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On Extensions of the Brunn–Minkowski and Prékopa–Leindler Theorems, Including Inequalities for Log Concave Functions, and with an Application to the Diffusion Equation

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We extend the Prékopa–Leindler theorem to other types of convex combinations of two positive functions and we strengthen the Prékopa–Leindler and Brunn–Minkowski theorems by introducing the notion of essential addition. Our proof of the Prékopa–Leindler theorem is simpler than the original one. We sharpen the inequality that the marginal of a log concave function is log concave, and we prove various moment inequalities for such functions. Finally, we use these results to derive inequalities for the fundamental solution of the diffusion equation with a convex potential.

1. INTRODUCTION

In this paper we give various extensions of the Brunn–Minkowski and Prékopa–Leindler theorems. The Brunn–Minkowski theorem for the convex addition $D = \lambda A + (1-\lambda)B = \{x \in \mathbf{R}^n \mid x = \lambda y + (1-\lambda)z, y \in A, z \in B\}$ of two nonempty, measurable sets $A, B \subset \mathbf{R}^n$ reads [1, 2]

$$\mu_n(D)^{1/n} \geqslant \lambda \mu_n(A)^{1/n} + (1-\lambda)\, \mu_n(B)^{1/n}, \tag{1.1}$$

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where μ_n means Lebesgue measure in $\mathbf{R}^n$. The requirement that A and B are nonempty is crucial.

The Prékopa–Leindler theorem [3, 4, 5] reads

$$\| k \|_1 \geqslant \| f \|_1^{\lambda} \, \| g \|_1^{1-\lambda}, \tag{1.2}$$

where

$$k(x \mid f, g) = \sup_{y \in \mathbf{R}^n} f\left(\frac{x-y}{\lambda}\right)^{\lambda} g\left(\frac{y}{1-\lambda}\right)^{1-\lambda}, \tag{1.3}$$

and f, g are nonnegative, measurable functions on $\mathbf{R}^n$. If f and g are the characteristic functions of A and B, respectively, k is the characteristic function of D. Thus, Eq. (1.2) states that $\mu_n(D) \geqslant 1$ if $\mu_n(A) = \mu_n(B) = 1$. By the scaling property, $\mu_n(\lambda A) = \lambda^n \mu_n(A)$. Thus Eq. (1.2) implies Eq. (1.1). In that sense, the Prékopa–Leindler theorem can be viewed as an extension of the Brunn–Minkowski theorem.

These theorems are extended here in the following ways.

EXTENSION 1. The sup in Eq. (1.3) is replaced by ess sup:

$$h(x \mid f, g) = \operatorname*{ess\,sup}_{y \in \mathbf{R}^n} f\left(\frac{x-y}{\lambda}\right)^{\lambda} g\left(\frac{y}{1-\lambda}\right)^{1-\lambda}. \tag{1.4}$$

The Prékopa–Leindler theorem strengthened in this way is contained in Theorems 3.2 and 3.3.

Our new version really is stronger than the old; generally, $\| h \|_1 \leqslant \| k \|_1$, and there are functions f and g such that h differs greatly from k. It is a fact, however, established in the Appendix, that f and g can always be replaced by functions f^* and g^* which differ only by null functions from f and g such that

$$h(x \mid f, g) = h(x \mid f^*, g^*) = k(x \mid f^*, g^*).$$

Thus, once one knows how to construct f^* and g^*, the strengthened Prékopa–Leindler theorem follows from the known one.

However, we prefer to work with the essential supremum h, because (1) $h(x)$ is unaltered if null functions are added to f and g, and (2) $h(x)$ is lower semicontinuous for any measurable f and g. The supremum k has neither property.

By taking characteristic functions for f and g, a stronger form of the Brunn–Minkowski theorem results; as above, it can be derived from the known theorem (see the Appendix). The proof given here of the Prékopa–Leindler theorem is based on the Brunn–Minkowski theorem; it is simpler than the original proof by Prékopa and Leindler. The idea of our proof is already contained in [6]. Another (rather involved) proof of the strengthened Prékopa–Leindler theorem is given by us in [7].

EXTENSION 2. Other types of convex combinations, h_α, of two functions, f and g are defined for $\alpha \in [-\infty, \infty]$; see Eqs. (2.1–2.3). The convex combination in Eq. (1.4) is the case $\alpha = 0$.

In Section 3 theorems of the Prékopa–Leindler type are given for general α (Theorems 3.1–3.3). A Brunn–Minkowski-like version of these theorems is contained in Corollary 3.4. For the case $\alpha = 0$ and with sup instead of ess sup, it was first given by Prékopa [2, 4]. A much simpler proof for that case was found by Rinott [8]; his proof is completely different from ours. Rinott also found the case $\alpha = -1/n$ in Corollary 3.4. Moreover, he found a converse of Corollary 3.4, saying that Eq. (3.8) for all A, B implies the existence of a log concave density function.

In Section 4 we consider log concave functions. A corollary of the Prékopa–Leindler theorem is that $\int F(x, y)\, dy$ is log concave in x if $F(x, y)$ is log concave in (x, y). This result is sharpened in Theorem 4.2. In Theorem 4.1 a Sobolev-type inequality for log concave measures is given.

Some theorems on log concave functions have counterparts for log convex functions (Theorems 4.3, 5.1, and 6.1). However, these counterparts are comparatively trivial; they essentially follow from the usual convexity arguments (Hölder's inequality). We stress that the log concave theorems and other Brunn–Minkowski and Prékopa–Leindler-like theorems do not follow trivially from Hölder's inequality.

In Section 5 we give inequalities for the moments of a Gaussian distribution, compared with the moments of the same distribution perturbed by a log concave (or log convex) function (Theorem 5.1).

In Section 6 we give an application to the diffusion equation in $\mathbf{R}^n$ with convex potential. More applications (the Ising model, the one dimensional Coulomb plasma) are given in [6].

2. NOTATION

Given nonnegative measurable functions $f(x)$, $g(x)$ on $\mathbf{R}^n$, we shall introduce various convex combinations of them, parametrized by the real number $\alpha \in [-\infty, \infty]$. With $0 < \lambda < 1$, we define

$$h_\alpha(x \mid f, g) = \operatorname*{ess\,sup}_{y \in \mathbf{R}^n} \left\{ \lambda f\left(\frac{x-y}{\lambda}\right)^\alpha \oplus (1-\lambda) g\left(\frac{y}{1-\lambda}\right)^\alpha \right\}^{1/\alpha}. \tag{2.1}$$

The symbol $\oplus$ differs from the ordinary addition $+$ in that for

$$f = 0 \quad \text{or} \quad g = 0, \quad \{\lambda f^\alpha \oplus (1-\lambda) g^\alpha\}^{1/\alpha} = 0. \tag{2.2}$$

Otherwise, $\oplus$ and $+$ are the same: For $f > 0$ and $g > 0$,

$$\begin{aligned} \{\lambda f^\alpha \oplus (1-\lambda) g^\alpha\}^{1/\alpha} & \\ = \{\lambda f^\alpha + (1-\lambda) g^\alpha\}^{1/\alpha}, &\quad \text{if } -\infty < \alpha < 0,\ 0 < \alpha < \infty; \\ = \min(f, g), &\quad \text{if } \alpha = -\infty; \\ = \max(f, g), &\quad \text{if } \alpha = \infty; \\ = f^\lambda g^{1-\lambda}, &\quad \text{if } \alpha = 0. \end{aligned} \tag{2.3}$$

Note, that $\oplus$ and $+$ are completely identical for $\alpha \leqslant 0$; however, for $\alpha > 0$ Eq. (2.2) makes them essentially different. Note further that

$$h_\alpha(x) \leqslant h_\beta(x), \quad \text{if } \alpha < \beta.$$

We shall often write $h_\alpha(f, g)$, $h_\alpha(x)$ or h_α if the dependence of $h_\alpha(x \mid f, g)$ on x, f and g, or both is obvious. The dependence on λ is not displayed, λ being held fixed.

