Convex Solutions to Nonlinear Elliptic and Parabolic Boundary Value Problems

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When are solutions to elliptic or parabolic boundary value problems given by convex functions? Extending a maximum principle derived in an earlier paper to a slightly larger class of elliptic equations and to related parabolic equations yields some new answers to this question as well as some new proofs of already known results.

In particular this paper contains:

(a) A proof that a function on a convex domain Ω whose graph makes zero contact angle with the bounding cylinder $\partial\Omega \times \mathbf{R}$ and which satisfies an elliptic equation of the appropriate type is convex.

(b) A generalization and direct proof of the Brascamp-Lieb result that the first eigenfunction of the Laplacian on a convex domain is Log concave (and so has convex level sets).

In order to get these results the nature of the maximum principle used requires strong constraints on the boundary behavior of u. The reason these constraints are necessary is illustrated with a simple counterexample.

The statements and proofs of the maximum principles are given in Section 1. They are used in Section 2 to prove the convexity results. The counterexample and some related unsolved problems are briefly discussed in Section 3.

I have learned that L. A. Caffarelli and J. Spruck have also (independently and concurrently) extended the results of the earlier paper [7]. Their work will apparently include much of what is presented here, as well as some further applications [2].

Section 1. Maximum principles. We introduce the concavity function and prove several maximum principles for it.

The concavity function $\mathscr{C}$ was first introduced in [7]. It is a natural way to measure by how much a function u fails to be convex. Let u be defined on the closure of a bounded domain Ω. Let $0 \leq \mu \leq 1$. Then for

$$y_1, y_3 \in \bar{\Omega} \quad \text{such that} \quad y_2 = \mu y_3 + (1-\mu)y_1 \in \bar{\Omega} \tag{1}$$

define

$$\mathscr{C}(y_1, y_3, \mu) = u(y_2) - \mu u(y_3) - (1-\mu)u(y_1). \tag{2}$$

Indiana University Mathematics Journal ©, Vol. 32, No. 4 (1983)

$\mathscr{C}(y_1,y_3,\mu)$ is the height difference between the graph S_u of u and the line segment joining $(y_1,u(y_1))$ to $(y_3,u(y_3))$, above the point y_2. The function u is (non-strictly) convex if and only if $\mathscr{C} \leq 0$ for all y_1, y_2, y_3 as above.

Remark 1.1. Although we allow $\mathscr{C}$ to depend on varying μ here, this is not necessary for the maximum principles of this section. Essentially the same proofs will hold for fixed $0 < \mu < 1$. We let μ vary because it is more convenient for the applications of Section 2.

Notice that $\mathscr{C}$ is defined on a closed subset of $\bar{\Omega} \times \bar{\Omega} \times [0,1]$ (1). Slightly abusing our notation we say that:

Definition 1.2. The triple (y_1,y_3,μ) is *in the interior* if each of y_1, y_2, $y_3 \in \Omega$. It is *on the boundary* if at least one of y_1, y_2, $y_3 \in \partial\Omega$.

The following theorem is the main result of this section for elliptic equations. It links the concavity function to the elliptic equation that u satisfies.

Theorem 1.3. (Concavity maximum principle). *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy the elliptic equation*

$$0 = Lu = a^{ij}(Du)u_{ij} - b(x,u,Du) \qquad \text{in } \Omega \tag{3}$$

where b satisfies

$$\frac{\partial b}{\partial u} \geq 0, \qquad b \text{ jointly concave with respect to } (x,u). \tag{4}$$

Then if $\mathscr{C}$ is anywhere positive, it attains its positive maximum on the boundary (Definition 1.2).

(In equation (3) and throughout this paper subscripts on functions denote differentiation and Du is the (spatial) gradient of u. Matrices are denoted with square brackets. We assume $[a^{ij}] > 0$ and $a^{ij} = a^{ji}$. Repeated indices are summed. The functions a^{ij} and b are assumed to depend smoothly on their arguments, although it is clear from the proof that this is not necessary if b is strictly increasing in u.)

