Mathematics Workshops

## Pythagorean Theorem in Game Form.

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## Suggestions and Tips for a Workshop on the Pythagorean Theorem and Thereabouts.

A series of puzzles serves as an introduction to the numerous facets of the Pythagorean Theorem. To begin with, the theorem is first examined in its traditional form and then in a more general way by replacing the squares by similar figures; in the final step, we proceed to the Theorems of Euclid and Pappus of Alexandria. The different facets of these theorems are made tangible through puzzles that take a more playful approach to the theorem and to its proof.

Aims of the workshop
The workshop aims at an introduction of a number of important geometric concepts through play. It deals with the theorem of Pythagoras in the first place, and then with Euclid's theorem, that usually precedes its proof, as well as with a rather surprising generalisation, due to Pappus of Alexandria (V century a. D.). The passage from squares to exhagons and stars makes possible to introduce the notion of similar figures, and to show that the areas of similar figures are proportional to the squares of corresponding sites.

We propose here some suggestions for the laboratory. We suggest a possible path, together with remarks concerning the management of the activities. These observations are distilled from the laboratories made at the museum. The reference classes are junior high schools, but with some adjustements the workshops can be adapted to high school pupils, for whom we suggest occasionally some possible additions.

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## 1.The Proposition

The workshop starts with the proposition of the Theorem, which is what two of the puzzles deal with. If the class is not yet familiar with the Pythagorean Theorem, the puzzles may be a way of getting the students to formulate the Theorem on their own. Should they be familiar with it, the workshop may begin by asking the students themselves to state the Theorem.
It is widely known that the Pythagorean Theorem may take different forms; of the two main ones, one is expressed geometrically and the other one algebraically. In the former case, it says

In a right-angled triangle, the squares drawn on the legs are equivalent to the square drawn on the hypotenuse.
or some equivalent formulation. In the latter, it may be said that
In a right-angled triangle of legs $a$ and $b$ and Hypotenuse $c$ we have $a^{2}+b^{2}=c^{2}$.
At times, school children miss the context and merely remember the $a^{2}+b^{2}=c^{2}$ formula; of course, it is then necessary to provide an explanation. One half-way measure speaks about areas and states that

In a right-angled triangle, the sum of the areas of the squares built upon the legs is equal to the area of the square built on the hypotenuse.

Once it has been established that these propositions are equivalent, we can use any one of them (for example, the third). We may wonder - what does this mean ?

You have three squares at stake whose sides are given respectively by the hypotenuse and by the two legs. The theorem states that the first square has the same area as the other two taken together. To verify this experimentally -mind you, not to prove it !-, you can proceed in different ways; for example, by measuring legs a and b and hypotenuse c and verify that $a^{2}+b^{2}=c^{2}$.

Another approach that will be used throughout the workshop involves cutting the square of the hypotenuse into a certain number of pieces and using them to form the squares of the legs.

Here, what is known as equidecomposability is applied - if two figures are made up of the same pieces arranged in a different way, then, they evidently have the same area. So, if you can build the two squares of the legs using the pieces of the square of the hypotenuse, it means that these two figures (the square of the hypotenuse, the two squares of the legs) are equivalent.

At this point, an initial observation may be made. Two equivalent figures are not always equidecomposable. For instance, it can be proven -and for those familiar with integral calculus this is not difficult to do - that the two figures below, i.e. the segment of a parabola and a rectangle, have the same area, yet they are not equidecomposable.


However, if the figures in question are polygons or composed of polygons, the opposite is also true: two polygons have the same area if and only if they can be cut into the same pieces.

Please, take note of -and possibly look further into- the fact that this result does not hold for polyhedra. Two equidecomposable polyhedra are obviously equivalent, i.e. they have the same volume, but two equivalent polyhedra are not always equidecomposable.

Now, let us examine puzzles related to the proposition, the first one having five and the second one seven pieces.

In the first puzzle the five pieces can be used to form either the two squares of the legs or the large square of the hypotenuse. This first apparently rather simple five-piece puzzle often turns out to be one of the trickiest, but don't lose heart! Anyway, here is the solution:


The second puzzle made up of seven pieces is analogous. Even though there are more pieces, a certain symmetry helps to recompose it. Here is the solution:


The solutions to these puzzles may be given to the class as a skills game. The class can be split up into two or more teams, depending on the number of students; they may work at the same time or one team at a time and an arbiter may be appointed to time the task. Once they are given the goahead, the teams first have to put the squares of the legs together. It is advisable to start with those , because they are easier than the square of the hypotenuse, which can be done afterwards. Whoever takes the least overall time wins.

