Symmetry and non–symmetry for the overdetermined Stekloff eigenvalue problem II*

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Abstract

We continue the study of the overdetermined Stekloff eigenvalue problem (1.1)–(1.3) below. In [1], we constructed a variety of non–symmetric planar domains for which a solution of (1.1)–(1.3) exists. Here, we consider the problem in dimension $n \geq 3$, and prove that if there is a solution of (1.1)–(1.3) that satisfies an additional integral condition, then the domain Ω must be a ball.

1 Introduction

This article is the continuation of the research [1] originated by a paper of Payne and Philippin [14] concerning the Stekloff eigenvalue problem:

$$\Delta u = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial \nu} = pu \quad \text{on } \partial\Omega. \tag{1.2}$$

Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, and ν denotes the exterior normal unit vector to $\partial\Omega$. It is well-known that this problem has infinitely many eigenvalues $0 = p_1 < p_2 \leq p_3 \leq \dots$ (see [16]).

In [14], the authors proved, for $n = 2$, that if there is an eigenfunction u of (1.1), (1.2), corresponding to the second eigenvalue p_2, which also satisfies the overdetermined condition:

$$|Du| = 1 \quad \text{on } \partial\Omega, \tag{1.3}$$

then u is linear and Ω must be a disk. They also pointed out that this result does not hold if $\partial\Omega$ is not of class C^2.

A natural question arises: if $\partial\Omega \in C^2$, suppose that for some $p > 0$ there exists a solution of (1.1), (1.2) satisfying (1.3); does this imply that Ω is a ball?

In [1], we examined the two–dimensional case and constructed a variety of non–symmetric domains for which a solution of (1.1)–(1.3) exists.

In the present paper, we are concerned with the case $n \geq 3$. The problem shows quite different features; in order to understand this, it is worth looking at solutions of (1.1)–(1.3) in the unit ball B_n of $\mathbb{R}^n$.

In this case, $\nu(x) = x$ on ∂B_n; by (1.2), since $x \cdot Du(x) - pu(x)$ is harmonic in B_n, we have that $x \cdot Du(x) = pu(x)$, $x \in B_n$, that is u must be a homogeneous harmonic polynomial

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of degree p. Therefore, (1.1)–(1.3) can be transformed into the problem:

$$\tilde{\Delta} u = g(u), \tag{1.4}$$

$$|\nabla u|^2 = f(u), \tag{1.5}$$

on $S^{n-1} = \partial B_n$, where $g(u) = -p(p+n-2)u$, $f(u) = 1 - p^2u^2$. Here, $\tilde{\Delta}$ and ∇ denote the Laplace–Beltrami operator and tangential gradient on S^{n-1}, respectively.

When $n = 2$, all solutions of (1.4), (1.5) are given by $\{\frac{1}{p}\cos(ps + s_0)\}_{p=1,2,\dots}$, where s is the arclength parameter on S^1. Note that the above set is complete in the space $\{v \in L^2(S^1) : \int_{S^1} v ds = 0\}$, and describes all the traces of the Stekloff eigenfunctions in the disk.

If $n \geq 3$, solutions of a system of type (1.4), (1.5), with f and g smooth, are well–known in the literature as *isoparametric functions.* Their level surfaces at regular values, the *isoparametric surfaces,* enjoy the nice geometric property of having all their principal curvatures constant.

Up to this date, a complete classification of these surfaces on the sphere is not available. Here, we want to stress the fact that they seem to be very rare. When $n = 3$, for example, it can be shown that the solutions of (1.4), (1.5) are just the restrictions to S^2 of linear functions on $\mathbb{R}^3$. More results and examples in this direction are contained in the works of E. Cartan [2], [3], who first considered the isoparametric surfaces on the sphere, Nomizu [12], [13], Munzner [11], Ferus–Karcher–Munzner [7], and Wang Q. M. [17], [18], who examined them on a complete Riemannian manifold. We refer the reader to [18] for a survey on the subject.

The main result of this paper is an analogue to Payne and Philippin's theorem.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a contractible bounded domain with boundary $\partial\Omega \in C^2$. Suppose that there exists a solution u of (1.1)–(1.3) which also satisfies:*

$$\int_{\partial\Omega} (u - x \cdot Du)\ u\ d\sigma = 0. \tag{1.6}$$

Then, Ω is a ball.