Notice that since $u \in C(\bar{\Omega})$, $\mathscr{C}$ is defined and continuous on a closed (hence compact) subset of $\bar{\Omega} \times \bar{\Omega} \times [0,1]$, so that it does attain its maximum value somewhere.

Theorem 1.3 can now be proven in two stages: The proof is simple if the constraint $\partial b/\partial u \geq 0$ in (4) is replaced by $\partial b/\partial u > 0$. We prove this case first. (This was essentially the case studied in [7], but we repeat the short proof here for completeness.) A perturbation argument then allows the extension to $\partial b/\partial u \geq 0$.

Lemma 1.4. *Let $u \in C^2(\Omega)$ satisfy (3), (4) and the stronger condition $\partial b/\partial u > 0$. Then $\mathscr{C}$ attains no local positive interior maximum (Definition 1.2) and must therefore attain any positive maximum on the boundary.*

Proof. Suppose $\mathscr{C}$ attains a local interior maximum at (x_1,x_3,λ). If x_1, x_2, x_3 are not distinct then $\mathscr{C} = 0$ and we are done. Hence we may assume they are distinct and in (the interior of) Ω. Calculus and $u \in C^1(\Omega)$ imply

$$(\nabla y_1 \mathscr{C})(x_1,x_3,\lambda) = (\nabla y_3 \mathscr{C})(x_1,x_3,\lambda) = 0. \tag{5}$$

But, using (2), (1)

$$\begin{aligned}(\nabla y_1 \mathscr{C})(x_1,x_3,\lambda) &= (1-\lambda)Du(x_2) - (1-\lambda)Du(x_1)\\ (\nabla y_3 \mathscr{C})(x_1,x_3,\lambda) &= \lambda Du(x_2) - \lambda Du(x_3)\end{aligned} \tag{6}$$

so that

$$Du(x_1) = Du(x_2) = Du(x_3). \tag{7}$$

The fact that the gradient of u is the same at these three points is crucial in allowing consideration of general nonlinear equations of the form (3), (4).

Consider now the restricted concavity function $\bar{\mathscr{C}}$ defined near (x_1,x_3,λ) by translating each of x_1, x_2, x_3 by the same vector v:

$$\bar{\mathscr{C}}(\mathrm{v}) = \mathscr{C}(x_1+\mathrm{v},x_3+\mathrm{v},\lambda) = u(x_2+\mathrm{v}) - \lambda u(x_3+\mathrm{v}) - (1-\lambda)u(x_1+\mathrm{v}). \tag{8}$$

Since $\bar{\mathscr{C}}$ has a local maximum at $\mathrm{v} = 0$ and since $u \in C^2(\Omega)$,

$$\nabla_{\mathrm{v}}\bar{\mathscr{C}}(0) = 0, \qquad [D^2_{\mathrm{v}}\bar{\mathscr{C}}(0)] \le 0. \tag{9}$$

(The symbols in (9) represent the gradient and Hessian of $\bar{\mathscr{C}}$ with respect to v.) Let a^{ij} and $b(x,u)$ be shorthand for $a^{ij}(Du)$ and $b(x,u,Du)$ at the common values of Du (7). Since $[a^{ij}] > 0$ and symmetric, (9) and linear algebra imply

$$a^{ij}[D^2_{\mathrm{v}}\bar{\mathscr{C}}(0)]_{ij} \le 0,$$

i.e.

$$a^{ij}(u_{ij}(x_2) - \lambda u_{ij}(x_3) - (1-\lambda)u_{ij}(x_1)) \le 0. \tag{10}$$

Using (10), (3) and then the joint concavity of b yields

$$\begin{aligned} b(x_2,u(x_2)) &\le \lambda b(x_3,u(x_3)) + (1-\lambda)b(x_1,u(x_1))\\ &\le b(\lambda(x_3,u(x_3)) + (1-\lambda)(x_1,u(x_1)))\\ &= b(x_2,\lambda u(x_3) + (1-\lambda)u(x_1)).\end{aligned} \tag{11}$$

The chain of inequalities (11) and b strictly increasing in u imply

$$u(x_2) \le \lambda u(x_3) + (1-\lambda)u(x_1)$$

i.e.

$$\mathscr{C}(x_1,x_3,\lambda) \le 0. \qquad \text{Q.E.D.}$$

In order to prove the result for $\partial b/\partial u \ge 0$ we need the following perturbation lemma. (See [4] for the same technique.)