As a particular case, take for f and g characteristic functions of measurable sets $A, B \subset \mathbf{R}^n$: $f = \chi_A$, $g = \chi_B$. Then by Eqs. (2.2, 2.3),

$$\{\lambda f^\alpha \oplus (1-\lambda) g^\alpha\}^{1/\alpha} = 0 \quad \text{or} \quad 1,$$

independent of α. Hence, there is a set C such that

$$h_\alpha(\chi_A, \chi_B) = \chi_C, \quad \forall \alpha.$$

We shall use the notation

$$C = \operatorname{ess}\{\lambda A \dotplus (1-\lambda)B\}.$$

To stress the difference with the ordinary Brunn–Minkowski addition we give appropriate definitions:

$$\begin{aligned} \lambda A \dotplus (1-\lambda)B &= \{x \in \mathbf{R}^n \mid (x - \lambda A) \cap (1-\lambda)B \neq \varnothing\}; \\ \operatorname{ess}\{\lambda A \dotplus (1-\lambda)B\} &= \{x \in \mathbf{R}^n \mid \mu_n[(x - \lambda A) \cap (1-\lambda)B] > 0\}. \end{aligned} \tag{2.4}$$

The ordinary addition results, if ess sup in Eq. (2.1) is replaced by sup. The ordinary and the essential additions may differ considerably, as can be seen by taking for A a single point. However, there always exist sets A^* and B^* which differ from A and B by null sets and such that

$$A^* + B^* = \mathrm{ess}(A^* + B^*) = \mathrm{ess}(A + B) \tag{2.5}$$

(see the Appendix). Equation (2.5) and the Brunn–Minkowski theorem, Eq. (1.1), immediately imply the strengthened Brunn–Minkowski theorem

$$\mu_n(C)^{1/n} \geqslant \lambda\mu_n(A)^{1/n} + (1-\lambda)\,\mu_n(B)^{1/n}, \tag{2.6}$$

if $\mu_n(A) > 0$, $\mu_n(B) > 0$.

In the next section we show how Eq. (2.6) extends to inequalities for $\| h_\alpha \|_1$ in terms of $\| f \|_1$ and $\| g \|_1$.

3. Inequalities for $\| h_\alpha \|_1$

The following theorem is basic.

Theorem 3.1. *Let f, g be nonnegative, measurable functions on* $\mathbf{R}$ *and define $h_{-\infty}$ as in Eqs.* (2.1–2.3):

$$h_{-\infty}(x) = \operatorname*{ess\,sup}_{y\in\mathbf{R}} \min\left\{ f\left(\frac{x-y}{\lambda}\right), g\left(\frac{y}{1-\lambda}\right)\right\}.$$

Let $\| f \|_\infty = \| g \|_\infty = m$. Then

$$\| h_{-\infty} \|_1 \geqslant \lambda \| f \|_1 + (1-\lambda) \| g \|_1 .$$

Proof. For $z > 0$, define the sets

$$A(z) = \{x \in \mathbf{R} \mid f(x) > z\},$$
$$B(z) = \{x \in \mathbf{R} \mid g(x) > z\},$$
$$D(z) = \{x \in \mathbf{R} \mid h_{-\infty}(x) > z\}.$$

Then

$$D(z) \supset \mathrm{ess}\{\lambda A(z) + (1-\lambda)\, B(z)\},$$

by the definitions of $h_{-\infty}$ and of the essential addition.

If $z < m$, $\mu_1(A(z)) > 0$ and $\mu_1(B(z)) > 0$. Thus, by Eq. (2.6)

$$\mu_1(D(z)) \geqslant \lambda\mu_1(A(z)) + (1-\lambda)\,\mu_1(B(z)).$$

Note, further, that $\mu_1(D(z)) = \mu_1(A(z)) = \mu_1(B(z)) = 0$ for $z \geqslant m$, and that

$$\| f \|_1 = \int_0^\infty \mu_1(A(z))\, dz, \quad \text{etc.}$$

This gives the desired result. Q.E.D.

By a simple rescaling, Theorem 3.1 immediately leads to

Theorem 3.2. *Let f, g be nonnegative measurable functions on* $\mathbf{R}$ *and define h_α as in Eqs.* (2.1–2.3). *Let $\| f \|_1 > 0$, $\| g \|_1 > 0$. Then, for $\alpha \geqslant -1$,*

$$\| h_\alpha \|_1 \geqslant \{\lambda \| f \|_1^\beta + (1-\lambda) \| g \|_1^\beta\}^{1/\beta}, \tag{3.1}$$

with $\beta = \alpha/(1+\alpha)$. In particular,

$$\| h_0 \|_1 \geqslant \| f \|_1^\lambda \| g \|_1^{1-\lambda}. \tag{3.2}$$

Proof. It is sufficient to consider bounded functions f and g, since any f, g can be approximated from below in L^1 by bounded functions. Now define

$$F(x) = f(x)/\| f \|_\infty ; \qquad G(x) = g(x)/\| g \|_\infty .$$

Let us first consider the case $\alpha \neq 0$. Then

$$h_\alpha(x \mid f, g) = \operatorname*{ess\,sup}_{y\in\mathbf{R}} \left\{\lambda \| f \|_\infty^\alpha F\left(\frac{x-y}{\lambda}\right)^\alpha + (1-\lambda) \| g \|_\infty^\alpha G\left(\frac{y}{1-\lambda}\right)^\alpha\right\}^{1/\alpha}$$
$$= [\lambda \| f \|_\infty^\alpha + (1-\lambda) \| g \|_\infty^\alpha]^{1/\alpha}$$
$$\times \operatorname*{ess\,sup}_{y\in\mathbf{R}} \left\{\theta F\left(\frac{x-y}{\lambda}\right)^\alpha + (1-\theta)\, G\left(\frac{y}{1-\lambda}\right)^\alpha\right\}^{1/\alpha},$$

with the obvious meaning of θ, $0 < \theta < 1$. Thus

$$h_\alpha(x \mid f, g) \geqslant [\lambda \| f \|_\infty^\alpha + (1-\lambda) \| g \|_\infty^\alpha]^{1/\alpha}\, h_{-\infty}(x \mid F, G),$$

and by Theorem 1

$$\| h_\alpha \|_1 \geqslant [\lambda \| f \|_\infty^\alpha + (1-\lambda) \| g \|_\infty^\alpha]^{1/\alpha} \left[\lambda \frac{\| f \|_1}{\| f \|_\infty} + (1-\lambda) \frac{\| g \|_1}{\| g \|_\infty}\right] \tag{3.3}$$

Now Eq. (3.1) for $-1 \leqslant \alpha < 0$ or $0 < \alpha \leqslant \infty$ follows by Hölder's inequality.