The proof of this result is based on Theorem 2.1, that essentially asserts that if u is a solution of equation (1.5) on a Riemannian manifold M, then the sets $\{x \in M : u(x) = c\}$, at critical values c, are smooth submanifolds of M. This quite surprising result is proved in [17], in a slightly different setting. In §2, we produce an alternative proof, based on some elementary arguments and with a more analytical flavour.

This paper is organized as follows. In §2, we state and prove Theorem 2.1. Section 3 is devoted to the proof of Theorem 1.1, as a consequence of Proposition 3.1 and Theorem 3.2.

2 Equation (1.5) on manifolds

We start with some preliminary notations. We consider a C^2 manifold M, without boundary, of dimension m, endowed with a Riemannian metric, which is represented by $\{g_{ij}(x)\}_{i,j=1,\dots,m}$ in the local coordinates $x = (x^1, \dots, x^m)$. If $v = (v^1, \dots, v^m)$ and $w = (w^1, \dots, w^m)$ are tangent vector fields on M, we define:

$$< v, w >= g_{ij}(x)v^iw^j, \quad |v| =< v, v >^{\frac{1}{2}} .$$

Here, we adopt the usual assumption on the sum over repeated indices.

Given a C^1 function on M, we introduce the gradient of u on M as

$$\nabla u = (\nabla_1 u, \dots, \nabla_m u), \qquad \nabla_i u = g^{ij}(x) u_{x^j}, \;\; i = 1, \dots, m;$$

here $\{g^{ij}(x)\}_{i,j=1,\dots,m}$ is the inverse of the matrix $\{g_{ij}(x)\}_{i,j=1,\dots,m}$.

We denote by $d : M \times M \to I\!R$ the geodetic distance on M; moreover, for any $x \in M$ and any closed subset $C \subset M$, it is well defined the number:

$$d(x, C) = \min\{d(x, y) : y \in C\}.$$

Let D be a bounded domain in M. We shall look at solutions of the following boundary value problem:

$$|\nabla\phi|^2 = f(\phi) \quad \text{in } D, \qquad 0 < \phi \le \Phi \quad \text{in } D, \tag{2.1}$$

$$\phi = 0 \quad \text{on } \partial D. \tag{2.2}$$

Here $f \in C^1((0, \Phi])$ is a function satisfying

$$f > 0 \quad \text{on } (0, \Phi), \qquad f(\Phi) = 0, \quad f'(\Phi) < 0. \tag{2.3}$$

Notice that (2.1) and (2.3) easily imply that

$$\max_{\overline{D}} \phi = \Phi. \tag{2.4}$$

Theorem 2.1. *Let $D \subset M$ be a bounded domain with boundary $\partial D \in C^2$. Let f be a $C^1((0, \Phi])$ function satisfying (2.3).*

If $\phi \in C(\overline{D}) \cap C^2(D)$ is a solution of (2.1)–(2.2), then for some integer h, $0 \le h \le m - 1$, the extremal level set

$$D_\Phi = \{x \in D : \phi(x) = \Phi\} \tag{2.5}$$

is an h–dimensional C^1 connected compact submanifold without boundary of M.

Moreover, D satisfies:

$$D = \{x \in M : d(x, D_\Phi) < L\}, \tag{2.6}$$

where

$$L = \int_0^\Phi \frac{ds}{\sqrt{f(s)}} \; . \tag{2.7}$$

Corollary 2.2. *If D is contractible, then D_Φ consists of a single point, and D is a geodetic ball centered at D_Φ.*

Remark. The thesis of Theorem 2.1 provides us with a nearly sufficient condition on D for the existence of a solution of (2.1)–(2.3). In fact, let $0 \le h \le m - 1$; given any C^2 h–dimensional compact submanifold without boundary D_Φ in M, let K be the maximum of the absolute value of all its principal curvatures. Then, for any $L < 1/K$, the domain D defined by (2.6) is such that a solution of (2.1)–(2.3) exists; take for instance

$$\phi(x) = \sqrt{L^2 - d(x, D_\Phi)^2}; \tag{2.8}$$

in this case we have $\Phi = L$ and $f(\phi) = L^2\phi^{-2} - 1$.

