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Starshapedness of Level Sets for Solutions of Nonlinear Elliptic Equations

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Abstract. We introduce a measure for the starshapedness of the level sets of solutions of certain nonlinear elliptic equations in a starshaped ring Ω of $\mathbb{R}^n$. We prove that a function which characterizes the starshapedness does not attain its minimum in Ω.

1. Introduction

Let T be a domain in $\mathbb{R}^n$ with C^1 boundary. We can measure the starshapedness of T with respect to the origin by considering for each point $x \in \partial T$ the angle $w(x)$ between the outer normal to ∂T at x and the radial direction x. T is starshaped with respect to the origin if $w(x) \leq \pi/2$ for every $x \in \partial T$ and we say that T is properly starshaped with respect to the origin if $w(x) < \pi/2$ for every $x \in \partial T$. For brevity we say that a set T is starshaped if ∂T is C^1 and T is starshaped with respect to the origin.

If T is a level set of a function u, $T = \{x \in \Omega : u(x) \geq c\}$, the normal direction to ∂T at x coincides with the direction of $Du(x)$.

At a maximum point of w the normal direction is as far as possible from the radial direction. We say that at such a point we have a minimum for the starshapedness.

We call a domain $\Omega \subset \mathbb{R}^n$ a starshaped ring if $\Omega = \Omega_0 \setminus \overline{\Omega}_1$, where Ω_0 and Ω_1 are open starshaped domains, $\overline{\Omega}_1 \subset \Omega_0$.

Consider a rotationally invariant and strictly elliptic differential equation

$$(1.1) \qquad G\left(r, u, |Du|^2, \sum_{i,j=1}^{n} u_i u_j u_{ij}, tr\,(D^2u), \ldots, tr\,\left((D^2u)^n\right)\right) = 0,$$

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where $Du = (u_1, \dots, u_n)$ and $D^2u = (u_{ij})_{i,j=1,\dots,n}$ are the gradient and the hessian matrix of a function u, $r = |x|$ and $G \in C^1(D)$, with $D = \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^{n+1}$.

We study the properties of starshapedness of level sets of solutions of (1.1) in a starshaped ring Ω with constant boundary values. We prove (with suitable assumptions on G), that the level sets of u are properly starshaped and the minimum starshapedness is achieved on $\partial\Omega$. To prove this, we apply the maximum principle to the angle $w(x)$ as is in [3] for the angle between $Du(x)$ and a fixed direction.

The same result was proved in [2] for solutions of the nonlinear Poisson equation $\Delta u = f(r, u)$, while in [1] the starshapedness of level sets was proved for solutions of the degenerate equation $\Delta_p u = f(u)$.

The applicability of our result is exhibited in several remarks at the end of the paper.

2. Minimum principle for starshapedness

Let us remark that

$$\frac{\partial G}{\partial u_{ij}}[u] = \frac{\partial G}{\partial q}[u]\, u_i u_j + \sum_{k=1}^{n} \frac{\partial G}{\partial t_k}[u] \frac{\partial t_k}{\partial u_{ij}}, \tag{2.1}$$

where, as in the following, $d = |Du|^2$, $q = \sum_{i,j=1}^n u_i u_j u_{ij}$, $t_k = tr\left(\left(D^2u\right)^k\right)$ and $[u] = (r, u, d, q, t_1, \dots, t_n)$.

We say that G is strictly elliptic in $w \in C^2(\Omega)$ if there exists a positive constant μ such that

$$\sum_{i,j=1}^{n} \frac{\partial G}{\partial u_{ij}}[w]\, \lambda_i \lambda_j \ \geq\ \mu\, |\lambda|^2 \tag{2.2}$$

for every $x \in \Omega$ and $\lambda \in \mathbb{R}^n$.

