega = \overline{\Omega_0^{(1)}(t) \cup \Omega_0^{(2)}(t)}, \quad (181) \]

where

\[ \Omega_0^{(1)}(t) = 0 \leq X < S(t), \quad (182) \]

\[ \Omega_0^{(1)}(t) = S(t) < X \leq L_0, \quad (183) \]

with \( X = S(t), S(0) = S_0 \), representing the free boundary (unknown) separating the domains.
Next, we assume that $\phi_0$ constant and $W = W_p = \mu = \text{constant}$. We thus have

- $X \in \Omega_1(t)$,

$$
\begin{align*}
\frac{\partial \phi}{\partial t} &= -2\rho_B^2 \mu \phi^2 \frac{\partial^2}{\partial X^2} \left[ \phi K(\phi) \frac{\partial}{\partial X} \left( \frac{1}{g^2 \phi} \right) \right], \quad 0 \leq X < S, \quad t > 0, \\
\frac{\partial g}{\partial t} &= \left( \frac{2\rho_B^2 \mu \phi_0}{\beta} \right) \frac{1}{g \phi}, \quad t > 0, \\
2\rho_B^2 \mu \phi_0 \phi K(\phi) \frac{\partial}{\partial X} \left( \frac{1}{g^2 \phi} \right) \bigg|_{X=0} &= -v^{\text{top}}(t), \quad t > 0, \\
\phi(S(t), t) &= \phi^*, \quad t > 0, \\
\phi(X, 0) &= \hat{\phi}_1(X), \quad 0 \leq X < S_0.
\end{align*}
$$

(184)

- $X \in \Omega_2(t)$,

$$
\begin{align*}
\frac{\partial \phi}{\partial t} &= -2\rho_B^2 \mu \phi^2 \frac{\partial^2}{\partial X^2} \left[ \phi K(\phi) \frac{\partial}{\partial X} \left( \frac{1}{g \phi} \right) \right], \quad S \leq X < L_0, \quad t > 0, \\
\frac{\partial}{\partial X} \left( \frac{1}{g \phi} \right) \bigg|_{X=L_0} &= 0, \quad t > 0, \\
\phi(S(t), t) &= \phi^*, \quad t > 0, \\
\phi(X, 0) &= \hat{\phi}_2(X), \quad S_0 \leq X < L_0.
\end{align*}
$$

(185)

- $X = S(t)$,

$$
\begin{align*}
g(S(t), t) &= 0, \quad t > 0, \quad (186)
\end{align*}
$$

and

$$
\begin{align*}
g(X, 0) &= 1, \quad 0 \leq X \leq S_0, \\
g(X, 0) &= 1, \quad S_0 < X \leq L_0.
\end{align*}
$$

(187)

Differentiating (186) with respect to time we get an evolution equation for the free boundary $S$, i.e.,

$$
\frac{\partial g}{\partial X} \frac{\partial S}{\partial t} + \frac{\partial g}{\partial t} = 0, \quad (188)
$$

which has to be coupled with (184) and (185).

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**References**
