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WHEN ARE SOLUTIONS TO NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS CONVEX?

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ABSTRACT

We extend and apply a concavity maximum principle from [10, 9, 7] to some nonlinear elliptic boundary problems and free boundary problems on convex domains $\Omega \subset \mathbb{R}^n$. In particular we extend "convex dead core" results from $n=2$ as in [4] to arbitrary n. We also show the convexity of the coincidence set in the obstacle problem under suitable assumptions.

INTRODUCTION

A few years ago N. Korevaar [10] found a nice way of proving that if u solves

(1) $\Delta u + \lambda u = 0 , \quad u > 0 \quad \text{in } \Omega, \ \lambda > 0 ,$

(2) $u = 0 \quad \text{on } \partial\Omega ,$

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with $\Omega \subset \mathbb{R}^n$ convex, then $v = -\log u$ is a convex function. His approach was extended by A. Kennington [9] and the author [7] to show that if u solves

$$(3) \qquad \Delta u + \lambda u^p = 0 \;, \quad u > 0 \qquad \text{in } \Omega,\ \lambda > 0\ ,\ 0 \le p < 1\ ,$$

$$(2) \qquad u = 0 \qquad \text{on } \partial\Omega\ ,$$

with $\Omega \subset \mathbb{R}^n$ convex, then $v = u^{(p-1)/2}$ is convex. In both cases the result was derived via a maximum principle for the concavity function C of v, which is defined by

$$(4) \qquad C(x_1,x_2) := v\left(\frac{x_1+x_2}{2}\right) - \frac{1}{2}v(x_1) - \frac{1}{2}v(x_2) \quad \text{in } \Omega\times\Omega\ ,$$

and using the appropriate differential equation for v. The motivation for this paper was to find fairly general assumptions on f and suitable functions $v = g(u)$ such that if u solves

$$(5) \qquad \Delta u + f(u) = 0\ , \qquad u > 0 \qquad \text{in } \Omega\ ,$$

$$(2) \qquad u = 0 \qquad \text{on } \partial\Omega\ ,$$

with $\Omega \subset \mathbb{R}^n$ convex, then $g(u(x))$ is convex in Ω. An answer to this problem will be stated in Corollary 3 below. If one substitutes $v = g(u)$ one derives from (5)

$$(6) \qquad \Delta v = -g'f + \frac{g''}{(g')^2}|\nabla v|^2 =: k(v,\nabla v)\ ,$$

Moreover in the cases (1) and (3) the function k satisfies

$$k(v,\nabla v) = h(v) \quad (K + |\nabla v|^2) \ , \tag{7}$$

where K is a suitable positive constant. For general f as in (5) it is therefore desirable that $-g'f$ and $g''/(g')^2$ are positive multiples of each other:

$$-g'f = \frac{g''}{(g')^2} K \ , \qquad K > 0 \ , \tag{8}$$

in which case (6) becomes

$$\Delta v = h(v) \quad (K + |\nabla v|^2) \quad , \qquad K > 0 \ . \tag{9}$$

L. Caffarelli and A. Friedman observed in [1], that the differential equation (8) for g' can be solved. Then they showed by a continuation method which is unrelated to our approach that $v = g(u)$ is even strictly convex. Their result was however restricted to dimension two, while our paper contains no such restriction.

In the first part of the paper we recall and slightly extend the main result of [10,9,7] on the convexity of solutions to equation (9). We want to point out that our extension of Kennington's theorem is small but crucial for our new applications.

In the second part we verify the assumptions of the main theorem for various examples which include problems of type (5) (2) and

$$\Delta u = f(u) \ , \qquad u < M \quad \text{in} \quad \Omega \ , \tag{10}$$

$$u = M \qquad\qquad \text{on} \quad \partial\Omega \ . \tag{11}$$

Problem (10) (11) was studied recently by A. Friedman and D. Phillips [4]. They derived the existence and convexity of a "dead core" $\{x \in \Omega \mid u(x) = 0\}$ for convex $\Omega \subset \mathbb{R}^2$ under suitable assumptions on f by a continuation method. We reduce problem (11) (10) to (5) (2) by substituting $w = M - u$ and derive the analogous result for arbitrary dimensions by maximum principles only.

Let us remark in passing that an extension of our results to parabolic equations is not a trivial exercise. It is however easily possible to weaken N. Korevaar's monotonicity assumption $\frac{\partial b}{\partial u} \geq 0$ [10, Thm. 1.6] in the parabolic case to $\frac{\partial b}{\partial u} \geq -M$, with M a nonnegative constant.