Theorem 2.1. *Let $\Omega = \Omega_0 \setminus \overline{\Omega}_1$ be a starshaped ring and $G \in C^1(D)$. Let $u \in C^3(\Omega) \cap C^1\left(\overline{\Omega}\right) \cap C^0(\Omega_0)$ be a solution of*

$$\begin{cases} G[u] = 0 & in \quad \Omega\,, \\ u = 1 & in \quad \Omega_1\,, \\ u = 0 & on \quad \partial\Omega_0\,, \end{cases}$$

such that G is strictly elliptic at u,

$$\frac{\partial G}{\partial u}[u] \leq 0 \quad for \ every \quad x \in \Omega \tag{2.3}$$

and

$$0 < u < 1 \quad in \quad \Omega\,. \tag{2.4}$$

Suppose that

$$(2.5)\quad 2\sum_{h=1}^{n} h\,\frac{\partial G}{\partial t_h}\,[u]\,t_h - r\,\frac{\partial G}{\partial r}\,[u] + 2\,|Du|^2\,\frac{\partial G}{\partial d}\,[u] + 4\sum_{i,j=1}^{n} u_i u_j u_{ij}\,\frac{\partial G}{\partial q}\,[u] \geq 0 \ \ in\ \ \Omega\,,$$

then the level sets of u are properly starshaped and $Du \neq 0$ in Ω. Moreover, unless Ω_0 and Ω_1 are concentric balls, the angle $w(x)$ between $Du(x)$ and the radial direction does not assume its maximum value in Ω (that is, the starshapedness does not assume its minimum value in Ω).

Proof. Let us first show that the level sets of u are starshaped. We introduce a function v defined by

$$v(x) \;=\; \langle x, Du(x)\rangle \;=\; \sum_{i=1}^{n} x_i u_i\,.$$

On $\partial\Omega_0$, where u attains its mimimum value, $Du(x)$ has the direction of the inner normal to $\partial\Omega_0$ at x and $v(x) \leq 0$; in the same way we can conclude that $v(x) \leq 0$ on $\partial\Omega_1$.

Let us show that v satisfies a linear elliptic equation.

Differentiating (1.1) with respect to x_k we obtain

$$(2.6)\qquad \begin{aligned} \sum_{i,j=1}^{n} \frac{\partial G}{\partial u_{ij}}\,[u]\,u_{ijk} \;=\; & -\frac{x_k}{r}\,\frac{\partial G}{\partial r}\,[u] - u_k\,\frac{\partial G}{\partial u}\,[u] \\ & -\; 2\,\frac{\partial G}{\partial d}\,[u]\sum_{i=1}^{n} u_i u_{ik} - 2\,\frac{\partial G}{\partial q}\,[u]\sum_{i,j=1}^{n} u_i u_{ij} u_{jk}\,, \end{aligned}$$

for every $x \in \Omega$.

Let L be the linear elliptic operator defined by

$$Lw \;=\; \sum_{i,j=1}^{n} \frac{\partial G}{\partial u_{ij}}\,[u]\,w_{ij}\,.$$

We can calculate

$$(2.7)\qquad Lv \;=\; 2\sum_{i,j=1}^{n} \frac{\partial G}{\partial u_{ij}}\,[u]\,u_{ij} + \sum_{k=1}^{n} x_k \left(\sum_{i,j=1}^{n} \frac{\partial G}{\partial u_{ij}}\,[u]\,u_{ijk} \right).$$

From (2.1), (2.6) and (2.7), and observing that

$$\sum_{i,j=1}^{n} \frac{\partial t_h}{\partial u_{ij}}\,u_{ij} \;=\; h \cdot t_h$$

and

$$\sum_{k=1}^{n} x_k u_{ki} \;=\; v_i - u_i\,,$$

we finally get

$$(2.8)\quad \begin{aligned} &Lv + 2\sum_{i=1}^{n}\left(u_i\,\frac{\partial G}{\partial d}[u] + \sum_{j=1}^{n} u_j u_{ij}\,\frac{\partial G}{\partial q}[u]\right) v_i + v\,\frac{\partial G}{\partial u}[u] \\ &= 2\sum_{h=1}^{n} h\,\frac{\partial G}{\partial t_h}[u]\,t_h - r\,\frac{\partial G}{\partial r}[u] + 2\,|Du|^2\,\frac{\partial G}{\partial d}[u] + 4\sum_{i,j=1}^{n} u_i u_j u_{ij}\,\frac{\partial G}{\partial q}[u]\,. \end{aligned}$$

By the assumptions (2.3) and (2.5) and the maximum principle, from $v \le 0$ on $\partial\Omega$ we conclude that $v < 0$ in Ω. Note that by the strong maximum principle, $Du \ne 0$ in Ω unless $v \equiv 0$ in $\overline{\Omega}$, but this is not possible since constant functions cannot be solutions of the Dirichlet problem. Since $Du \ne 0$ in Ω, the angle w is well defined in Ω and $w(x) < \pi/2$, so the level sets of u are properly starshaped.