Lemma 1.5. *Let $\Omega' \subset\subset \Omega$, $\partial\Omega' \in C^\infty$, $u \in C^2(\Omega)$ satisfying (3), (4). Then* *for small enough* $0 < \varepsilon < \varepsilon_1$ *we can solve the perturbed problem*

$$(12)\qquad \begin{aligned} a^{ij}(Dv^{\varepsilon})v^{\varepsilon}_{ij} &= b(x,v^{\varepsilon},Dv^{\varepsilon}) + \varepsilon v^{\varepsilon} \qquad \textit{in } \Omega' \\ v^{\varepsilon} &= u \qquad \textit{on } \partial\Omega' \end{aligned}$$

where $\exists M > 0$ *so that* v^{ε} *has the form*

$$(13)\qquad v^{\varepsilon} = u + \varepsilon w^{\varepsilon}, \qquad \|w^{\varepsilon}\|_{C^{2,\alpha}(\Omega')} < M, \qquad \textit{independently of } \varepsilon.$$

Proof. Using (13) to expand (12) in powers of ε and using the smoothness of a^{ij} and b yields:

$$(14)\quad a^{ij}(Du)u_{ij} + \varepsilon\left(a^{ij}(Du)w^{\varepsilon}_{ij} + \left(u_{ij}\frac{\partial}{\partial p^k}a^{ij}(Du) - \frac{\partial}{\partial p^k}b(x,u,Du)\right)w^{\varepsilon}_k - \frac{\partial b}{\partial u}(x,u,Du)w^{\varepsilon}\right) = b(x,u,Du) + \varepsilon u + \varepsilon^2 G(w^{\varepsilon},Dw^{\varepsilon}D^2w^{\varepsilon}).$$

Here G is a smooth function of its arguments (depending on u) and we have used $p = (p^1,\ldots,p^n)$ for the gradient argument in a^{ij} and b.

Since u satisfies (3) the expression (14) simplifies to the following almost linear equation for w^{ε} (and we write w for w^{ε}):

$$(15)\quad \tilde{L}w = a^{ij}(Du)w_{ij} + \left(u_{ij}\frac{\partial}{\partial p^k}a^{ij}(Du) - \frac{\partial}{\partial p^k}b(x,u,Du)\right)w_k - \left(\frac{\partial b}{\partial u}\right)(x,u,Du)w = u + \varepsilon G(w,Dw,D^2w).$$

Notice that the operator $\tilde{L}$ on the left of (15) is a uniformly elliptic linear operator on Ω', and the coefficient of w is nonpositive (by (4)). For operators $\tilde{L}$ of this form it is well known that for a given f, $\tilde{L}w = f$ can be solved, and in addition [6]:

$$(16)\qquad \left.\begin{aligned} Lw &= f \quad \text{in } \Omega' \\ w &= 0 \quad \text{on } \partial\Omega' \end{aligned}\right\} \Rightarrow \|w\|_{2,\alpha,\Omega'} \le K_1\|f\|_{0,\alpha,\Omega'}.$$

($\|\cdot\|_{2,\alpha,\Omega'}$ is the $C^{2,\alpha}(\Omega')$ norm).

Also, the fact that G is C^1 implies, given any K_2, there exists K_3 depending on K_2 so that

$$(17)\qquad \|v\|_{2,\alpha,\Omega'} \le K_2 \Rightarrow \|G(v,Dv,D^2v)\|_{0,\alpha,\Omega'} \le K_3.$$

Using (16) and (17) one can solve for w in (15) by iterating: Let $w^1 = 0$ and solve for $k \ge 1$:

$$\begin{aligned} \tilde{L}w^{k+1} &= u + \varepsilon G(w^k,Dw^k,D^2w^k) \qquad \text{in } \Omega' \\ w^{k+1} &= 0 \qquad \text{on } \partial\Omega'. \end{aligned}$$

It is easy to find an $\varepsilon_0 > 0$ and a K_4 so that $0 < \varepsilon < \varepsilon_0$ implies