For $\alpha = 0$,

$$h_0(f, g) = \|f\|_\infty^\lambda \|g\|_\infty^{1-\lambda} h_0(F, G) \geqslant \|f\|_\infty^\lambda \|g\|_\infty^{1-\lambda} h_{-\infty}(F, G).$$

Then Theorem 1 gives

$$\|h_0\|_1 \geqslant \|f\|_\infty^\lambda \|g\|_\infty^{1-\lambda} \left[\lambda \frac{\|f\|_1}{\|f\|_\infty} + (1-\lambda) \frac{\|g\|_1}{\|g\|_\infty}\right], \tag{3.4}$$

and Eq. (3.2) follows by the arithmetic-geometric mean inequality. Q.E.D.

Remarks. 1. Equation (3.3) (supplemented with Eq. (3.4) for $\alpha = 0$) holds for all $\alpha \in [-\infty, \infty]$. The restriction $\alpha \geqslant -1$ arises from the final application of Hölder's inequality.

2. Theorem 3.2 does not hold if $\alpha > 0$, $\|f\|_1 = 0$, $\|g\|_1 > 0$; in that case $h_\alpha = 0$. Analogously, the extended Brunn–Minkowski theorem [Eq. (2.6)] is not true if A or B has measure zero.

The n-dimensional version of Theorem 3.2 reads thus.

THEOREM 3.3. *Let f, g be nonnegative measurable functions on $\mathbf{R}^n$ and define h_α as in Eqs.* (2.1–2.3). *Let $\|f\|_1 > 0$, $\|g\|_1 > 0$. Then for $\alpha \geqslant -1/n$,*

$$\|h_\alpha\|_1 \geqslant \{\lambda \|f\|_1^\gamma + (1-\lambda)\|g\|_1^\gamma\}^{1/\gamma}, \tag{3.5}$$

with $\gamma = \alpha/(1 + n\alpha)$. In particular,

$$\|h_0\|_1 \geqslant \|f\|_1^\lambda \|g\|_1^{1-\lambda}.$$

Proof. Write $\mathbf{R}^n \ni x = (y, z)$, with $y \in \mathbf{R}$, $z \in \mathbf{R}^{n-1}$. Define

$$F(z) = \int dy\, f(y, z); \qquad G(z) = \int dy\, g(y, z). \tag{3.6}$$

Since

$$h_\alpha(y, z \mid f, g) = \operatorname*{ess\,sup}_{w \in \mathbf{R}^{n-1}} \operatorname*{ess\,sup}_{v \in \mathbf{R}} \left\{\lambda f\left(\frac{y-v}{\lambda}, \frac{z-w}{\lambda}\right)^\alpha \oplus (1-\lambda) g\left(\frac{v}{1-\lambda}, \frac{w}{1-\lambda}\right)^\alpha\right\}^{1/\alpha},$$

it follows from Theorem 3.2 that

$$\int dy\, h_\alpha(y, z \mid f, g) \geqslant h_\beta(z \mid F, G), \tag{3.7}$$

with $\beta = \alpha/(\alpha + 1)$. Note, that we used that

$$\int dy \operatorname*{ess\,sup}_w \geqslant \operatorname*{ess\,sup}_w \int dy.$$

Note further, that Theorem 3.2 does not apply, if z and w are such that $F((z - w)/\lambda) = 0$ or $G(z/(1-\lambda)) = 0$. However, Eq. (3.7) is saved by the $\oplus$ sign in the definition of h_β [cf. Eq. (2.2)].

If we assume Theorem 3.3 to be true for $n - 1$, we have that

$$\|h_\beta(F, G)\|_1 \geqslant \{\lambda \|F\|_1^\gamma + (1-\lambda)\|G\|_1^\gamma\}^{1/\gamma},$$

with $\gamma = \beta/[1 + (n-1)\beta] = \alpha/(1 + n\alpha)$.

With Eqs. (3.6, 3.7) and Fubini's theorem, this leads to Eq. (3.5). Thus Theorem 3.3 is proved by induction. Q.E.D.

As an introduction to two corollaries of Theorem 3.3, let us define the classes of functions $K_\alpha(\mathbf{R}^n)$.

DEFINITION. $K_\alpha(\mathbf{R}^n)$ consists of the nonnegative, measurable functions F on $\mathbf{R}^n$ such that for all $\lambda \in (0, 1)$

$$F = h_\alpha(F, F) \quad \text{a.e.}$$

In more pedestrian terms, this means that F has the following convexity properties (apart from null functions).

$\alpha = -\infty$: F is unimodal, i.e., the sets $\{z \mid F(x) > z\}$ are convex.

$-\infty < \alpha < 0$: F^α is convex.

$\alpha = 0$: F is logarithmically concave, i.e.,

$$F(\lambda x + (1-\lambda)y) \geqslant F(x)^\lambda F(y)^{1-\lambda}.$$

$0 < \alpha < \infty$: F^α is concave on a convex set, and $F(x) = 0$ outside this set.

$\alpha = \infty$: $F(x) = $ const. on a convex set, and $F(x) = 0$ outside this set.

Note, that $K_\alpha \subset K_\beta$ if $\alpha > \beta$. This follows from Jensen's inequality.

COROLLARY 3.4. *Let A, B be measurable sets in $\mathbf{R}^n$ of positive measure, and let*

$$C = \operatorname{ess}\{\lambda A + (1-\lambda)B\}.$$

Let $F \in K_\alpha(\mathbf{R}^n)$, $\alpha \geqslant -1/n$, *and let*

$$\mu_F(A) = \int_A F(x)\, dx.$$

Then, with $\gamma = \alpha/(1 + n\alpha)$,

$$\mu_F(C) \geqslant \{\lambda \mu_F(A)^\gamma + (1-\lambda)\, \mu_F(B)^\gamma\}^{1/\gamma}.$$

In particular, if F is log concave,

$$\mu_F(C) \geqslant \mu_F(A)^\lambda\, \mu_F(B)^{1-\lambda}. \tag{3.8}$$

Proof. Let $f = F\chi_A$ and $g = F\chi_B$. Then $h_\alpha(f, g) \leqslant \chi_C h_\alpha(F, F) = \chi_C F$. Apply Theorem 3.3 to complete the proof. Q.E.D.

Examples. (1) Let $F(x) \equiv 1 \in K_\infty$. Then $\gamma = 1/n$ and we recover the Brunn–Minkowski theorem, Eq. (2.6).