Let us prove that w achieves its maximum value on $\partial\Omega$. Maximum points of w are maximum points of $\Phi(x) = \tan w(x)$, which can be written as

$$\Phi = -\frac{h}{v}\,,$$

where

$$h = \left[\frac{1}{2}\sum_{k,l=1}^{n}(x_k u_l - x_l u_k)^2\right]^{1/2}.$$

We have proved that Φ is positive and differentiable in $\Omega' = \Omega \setminus \{x \in \Omega : h(x) = 0\}$. In Ω' we have

$$\Phi_{ij} = -\frac{h_{ij}}{v} - \frac{\Phi}{v}v_{ij} + \frac{1}{v}(v_j\Phi_i + v_i\Phi_j)$$

and

$$(2.9)\quad L\Phi + \frac{2}{v}\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\frac{\partial G}{\partial u_{ij}}[u]\,v_i\right)\Phi_j = -\frac{1}{v}\,Lh - \frac{\Phi}{v}\,Lv\,.$$

Since $\left(\frac{\partial G}{\partial u_{ij}}[u]\right)_{i,j=1,\dots,n}$ is positively defined, we can apply the Schwarz inequality and obtain

$$(2.10)\quad \begin{aligned} 2hLh &\ge \sum_{i,j=1}^{n}\frac{\partial G}{\partial u_{ij}}[u]\sum_{k,l=1}^{n}(x_k u_l - x_l u_k)\,\frac{\partial^2}{\partial x_i\partial x_j}\,(x_k u_l - x_l u_k) \\ &= 4\sum_{i,j,l=1}^{n}\frac{\partial G}{\partial u_{ij}}[u]\,u_{lj}(x_i u_l - x_l u_i) \\ &\quad + \sum_{k,l=1}^{n}(x_k u_l - x_l u_k)\left(x_k\sum_{i,j=1}^{n}\frac{\partial G}{\partial u_{ij}}[u]\,u_{ijl} - x_l\sum_{i,j=1}^{n}\frac{\partial G}{\partial u_{ij}}[u]\,u_{ijk}\right). \end{aligned}$$

Observe that

$$\sum_{j=1}^{n}\frac{\partial G}{\partial u_{ij}}[u]\,u_{lj} = \frac{\partial G}{\partial q}[u]\sum_{j=1}^{n}u_i u_j u_{lj} + \sum_{h,j=1}^{n}\frac{\partial G}{\partial t_h}[u]\,\frac{\partial t_h}{\partial u_{ij}}\,u_{lj}\,.$$

Since

$$\left(\sum_{j=1}^{n} \frac{\partial t_h}{\partial u_{ij}}\, u_{lj}\right)_{i,l=1,\dots,n}$$

is a symmetric matrix, while $(x_i u_l - x_l u_i)_{i,l=1,\dots,n}$ is antisymmetric,

$$\sum_{i,l=1}^{n}\left(\sum_{j=1}^{n} \frac{\partial t_h}{\partial u_{ij}}\, u_{lj}\right)(x_i u_l - x_l u_i) \;=\; 0\,,$$

hence

$$(2.11) \qquad 4\sum_{i,j,l=1}^{n} \frac{\partial G}{\partial u_{ij}}\,[u]\, u_{lj}(x_i u_l - x_l u_i) \;=\; 4\,\frac{\partial G}{\partial q}\,[u] \sum_{i,j,l=1}^{n} u_i u_j u_{jl}(x_i u_l - x_l u_i)\,.$$

From (2.10), (2.11) and (2.6) and after some calculations we conclude that

$$(2.12) \quad 2hLh \;\geq\; -2h^2\,\frac{\partial G}{\partial u}\,[u] - 4h\,\frac{\partial G}{\partial d}\,[u]\sum_{j=1}^{n} u_j h_j - 4h\,\frac{\partial G}{\partial q}\,[u]\sum_{i,j=1}^{n} u_i u_{ij} h_j\,.$$