(18) $$\|w^k\|_{2,\alpha,\Omega'} \le K_4,$$

and using this fact, that $\exists \varepsilon_1 > 0$ such that $0 < \varepsilon < \varepsilon_1$ implies

(19) $$\|w^{k+1} - w^k\|_{2,\alpha,\Omega'} \le \rho \|w^k - w^{k-1}\|_{2,\alpha,\Omega'} \qquad \rho < 1.$$

Both steps follow from estimates (16), (17), first applied (inductively) to $\tilde{L}w^k$ and then to $\tilde{L}(w^{k+1} - w^k)$. We omit the straightforward details.

But (18) and (19) imply that the sequence $\{w^k\}$ converges in $C^{2,\alpha}(\Omega')$ to a solution w of (15) satisfying

$$\|w\|_{2,\alpha,\Omega'} \le K_4.$$ Q.E.D.

We can now prove the entire Theorem 1.3: Pick an increasing sequence of C^∞ domains $\{\Omega^m\}$ whose union is Ω and such that $d(\partial\Omega,\partial\Omega^m) \to 0$. For any fixed Ω^m and small enough $\varepsilon > 0$, Lemmas 1.4 and 1.5 imply that the solution v^ε to (12) attains its maximum concavity (if positive) on the boundary. As $\varepsilon \to 0$, $v^\varepsilon \to u$ uniformly (13). Hence any positive maximum concavity of u for the domain Ω^ε is attained on the boundary.

Letting $m \to \infty$, and since $u \in C(\bar{\Omega})$, the maximum concavity of u for the domain Ω (if positive) is attained on the boundary. Q.E.D.

As was kindly shown to me by L. C. Evans (in the form of a proof of Theorem 2.8 for the heat equation), the concavity maximum principle extends naturally to parabolic equations. Given a function $u(x,t)$ defined on $\bar{\Omega} \times [0,T]$, define $\mathscr{C}(t,y_1,y_3,\mu)$ to be $\mathscr{C}(y_1,y_3,\mu)$ for the function $u(x,t)$ of the section $\bar{\Omega} \times \{t\}$. Then if u satisfies an appropriate parabolic equation, the maximum concavity is attained either initially or on the boundary.

Theorem 1.6. (Concavity maximum principle for parabolic equations). *Let $u \in C(\bar{\Omega} \times [0,T])$ be such that*

$$u(x,t) \in C^2(\Omega) \qquad \forall t \in (0,T]$$
$$u(x,t) \in C^1((0,T]) \qquad \forall x \in \Omega.$$

Suppose u satisfies the equation

(20) $$u_t = Lu$$

in Ω, where for fixed t, L has the form (3), (4). (In other words, a^{ij} and b may have any t-dependence.) Then if $\mathscr{C}(t,y_1,y_3,\mu)$ is anywhere positive, its positive maximum value is attained at some (t_0,x_1,x_3,λ) satisfying

(21) $$t_0 = 0 \quad \text{or one of} \quad x_1, x_2, x_3 \in \partial\Omega.$$

Proof. We first give the proof for $\partial b/\partial u > 0$, then extend to $\partial b/\partial u \ge 0$. The extension for parabolic equations is much easier than for elliptic ones.

Let u satisfy (20) with $\partial b/\partial u > 0$. Let $\mathscr{C}$ have a local maximum at (t_0,x_1,x_3,λ) not satisfying (21). As in Lemma 1.4 we show that this maximum value cannot be positive. We may assume

$$(22) \qquad x_1, x_2, x_3 \in \Omega, \qquad \text{distinct}, \quad 0 < t_0 \leq T.$$

Arguing exactly as in Lemma 1.4, from (5)–(10), and using the fact that u satisfies (20) instead of (3) one arrives at the analog of (11):

$$(23) \quad u_t(x_2) - \lambda u_t(x_3) - (1 - \lambda)u_t(x_1) + (b(x_2,u(x_2)) - \lambda b(x_3,u(x_3)) - (1 - \lambda)b(x_1,u(x_1))) \leq 0.$$