(2) Let $G(x) = \exp(-x^2) \in K_0$. Then in any $\mathbf{R}^n$

$$\mu_G(C) \geqslant \mu_G(A)^\lambda\, \mu_G(B)^{1-\lambda}.$$

(3) Let $L(x) = (1 + x^2)^{-1} \in K_{-1/2}$. Then

$$\mu_L(C) \geqslant \{\lambda \mu_L(A)^{-1} + (1-\lambda)\, \mu_L(B)^{-1}\}^{-1}, \quad \text{in } \mathbf{R},$$
$$\mu_L(C) \geqslant \min\{\mu_L(A), \mu_L(B)\}, \quad \text{in } \mathbf{R}^2.$$

Corollary 3.5. *Let* $F(x, y) \in K_\alpha(\mathbf{R}^{m+n})$, $x \in \mathbf{R}^m$, $y \in \mathbf{R}^n$. *Let*

$$G(x) = \int_{\mathbf{R}^n} F(x, y)\, dy.$$

Then $G \in K_\gamma(\mathbf{R}^m)$, $\gamma = \alpha/(1 + n\alpha)$. ***In particular, if F is log concave, so is G.***

Proof. Since $F(x, y) > 0$ on a convex set in $\mathbf{R}^{m+n}$, $G(x) > 0$ on a convex set in $\mathbf{R}^m$. Now fix points x_0, x_1 in this set, and define $f(y) = F(x_1, y)$, $g(y) = F(x_0, y)$. Then

$$F(\lambda x_1 + (1-\lambda)\, x_0, y) \geqslant h_\alpha(y \mid f, g).$$

Now apply Theorem 3.3 to $h_\alpha(y \mid f, g)$. Q.E.D.

4. Log Concave Functions and Measures

In this section we prove a Sobolev-type inequality (Theorem 4.1) for log concave measures (i.e., measures given by a log concave density function). We shall write $F(x) = \exp[-f(x)]$, $x \in \mathbf{R}^n$; $F(x)$ is log concave iff $f(x)$ is convex. If $f(x)$ is twice continuously differentiable, this means that the second derivatives matrix, f_{xx}, is nonnegative.

It is often convenient to write $\mathbf{R}^{n+m} \ni x = (y, z)$, $y \in \mathbf{R}^m$, $z \in \mathbf{R}^n$. The matrix f_{xx} is then partitioned in an obvious way as

$$f_{xx} = \begin{pmatrix} f_{yy} & f_{yz} \\ f_{zy} & f_{zz} \end{pmatrix}. \tag{4.1}$$

We shall often encounter

$$G(y) = \exp[-g(y)] = \int_{\mathbf{R}^n} F(y, z)\, dz. \tag{4.2}$$

Then $G(y)$ is log concave by Corollary 3.5. A sharper form of this result will be given in Theorem 4.2.

With F as a density function, define

$$\begin{aligned} \langle A \rangle &= \int_{\mathbf{R}^n} A(x)\, F(x)\, dx \Big/ \int_{\mathbf{R}^n} F(x)\, dx, \\ \operatorname{var} A &= \langle | A - \langle A \rangle |^2 \rangle, \\ \operatorname{cov}(A, B) &= \langle (\bar{A} - \langle \bar{A} \rangle)(B - \langle B \rangle) \rangle. \end{aligned} \tag{4.3}$$

If $x = (y, z)$, $y \in \mathbf{R}^m$, $z \in \mathbf{R}^n$, we write

$$\begin{aligned} \langle A \rangle_z (y) &= \int_{\mathbf{R}^n} A(y, z)\, F(y, z)\, dz \Big/ \int_{\mathbf{R}^n} F(y, z)\, dz, \\ \langle B \rangle_y &= \int_{\mathbf{R}^m} B(y)\, G(y)\, dy \Big/ \int_{\mathbf{R}^m} G(y)\, dy, \end{aligned} \tag{4.4}$$

so that $\langle A \rangle = \langle \langle A \rangle_z \rangle_y$. In analogy with Eq. (4.3), var_y, cov_y, var_z, and cov_z are defined.

Theorem 4.1. *Let* $F(x) = \exp[-f(x)]$, $x \in \mathbf{R}^n$, *let f be twice continuously differentiable and let f be strictly convex. Let f have a minimum, so that F decreases exponentially in all directions; then*

$$\int_{\mathbf{R}^n} F(x)\, dx < \infty.$$

Let $h \in C^1(\mathbf{R}^n)$, *and let* $\operatorname{var} h < \infty$. *Then*

$$\operatorname{var} h \leqslant \langle (h_x, (f_{xx})^{-1} h_x) \rangle, \tag{4.5}$$

where the inner product is with respect to $\mathbf{C}^n$, *and* h_x *denotes the gradient of* h.

It is convenient to postpone the proof of Theorem 4.1 a moment. We prefer to give an immediate corollary first.

THEOREM 4.2. *Let* $F(x) = F(y, z) = \exp[-f(y, z)]$, $y \in \mathbf{R}^m$, $z \in \mathbf{R}^n$, *satisfy the assumptions of Theorem* 4.1. *Moreover, let the integrals*

$$\int_{\mathbf{R}^n} (\phi, f_{yy}\phi) F\, dz, \qquad \int_{\mathbf{R}^n} (\phi, f_y)^2 F\, dz \tag{4.6}$$

converge uniformly in y *in a neighborhood of a given point* $y_0 \in \mathbf{R}^m$, *for all vectors* $\phi \in \mathbf{R}^m$. *Then, with the notation of Eqs.* (4.1, 4.2, 4.4), $g(y)$ *is twice continuously differentiable near* y_0, *and*

$$g_{yy} \geqslant \langle f_{yy} - f_{yz}(f_{zz})^{-1} f_{zy} \rangle_z \tag{4.7}$$

as a matrix inequality.

Proof. We denote differentiation in a direction t at y_0 by a subscript t. Then Eq. (4.7) is equivalent to saying that for all directions t

$$g_{tt} \geqslant \langle f_{tt} - f_{tz}(f_{zz})^{-1} f_{zt} \rangle_z .$$

By differentiating $g(y) = \log \int F(y, z)\, dz$, one gets

$$g_{tt} = \langle f_{tt} \rangle_z - \operatorname{var}_z f_t . \tag{4.8}$$

The differentiation can be done under the integral sign by the uniform convergence of the integrals (4.6), which also ensures the continuity of g_{tt}.

The result (4.7) follows by applying Theorem 4.1 with $h(z) = f_t(y_0, z)$. Q.E.D.

Remark. Even though F is assumed to be a log concave function, decreasing exponentially in all directions, the convergence of the integrals (4.6) does not follow automatically. For example, define the convex function $\phi(x)$, $x \in \mathbf{R}$, by $\phi(0) = \phi'(0) = 0$, and

$$\phi''(x) = \sum_{n \neq 0} a_n \delta(x - n), \qquad a_n > 0, \quad a_n = a_{-n} .$$

Then

$$\int \phi''(x) \exp[-\phi(x)]\, dx = 2 \sum_{n=1}^{\infty} a_n \exp\left[-\sum_{k=1}^{n-1} (n - k)\, a_k\right],$$

which can be made divergent by an appropriate recursive definition of a_n. If we take

$$f(y, z) = y^2 + \phi(y + z), \qquad y, z \in \mathbf{R},$$

the integrals (4.6) obviously diverge for all y.