LEMMA 2.3. *The closed set D_Φ defined in (2.5) has no interior points.*

Proof. Suppose by contradiction that $int(D_\Phi)$ is not empty.

Pick a point $x_0 \in \overline{int(D_\Phi)} \setminus int(D_\Phi)$; then $x_0 \in \partial D_\Phi$, since $\overline{int(D_\Phi)} \subseteq D_\Phi$. Let U be a coordinate neighborhood of x_0 and let $x \in (D \setminus D_\Phi) \cap U$. By (2.1), for every $k = 1, \ldots, m$, we have:

$$f'(\phi(x))\phi_{x^k}(x) = \partial_{x^k}|\nabla\phi|^2 = 2\phi_{x^i x^k}(x)g^{ij}(x)\phi_{x^j} + g^{ij}_{x^k}(x)\phi_{x^i}(x)\phi_{x^j}(x).$$

Since $\{g^{ij}(x)\}$ is uniformly positive definite, by the boundedness of the $g^{ij}_{x^k}(x)$'s and the Schwarz inequality, we may find positive constants c_1 and c_2 such that

$$[\sum_{i,j=1}^{m} \phi_{x^i x^j}(x)^2]^{\frac{1}{2}} \geq c_1|f'(\phi(x))| - c_2|\nabla\phi(x)|,$$

for all $x \in (D \setminus D_\Phi) \cap U$. Hence,

$$\lim_{D\setminus D_\Phi \ni x \to x_0} [\sum_{i,j=1}^{m} \phi_{x^i x^j}(x)^2]^{\frac{1}{2}} \geq c_1|f'(\Phi)| > 0,$$

whereas, obviously

$$\lim_{int(D_\Phi)\ni x \to x_0} [\sum_{i,j=1}^{m} \phi_{x^i x^j}(x)^2]^{\frac{1}{2}} = 0.$$

This is a contradiction. ◻

Let us set now:

$$\begin{aligned} F(t) &= \int_0^t \frac{ds}{\sqrt{f(s)}}, \quad t \in [0, \Phi], \\ \delta(x) &= F(\phi(x)), \quad x \in M. \end{aligned} \tag{2.9}$$

Notice that $\delta \in C(\overline{D}) \cap C^2(D \setminus D_\Phi)$, and also

$$\begin{aligned} &|\nabla\delta| = 1 \quad \text{in } D \setminus D_\Phi, \\ &\delta = 0 \quad \text{on } \partial D, \ \ \delta = L \quad \text{on } D_\Phi, \\ &0 < \delta < L \quad \text{in } D \setminus D_\Phi, \end{aligned} \tag{2.10}$$

where L is given by (2.7).

The next lemma shows the relationship between δ and the distance function. We shall use the following definition.

DEFINITION 2.4. Let $x \in D \setminus D_\Phi$. The *stream line $\gamma(x;\cdot)$ of δ passing through x* is the maximal solution of the initial value problem:

$$\gamma'(x;t) = \nabla\delta(\gamma(x;t)), \quad \gamma(x;0) = x. \tag{2.11}$$

We denote by $(\alpha(x), \beta(x))$, $\alpha(x) < 0 < \beta(x)$, the maximal existence interval for $\gamma(x;t)$.

LEMMA 2.4. *For any $x \in D \setminus D_\Phi$, we have:*

$$\alpha(x) = -\delta(x), \quad \beta(x) = L - \delta(x); \tag{2.12}$$

moreover

$$\lim_{t \to -\delta(x)^+} \gamma(x;t) = x_0, \qquad \lim_{t \to [L-\delta(x)]^+} \gamma(x;t) = x_L, \tag{2.13}$$

for some $x_0 \in \partial D, x_L \in D_\Phi$, and

$$\lim_{t \to -\delta(x)^+} \gamma'(x;t) = \xi_0, \qquad \lim_{t \to [L-\delta(x)]^+} \gamma'(x;t) = \xi_L, \tag{2.14}$$

for some ξ_0, ξ_L such that $|\xi_0|, |\xi_L| = 1$.

The stream line $\gamma(x;t)$, $t \in [-\delta(x), 0]$ (resp. $t \in [0, L - \delta(x)]$) is the unique minimal geodesic joining x_0 to x (resp. x to x_L).