(t_0 and $Du = Du(x_1) = Du(x_2) = Du(x_3)$ are suppressed as before.) But $0 < t_0 \leq T$ implies

$$u_t(x_2) - \lambda u_t(x_3) - (1 - \lambda)u_t(x_1) = \frac{\partial}{\partial t}\mathscr{C}(t,x_1,x_3,\lambda) \geq 0$$

so that

$$(24) \qquad b(x_2,u(x_2)) - \lambda b(x_3,u(x_3)) - (1 - \lambda)b(x_1,u(x_1)) \leq 0.$$

Hence as in (11)

$$(25) \qquad \mathscr{C}(t_0,x_1,x_3,\lambda) \leq 0,$$

as was to be shown.

As in the elliptic case $\mathscr{C}$ attains its maximum somewhere, so any positive maximum must therefore be attained at points satisfying (21).

If $b(x,u,Du)$ only satisfies $\partial b/\partial u \geq 0$ consider the function

$$(26) \qquad \mathrm{v}(x,t) = e^{-\varepsilon t}u(x,t)$$

for small positive ε. (20) and (26) imply that v satisfies

$$\mathrm{v}_t = a^{ij}(e^{\varepsilon t}D\mathrm{v})\mathrm{v}_{ij} - (e^{-\varepsilon t}b(x,e^{\varepsilon t}\mathrm{v},e^{\varepsilon t}D\mathrm{v}) + \varepsilon \mathrm{v})$$

which is of the form just discussed. Thus the maximum concavity of v, if positive, is attained at some (t_0,x_1,x_3,λ) satisfying (21). As $\varepsilon \to 0$, $\mathrm{v} \to u$ uniformly, implying that Theorem 1.6 holds for u. Q.E.D.

Remark 1.7. The same proof of Theorem 1.6 holds for the more general equation

$$(27) \qquad f(t,x,u,u_t,Du) = a^{ij}(t,Du)u_{ij}$$

where $[a^{ij}] > 0$, $a^{ij} = a^{ji}$ and

$$(28) \qquad \frac{\partial f}{\partial u} \geq 0, \quad \frac{\partial f}{\partial u_t} > 0, \quad f \text{ jointly concave with respect to } (x,u,u_t).$$

There is also no need to restrict to "tubular" domains $\bar{\Omega} \times [0,T]$ in the (x,t) plane.

Section 2. Boundary conditions and applications. In order to prove convexity for solutions of the equations discussed in Section 1 we must find boundary conditions for u that prevent $\mathscr{C}$ from attaining its (positive) maximum on the boundary. As a first step we require Ω, which was only bounded in Section 1, to be (smooth and) strictly convex. This prevents x_2 from being a boundary point.

If u is C^1 then it is convex if and only if its graph S_u lies above all of its tangent planes. Conversely, the lemma below shows that if this property can be verified for all "boundary" tangent planes π_x (planes tangent to S_u above $x \in \partial\Omega$), then $\mathscr{C}$ cannot attain a positive maximum on the boundary.

Lemma 2.1. *Let Ω be strictly convex, smooth and bounded. Let u be such that S_u has tangent planes π_x $\forall x \in \partial\Omega$. If each of these boundary planes lies beneath S_u (contacting it only at $(x,u(x))$), then $\mathscr{C}$ does not attain any positive maximum concavity on the boundary (Definition 1.2). (We also allow π_x to be vertical as long as S_u makes contact angle zero with $\partial\Omega \times \mathbf{R}$ there, and not contact angle π).*

(Recall that the contact angle at the intersection of two surfaces is defined to be the angle between their normals there. In particular, the contact angle between S_u and the cylinder $\partial\Omega \times \mathbf{R}$ at a point of contact is the angle between the downward normal to S_u and the exterior normal to $\partial\Omega \times \mathbf{R}$.)