The function ϕ can be approximated by a C^2 function without changing the conclusion.

Proof of Theorem 4.1. We can obviously restrict h to be real valued. Let us first give the proof for $\mathbf{R}^1$. If $f(x)$ has its unique minimum at $x = a$, write

$$h(x) - h(a) = f'(x)\, k(x).$$

Then $k(x)$ is continuously differentiable, except possibly at $x = a$. However, if we set $k(a) = h'(a)/f''(a)$, k is continuous at $x = a$. Now

$$\begin{aligned}\int (h')^2/f'' F\, dx &= \int [(k'f')^2/f'' + 2kk'f' + k^2 f''] F\, dx \\ &= \int [(k'f')^2/f'' + (kf')^2] F\, dx + [k^2 f' F]_{-\infty}^{a} + [k^2 f' F]_a^{\infty} \\ &\geqslant \int [h(x) - h(a)]^2 F(x)\, dx.\end{aligned}$$

Equation (4.5) follows by noting that

$$\operatorname{var} h \leqslant \langle [h - h(a)]^2 \rangle.$$

Now assume that Theorem 4.1 has been proved for $x \in \mathbf{R}^{n-1}$. Hence we also have Theorem 4.2 for $z \in \mathbf{R}^{n-1}$ at our disposition.

Write $\mathbf{R}^n \ni x = (y, z)$, $y \in \mathbf{R}$, $z \in \mathbf{R}^{n-1}$. Then

$$\operatorname{var} h = \langle \operatorname{var}_z h \rangle_y + \operatorname{var}_y \langle h \rangle_z ,$$

with the notation of Eqs. (4.3, 4.4).

Let us first restrict ourselves to functions h with compact support. This has the advantage that F can be modified outside the support of h in such a way, that it satisfies all the assumptions of Theorem 4.2

for all y. Then $G(y) = \int F(y, z)\, dz$ satisfies the assumptions of Theorem 4.1, so that

$$\operatorname{var}_y \langle h \rangle_z \leqslant \langle ((d/dy)\langle h \rangle_z)^2 / g'' \rangle_y .$$

Now all differentiations can be carried out under the integral signs, since h has compact support and F has been appropriately modified. Thus we find (cf. Eq. (4.8))

$$\operatorname{var} h \leqslant \langle B \rangle_y ,$$

$$B = \operatorname{var}_z h + \frac{[\langle h_y \rangle_z - \operatorname{cov}_z(h, f_y)]^2}{\langle f_{yy} \rangle_z - \operatorname{var}_z f_y} . \tag{4.9}$$

Applying Theorem 4.1 for $z \in \mathbf{R}^{n-1}$, with fixed $y \in \mathbf{R}$, we have

$$\operatorname{var}_z H \leqslant \langle (H_z , f_{zz}^{-1} H_z) \rangle_z .$$

Since this is true for

$$H = \lambda h + \mu f_y$$

with arbitrary λ and μ, we get

$$B \leqslant \langle (h_z , f_{zz}^{-1} h_z) \rangle_z + \frac{\langle h_y - (h_z , f_{zz}^{-1} f_{zy}) \rangle_z^2}{\langle f_{yy} - (f_{yz} , f_{zz}^{-1} f_{zy}) \rangle_z} .$$

Since f is convex, the denominator above is positive and we can use Schwarz's inequality to obtain

$$B \leqslant \left\langle (h_z , f_{zz}^{-1} h_z) + \frac{[h_y - (h_z , f_{zz}^{-1} f_{zy})]^2}{f_{yy} - (f_{yz} , f_{zz}^{-1} f_{zy})} \right\rangle_z$$

$$= \langle (h_x , f_{xx}^{-1} h_x) \rangle_z . \tag{4.10}$$

Eq. (4.5) follows by combining Eqs. (4.9) and (4.10).

Now only the restriction that h has compact support remains to be removed. As an intermediate step, let us show that for all h and F satisfying the assumptions of Theorem 4.1

$$\operatorname{var}_S h \leqslant \langle (h_x , f_{xx}^{-1} h_x) \rangle_S , \tag{4.11}$$

where the averages are taken over a ball with radius S centered at the origin, instead of over all $\mathbf{R}^n$.

Modify h outside the ball smoothly to a function k with compact support, and let

$$f^{(N)}(x) = f(x), \qquad \text{if } |x| \leqslant S;$$

$$f^{(N)}(x) = f(x) + N(|x| - S)^4, \qquad \text{if } |x| \geqslant S.$$

By our results until now, we have that

$$\operatorname{var}_N k \leqslant \langle (k_x , (f_{xx}^{(N)})^{-1} k_x) \rangle_N ,$$

with averages with respect to the weight $\exp[-f^{(N)}(x)]$. Equation (4.11) is proved by taking the limit $N \to \infty$ and using the monotone convergence theorem.

Now let $S \to \infty$ in Eq. (4.11). Then $\operatorname{var}_S h \to \operatorname{var} h$, and

$$\int_S (h_x , f_{xx}^{-1} h_x) F\, dx$$

increases (it may actually increase to ∞). This concludes the proof. Q.E.D.

Examples. 1. Let $M_{ij} = \operatorname{cov}(x_i , x_j)$. Then we have the matrix inequality

$$M \leqslant \langle (f_{xx})^{-1} \rangle, \tag{4.12}$$

as can be seen by taking $h(x) = (\phi, x)$ for any $\phi \in \mathbf{R}^n$ in Theorem 4.1. As a curiosity, compare (4.12) with the one dimensional inequality

$$\operatorname{var} x \geqslant \langle f'' \rangle^{-1}, \tag{4.13}$$

which holds for general weights F. The proof is

$$1 = [\operatorname{cov}(x, f')]^2 \leqslant \operatorname{var} f' \operatorname{var} x = \langle f'' \rangle \operatorname{var} x,$$

with Schwarz's inequality and two integrations by parts.

2. For the Gaussian weight $F(x) = \exp[-(x, Ax)]$,

$$\operatorname{var} h \leqslant \langle (h_x , (2A)^{-1} h_x) \rangle. \tag{4.14}$$

In particular, if $F(x) = \exp[-(x, x)/2]$,

$$\operatorname{var} h \leqslant \langle |h_x|^2 \rangle. \tag{4.15}$$

3. If $F(x) = \exp[-(x, Ax)]$, $M = (2A)^{-1}$, and thus the inequality in (4.12) holds as an equality.

4. The analog of Example 3 in the setting of Theorem 4.2 concerns the Gaussian

$$\Phi(x, y) = \exp\left[-(x, y)\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right], \qquad (x, y) \in \mathbf{R}^m \times \mathbf{R}^n, \quad (4.16)$$

with a real, positive matrix $\left(\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right)$. Then

$$\int \Phi(x, y)\, dy = \text{const.} \exp[-(x, Dx)], \qquad (4.17)$$

with

$$D = B - BC^{-1}B^*. \qquad (4.18)$$

Thus for Gaussians the equality sign in Eq. (4.7) holds.