Finally, we have:

$$\delta(x) = d(x, \partial D), \quad L - \delta(x) = d(x, D_\Phi), \quad x \in \overline{D}. \tag{2.15}$$

REMARK. Solutions of the eikonal equation in (2.10) have been studied by several authors and from different viewpoints (see [9], and the references therein). For instance, it is not difficullt to prove (2.15) in a small one–sided neighborhood of a smooth hyprsurface (see e. g. [4], II.9). The above lemma gives a global version of this type of result adapted to the specific problem (2.10). Observe that, in this case, no smoothness is required on ∂D or D_Φ.

Proof. By (2.10) and (2.11), we have $\frac{d}{dt}\delta(\gamma(x;t)) = 1$, for any $t \in (\alpha(x), \beta(x))$, that is

$$\delta(\gamma(x;t)) = \delta(x) + t, \quad t \in (\alpha(x), \beta(x)). \tag{2.16}$$

Since (2.11) is an autonomous system with bounded right–hand side, $\gamma(x;t)$ is defined for all (possible) t's, that is for all $-\delta(x) < t < L - \delta(x)$, since $0 < \delta < L$ in $D \setminus D_\Phi$. This implies (2.12).

The existence of the limits in (2.13) is a consequence of the fact that $\gamma(x;t)$ is uniformly Lipschitz continuous in t, by (2.10) and (2.11). Therefore, $\gamma(x;t)$ can be extended continuously to $[-\delta(x), L - \delta(x)]$, and also, by (2.16), we obtain that $x_0 \in \partial D$ and $x_L \in D_\Phi$, since $\delta = 0$ only on ∂D and $\delta = L$ only on D_Φ.

Let $r = d(x, \partial D)$ and let $x_0^* \in \partial D$ be such that $d(x, x_0^*) = r$. Since $\gamma(x;t) \to x_0$ as $t \to -\delta(x)^+$, we obtain that $d(\gamma(x;t), \partial D) \to 0$ as $t \to -\delta(x)^+$; thus,

$$\liminf_{t \to -\delta(x)^+} d(x, \gamma(x;t)) \geq d(x, \partial D) = r.$$

Notice that (2.10) and (2.11) imply that $\gamma(x;\cdot)$ is parametrized by arclength, and hence $|t| \geq d(x, \gamma(x;t))$, for every $t \in (-\delta(x), L - \delta(x))$, thus, letting $t \to -\delta(x)$, we obtain $\delta(x) \geq r$.

Let $\tilde{\gamma}$ be the minimal geodesic joining x to x_0^*, parametrized by arclength as follows: $\tilde{\gamma} = \tilde{\gamma}(t)$, $-r \leq t \leq 0$, $\tilde{\gamma}(-r) = x_0^*$, $\tilde{\gamma}(0) = x$. We have:

$$r \leq \delta(x) = \int_{-r}^{0} < \nabla\delta(\tilde{\gamma}(t), \tilde{\gamma}'(t) > \; dt \leq \int_{-r}^{0} |\tilde{\gamma}'(t)| \; dt = r.$$

Consequently, $\delta(x) = r$ and also $\tilde{\gamma}'(t) = \nabla\delta(\tilde{\gamma}(t))$, for all $t \in (-r, 0)$. hence, the first formula in (2.15) holds for $x \in D \setminus D_\Phi$, and by lemma 2.3, for all $x \in \overline{D}$. The geodesic $\tilde{\gamma}(t)$ is uniquely determined and coincides with $\gamma(x;t)$ when $t \in [-\delta(x), 0]$. This also implies that $x_0^* = x_0$ and that $\gamma(x;t)$, with $t \in [-\delta(x), 0]$, is the minimal geodesic joining x_0 to x. Thus, $\gamma(x;t)$, $t \in [-\delta(x), 0]$, solves the second order differential equation for geodesics (see e.g. [5], Ch. 3), which has continuous coefficients, M being C^2-smooth. We deduce that $\gamma'(x;t)$ is uniformly Lipschitz continuous and (2.14) follows. Likewise, we obtain the latter formulas in (2.12), (2.13), (2.14), and (2.15).