Proof. Let $\mathscr{C}$ attain a positive maximum at the (therefore distinct) points x_1, x_2, x_3. Since Ω is strictly convex we need only show that neither x_1 nor $x_3 \in \partial\Omega$. Suppose $x_1 \in \partial\Omega$. Consider the graph of u above the line segment $\overline{x_1x_3}$. By hypothesis $(x_3,u(x_3))$ lies above the tangent line to this curve through $(x_1,u(x_1))$. Thus by keeping x_2 and x_3 fixed and moving x_1 to $\tilde{x}_1$, a little nearer x_3 on the same line segment, the height u will be roughly the height of the tangent line through $(x_1,u(x_1))$. Thus (at x_2) the line segment joining $(\tilde{x}_1,u(\tilde{x}_1))$ to $(x_3,u(x_3))$ will be lower than the one joining $(x_1,u(x_1))$ to $(x_3,u(x_3))$. Thus the concavity for $\{\tilde{x}_1,x_2,x_3\}$ will be greater than the concavity for $\{x_1,x_2,x_3\}$. This is a contradiction. The same argument shows that x_3 is not in $\partial\Omega$. Q.E.D.

Combined with the results of Section 1 (Theorems 1.3 and 1.6) Lemma 2.1 immediately yields:

Theorem 2.2. *Let Ω be a C^1, strictly convex bounded domain. Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy an equation of the form (3), (4). Suppose S_u makes contact angle zero with $\partial\Omega \times \mathbf{R}$. Then u is a convex function.*

Theorem 2.3. *Let Ω be as in Theorem 2.2 and let u satisfy the hypotheses of Theorem 1.6. If for every fixed $0 < t \leq T$ the graph of u makes zero contact angle with its bounding cylinder and if $u_0 = u(x,0)$ is convex, then $u(x,t)$ is a convex function of x for every fixed $0 \leq t \leq T$.*

A natural example of Theorem 2.2 applying is in capillarity, where the contact angle boundary condition is natural. In this case $[a^{ij}]$ is a multiple of the mean curvature operator:

$$[a^{ij}(p)] = \frac{1}{(1 + |p|^2)^{3/2}}[(1 + |p|^2)I - pp^t]$$

and

$$b(x,u,Du) = \begin{cases} \kappa u + M & \kappa > 0 \text{ in a (downward-pointing) gravitational field} \\ M & \text{in no gravitational field.} \end{cases}$$

In [7] we mentioned this result for capillary surfaces in gravitational fields. With the strengthened maximum principle we get it for the constant mean curvature case too.

J. T. Chen and W. Huang [3] have recently found a cute comparison technique to prove the constant mean curvature result in two dimensions. It does not seem to generalize to more dimensions or other equations, however.

In [7] we showed that if the constant contact angle is not zero, there exist convex domains for which the solution to the (gravitational) capillary problem is not a convex function. Finn has recently found a counterexample for the constant mean curvature case [5].

What other boundary conditions on u do enable one to verify Lemma 2.1? The counterexamples for constant nonzero contact angle that were just mentioned, as well as the counterexample for constant Dirichlet data that is discussed in Section 3 indicate that for general convex domains there are not many suitable boundary conditions.

At least some do exist though. They arise by transforming functions with constant Dirichlet data and lead to a new proof that the first eigenfunction of the Laplacian on a convex domain is Log concave (i.e. its logarithm is a concave function) (Theorem 2.5 below).

Lemma 2.4. *Let Ω be smooth, bounded and strongly convex (i.e. all the principal curvatures of $\partial\Omega$ are positive). Let $u \in C^2(\bar{\Omega})$ satisfy*

$$u = 0 \quad on\ \partial\Omega, \qquad u > 0 \quad in\ \Omega, \qquad Du \cdot \nu > 0 \quad on\ \partial\Omega, \tag{29}$$

where ν is the interior normal to $\partial\Omega$. Let

$$\Omega_\delta = \{x \in \Omega \quad such\ that \quad d(x,\partial\Omega) > \delta\} \tag{30}$$

and let $\mathrm{v} = -\log u$. Then for small enough $\delta > 0$ the function v satisfies Lemma 2.1 on the domain Ω_δ. More generally, this holds for any smooth transformation $\mathrm{v} = f(u)$ where f is defined for positive u and satisfies

$$\begin{gathered} (i)\ f' < 0, \qquad (ii)\ \lim_{u\to 0^+} f'(u) = -\infty, \qquad (iii)\ f'' > 0, \\ (iv)\ \lim_{u\to 0^+} \frac{f'}{f''} = 0, \qquad (v)\ \lim_{u\to 0^+} \frac{f}{f'} = 0. \end{gathered} \tag{31}$$