I. THE CONCAVITY MAXIMUM PRINCIPLE

A continuous function v defined on a convex set $\Omega \subset \mathbb{R}^n$ is obviously convex if and only if its concavity function C defined by (4) is nonpositive in $\Omega \times \Omega$. In order to formulate the following theorem we have to define harmonic concavity. We call a function h defined on an interval $I \subset \mathbb{R}$ <u>harmonic concave</u> if and only if

$$(12) \qquad h(y_1)h(y_2) \leq h\left(\frac{y_1 + y_2}{2}\right) \cdot \frac{1}{2}(h(y_1) + h(y_2)) \quad \text{for every } y_1, y_2 \in I \ .$$

If h happens to be positive in I, then h is harmonic concave if and only if $1/h$ is convex. In the following theorem $R(v)$ denotes the closure in $\mathbb{R}$ of the range of v in Ω and $\tilde{R}(v)$ denotes $R(v)$ minus its left end point.

Theorem 1

Let $\Omega \subset \mathbb{R}^n$ be convex

(13) $\Delta v = k(v, \nabla v)$

Suppose that a) b) and c

a) $C(x_1, x_2)$ cannot becom

b) $k \geq 0$ on $R(v) \times \mathbb{R}^n$ an

c) For every $\xi \in \mathbb{R}^n$ the

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$J_\xi := \{t \in R(v) \mid k(t,\xi)$

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II. APPLICATIONS

A) Let $u : \bar{\Omega} \to \mathbb{R}$ be a class

(5) $\Delta u + f(u) = 0$,

(2) $u = 0$

Theorem 1

Let $\Omega \subset \mathbb{R}^n$ *be convex and let* $v \in C^2(\Omega)$ *be a solution of*

$$\Delta v = k(v, \nabla v) \quad \text{in} \quad \Omega . \tag{13}$$

Suppose that a) b) *and* c) *hold:*

a) $C(x_1, x_2)$ *cannot become positive as* (x_1, x_2) *approaches* $\partial(\Omega \times \Omega)$.

b) $k \geq 0$ *on* $R(v) \times \mathbb{R}^n$ *and* $k > 0$ *in* $\tilde{R}(v) \times \mathbb{R}^n$.

c) *For every* $\xi \in \mathbb{R}^n$ *the mapping defined by* $t \to k(t, \xi)$ *is positive, monotone increasing and harmonic concave in*

$J_\xi := \{t \in R(v) \mid k(t, \xi) > 0\}$.

Then v *has to be convex in* Ω .

A. Kennington gave a proof of this theorem [9] under the slightly stronger assumption that k is monotone increasing and harmonic concave on R(v), but one sees from the proof in [7, Appendix] that R(v) can be replaced by J_ξ. This will be crucial for the applications in the second part of our paper. Let us also point out that assumption a) is global in nature in the sense that $C(x_1, x_2)$ should be nonpositive for every x_1 approaching $\partial\Omega$ and every fixed $x_2 \in \Omega$

II. APPLICATIONS

A) Let $u : \bar{\Omega} \to \mathbb{R}$ be a classical solution of

$$\Delta u + f(u) = 0 , \qquad u > 0 \qquad \text{in} \quad \Omega , \tag{5}$$

$$u = 0 \qquad \text{on} \quad \partial\Omega , \tag{2}$$

where $\Omega \in \mathbb{R}^n$ is a strictly convex, bounded domain which satisfies the interior sphere condition. Recall that Ω is strictly convex if and only if $(x_1 + x_2)/2 \in \Omega$ for every $x_1 \in \bar{\Omega}$, $x_2 \in \bar{\Omega}$ with $x_1 \neq x_2$. Furthermore let $F(t) := \int_0^t f(s)ds$ and let $\overset{o}{R}(u)$ be the interior of the range $R(u)$.

Corollary 2

Let $f: R(u) \to [0,\infty)$ *be in* $C^0(R(u)) \cap C^2(\overset{o}{R}(u))$ *and suppose that* f *satisfies the conditions* i) ii) *and* iii):

i) $f(t) > 0$ for $t \in \overset{o}{R}(u)$.

ii) $\dfrac{f^2(t)}{2F(t)} - f'(t) > 0$ for $t \in \overset{o}{R}(u)$.

iii) $2(f'(t))^2 - f(t)f''(t) - \dfrac{(f(t))^2 f'(t)}{F(t)} \geq 0$ for $t \in \overset{o}{R}(u)$.