LEMMA 2.5. *The set D_Φ is a deformation retract of D.*

Proof. It suffices to verify that the mapping $\tau : D \times [0,1] \to D$, defined by

$$\tau(x,r) = \begin{cases} \gamma(x; r[L - \delta(x)]), & \text{for } (x,r) \in (D \setminus D_\Phi) \times [0,1], \\ x, & \text{for } (x,r) \in D_\Phi \times [0,1], \end{cases}$$

is continuous. In fact, since $\tau(\cdot, 0) = \mathrm{id}_D$, $\tau(D, r) = D_\Phi$, and $\tau(\cdot, r) = \mathrm{id}_{D_\Phi}$, for all $r \in [0,1]$, we have that $\tau(\cdot, 1) : D \to D_\Phi$ is retraction homotopic to id_D.

REMARK. By the above lemma and [8], Ch. 1, we obtain:
(i) D_Φ is connected,
(ii) if D is contractible, then also D_Φ is contractible.

Proof of Corollary 2.2. By Theorem 2.1 and the above lemma and remark, if D is contractible, then D_Φ is contractible compact manifold without boundary. Classical results imply that $h = \dim D_\Phi = 0$ (see [10], theorem 4.1 and example p. 21, and [6], corollary 17.6.1).

Proof of Theorem 2.1. It is enough to prove that D_Φ is locally a submanifold of M, since D_Φ is connected by the above remark and is compact by (2.10).

Let $P \in D_\Phi$ and fix local coordinates $x^1, \ldots, x^m$ near P such that $P = (0, \ldots, 0)$ and $g_{ij}(P) = \delta_{ij}$, the Krönecker delta.

Observe that, by (2.15), and by rephrasing the arguments of lemma 2.5, if we choose $Q \in \partial D$ such that $d(P,Q) = d(P, \partial D)$, then there exists a unique minimal geodesic γ joining P to Q which is a stream line of δ (and also of ϕ, by (2.9)).

Let $T_P(M)$ be the tangent space to M at P and define:

$$\begin{aligned} \Xi(P) = &\{\xi \in T_P(M) : |\xi| = 1, \\ &\quad \text{and } \exists \text{ a stream line } \gamma(t), t \in [0, L] \text{ of } \delta : \gamma(L) = P, \gamma'(L) = \xi\}; \end{aligned}$$

this set is not empty by (2.14). Let us continue each $\xi \in \Xi(P)$ as a constant vector field in a neighborhood V of Q, with respect to the chosen coordinates $x^1, \ldots, x^m$.

Pick $\xi \in \Xi(P)$ and differentiate (2.1) along ξ; in a neighborhood of P, we have:

$$2g^{ij}\phi_{x^i\xi}\phi_{x^j} + g^{ij}_\xi \phi_{x^i}\phi_{x^j} = f'(\phi)\phi_\xi.$$

We obtain the same formula (with ξ replaced by η) by differentiating (2.1) along any direction η orthogonal to ξ.

Restricting these formulas to a stream line $\gamma(t)$ of δ through P with $\gamma'(L) = \xi$, dividing by $|\nabla\phi(\gamma(t))|$, and letting $t \to L$, yield:

$$\phi_{\xi\xi}(P) = \frac{1}{2} f'(\Phi) < 0, \quad \forall \xi \in \Xi(P), \tag{2.17}$$

$$\phi_{\xi\eta}(P) = 0, \qquad \forall \xi \in \Xi(P), \forall \eta, < \eta, \xi >= 0. \tag{2.18}$$

Let $k = k(P)$ be the maximum number of linearly indipendent elements in $\Xi(P)$ and let $\xi_1, \ldots, \xi_k \in \Xi(P)$ be a choice of such elements. By (2.17) and (2.18), we get for $i, j = 1, \ldots, k$

$$\phi_{\xi_i \xi_j}(P) =< \xi_i, \xi_j > \phi_{\xi_i \xi_i}(P) = \frac{1}{2} < \xi_i, \xi_j > f'(\Phi). \tag{2.19}$$

Therefore, the C^1 mapping $\psi : V \to \mathbb{R}^k$ defined by $\psi(x) = (\phi_{\xi_1}(x), \ldots, \phi_{\xi_k}(x))$ has rank k at P, by (2.19) and (2.18), and also $\psi(P) = (0, \ldots, 0)$ since $P \in D_\Phi$. By the implicit function theorem, we may find a neighborhood $U \subseteq V$ of P such that $N = \{x \in U : \psi(x) = (0, \ldots, 0)\}$ is a C^1 submanifold. Furthermore, the normal space to N at P is spanned by $\xi_1, \ldots, \xi_k$.

