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Symmetry and non-symmetry for the overdetermined Stekloff eigenvalue problem

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1. Introduction

In a simply connected plane domain Ω with sufficiently smooth boundary $\partial\Omega$, consider the Stekloff eigenvalue problem

$$\Delta u = 0 \qquad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial n} = pu \qquad \text{on } \partial\Omega, \tag{1.2}$$

where n denotes the exterior normal unit vector to $\partial\Omega$. This problem has infinitely many eigenvalues $0 = p_1 < p_2 \leq p_3 \leq \cdots$ (see [S]). Payne and Philippin [P-P] have recently proven that if u is an eigenfunction corresponding to the second eigenvalue p_2 and satisfies the overdetermined condition

$$|\nabla u| = 1 \qquad \text{on } \partial\Omega, \tag{1.3}$$

then Ω is a disk of radius $R = p_2$. They also show that, for any $h \geq 0$, the domain

$$\Omega_h = \{(x, y) \in \mathbb{R}^2 : |x| < 1, -h/2 - \sqrt{1 - x^2} < y < h/2 + \sqrt{1 - x^2}\}$$

is such that $u(x, y) = x$ satisfies (1.1)–(1.3); however, u is not the second eigenfunction unless $h = 0$.

Thus, they conjecture that, if $\partial\Omega$ is of class C^2, then for any non-trivial eigenvalue p and for any dimension, the existence of a solution u of (1.1)–(1.3) implies that Ω is a ball.

In this paper, we treat the two-dimensional case when $\partial\Omega$ is of class C^2 and $u \in C^2(\bar{\Omega})$. Observe that if $\partial\Omega$ is in $C^{2,\alpha}$, then any solution of (1.1)–(1.2) belongs to $C^{2,\alpha}(\bar{\Omega})$.

Let us introduce the conjugate harmonic function v to u in Ω, chosen in such a way that

$$\int_{\partial\Omega} v \, ds = 0, \tag{1.4}$$

here ds denotes the arclength element. We let

$$F = u + iv \tag{1.5}$$

be the *complex potential associated to u*. We shall first prove the following necessary and sufficient condition for the Payne and Philippin conjecture to hold.

Theorem 1.1. Let $u \in C^2(\bar{\Omega})$ be a solution of (1.1)–(1.3), and let F be the complex potential associated to u, as defined by (1.4) and (1.5).

Then, Ω is a disk if and only if F vanishes at only one point in Ω.

We stress the fact that the conclusion of Theorem 1.1 does not involve the vanishing rate of F at its zero. In §2, we shall give a proof of Theorem 1.1 through a sequence of statements which may be of some interest of their own. In particular, Theorem 2.2 shows an interesting connection with another symmetry problem, involving Green's function, which has already been treated by Payne and Schaefer [P-S] and Lewis and Vogel [L-V].

The combination of Theorem 1.1 and of Theorem 1.2 below shows that Payne's and Philippin's conjecture fails to be true: the disk is not the only domain with C^2 boundary for which a solution of (1.1)–(1.3) exists.

Theorem 1.2. Given the integers $K > 1$, and $m_1, \dots, m_K \geq 1$, there exist a simply connected domain Ω with analytic boundary, and a function F holomorphic in Ω such that $u = \mathrm{Re}(F)$ satisfies (1.1)–(1.3) and F has exactly K distinct zeros $z_1, \dots, z_K \in \Omega$ with respective multiplicities $m_1, \dots, m_K$.

Section 3 contains a constrictive proof of Theorem 1.2, which is based on the classical method of conformal mappings; such a proof can also be adapted to provide an alternative proof of Theorem 1.2.