Then the function

$$g(u(x)) := -\int^{u(x)} [F(s)]^{-1/2} ds$$

is convex in Ω *and consequently the level sets* $\Omega_c := \{x \in \Omega \mid u(x) \geq c\}$ *of* u *are convex.*

The proof of this corollary follows from Theorem 1. In fact v satisfies

$$(14) \quad \Delta v = h(v)\ (2 + |\nabla v|^2),$$

where h(v) is implicitly defined by the relation

(15) $h(v) = \frac{1}{2} F(u)^{-1/2} f$

Condition c) of Theorem 1

rule from noting that

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Condition b) of Theorem 1

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Lemma 3

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$g'(u) < 0$ *for* $u > 0$

Finally suppose that C *canno*

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Then

$\limsup_{(x_1,x_2) \to \partial(\Omega \times \Omega)}$ C

(15) $\quad h(v) = \frac{1}{2} F(u)^{-1/2} f(u)$.

Condition c) of Theorem 1 can easily be established by the chain rule from noting that

$$k > 0 \iff \text{i)} \ ,$$
$$\frac{\partial k}{\partial v} > 0 \iff \text{ii)} \ ,$$
$$\frac{\partial^2}{\partial v^2}\left(\frac{1}{k}\right) \geq 0 \iff \text{iii)} \ .$$

Condition b) of Theorem 1 holds by assumptions. Under conditions b) and c) the concavity function C cannot attain a positive local maximum in $\Omega \times \Omega$ and it remains to verify condition a). But this is a consequence of the following "boundary point lemma":

Lemma 3

Let u *be a solution of* (5) (2), *let* $\Omega \subset \mathbb{R}^n$ *be bounded and strictly convex with boundary of class* C^1. *Suppose that* $\frac{\partial u}{\partial n} < 0$ *on* $\partial\Omega$, *where* n *denotes the exterior normal to* $\partial\Omega$, *and suppose that* g(u) *satisfies*

$$g'(u) < 0 \ \text{for} \ u > 0 \quad \text{and} \quad \lim_{u \to 0^+} g'(u) = -\infty \ .$$

Finally suppose that C *cannot attain a positive local maximum in* $\Omega \times \Omega$.

Then

$$\limsup_{(x_1,x_2) \to \partial(\Omega \times \Omega)} C(x_1,x_2) \leq 0 \ .$$

Under additional assumptions on g and on $\partial\Omega$ this Lemma is due to N. Korevaar [10]. A proof for the version presented here can be found in [8, p. 132-134].

Remark 4

A slightly weaker version of ii) is $f^2 - 2Ff' \geq 0$, which is equivalent to the condition that $F^{1/2}(t)$ is concave in t. If $f' > 0$ and $f'' \geq 0$, then iii) implies $f' - f^2/2F \geq 0$ and contradicts ii).

Remark 5

If (in addition to i)) we have $f(t) > 0$ on $R(u)$, then it follows from the work [11] of N. Korevaar and J.L. Lewis that the rank of the Hessian matrix of $g(u(x))$ is constant in Ω.

Remark 6

Notice that $g(u(x))$ may or may not be finite on $\partial\Omega$. The proof of Lemma 3 takes care of both cases.

Remark 7

Using the continuous dependence of u on the domain Ω one can sometimes weaken the strict convexity assumption on Ω to mere convexity.

Remark 8

Known examples of nonlinear problems (5) (2) which have solutions with convex level sets are provided by $f(u) = \lambda u^p$, $0 \leq p \leq 1$, $\lambda > 0$. A new example seems to be

$$(16) \qquad f(u) = -\lambda u + 1 \;, \qquad \lambda \geq 0 \;,$$

for which $\arcsin(1 - \lambda u)$ is a convex function.

Remark 9 on open problems

If Ω is convex and u i

(17) $f(u) = \lambda u^p$

(18) $f(u) = \lambda(u-1)^+$

(19) $f(u) = 1 - \lambda u^p$

(20) $f(u) = \lambda u + 1$

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B) The problem

(10) $\Delta u = f(u)$, u

(11) $u = M$,

Remark 9 on open problems.