Obviously $D_\Phi \cap U \subseteq N$, since $|\nabla \phi| = 0$ on D_Φ. In order to conclude the proof, we need to show that $D_\Phi \cap U = N$, by possibly restricting U. The number h in the statement of the theorem will be given by $m - k$.

Let us denote by $B_\varepsilon(P)$ the geodetic ball in M centered at P and of radius $\varepsilon > 0$. Suppose by contradiction that for any $\varepsilon > 0$ there exists $P_\varepsilon \in (N \setminus D_\Phi) \cap B_\varepsilon(P)$. Let $P_\varepsilon^* \in D_\Phi$ be such that $d(P_\varepsilon, P_\varepsilon^*) \leq d(P_\varepsilon, P) < \varepsilon$, and hence $d(P, P_\varepsilon^*) < 2\varepsilon$.

Let γ_ε be the stream line of δ through P_ε; by lemma 2.5 we have that its endpoint on D_Φ is P_ε^*. Let us parametrize γ_ε by arclength in such a way that $\gamma_\varepsilon(L) = P_\varepsilon^*$, and let $\xi_\varepsilon = \gamma_\varepsilon'(L)$; ξ_ε is a unit vector in $T_{P_\varepsilon^*}(M)$. By possibly passing to subsequences, P_ε and $P_\varepsilon^* \to P$, and $\xi_\varepsilon \to \xi$ as $\varepsilon \to 0$, where $\xi \in T_P(M)$ is a unit vector. By the continuous dependence on the Cauchy data, for all $t \in [0, L]$, $\gamma_\varepsilon(t)$ converges to $\gamma(t)$, where γ is a geodesic such that $\gamma(L) = P$ and $\gamma'(L) = \xi$. Thus, γ is a stream line of δ, since all γ_ε's are stream lines of δ. In particular, we deduce $\xi \in \Xi(P)$.

We will show now that $\xi \in T_P(N)$, contradicting the fact that $\Xi(P)$ is contained in the normal space to N at P. We choose a local coordinate system near P such that N is represented by the equations $x^{h+1} = \cdots = x^m = 0$. In this system $P_\varepsilon = (x_\varepsilon^1, \ldots, x_\varepsilon^h, 0, \ldots, 0)$, and analogously for P_ε^*. Since by Taylor's formula, we have:

$$P_\varepsilon^* = P_\varepsilon + d(P_\varepsilon^*, P_\varepsilon)\xi_\varepsilon + o(d(P_\varepsilon^*, P_\varepsilon)), \quad \text{as } \varepsilon \to 0,$$

we obtain that $\xi_\varepsilon^i \to 0$ as $\varepsilon \to 0$, $\forall i = h+1, \ldots, m$.

This means that $\xi^i = 0, i = h+1, \ldots, m$, that is $\xi \in T_P(N)$.

Finally (2.6) and (2.7) follow from (2.10) and (2.15).

3 Overdetermined Stekloff eigenfunctions

In the sequel, we will denote by $x = (x', x^n)$ a point of $\mathbb{R}^n$, where $x' \in \mathbb{R}^{n-1}$ has coordinates $(x^1, \ldots, x^{n-1})$; $\nu = (\nu_1, \ldots, \nu_n)$ will indicate the exterior normal unit vector to $\partial\Omega$.

We begin with the following result, which has its own interest.

PROPOSITION 3.1 *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy (1.1) and (1.3). If*

$$\int_{\partial\Omega} [u - x \cdot Du] \, \frac{\partial u}{\partial \nu} \, d\sigma = 0, \tag{3.1}$$

then u is linear.