2. Proof of Theorem 1.1

We shall denote by $|\partial\Omega|$ the perimeter of Ω, and by $z = z(s)$, $0 \leq s \leq |\partial\Omega|$, the arclength parametrization of $\partial\Omega$ taken with the counterclockwise orientation, so that $\dot{z}(s)$ and $-i\dot{z}(s)$ are respectively the tangent and normal unit vector to $\partial\Omega$. As usual, a prime will denote the derivative with respect to the complex variable z, while we chose to indicate by the subscripts, s, n respectively the tangential and normal partial derivatives at points of $\partial\Omega$.

Theorem 2.1. Let $u \in C^2(\bar{\Omega})$ satisfy (1.1)–(1.3). Then, there exists a positive integer N such that we have:

$$p = \frac{2\pi N}{|\partial\Omega|}, \tag{2.1}$$

$$F(z(s)) = F(z(0))\, e^{ips}, \qquad 0 \le s \le |\partial\Omega|, \tag{2.2a}$$

$$|F(z(0))| = \frac{1}{p}. \tag{2.2b}$$

Moreover, F and F' have respectively N and $N-1$ zeros in Ω, when counted according to their multiplicities.

The proof of this Theorem is mainly of computational character and is left to the end of this section.

Remark. Notice that Theorem 2.1 can be interpreted as follows: if a solution u of (1.1)–(1.3) exists in Ω, then p is a Stekloff eigenvalue for a disk B_R which has the same perimeter as Ω; also, the restriction of u to $\partial\Omega$, as a function of the arclength s, coincides with the restriction to ∂B_R of a Stekloff eigenfunction for B_R.

Corollary 2.1. There exists $u \in C^2(\bar{\Omega})$ satisfying (1.1)–(1.3) if and only if there exists a holomorphic function F in Ω such that

$$|F| = \frac{1}{p} \qquad \text{on } \partial\Omega, \tag{2.3a}$$

$$|F'| = 1 \qquad \text{on } \partial\Omega. \tag{2.3b}$$

Moreover, F is the complex potential defined in (1.5).

Proof. If u satisfies (1.1)–(1.3), then by Theorem 2.1 the associated complex potential F satisfies (2.3). Vice versa, conditions (2.3) imply that $\partial\Omega$ is analytic (see [F]) and F is analytic up to $\partial\Omega$. Let

$$N = \frac{1}{2\pi i}\int_0^{|\partial\Omega|} \frac{F'(z(s))}{F(z(s))}\, \dot{z}(s)\, ds = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{F'(z)}{F(z)}\, dz.$$

By the logarithmic residue formula, N is the number of zeros of F in Ω; N is not zero, otherwise $|F| = 1/p$ in Ω, which would imply $F' \equiv 0$, contrary to (2.3b). By differentiating (2.3a) by s, we have $\mathrm{Re}(F'\dot{z}\bar{F}) = 0$ on $\partial\Omega$, thus, by (2.3) we obtain $F'\dot{z} = \pm ipF$ on $\partial\Omega$. This implies $2\pi N = \pm p|\partial\Omega|$, so that we must choose the positive sign. Therefore, by setting $F = u + iv$, we have

$$u_s + iu_n = F'\dot{z} = ipF = ip(u + iv),$$

hence $u_n = pu$ on $\partial\Omega$, and also $|\nabla u| = |F'| = 1$ on $\partial\Omega$, that is u satisfies (1.1)–(1.3).

Theorem 2.2. Let F be a holomorphic function in Ω satisfying (2.3) and suppose F vanishes at only one point in Ω.

Then Ω is a disk $B_R(z_0)$ and, for some positive integer N, we have:

$$F(z) = \alpha(z - z_0)^N,$$
$$p = N/R, \tag{2.4}$$
$$|\alpha| = 1/NR^{N-1}.$$

Proof. Let z_0 be the only zero of F in Ω, and let N be its multiplicity. We may factor $F(z) = (z - z_0)^N \Phi(z)$ where Φ is holomorphic which never vanishes in Ω. Then the function $w(z) = \log p|F(z)|$ has the following properties: $\Delta w = 2\pi N\delta(\cdot - z_0)$ in Ω, $w = 0$ on $\partial\Omega$, and also $|\nabla w| = |F'|/|F| = p$ on $\partial\Omega$.

