Analisi Matematica I: Calcolo differenziale, CdL Informatica, a.a. 1998-98.
Soluzioni al test del novembre 1998

(1a)

$$\lim_{n\to+\infty}\left(\sqrt[3]{n^2+3n}-\sqrt[3]{n^2-7n+9}\right)=$$
$$\lim_{n\to+\infty}\frac{(n^2+3n)-(n^2-7n+9)}{\left[(\sqrt[3]{n^2+3n})^2+(\sqrt[3]{n^2+3n})(\sqrt[3]{n^2-7n+9})+(\sqrt[3]{n^2-7n+9})^2\right]}=$$
$$\lim_{n\to+\infty}\frac{n}{n^{4/3}}\cdot\frac{10}{1+1+1}=0.$$

(1b)

$$\lim_{n\to+\infty}\left(\sqrt[4]{n^3-3n^2+1}-\sqrt[4]{n^3+7n^2+9}\right)\sqrt[4]{n+7}=$$
$$\lim_{n\to+\infty}\left(\sqrt[4]{(n+7)(n^3-3n^2+1)}-\sqrt[4]{(n+7)(n^3+7n^2+9)}\right)=$$
$$\lim_{n\to+\infty}\left(\sqrt[4]{a_n}-\sqrt[4]{b_n}\right)=$$
$$\lim_{n\to+\infty}\frac{(n+7)(n^3-3n^2+1)-(n+7)(n^3+7n^2+9)}{a_n^3+a_n^2b_n+a_nb_n^2+b_n^3}=$$
$$\lim_{n\to+\infty}\frac{n^3}{n^3}\cdot\frac{-10}{1+1+1+1}=-\frac{5}{2}.$$

(2a) Scegliendo $n_k=2k$ e $n'_k=2k+1$ si dimostra che il limite non esiste.

(2b)

$$\left|\frac{n\cos(n+1/n)\pi}{\sqrt{n^4+3}}\right|\le\frac{n}{\sqrt{n^4+3}}\le\frac{1}{n}\to 0.$$

(3a)

$$\lim_{n\to+\infty}\frac{7^n+2^n}{3^n+5^n}=\lim_{n\to+\infty}(7/5)^n\frac{1+(2/7)^n}{(3/5)^n+1}=+\infty$$

(3b)

$$\lim_{n\to+\infty}\frac{a^n+b^n}{c^n+d^n}=$$
$$\lim_{n\to+\infty}\frac{\max(a,b)^n}{\max(c,d)^n}=\begin{cases}+\infty, & \max(a,b)>\max(c,d),\\ 0, & \max(a,b)<\max(c,d).\end{cases}$$

(4a)

$$\lim_{n\to+\infty}\frac{\log_2(n^2+7^n)}{\log_3(n^3+2^n)}=\lim_{n\to+\infty}\frac{n\log_2 7+\log_2(n^2/7^n+1)}{n\log_3 2+\log_3(n^3/2^n+1)}=\frac{\log_2 7}{\log_3 2}.$$

(4b) $$\lim_{n\to+\infty}\frac{\log_a(n^\alpha+A^n)}{\log_b(n^\beta+B^n)}=\lim_{n\to+\infty}\frac{n\log_a A+\log_a(n^\alpha/A^n+1)}{n\log_b B+\log_b(n^\beta/B^n+1)}=\frac{\log_a A}{\log_b B}.$$

(5a) $$\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}=\frac{(1+1)^n}{n}=\frac{2^n}{n}\to+\infty.$$

(5b) $$n^n\sum_{k=1}^{n}(-1)^k\binom{n}{k}=n^n\cdot(1-1)^n=0\to 0.$$

(6a) Sia $a_n=\binom{2n}{n}$; poiché $a_{n+1}/a_n\to 4$, allora $\sqrt[n]{a_n}\to 4$.
(6b) Il risultato di (6a) implica che il limite (6b) vale $+\infty$.

(7a) $$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}k\,[\log_2(k+1)-\log_2 k]=\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}a_n=\lim_{n\to+\infty}a_n,$$

dove $a_n=n\,[\log_2(n+1)-\log_2 n]\to\log_2 e$.

(7b) $$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\log_3(k+1)}=\lim_{n\to+\infty}\frac{1}{\log_3(n+1)}=0.$$

(8a) $$2^{a\cdot n}=(1+2^a-1)^n\geq 1+n(2^a-1);$$
$$1+n(2^a-1)\geq 1+n \quad \text{se e solo se } a\geq 1.$$

(8b) Se $\alpha=0$, allora $a_n=0, n=1,2,\ldots$. Se $0<\alpha\leq 1$, allora la successione $\{a_n\}$ è data da:

$$\alpha,-\alpha/(1+\alpha),\alpha,-\alpha/(1+\alpha),\alpha,-\alpha/(1+\alpha),\ldots$$

e quindi a_n non converge.

(9a) $$\left(\frac{1}{2}-\frac{n}{\sqrt{n^2+3}}\right)>-\frac{1}{2}$$
$$\text{e}\quad\lim_{n\to+\infty}\left(\frac{1}{2}-\frac{n}{\sqrt{n^2+3}}\right)=-\frac{1}{2};$$

la risposta è quindi $-1/2$.

(9b) $$\frac{n\cos(n+1/n)\pi}{\sqrt{n^2+3}}<1$$
$$\text{e}\quad\lim_{k\to+\infty}\frac{2k\cos(2k+1/(2k))\pi}{\sqrt{4k^2+3}}=1;$$

la risposta è quindi 1.
(10a) (2)
(10b) (3).