THEOREM 4.3. *With the notation of Eqs.* (4.16–4.18), *let* $G(x)$ *be defined by*

$$\int \Phi(x, y)\, F(x, y)\, dy = G(x) \exp[-x, Dx)].$$

Then, if $F(x, y)$ *is log concave,* $G(x)$ *is log concave; if* $F(x, y)$ *is log convex,* $G(x)$ *is log convex.*

Proof. Write

$$\Phi(x, y) = \exp[-(x, Dx) - (y', Cy')],$$
$$y' = y + C^{-1}B^*x.$$

Then

$$G(x) = \int \exp[-(y, Cy)]\, F(x, y - C^{-1}B^*x)\, dy. \qquad (4.19)$$

If $F(x, y)$ is log concave, the integrand in Eq. (4.19) is log concave. Then $G(x)$ is log concave by Corollary 3.5. If $F(x, y)$ is log convex, the integrand is log convex in x for all fixed y. Then $G(x)$ is log convex by Hölder's inequality. Q.E.D.

Note, that the log concave part of Theorem 4.3 also follows from Theorem 4.2.

5. MOMENT INEQUALITIES

THEOREM 5.1. *Let* $F(x)$ *be a nonnegative function on* $\mathbf{R}^n$, *and let* A *be a real, positive definite,* $n \times n$ *matrix. Assume* $\exp[-(x, Ax)]\, F(x) \in L^1$ *and define*

$$\langle k \rangle_F = \int k(x) \exp[-(x, Ax)]\, F(x)\, dx \Big/ \int \exp[-(x, Ax)]\, F(x)\, dx.$$

If $F(x) \equiv 1$ *we write* $\langle \cdot \rangle_1$. *Let* $\phi \in \mathbf{R}^n$, $a \in \mathbf{R}$. *Then*

$$\langle |(\phi, x) - \langle (\phi, x) \rangle_F|^\alpha \rangle_F \leqslant \langle |(\phi, x)|^\alpha \rangle_1,$$

when F *is log concave and* $\alpha \geqslant 1$;

$$\langle |(\phi, x) - a|^\alpha \rangle_F \leqslant \langle |(\phi, x)|^\alpha \rangle_1, \qquad \text{if} \quad \alpha > 0,$$
$$\langle |(\phi, x) - a|^\alpha \rangle_F \geqslant \langle |(\phi, x)|^\alpha \rangle_1, \qquad \text{if} \quad -1 < \alpha < 0,$$

when F *is log convex.*

Proof. By a linear transformation such that $(\phi, x) \to x_1$ and by Theorem 4.3 it suffices to prove Theorem 5.1 for the one-dimensional case. This will be done in Lemmas 5.2 and 5.3. Q.E.D.

LEMMA 5.2. *Let* $F(x)$ *be a log convex function on* $\mathbf{R}$, *and let the averages* $\langle \cdot \rangle_F$ *and* $\langle \cdot \rangle_1$ *be computed with the weights* $\exp(-x^2)\, F(x)$ *and* $\exp(-x^2)$, *respectively. Let* $a \in \mathbf{R}$. *Then*

$$\langle |x - a|^\alpha \rangle_F \geqslant \langle |x|^\alpha \rangle_1, \qquad \text{if} \quad \alpha > 0; \qquad (5.1)$$
$$\langle |x - a|^\alpha \rangle_F \leqslant \langle |x|^\alpha \rangle_1, \qquad \text{if} \quad -1 < \alpha < 0. \qquad (5.2)$$

Proof. Note that

$$\langle |x - a|^\alpha \rangle_F = \langle |x|^\alpha \rangle_G = \langle |x|^\alpha \rangle_H,$$

where

$$G(x) = F(x + a) \exp(-2ax),$$
$$H(x) = G(x) + G(-x).$$

Since F is log convex, G and H are log convex; moreover, H is even. Thus, for $\alpha > 0$, it has to be shown that

$$\langle x^\alpha H(x) \rangle \geqslant \langle x^\alpha \rangle \langle H(x) \rangle, \qquad (5.3)$$

with the averages computed over $x > 0$ with the weight $\exp(-x^2)$. But this is equivalent to the inequality

$$\int_0^\infty \int_0^\infty dx\, dy \exp(-x^2 - y^2)[H(x) - H(y)](x^\alpha - y^\alpha) \geqslant 0, \qquad (5.4)$$

which is obvious, since $H(x)$ and x^α are increasing functions for $x > 0$.

If $-1 < \alpha < 0$, x^α is decreasing for $x > 0$, and hence

$$\langle x^\alpha H(x)\rangle \leqslant \langle x^\alpha\rangle\langle H(x)\rangle.$$

This proves Eq. (5.2). Q.E.D.

Lemma 5.3. *Let $F(x)$ be a log concave function on* **R**. *Then, with the notation of Lemma* 5.2,

$$\langle | x - \langle x\rangle_F |^\alpha\rangle_F \leqslant \langle | x |^\alpha\rangle_1, \qquad if \quad \alpha \geqslant 1. \qquad (5.5)$$

Proof. Write

$$\langle | x - \langle x\rangle_F |^\alpha\rangle_F = \langle | x |^\alpha\rangle_G,$$

with

$$G(x) = F(x + \langle x\rangle_F) \exp(-2x\langle x\rangle_F).$$

Then $G(x)$ is log concave, and $\langle x\rangle_G = 0$. By approximation, it is sufficient to assume $G \in C^1$. Hence

$$\int dx \exp(-x^2)\, G'(x) = 2 \int dx\, x \exp(-x^2)\, G(x) = 0. \qquad (5.6)$$

Moreover, there must exist a number K such that $G(x)$ is increasing for $x < K$; decreasing for $x > K$. By Eq. (5.6) K must be finite and we can assume that $K \geqslant 0$, say. Then $G'(x) \geqslant 0$ for $x < 0$, and Eq. (5.6) implies that

$$\int_0^\infty dx \exp(-x^2)\, G'(x) \leqslant 0. \qquad (5.7)$$

It has to be shown that

$$\langle x^\alpha[G(x) + G(-x)]\rangle \leqslant \langle x^\alpha\rangle\langle G(x) + G(-x)\rangle, \qquad (5.8)$$

where the averages are with respect to $\exp(-x^2)$, $x > 0$.

We assumed, that $G'(x) \geqslant 0$ for $x < 0$, and thus (cf. Eqs. (5.3, 5.4)]

$$\langle x^\alpha G(-x)\rangle \leqslant \langle x^\alpha\rangle\langle G(-x)\rangle.$$

We wish to show the same inequality for the $G(x)$ part in Eq. (5.8), which is equivalent to

$$\int_0^\infty dx \int_0^x dy \exp(-x^2 - y^2)[G(x) - G(y)](x^\alpha - y^\alpha) \leqslant 0. \qquad (5.9)$$

If we write

$$G(x) - G(y) = \int_y^x G'(z)\, dz,$$

Eq. (5.9) becomes

$$\int_0^\infty dz\, \psi(z) \exp(-z^2)\, G'(z) \leqslant 0, \qquad (5.10)$$

$$\psi(z) = \exp(z^2) \int_z^\infty dx \int_0^z dy \exp(-x^2 - y^2)(x^\alpha - y^\alpha). \qquad (5.11)$$

If we manage to show that $\psi(z)$ is an increasing function for $z > 0$, Eq. (5.10) follows from Eq. (5.7) and the fact that $G'(x) \geqslant 0$ for $0 < x < K$; $G'(x) \leqslant 0$ for $x > K$, and Lemma 5.3 is proved.