In other words, $w(z) = -2\pi NG(z, z_0)$ where $G(z, z_0)$ is the Green's function for Ω with pole at z_0. Therefore, $|\nabla G(\cdot, z_0)| = p/2\pi N$ on $\partial\Omega$, and, by Theorems III.1, III.2 in [P-S], we have that Ω is a disk centered at z_0, and (2.4) follows easily.

Remark. We observe that another proof of the spherical symmetry for the above mentioned overdetermined problem for the Green's function can be found in Lewis and Vogel [L-V]. As is observed in [P-S], still another proof could be obtained by the method of moving parallel planes of Serrin [Se].

Proof of Theorem 1.1. Let Ω be a disk of radius R centered at z_0. As is well-known (see [B]), the Stekloff eigenfunctions of Ω are given by the real or the imaginary part of the holomorphic functions $\alpha(z - z_0)^N$, where N is an integer and α is a complex number. Therefore, the complex potential associated to a solution of (1.1)–(1.3) in Ω takes the form $F = \alpha(z - z_0)^N$ with $|\alpha| = 1/(NR^{N-1})$.

Vice versa, if u satisfies (1.1)–(1.3) in Ω and the function F in (1.5) vanishes only at one point $z_0 \in \Omega$, then Corollary 2.1 and Theorem 2.2 imply that Ω is a disk centered at z_0.

Remark. We point out an interesting connection between the overdetermined problem (1.1)–(1.3) and the field of quadrature identities.

From Corollary 2.1 and by the arguments of Theorem 2.2, we readily see that, if u satisfies (1.1)–(1.3), and F has K distinct zeros $z_1, \ldots, z_K$ with respective multiplicities $m_1, \ldots, m_k$, then the function $w = \log p|F|$ satisfies $\Delta w = 2\pi \sum_{k=1}^{K} m_k \delta(\cdot - z_k)$ in Ω, $w = 0$ on $\partial\Omega$, and also $|\nabla w| = p$ on $\partial\Omega$. Let f be any holomorphic function in a neighborhood of $\bar{\Omega}$. By applying Green's identity for f and w, we have:

$$\int_{\partial\Omega} f\, ds = \sum_{k=1}^{K} \frac{2\pi m_k}{p} f(z_k).$$

A standard density argument allows to extend the validity of the above identity to any holomorphic function f in Ω, whose non-tangential limit at the boundary exists in the space $L^1(\partial\Omega)$.

An identity of this kind is known as a "quadrature identity for the arclength" (see [G]).

Proof of Theorem 2.1. Let $f(z) = F'(z)$, $z \in \Omega$. We have $f = u_x - iu_y$ in Ω, and also $f(z(s))\dot{z}(s) = u_s + iu_n$ on $\partial\Omega$. Differentiating by s in (1.2), gives:

$$\operatorname{Im}[(2ifk + f')\dot{z}] = u_{ns} = pu_s = p \operatorname{Re}[f\dot{z}], \qquad \text{on } \partial\Omega. \tag{2.5}$$

Here $k = k(s)$ denotes the curvature of $\partial\Omega$ at $z(s)$, or in other words $k = -i\ddot{z}/\dot{z}$. Formula (2.5) can be rearranged as

$$\operatorname{Re}\{[(p-k)f + if'\dot{z}]\dot{z}\} = 0, \qquad \text{on } \partial\Omega. \tag{2.6}$$

Since (1.3) can be rewritten as $|f| = 1$ on $\partial\Omega$, we have:

$$0 = \frac{1}{2}\frac{d}{ds}|f|^2 = \operatorname{Re}[f'\dot{z}\bar{f}] = -\operatorname{Im}\left[i\frac{f'\dot{z}}{f}\right].$$