Using (2.8), (2.9) and (2.12) we can see that Φ satisfies the inequality

$$L\Phi + 2\sum_{i=1}^{n} c_i \Phi_i + \frac{g}{v}\Phi \;\geq\; 0\,,$$

where

$$c_i \;=\; \frac{1}{v}\sum_{j=1}^{n} \frac{\partial G}{\partial u_{ij}}\,[u]\, v_j + \frac{\partial G}{\partial d}\,[u]\, u_i + \frac{\partial G}{\partial q}\,[u]\sum_{j=1}^{n} u_j u_{ij}$$

and

$$g \;=\; 2\sum_{h=1}^{n} h\,\frac{\partial G}{\partial t_h}\,[u]\, t_h - r\,\frac{\partial G}{\partial r}\,[u] + 2\,|Du|^2\,\frac{\partial G}{\partial d}\,[u] + 4\sum_{i,j=1}^{n} u_i u_j u_{ij}\,\frac{\partial G}{\partial q}\,[u]\,.$$

By the assumption (2.5) and since $v < 0$ in Ω, we have $\frac{g}{v} \leq 0$. By the maximum principle, w does not assume its positive maximum in Ω unless it is constant. Let us remark that the only admissible constant for w is zero (remember that level surfaces are closed surfaces since level sets are starshaped). In this the case the level sets of u are balls with centre at the origin and this is possible only if Ω_0 and Ω_1 are balls with centre at the origin. □

3. Remarks

1. Notice that (2.4) is assured if we suppose that G is strictly elliptic in tu for $t \in [0,1]$, $\frac{\partial G}{\partial u}\,[tu] \geq 0$ in Ω and $G[0] = 0$. Indeed applying the mean value theorem to

$$G\,[u] - G\,[0] \;=\; 0$$

we can see that u is a solution of a linear elliptic equation, hence, by the maximum principle, 2.4 follows.

2. Let us show some example of equations to which Theorem 2.1 can be applied.

2a. Consider a solution of

$$\begin{cases} \Delta_p u = f(r,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega_0\,, \\ u = 1 & \text{in } \Omega_1\,, \end{cases}$$

where $\Delta_p u = \operatorname{div}\left(|Du|^{p-2} Du\right)$, $p \geq 2$ is the $p-$Laplace operator.

Suppose that $|Du| > 0$ (if $p > 2$) and $\frac{\partial f}{\partial u} \geq 0$ in Ω. In this case, the assumption (2.5) becomes

$$pf + r\frac{\partial f}{\partial r} \geq 0 \quad \text{in} \quad \Omega\,. \tag{3.1}$$

Observe that for the Poisson equation $\Delta u = f(r,u)$ the assumption (3.1) is the same as in [2].

2b. For solutions of

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = f(r,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega_0\,, \\ u = 1 & \text{in } \Omega_1\,, \end{cases}$$

Theorem 2.1 holds if

$$\begin{aligned} &\frac{\partial f}{\partial u}(r,u) \geq 0 \quad \text{in} \quad \Omega\,, \\ &0 < u < 1 \quad \text{in} \quad \Omega\,, \end{aligned}$$

and

$$2\left(1+|Du|^2\right)^{-5/2} \sum_{i,j=1}^{n} u_i u_j u_{ij} - \frac{2+|Du|^2}{1+|Du|^2}\, f - r\,\frac{\partial f}{\partial r} \leq 0 \quad \text{in} \quad \Omega\,.$$

For the minimal surface equation ($f \equiv 0$) this is true if

$$\sum_{i,j=1}^{n} u_i u_j u_{ij} \leq 0\,.$$

In particular the above relation holds when u is concave in the direction of Du.