If Ω is convex and u is a solution of problem (5) (2) with

(17) $f(u) = \lambda u^p$, with $\lambda > 0$, $1 < p < \frac{n+2}{n-2}$, or

(18) $f(u) = \lambda(u-1)^+$, with $\lambda > 0$, or

(19) $f(u) = 1 - \lambda u^p$, with $\lambda > 0$, $p > 0$, or

(20) $f(u) = \lambda u + 1$, with $\lambda > 0$,

then it is believed that the level sets Ω_c of u are convex, i.e. that u is quasi concave. Problem (17) was pointed out to me by P. Sacks. Problem (18) was attacked in [2] by a method due to A. Acker and it was shown that one out of possibly many solutions is quasiconcave. Problem (19) is a generalization of (16) and supported by numerical experiments of R. Rannacher. Problem (20) was posed by J. Hersch [5]. Its solution is relevant for the proof of a conjectured isoperimetric inequality. Furthermore the following conjecture can be reduced to (20). Suppose that Ω is not a convex domain and that v is the solution of (1) (2). Locally near a maximum of v the level sets of v have components that are close to ellipsoids. Suppose there is a positive number c_o such that the set $\Omega_{c_o} := \{x \in \Omega \mid v(x) \geq c_o\}$ has a strictly convex component, say D. Then for $c > c_o$ the sets $D_c := \{x \in D \mid v(x) > c\}$ should all be convex. If one substitutes $u = (v - c_o)/\lambda c_o$, then u is a solution to Problem (20).

B) The problem

(10) $\Delta u = f(u)$, $u < M$ in Ω ,

(11) $u = M$, on $\partial\Omega$,

can be reduced to the previous one (5) (2) by the substitution $W = M - u$. For the reader's convenience let us give the analogue of Corollary 2.

Corollary 10

Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution to (10) (11). Let $\Omega \subset \mathbb{R}^n$ be a bounded strictly convex domain which satisfies the interior sphere condition. Suppose that $f \in C(R(u)) \cap C^2(\overset{o}{R}(u))$ satisfies i) ii) and iii):

i) $\quad f(t) > 0 \qquad$ for $t \in \overset{o}{R}(u)$,

ii) $\quad f'(t) + \dfrac{f^2(t)}{2F(M) - 2F(t)} > 0 \qquad$ for $t \in \overset{o}{R}(u)$,

iii) $\quad 2(f'(t))^2 - f(t)f''(t) + \dfrac{(f(t))^2 f'(t)}{F(M) - F(t)} \geq 0 \quad$ for $t \in \overset{o}{R}(u)$.

Then the function

$$g(u(x)) := \int^{u(x)} \frac{ds}{\sqrt{2F(M) - 2F(s)}}$$

is convex in Ω and consequently the level sets $\Omega^c : \{x \in \Omega \mid u(x) \leq c\}$ of u are convex.

Remark 11

It is sufficient for the validity of i) ii) and iii) thet f is positive, monotone increasing and harmonic concave.

Remark 12

The previous result [1] differs from ours in the following points. There is a restriction to $n = 2$, instead of ii) one requires

$$f'(t) + \frac{f(t)}{M - t} > 0$$

and then $g(u(x))$ is strictly

Main results

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B2) Let $u \in C^{1,1}(\Omega)$ be a solu

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$$-\Delta u \geq 0$$

$$u - \psi \geq 0$$

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$$f'(t) + \frac{f(t)}{M-t} > 0 \qquad \text{for} \quad t \in \overset{\circ}{R}(u)$$

and then $g(u(x))$ is strictly convex in the support of u.

<u>Main results</u>

B1) In problem (10) (11) let $M = 1$ and $f(u) = \lambda(u^+)^p$, $0 < p < 1$, $0 < \lambda$. This problem was studied in [4]. It is known that for sufficiently large λ there exists a dead core $D = \{x \in \Omega \mid u(x) = 0\}$ with positive measure and that for $n = 2$ the dead core is convex. We can now show that <u>the dead core is convex for arbitrary</u> $n \geq 2$, since this follows from Corollary 10.

B2) Let $u \in C^{1,1}(\Omega)$ be a solution to the obstacle problem

$$(u-\psi)\Delta u = 0 \quad \text{a.e.} \quad \text{in } \Omega,$$

$$-\Delta u \geq 0 \quad \text{a.e.} \quad \text{in } \Omega,$$

$$u - \psi \geq 0 \quad \text{a.e.} \quad \text{in } \Omega,$$

and

$$u - \psi = 1 \qquad \text{on } \partial\Omega .$$

For strictly convex Ω and if $\Delta\psi = -C$ for some positive constant C we can conclude that <u>the coincidence set</u> $\{x \in \Omega \mid u(x) = \psi(x)\}$ <u>is convex</u>. This result was previously known for $n = 2$ [4]. For the proof one introduces $U := u - \psi$ as a new function and notes that this reduces the problem to B1) with $f(U) = C \cdot$ Heaviside function of U as a limiting case of $(U^+)^p$, $p \to 0^+$. Finally one has to realize that quasiconvexity, i.e. the property that the level sets Ω^c are convex, is preserved under pointwise convergence.

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