Therefore, we obtain:

$$\left[(p-k) + i\frac{f'}{f}\dot{z}\right]\operatorname{Re}[f\dot{z}] = 0 \qquad \text{on } \partial\Omega. \tag{2.7}$$

Let A be the subset of $\partial\Omega$ where $(p-k) + i\dot{z}f'/f = 0$, and let B be the subset where $u_s = \operatorname{Re}[f\dot{z}] = 0$. By continuity, A and B are closed subsets of $\partial\Omega$, and (2.7) gives $A \cup B = \partial\Omega$. Notice that B cannot coincide with $\partial\Omega$ otherwise, by the maximum principle, u would be constant in Ω, violating (1.3). Moreover, in B we have $f = \pm i\bar{\dot{z}}$ thus, since $\partial\Omega \backslash A$ is in the interior of B, by differentiation we obtain:

$$f'\dot{z} = \pm i\bar{\ddot{z}} = \pm k\bar{\dot{z}} = -ikf, \qquad \text{in } \partial\Omega\backslash A.$$

Consequently, we have:

$$\frac{f'}{f}\dot{z} = i(p-k) \qquad \text{in } A, \tag{2.8}$$

$$\frac{f'}{f}\dot{z} = -ik \qquad \text{in } \partial\Omega\backslash A. \tag{2.9}$$

If $A \neq \partial\Omega$, then (2.8), (2.9) would hold simultaneously at some point, which is impossible because $p > 0$, by (1.2), (1.3). Thus, $A = \partial\Omega$ and (2.8) holds in Ω.

By the logarithmic residue formula, we obtain:

$$\frac{1}{2\pi i}\int_0^{|\partial\Omega|}\frac{f'}{f}\dot{z}\,ds = \sum_{j=1}^{L}\mu_j, \tag{2.10}$$

where $\mu_1, \ldots, \mu_L$ are the multiplicities of the zeros of f in Ω (possibly, there are none!). By (2.8), we deduce:

$$\frac{1}{2\pi}\int_0^{|\partial\Omega|}(p-k)\,ds = \sum_{j=1}^{L}\mu_j,$$

and since $\int_0^{|\partial\Omega|} k\,ds = 2\pi$, we obtain (2.1) with

$$N = (\text{number of zeros of } F') + 1. \tag{2.11}$$

Now, writing (2.8) as $f'\dot{z}/f = ip + \ddot{z}/\dot{z}$ and integrating, we have:

$$f(z(s))\dot{z}(s) = f(z(0))\dot{z}(0)\,e^{ips},\ 0 \le s \le |\partial\Omega|.$$

By further integrating and noticing that

$$\int_0^{|\partial\Omega|}F(z(s))\,ds = \int_0^{|\partial\Omega|}u\,ds + i\int_0^{|\partial\Omega|}v\,ds = \frac{1}{p}\int_0^{|\partial\Omega|}u_n\,ds + i\int_0^{|\partial\Omega|}v\,ds = 0,$$

we arrive at

$$F(z(s)) = \frac{f(z(0))\dot{z}(0)}{ip}e^{ips}, \qquad 0 \le s \le |\partial\Omega|,$$

and (2.2) follows.

Finally, we notice that the number of zeros of F in Ω is given by

$$\frac{1}{2\pi i}\int_{\partial\Omega}\frac{F'}{F}dz = \frac{1}{2\pi i}\int_0^{|\partial\Omega|}\frac{f\dot{z}}{F}ds = \frac{p}{2\pi}\int_0^{|\partial\Omega|}ds = N.$$

3. Proof of Theorem 1.2

We fix distinct points $\zeta_1, \ldots, \zeta_K$, $K > 1$, in the unit disk $B_1(0)$, and positive integers $m_1, \ldots, m_k$. Our aim is to find a univalent function $\chi\colon B_1(0) \to \Omega$ and a holomorphic function Φ on B_1 such that the function $F = \Phi \circ \chi^{-1}$, defined on Ω, satisfies conditions (2.3) and vanishes at $z_1, \ldots, z_K$, $z_k = \chi(\zeta_k)$, $k = 1, \ldots, K$, with respective multiplicities $m_1, \ldots, m_K$.