## 2. Hexagons and Stars

After finishing the games dealing with the traditional Pythagorean Theorem, unless you find yourself obliged to stop them, promising to make the puzzles available at break-time to those who are eager to take up the challenge posed by the impenetrability of bodies, you can go on to the following step.

Taking advantage of the fact that, by now, your students will have become warmed up by their attempts at tackling the traditional theorem, you can get them to state the proposition of the Theorem once again as a chorus - repeating never does any harm. If you notice that they have reached a higher level of awareness when saying words such area, squares, triangles, legs, etc., in a row, and if you see them looking at the puzzles they have been doing for confirmation and support, then your first goal has been achieved.

At this point, after the traditional Theorem, a new challenge can be presented to them: "what if I said 'hexagon of the hypotenuse....'?"
Some classes have already seen the extension and the best students will say: " That goes for all the regular polygons."

Then, be even more daring and say : "What about stars?"


And if they do not appear surprised, you will wrong-foot them by saying : "What if I put the figure of a lion (or whatever comes into your mind - the most unsymmetrical one possible) there?" The proposition of the Theorem remains valid whatever figure you might choose (and this generally strikes them as unexpected) as long as the very same figure is placed on every side, after having enlarged or reduced it as needed.

In other words - and here you may risk repeating the same thing in more geometric terms by talking about similarity instead of enlargement or perhaps by eliciting the word similarity from them. Given the three sides of the right-angled triangle, in the case of the traditional proposition, it is easy to understand how large the three squares in the game have to be : their sides will be the same length as the two legs and the hypotenuse. When generalizing, someone may find it hard to understand how the figures are enlarged or reduced.

When dealing with hexagons and regular polygons in general, this is still quite immediate; here, too, three figures with long sides respectively like the two legs and the hypotenuse may be taken. As regards stars, this is already less apparent. In the puzzle given, it is the distance between two points, meaning a segment that is not part of the figure needs to coincide with the legs or the hypotenuse every time. Generally, any segment may be chosen, also in case of regular squares or polygons, not necessarily one side, but also a diagonal, an apothem or any segment. The important thing is to always take corresponding segments for the three figures: always diagonals, or always apothems, or whatever.

In the figure below, on the left, the sides of the right-angled triangle are equal to three corresponding segments in the hexagons that can be seen in the figure on the right. Thus, the Pythagorean Theorem is applicable to the three figures on the left : the area of the figure of the hypotenuse is equal to the sum of the areas of the figures of the legs. Of course, the Theorem also holds for corresponding hexagons.


If the class has followed you up to this point, you can go on to "Why is it applicable?", proceeding to explain that in similar figures, the areas are proportional to the squares of corresponding segments and this brings us back to the traditional Pythagorean Theorem.

In other words, if we have three similar figures in which $a, b, c$ are the lengths of corresponding segments, e.g. the sides of three regular hexagons or the three segments in the figure above, the areas of the three figures will be $k a^{2}, k b^{2}$ and $k c^{2}$. If $a, b, c$ are the legs and the hypotenuse of a right-angled triangle, the equation will be $a^{2}+b^{2}=c^{2}$. And therefore also $k a^{2}+k b^{2}=k c^{2}$.

Now, back to the puzzles. The first one gives the version of the Pythagorean Theorem with hexagons, the second one is with stars.
If the students enjoyed competing, the puzzle can be done by the same teams as before. Here are the solutions:


## 3. Proof

It is not known whether or how Pythagoras himself managed to prove his Theorem, that is to say whether he actually worked out a convincing argument for the validity of his discovery. However, after him, a number of proofs were provided by different scholars.

One of the simplest ones known is proposed as a puzzle. Four copies of the right-angled triangle at issue are placed in a large square whose side is equal to the sum of the legs. This is done in two different ways. In the first, the square of the hypotenuse remains uncovered while in the second it is the squares of the two legs that are not covered. Since the square is large and the four triangles are always the same, it may be deduced that the area of the uncovered figures in the first and the second case is identical.


This proof is sometimes already known to students, and therefore they will be able to work out the configuration once again. Seldom, though, will they be able to explain why the two configurations result in the Theorem. Try guiding them one step at a time to identify the two small squares and the large one, explaining what makes them equivalent.

If you deem it fitting, when working with older students, you may also raise the question as to whether the square of the hypotenuse in the first configuration really is a square. This stage is often passed over, but it is necessary to complete the proof without relying solely on the visual aspect.

Thus, if you want to prove that the first square is indeed a square, you may first of all observe that it has four equal sides since they are all equal to the hypotenuse of the right-angled triangle given. Therefore, it undoubtedly is a rhombus, but to infer that it is a square, the angles have to be right angles. This may be done by observing that each of the white angles is equal to a flat angle minus the two yellow angles and that, since the two are angles of a right-angled triangle, they form a right angle. Consequently, every white angle is a right angle and thus we do have a square.