After some manipulation, we find that

$$\psi'(z) = \int_z^\infty dx \exp(-x^2)(x^\alpha - z^\alpha)$$
$$+ z \exp(z^2) \int_z^\infty dx \int_0^\infty dy \exp(-x^2 - y^2)[(\alpha - 1)\, x^{\alpha-2} + y^\alpha x^{-2}].$$

Thus, if $\alpha \geqslant 1$, $\psi'(z) > 0$. Q.E.D.

Remark. Here, as well as in Theorem 4.3, the log convex case is much simpler than the log concave case. We leave as an open question, the correct generalization of Eq. (5.5) when $-1 < \alpha < 1$. If $F(x)$ is symmetric decreasing, which implies that $\langle x\rangle_F = 0$ but does not imply that F is log concave, then Eq. (5.5) trivially generalizes to

$$\langle | x |^\alpha\rangle_F \leqslant \langle | x |^\alpha\rangle_1, \qquad \text{if} \quad \alpha > 0;$$
$$\langle | x |^\alpha\rangle_F \geqslant \langle | x |^\alpha\rangle_1, \qquad \text{if} \quad -1 < \alpha < 0;$$

cf. the proof of Lemma 4.2 and especially Eqs. (5.3, 5.4).

Theorem 5.4. *Under the assumptions of Theorem* 5.1, *let M be the covariance matrix*

$$M_{ij} = \langle x_i x_j\rangle_F - \langle x_i\rangle_F \langle x_j\rangle_F.$$

Then

$$M \leqslant \langle (2A + f_{xx})^{-1}\rangle_F \leqslant (2A)^{-1}, \qquad \textit{if} \quad F = \exp(-f) \textit{ is log concave};$$
$$M \geqslant (2A)^{-1}, \qquad \textit{if} \quad F \textit{ is log convex}. \qquad (5.12)$$

Proof. Setting $\alpha = 2$ in Theorem 5.1 leads to $M \leqslant (2A)^{-1}$ resp. $M \geqslant (2A)^{-1}$. The stronger inequality (5.12) is obtained from Theorem 4.1 by taking $h(x) = (\phi, x)$ and replacing the weight $F(x)$ by $\exp[-(x, Ax)]\, F(x)$. Q.E.D.

6. The Diffusion Equation

Consider the diffusion equation in $\mathbf{R}^n$

$$\partial\psi/\partial t = -H_A\psi \tag{6.1}$$

with the Hamiltonian

$$(H_A\psi)(x) = -\tfrac{1}{2}(\Delta\psi)(x) + V(x)\,\psi(x), \tag{6.2}$$

defined on an open, connected region $A \subset \mathbf{R}^n$, with zero boundary conditions. The potential $V(x)$ is assumed to be convex; in particular, $V(x)$ may be ∞ outside a convex set D. Further we assume the region A to be such that

$$\int_A \exp[-tV(x)]\, dx < \infty, \qquad \forall t > 0. \tag{6.3}$$

(This means that A is bounded in the directions, for which $V(x)$ does not go to ∞ as $|x| \to \infty$.)
The fundamental solution $G_A(x, y; t)$ of Eq. (6.1) is defined by

$$\begin{aligned}
((\partial/\partial t) - H_{A,x})\, G_A(x, y; t) &= 0, && x, y \in A \cap D,\ t > 0;\\
G_A(x, y; 0) &= \delta(x - y), && x, y \in A \cap D;\\
G_A(x, y; t) &= 0, && x \in \partial(A \cap D);\\
G_A(x, y; t) &= 0, && x \notin A \cap D \text{ or } y \notin A \cap D.
\end{aligned}$$

We could, of course, replace A by $A \cap D$ without changing G_A, but the point is that in Theorem 6.2 we want to vary A while keeping D fixed.
Using the Trotter product formula, we can write

$$G_A(x, y; t) = \lim_{M\to\infty} \left(\frac{2\pi t}{M}\right)^{-nM/2} \int_A dx_1 \cdots \int_A dx_{M-1} \times \prod_{j=1}^{M} \exp\left[-\frac{M}{2t}(x_j - x_{j-1})^2 - \frac{t}{M} V(x_j)\right], \tag{6.4}$$

where $x_0 = x$, $x_M = y$.

Define the partition function by

$$Z_A(t) = \operatorname{Tr} \exp(-tH_A) = \int_A G_A(x, x; t)\, dx. \tag{6.5}$$

Then Eq. (6.3) guarantees, that $Z_A(t) < \infty$ for all $t > 0$, so that H_A has a pure point spectrum. In fact, Hölder's inequality applied to Eqs. (6.4, 6.5) gives that

$$\begin{aligned}
Z_A(t) &\leqslant \int_A G^0(x, x; t) \exp[-tV(x)]\, dx\\
&= (2\pi t)^{-n/2} \int_A \exp[-tV(x)]\, dx,
\end{aligned}$$

where G^0 is the fundamental solution of Eq. (6.1) with $V(x) = 0$. Moreover the ground state is nondegenerate and the corresponding eigenfunction is nonnegative [9].

Theorem 6.1. *Let $A = \mathbf{R}^n$, and let the potential be of the form*

$$V(x) = \tfrac{1}{2}\omega^2 x^2 + W(x), \qquad \omega \geqslant 0, \tag{6.6}$$

with a convex function $W(x)$. Then the ground state wave function $\psi_0(x)$ is of the form

$$\psi_0(x) = \exp(-\tfrac{1}{2}\omega x^2)\, \phi(x),$$

where $\phi(x)$ is log concave.

Proof. Let $G_\omega(x, y; t)$ be the fundamental solution of Eq. (6.1) for $V(x) = \frac{1}{2}\omega^2 x^2$. Then the fundamental solution for the potential (6.6) is of the form

$$G(x, y; t) = G_\omega(x, y; t)\, H(x, y; t),$$

where $H(x, y, t)$ is log concave in (x, y) for all t. This follows directly from Theorem 4.3 applied to Eq. (6.4).
If ϵ is the ground state energy,

$$\psi_0(x)\, \psi_0(y) = \lim_{t\to\infty} G(x, y; t) \exp(\epsilon t).$$

Since the pointwise limit of log concave functions is log concave, the theorem follows. Q.E.D.

Remark. If $W(x)$ is concave instead of convex, (but such that Eq. (6.3) still holds), the log convex part of Theorem 4.3 implies in the same way as above that $\phi(x)$ is log convex.