Let $\Psi(\zeta) = F'(\chi(\zeta))$, $\zeta \in B_1(0)$. We have:

$$\Phi'(\zeta) = F'(\chi(\zeta))\chi'(\zeta) = \Psi(\zeta)\chi'(\zeta),$$

hence

$$\chi'(\zeta) = \frac{\Phi'(\zeta)}{\Psi(\zeta)}.$$

On $\partial B_1(0)$, we impose:

$$|\Phi| = \frac{1}{p} \quad \text{and} \quad |\Psi| = 1,$$

so that (2.3) will be satisfied. We also require that Φ vanishes at the points $\zeta_1, \ldots, \zeta_k$ with multiplicities $m_1, \ldots, m_K$. It follows that Ψ should vanish at $\zeta_1, \ldots, \zeta_k$, with multiplicities $m_1 - 1, \ldots, m_K - 1$ and at some other points $\eta_1, \ldots, \eta_L$ with multiplicities $n_1, \ldots, n_L$ such that $\sum_{l=1}^{L} n_l = K - 1$.

Thus, up to constant rotations Φ and Ψ must have the following Blaschke product representation:

$$\Phi(\zeta) = \frac{1}{p} \prod_{k=1}^{K} \left(\frac{\zeta - \zeta_k}{1 - \zeta\bar{\zeta}_k} \right)^{m_k}, \tag{3.1}$$

$$\Psi(\zeta) = \prod_{k=1}^{K} \left(\frac{\zeta - \zeta_k}{1 - \zeta\bar{\zeta}_k} \right)^{m_k - 1} \prod_{l=1}^{L} \left(\frac{\zeta - \eta_l}{1 - \zeta\bar{\eta}_l} \right)^{n_l}. \tag{3.2}$$

Therefore

$$\Phi'(\zeta) = \frac{1}{p} \prod_{k=1}^{K} \left(\frac{\zeta - \zeta_k}{1 - \zeta\bar{\zeta}_k} \right)^{m_k} \sum_{k=1}^{K} m_k \frac{1 - |\zeta_k|^2}{(\zeta - \zeta_k)(1 - \zeta\bar{\zeta}_k)}.$$

Consequently

$$\chi'(\zeta) = \frac{1}{p} \prod_{k=1}^{K} \frac{\zeta - \zeta_k}{1 - \zeta\bar{\zeta}_k} \prod_{l=1}^{L} \left(\frac{1 - \zeta\bar{\eta}_l}{\zeta - \eta_l} \right)^{n_l} \sum_{k=1}^{K} m_k \frac{1 - |\zeta_k|^2}{(\zeta - \zeta_k)(1 - \zeta\bar{\zeta}_k)}.$$

Denoting by P the following polynomial

$$P(\zeta) = \left[\prod_{k=1}^{K} (\zeta - \zeta_k)(1 - \zeta\bar{\zeta}_k) \right] \sum_{k=1}^{K} m_k \frac{1 - |\zeta_k|^2}{(\zeta - \zeta_k)(1 - \zeta\bar{\zeta}_k)}, \tag{3.3}$$

we may factor χ' as follows

$$\chi'(\zeta) = \frac{1}{p} P(\zeta) \prod_{k=1}^{K} (1 - \zeta\bar{\zeta}_k)^{-2} \prod_{l=1}^{L} \left(\frac{1 - \zeta\bar{\eta}_l}{\zeta - \eta_l} \right)^{n_l}. \tag{3.4}$$

The degree d of P is at most $2(K - 1)$, and P has the following property:

$$P\left(\frac{1}{\bar{\zeta}} \right) = (\bar{\zeta})^{2(1-K)} \overline{P(\zeta)}. \tag{3.5}$$

Therefore, if $\xi \neq 0$ is a root of P, also $\bar{\xi}^{-1}$ is a root of P. Notice also that P does not vanish on ∂B_1. In fact, if $|\zeta| = 1$, then $\chi'(\zeta) \neq 0$, since $\Psi(\zeta)$ is bounded and $\Phi'(\zeta) \neq 0$.