One may also notice that the figure on the right shows the algebraic formula

$$
(a+b)^{2}=a^{2}+b^{2}+2 a b
$$

As a matter of fact, if $a$ and $b$ are the legs of the right-angled triangle, then the large square has the
sides $a+b$ and its area $(a+b)^{2}$ is the sum of the areas of the squares of the legs $\left(a^{2}+b^{2}\right)$, plus the areas of the two rectangles with sides $a$ and $b$.

## 4. Euclid

The oldest known proof of the Pythagorean Theorem is due to Euclid and can be found in the first book of his Elements.


Euclid approached one part of the problem at a time, and starting with just one of the squares of the legs, he saw that .. At this point, the corresponding puzzle can be given to the students. "Are you already familiar with Euclid's Theorem? Well, then, let's check. Otherwise, solve the puzzle to discover it".

Using the pieces provided, compose first the square of the leg and then the rectangle that is equal to it. Then you may see that in a right triangle, the square built on a leg is equivalent to the rectangle whose sides are the hypotenuse and the projection of the leg on the hypotenuse.

This result enables Euclid to prove Pythagoras' Theorem.
What happens if the square of the other leg is taken into consideration as well?

## 5. Euclid's Second Theorem

Let us digress for a moment on a small tangent regarding Euclid's second Theorem:
In a right-angled triangle, the square built on the height corresponding to the hypotenuse is equal to the rectangle whose sides are the projections of the two legs on the hypotenuse.

As a puzzle, Euclid's second theorem is broken up exactly as the first one was, that is to say, a puzzle corresponding to the first theorem works for the second theorem as well, and vice versa.


Consider the right-angled triangle $a b c$ where $b d$ is the height on the $a c$ hypotenuse. The second Euclidean Theorem states that the square befd on the height is equivalent to the rectangle adhg, whose sides are the projections of the legs on the hypotenuse.

At this point, let us examine another right-triangle with the hypotenuse $D C$ equal to the projection $d c$ of the previous triangle's greater leg and a leg $B D$ equal to the height $b d$. The first Euclidean Theorem states that the square of the leg $B D$ is equal to the rectangle whose sides are the hypotenuse $D C$ and the projection $D A$ of the leg $B D$ on the hypotenuse.

Since the squares bedf and $B E D F$ are the same by construction, the rectangles $a d h g$ and $D A H G$ must be equivalent. But these rectangles have heights $a g=D G$, and therefore the bases $a d$ and $D A$ are the same. As a result, the two rectangles are congruent. If we decompose the rectangle $D A H G$ so that the pieces make up the square $B E D F$, the same decomposition works for the rectangle adhg and the square bedf.

## 6. Pappus of Alexandria.

Some may wonder what happens when the triangle is not right-angled. This question may arise when the students repeat the Proposition; someone will inevitably forget to say that the triangle has to be right-angled.

A few centuries after Euclid, Pappus of Alexandria ( $4^{\text {th }}$ c. A.D.) found a theorem analogous to the first Euclid's, except for the fact that triangle $A B C$ is not right-angled and, instead of a square, there is an arbitrary parallelogram .

Given a triangle $A B C$, we build a parallelogram $A B G F$ on the
 side $A B$, and on the extension of side $F G$ we take any point $H$. We join $H$ and $B$ and prolonge the segment $H B$; on this extension, which meets side $A C$ in $I$, we take $I L=H B$ and complete parallelogram AILM. Pappus' Theorem states that parallelogram AILM is equivalent to the initial parallelogram $A B F G$.

This is the result illustrated in the last puzzle. As for Euclid's Theorem, once again we now need to check the equivalence between the two parallelograms by using the same pieces to buildi the parallelograms in question.

Proving Pappus' Theorem is not difficult and this activity may be proposed to older students. The important thing is to remember that two parallelograms that have the same base and are between the same parallels, that is have the same height, are equivalent.

To prove this equivalence, the trick is to add the dotted lines $A P$ and $P Q$ and to take into consideration parallelogram $A P Q I$. In the figure, you immediately notice that parallelograms $A B G F$ and $A B H P$ are equivalent, because they have the same base $A B$ and are between parallels $A B$ and $F H$. Parallelograms $A B H P$ and $A P Q I$ are equivalent in the same way, as they have the same base $A B$ and they are between parallels $A P$ and $I H$. Consequently, parallelograms $A B G F$ and $A P Q I$ are equivalent. If you then observe that $P A$ is equal to $H B$, being the opposite sides of a parallelogram, then you can deduce that parallelograms APQI and AILM are equal, and therefore the latter is equivalent to $A B G F$.