THEOREM 6.2. *Let A and B be open, connected regions, let $C = \lambda A + (1-\lambda)B$, and let $V(x)$ be convex. Then*

$$Z_C(t) \geqslant Z_A(t)^\lambda Z_B(t)^{1-\lambda}; \tag{6.7}$$

$$\epsilon_C \leqslant \lambda\epsilon_A + (1-\lambda)\,\epsilon_B, \tag{6.8}$$

where $\epsilon_A(\epsilon_B, \epsilon_C)$ is the ground state energy of $H_A(H_B, H_C)$.

Proof. Equations (6.4, 6.5) together give an expression for the partition function. We note, that we can apply Corollary 3.4 to the sets A^M, B^M, and C^M. This proves Eq. (6.7). Further

$$\epsilon_A = -\lim_{t\to\infty} t^{-1} \log Z_A(t),$$

which gives Eq. (6.8). Q.E.D.

APPENDIX

THEOREM A.1. *For measurable sets A and $B \subset \mathbf{R}^n$, define the essential sum $C = \mathrm{ess}\{A+B\}$ as in Eq. (2.4). Then C is open, and*

$$\mu_n(C)^{1/n} \geqslant \mu_n(A)^{1/n} + \mu_n(B)^{1/n}. \tag{A.1}$$

THEOREM A.2. *For nonnegative, measurable functions $f(x)$ and $g(x)$ on $\mathbf{R}^n$, define*

$$H_\alpha(x \mid f, g) = \operatorname*{ess\,sup}_{y\in\mathbf{R}^n}\{f(x-y)^\alpha \oplus g(y)^\alpha\}^{1/\alpha}, \tag{A.2}$$

cf. Eqs. (2.1–2.3). *Then $H_\alpha(x)$ is lower semicontinuous in x for all α.*

Proof of Theorem A.1. All the above facts are based on the following observation: For an arbitrary measurable set $A \subset \mathbf{R}^n$, define

$$A^* = \{x \in \mathbf{R}^n \mid \mu_n[A \cap V(\epsilon, x)]/W_n(\epsilon) \to 1 \quad \text{for} \quad \epsilon \downarrow 0\}, \tag{A.3}$$

where $V(\epsilon, x)$ is the open ball of radius ϵ centered at x, and $W_n(\epsilon)$ is its volume. Then A^* is measurable and $\mu_n(A^* \varDelta A) = 0$, where $\varDelta$ means symmetric difference [2, Theorem 2.9.11]. Hence

$$\mathrm{ess}(A+B) = \mathrm{ess}(A^* + B^*), \tag{A.4}$$

and it is sufficient to prove the theorem when A and B are replaced by A^* and B^*.

Let $x \in A^* + B^*$, i.e., there is a point $y \in A^* \cap (x - B^*)$. Notice, that $A^{**} = A^*$; thus for some $\epsilon > 0$,

$$\mu_n[A^* \cap V(\epsilon, y)] \geqslant \tfrac{3}{4} W_n(\epsilon);$$

$$\mu_n[(x - B^*) \cap V(\epsilon, y)] \geqslant \tfrac{3}{4} W_n(\epsilon).$$

Hence, $\mu_n[A^* \cap (v - B^*)] > 0$ for all v in some neighborhood $V(\delta, x)$, which implies that $A^* + B^*$ is open, and that

$$A^* + B^* = \mathrm{ess}(A^* + B^*). \tag{A.5}$$

Equation (A.1) now follows from Eqs. (A.4, A.5) and the Brunn–Minkowski theorem, Eq. (1.1). Q.E.D.

Proof of Theorem A.2. For a nonnegative, measurable function f, let

$$A_f = \{(x, z) \in \mathbf{R}^{n+1} \mid 0 < z < f(x)\}. \tag{A.6}$$

Define A_f^* as in (A.3). If $(x, z) \in A_f^*$, $(x, t) \in A_f^*$ for all t, $0 < t < z$. Thus it makes sense to define

$$f^*(x) = \sup\{z \mid (x, y) \in A_f^*\}. \tag{A.7}$$

The supremum over the empty set is taken to be zero. Given f^*, define A_{f^*} according to definition (A.6). Clearly A_f, A_f^* and f^* are all measurable. By (A.6) and (A.7), $A_f^* \supset A_{f^*}$. Since

$$A_f^* \backslash A_{f^*} \subset G = \{(x, f^*(x)) \mid x \in \mathbf{R}^n\},$$

and since $\mu_{n+1}(G) = 0$, it follows that $\mu_{n+1}(A_f^* \backslash A_{f^*}) = 0$. In general, $\int p = \mu_{n+1}(A_p)$. Therefore

$$\begin{aligned} \int |f^* - f| \, dx &= \mu_{n+1}(A_{f^*} \varDelta A_f) \\ &= \mu_{n+1}(A_f^* \varDelta A_f) = 0. \end{aligned} \tag{A.8}$$

As a consequence of (A.8),

$$H_\alpha(f, g) = H_\alpha(f^*, g^*). \tag{A.9}$$

Now consider the function

$$K_\alpha(x \mid f, g) = \sup_{y\in\mathbf{R}^n}\{f(x-y)^\alpha \oplus g(y)^\alpha\}^{1/\alpha}. \tag{A.10}$$

Note that generally $K_\alpha(x) \geqslant H_\alpha(x)$. Let

$$D(z) = \{x \in \mathbf{R}^n \mid K_\alpha(x \mid f^*, g^*) > z\}, \qquad z \geqslant 0. \tag{A.11}$$

Choose $z \geqslant 0$, $x \in D(z)$. By definitions (A.10) and (A.11), there is a $y \in \mathbf{R}^n$, and numbers $b, c > 0$ such that

$$z \leqslant (b^\alpha + c^\alpha)^{1/\alpha},$$
$$f^*(x - y) > b, g^*(y) > c.$$

In other words

$$\beta = (x - y, b) \in A_{f^*}, \qquad \gamma = (y, c) \in A_{g^*}.$$

Then for all $\delta > 0$ there exist balls $V(\epsilon, \beta)$ and $V(\epsilon, \gamma)$ in $\mathbf{R}^{n+1}$ such that, in the notation of (A.3),

$$\mu_{n+1}(A_{f^*} \cap V(\epsilon, \beta)) \geqslant (1 - \delta)\, W_{n+1}(\epsilon),$$
$$\mu_{n+1}(A_{g^*} \cap V(\epsilon, \gamma)) \geqslant (1 - \delta)\, W_{n+1}(\epsilon).$$

If δ is small enough, it follows that the sets

$$\{v \in V(\epsilon, x - y) \mid f^*(v) > b\},$$
$$\{w \in V(\epsilon, y) \mid g^*(w) > c\}$$

have measure at least equal to $\frac{3}{4} W_n(\epsilon)$. This implies (1) that $H_\alpha(x \mid f^*, g^*) > z$, so that in fact

$$H_\alpha(f^*, g^*) = K_\alpha(f^*, g^*), \tag{A.12}$$

and (2) that $D(z)$ contains a neighborhood of x, such that $D(z)$ is open. Hence $K_\alpha(f^*, g^*)$ is lower semicontinuous. By Eqs. (A.9, A.12), so is $H_\alpha(f, g)$. Q.E.D.

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