Consequently, we may factor P as follows:

$$P(\zeta) = A\zeta^{\varrho} \prod_{j=1}^{J} [(\zeta - \xi_j)(1 - \zeta\bar{\xi}_j)]^{v_j},$$

where: $A \neq 0$, $\xi_j \in B_1(0)$, $\xi_j \neq 0$ $j = 1, \ldots, J$, and $\varrho + 2\sum_{j=1}^{J} v_j = d$.

By (3.5), letting $\zeta \to \infty$ yields: $-\varrho = 2(1-K) + d$, and hence $\varrho + \sum_{j=1}^{J} v_j = K - 1$.

We choose $n_1, \ldots, n_L, \eta_1, \ldots, \eta_L$ as functions of $m_1, \ldots, m_K$, $\zeta_1, \ldots, \zeta_K$ by imposing

$$\begin{aligned} &L = J + 1, \qquad n_j = v_j, \qquad j = 1, \ldots, J, \qquad n_L = \varrho, \\ &\eta_j = \xi_j, \qquad j = 1, \ldots, J, \qquad \eta_L = 0. \end{aligned} \tag{3.6}$$

With such a choice, we have:

$$\chi'(\zeta) = \frac{A}{p} \prod_{k=1}^{K} (1 - \zeta\bar{\zeta}_k)^{-2} \prod_{j=1}^{J} (1 - \zeta\bar{\xi}_j)^{2v_j}, \tag{3.7}$$

which is regular and never vanishes on all $\overline{B_1(0)}$.

By Nehari's Theorem (see [D], Theorem 8.12), if $\zeta_1, \ldots, \zeta_K$ are close enough to zero, the primitive $\chi(\zeta)$ of the rational function in (3.7) is univalent and provides us with the desired conformal mapping. Thus, by setting

$$\Omega = \chi(B_1(0)), \qquad z_k = \chi(\zeta_k), \qquad k = 1, \ldots, K, \qquad F = \Phi \circ \chi^{-1},$$

the proof of Theorem 1.2 is completed.

Another Proof of Theorem 1.1. Given Ω, if F has only one zero, say z_1, with multiplicity m_1, then $K = 1$ and $N = m_1$. We can choose the mapping χ to be such that $\chi(0) = z_1$, that is $\zeta_1 = 0$. Hence, (3.7) becomes $\chi'(\zeta) = N/p$, so that $\chi(\zeta) = (N/p)\zeta + z_1$, that is $\Omega = \chi(B_1(0))$ is a disk centered at z_1 with radius $N/p = |\partial\Omega|/2\pi$.

Vice versa, if Ω is a disk centered at z_0 and with radius R, up to rotations, the univalent function $\chi\colon B_1(0) \to \Omega$ is given by $\chi(\zeta) = z_0 + R\zeta$. Therefore $\chi'(\zeta) = R$, and hence $\zeta_k = 0$, $k = 1, \ldots, K$, that is F vanishes only at z_0.

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Abstract

We consider the Stekloff eigenvalue problem (1.1)–(1.2); Payne and Philippin conjectured that if u is an eigenfunction which satisfies the overdetermined condition $|\nabla u| = 1$ on $\partial\Omega$, then Ω should be a disk. In this paper we show that this conjecture holds if and only if the complex potential F associated to u vanishes only at one point. Then we show how to construct non-symmetric domains in the case where F vanishes at more than one point.

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