Proof. By Rellich's identity (see [15]),

$$\int_{\partial\Omega} \{2(x \cdot Du)\frac{\partial u}{\partial \nu} - |Du|^2(x \cdot \nu)\} \, d\sigma = \int_{\Omega} \{2(x \cdot Du)\Delta u + (2 - N)|Du|^2\} \, dx.$$

By (1.1) and (1.3), we obtain via the divergence theorem:

$$\int_{\partial\Omega} \{2(x \cdot Du)\frac{\partial u}{\partial \nu} - (x \cdot \nu)\}\ d\sigma = 2\int_{\partial\Omega} u\ \frac{\partial u}{\partial \nu}\ d\sigma - N\int_{\Omega} |Du|^2\ dx.$$

Thus, (3.1) yields:

$$N\int_{\Omega} |Du|^2\ dx = \int_{\partial\Omega} x \cdot \nu\ d\sigma = N|\Omega|, \tag{3.2}$$

where $|\Omega|$ is the Lebesgue measure of Ω.

Since $|Du|^2$ is subharmonic in Ω, by (1.3), we have $|Du| \le 1$ in Ω, so that (2.2) implies $|Du| \equiv 1$ in Ω. Therefore, $2\sum_{i,j=1}^n u_{ij}^2 = \Delta|Du|^2 \equiv 0$ in Ω, and hence u is linear in Ω.

Theorem 1.1 will be a consequence of the following more general result.

THEOREM 3.2 *Let $\Omega \subset I\!\!R^n$ be a bounded domain with boundary $\partial\Omega \in C^2$. Suppose that u is a linear solution of (1.1)–(1.3).*

Then, up to a rigid change of coordinates, for some $h = 0, 1, \dots, n-2$, there exists a C^1 h-dimensional submanifold $D_\Phi \subset \{x \in I\!\!R^n : x^n = 0\}$, such that

$$\Omega = \{x \in I\!\!R^n : \mathrm{dist}(x, D_\Phi) < \frac{1}{p}\}. \tag{3.3}$$

Furthermore, if Ω is contractible, then Ω is a ball.

Proof. Up to a rigid change of coordinates, we may assume that $u(x) = x^n$. By (1.2), we have:

$$\nu_n = px^n \quad \text{on } \partial\Omega. \tag{3.4}$$

If we consider $\Gamma = \{x \in \partial\Omega : x^n > 0\}$, we have that $\nu_n > 0$ on Γ, hence Γ is the graph of a function $\phi = \phi(x')$, where x' ranges over $D = \{x \in \Omega : x^n = 0\}$. The vector ν is then given by $\{1 + |\nabla\phi|^2\}^{-1/2}(-\nabla\phi, 1)$ on D, where ∇ denotes the gradient in the variable $x' \in D$.

Therefore, (3.4) yields $\{1 + |\nabla\phi|^2\}^{-1/2} = px^n = p\phi$, that is

$$\begin{aligned} &|\nabla\phi|^2 = \frac{1}{p^2\phi^2} - 1 \quad \text{in } D, \\ &\phi = 0 \qquad \text{on } \partial D. \end{aligned} \tag{3.5}$$

Since $\partial\Omega \in C^2$, we also have that $\phi \in C(\overline{D}) \cap C^2(D)$; hence, by setting $m = n - 1$, theorem 3.1 applies to the (flat) domain D. Note that $D_\Phi = \{x' \in D : \phi(x') = \frac{1}{p}\}$ and that

$$\phi(x') = \sqrt{\frac{1}{p^2} - \mathrm{dist}(x', D_\Phi)^2}. \tag{3.6}$$

Therefore,

$$\{x \in \Omega : x^n > 0\} = \{x \in I\!\!R^n : x^n > 0, \mathrm{dist}(x, D_\Phi) < \frac{1}{p}\}$$

and, by the same argument,

$$\{x \in \Omega : x^n < 0\} = \{x \in I\!\!R^n : x^n < 0, \mathrm{dist}(x, D_\Phi) < \frac{1}{p}\}.$$

Consequently, we obtain (3.3).

Finally, one easily sees that D is a deformation retract of Ω, and hence D is contractible, if Ω is so. Corollary 2.2 implies that D is an $(n-1)$–dimensional ball, that is, by (3.6), Ω is an n–dimensional ball

Proof of Theorem 1.1. By (1.1)–(1.3), (1.6), and proposition 3.1, we have that u is linear, and hence Theorem 3.2 applies.

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