2c. If we indicate as $\lambda_1(D^2u) \leq \lambda_2(D^2u) \leq \cdots \leq \lambda_n(D^2u)$ the eigenvalues of the symmetric matrix D^2u, rotationally invariant operators of the form

$$\sum_{i=1}^{n} a_i \lambda_i^k (D^2 u) = f\left(r, u, |Du|^2\right)$$

are elliptic when the a_i's are positive constants and either $k = 1$ or k is even and D^2u is not singular. For such operators the assumption (2.5) becomes

$$2kf + r\,\frac{\partial f}{\partial r} - 2\,|Du|^2\,\frac{\partial f}{\partial d} \geq 0\,,$$

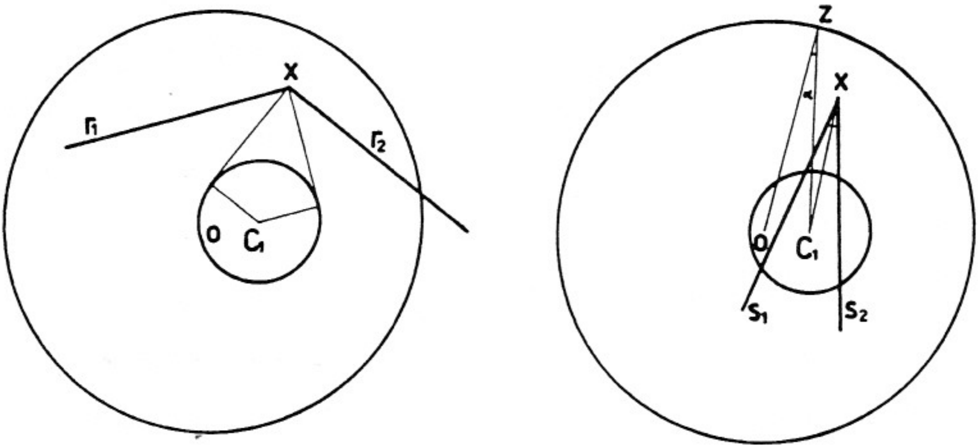

Figure 1: A and B

and this holds when $f(r,u,d) = e^u + r^2 d$, for example. For $k = 1$ this case includes extremal operators (see [4]).

3. The assumption that G is rotationally invariant cannot be removed. Suppose that Ω_0 and Ω_1 are balls with centre at the origin. Consider the following problem:

$$\begin{cases} a\,\dfrac{\partial^2 u}{\partial x^2} + b\,\dfrac{\partial^2 u}{\partial y^2} \; = \; 0 & \text{in} \quad \Omega_0\,, \\ u \; = \; 1 & \text{in} \quad \Omega_1\,, \\ u \; = \; 0 & \text{on} \quad \partial\Omega_0\,, \end{cases}$$

where a and b are positive constants with $a \neq b$. The smoothness of the data implies the existence of a solution. Since $\Phi = 0$ on $\partial\Omega$, our theorem would give $\Phi \equiv 0$ in Ω and the solution would be radial. However, it can easily be seen that no radial solution exists for $a \neq b$.

4. If $\frac{\partial G}{\partial r} \equiv 0$ we can consider the starshapedness of level sets with respect to any other point of Ω_1. This gives more information about the shape of the level sets of u. If Ω_0 and Ω_1 are starshaped with respect to each point of a set $K \subset \Omega_1$, the level sets of u will still be starshaped with respect to each point of K. In this case we can consider for each $y \in K$ the function $w_y(x)$ which represents the angle between $Du(x)$ and the direction $x - y$. Let $M(y)$ be the maximum of w_y on $\partial\Omega$ and let $C(y)$ be the cone with axis parallel to $x - y$ and angle $M(y)$ with this axis. For a fixed point $x \in \Omega$,

$$Du(x) \in \bigcap_{y \in K} C(y)\,.$$

Observe that the maximum principle for w gives sharper information than starshapedness with respect to K. For example, let Ω_1 and Ω_0 be nonconcentric balls in $\mathbb{R}^2$. The level sets of u are starshaped with respect to each point of Ω_1, so $Du(x)$ belongs to the intersection of half-planes orthogonal to the directions $y - x$ for every $y \in \Omega_1$, a cone E_1 bounded by half-lines r_1 and r_2 (see Fig. 1A). On the other hand, if we consider the minimum principle for starshapedness with respect to the centre C_1 of Ω_1, we see that the angle between $Du(x)$ and the radial direction assumes its maximum α at some point z on $\partial\Omega_0$. Hence $Du(x)$ belongs to the cone E_2 bounded by the half-lines s_1 and s_2 indicated in Fig. 1B, which is strictly contained in E_1.

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