The Euclidean Theorem is a special case of Pappus's Theorem .


In fact, if triangle $A B C$ is right-angled and parallelogram $A B G F$ is a square, by choosing $H$ so that stright line $H B$ is perpendicular to hypotenuse $A C$, you get a segment HB that is equal to $A C$. Indeed, the angle $G H B$ is equal to the angle $A C B$ as their sides are perpendicular, the angles $H G B$ and $A B C$ are right and segments $A B$ and $B G$ are also equal since $A B G F$ is a square. Parallelogram AILM, which according to the Pappus's Theorem is equivalent to the square $A B G F$, is therefore the rectangle that has as its sides the segment $I L$ (which by construction is equal to $H B$ and it is thus also equal to hypotenuse $A C$ ) and the projection $A I$ of $A B$ on the hypotenuse, as stated by the first Euclidean Theorem.

Then, if two parallelograms are constructed, one for each of the two sides, and if as a point $H$ you take the one where the prolongation of sides
$F G$ and $D E$ meet, by drawing a straight line and completing the Pappus construction on both sides, what you get is parallelogram $A C N M$, built on base $A C$, with side $A M$ equal and parallel to segment $H B$. Thus, Pappus's Theorem tells us that its area is equal to the sum of the areas of the initial parallelograms.

This is an extension of the Pythagorean Theorem that you find in the particular case in which the triangle is right-angled and the two parallelograms are squares.


As a matter of fact, in this case segment $H I$ is perpendicular to $A C$, which can be demonstrated by means of the figure on the right. Rectangle $H E B G$ is equal to rectangle $A B C T$ since by construction we have $H E=A B$ and $B E=B C$. As a result, triangles $H E B$ and $A B C$, each one being half of the respective rectangle, will also be equal, and in particular angles $I H E$ and $C A E$ will be equal.
Since $H E$ is perpendicular to $A E$, also $H I$ will be perpendicular to $A C$.
Therefore, we may apply the Euclidean Theorem to squares $A B G F$ and $B C D E$ and obtain the Pythagorean Theorem.

## 7. A More In-Depth Look for Grown-Ups



From the Pappus's Theorem, it follows that in an acute triangle, the square on the greater side, i.e. the hypotenuse in the case of a right-angled triangle, is less than the sum of the squares of the other two sides, i.e. the legs if the triangle is right-angled. When the triangle is obtuse, the square is greater.

Let us take an acute triangle $A B C$, whose greater side is $A C$. From vertex B , we draw $B I$ perpendicular to $A C$ and on its extension we take point $H$ so that $H B=A C$. From $H$ we draw $H F$ parallel to $A B$ and $H D$ parallel to $B C$, and then we build rectangles $A B G F$ and $B C D E$. According to Pappus's Theorem, they will be equivalent to rectangles $A I L M$ and $I C N L$, respectively; therefore, together they will be equivalent to square ACNM.

At this point, we may ask: Will rectangle $B C D E$ be greater or lesser than the square on $B C$ ?
To respond, we draw $A Q$ perpendicular to $B C$.


Since angle $\gamma$ is acute, point $Q$ will fall between $C$ and $B$. Triangles $A Q C$ and $H E B$ are equal, indeed they are right-angled with $H B=A C$; besides, angles $E H B$ and $A C Q$ are equal since their sides are perpendicular. As a result, $B E=C Q<B C$ and therefore, rectangle $B C D E$ is lesser than the square on $B C$. Similarly, rectangle $A B G F$ is proven to be lesser than the square on $A B$ and thus the square on the greater side $A C$ is less than the sum of the squares on the other two sides.

If instead angle $\gamma$ is obtuse, point $Q$ falls outside segment $B C$ and thus $B E$ will be greater than side $B C$ and rectangle $B C D E$ will be greater than the square on $B C$. As a result, the square on side $A C$ will be greater than the sum of the squares on the other two.

Using a bit of trigonometry, it is possible to specify by how much the square on a side deviates from the sum of the other two. Since $C Q=B C-B Q=b-a \cos \gamma$, the area of rectangle $B C D E$ will be $b(b-a \cos \gamma)$.
Likewise, the area of rectangle $A B G F$ is found to be $a(a-b \cos \gamma)$.
Therefore, by the Theorem of Pappus we have $c^{2}=a(a-b \cos \gamma)+b(b-a \cos \gamma)$, that is to say

$$
c^{2}=a^{2}+b^{2}-2 a b \cos \gamma
$$

This is known as the Carnot Theorem.