Proof. Let $x \in \partial\Omega_\delta$ and let A_x be the set of points y for which π_x does not lie beneath S_u:

$$A_x = \{y \in \Omega_\delta \quad \text{such that} \quad \pi_x(y) \geq u(y)\}.$$

We must show that for small enough, δ, $A_x = \emptyset\ \forall x \in \partial\Omega_\delta$. First we show that for points x near enough $\partial\Omega$, A_x is also near $\partial\Omega$:

Fact 1. Given $\varepsilon > 0\ \exists\delta_0 > 0$ such that for $0 < \delta < \delta_0$ and $x \in \partial\Omega_\delta$, we have $A_x \cap \Omega_\varepsilon = \emptyset$. Since $\mathrm{v} = f(u)$,

$$Dv(x) = f'(u)Du(x). \tag{32}$$

From (29) and (31) it follows that for x near $\partial\Omega$, π_x is practically vertical and its gradient practically points in the exterior normal direction. (Extend ν smoothly in a neighborhood of $\partial\Omega$ and you may talk about normal directions in the entire neighborhood.) Fact 1 now follows from elementary geometry, the convexity of Ω and equations (29), (31(ii)(v)), and (32).

Now we show that v is convex in a boundary strip about $\partial\Omega$:

Fact 2. $\exists\varepsilon > 0$ such that $x \in \Omega\backslash\Omega_\varepsilon \Rightarrow [D^2\mathrm{v}(x)] > 0$. To show this we study the two terms comprising $[D^2\mathrm{v}]$:

$$[D^2\mathrm{v}] = f'(u)[D^2u] + f''(u)[(Du)(Du)^t]. \tag{33}$$

The matrix $[(Du)(Du)^t]$ is positive semidefinite. If $x \in \partial\Omega$, Du is a positive multiple of the interior normal $\nu(x)$ (29). Hence for $x \in \partial\Omega$, $[(Du)(Du)^t]$ is positive in a direction η if and only if η is nontangential (to $\partial\Omega$).

On the other hand, if $x \in \partial\Omega$ the matrix $[D^2u]$ is negative definite in all tangential directions. (This follows from the strong convexity of $\partial\Omega$ and conditions (29). The calculation is straightforward.)

Extending the normal vector field $\nu(x)$ smoothly into a strip about $\partial\Omega$ one can continue to talk about tangential ($\nu(x)\cdot\eta = 0$) and nontangential directions. Since D^2u is continuous it follows that for directions η sufficiently close to tangential and for x in a sufficiently narrow strip $\Omega\backslash\Omega_\varepsilon$, $[D^2u]$ is negative in those directions η. Hence $f'(u)[D^2u]$ is positive in those directions.

Because D^2u is continuous one may then pick a possibly narrower strip $\Omega\backslash\Omega_\varepsilon$ on which $f'(u)/f''(u)$ is small enough (31(iv)) and on which Du is close enough to the normal direction, so that for all other directions η

$$f''(u)\eta^t[(Du)(Du)^t]\eta > f'(u)\eta^t[D^2u]\eta.$$

On this strip we have shown that $[D^2\mathrm{v}]$ is positive in any direction η.

Taken together Facts 1 and 2 imply Lemma 2.4: Pick $\varepsilon > 0$ so that $[D^2\mathrm{v}(x)] > 0$ for $x \in \Omega\backslash\Omega_\varepsilon$. For that ε pick δ_0 so that for $0 < \delta < \delta_0$ and $x \in \partial\Omega_\delta$, $A_x \cap \Omega_\varepsilon = \emptyset$. But because $[D^2\mathrm{v}]$ is positive in $\Omega\backslash\Omega_\varepsilon$, $A_x \cap (\Omega_\delta\backslash\Omega_\varepsilon) = \emptyset$ too. Thus for $0 < \delta < \delta_0$ $A_x = \emptyset$ as was to be shown. Q.E.D.

In particular we consider the transformation $\mathrm{v} = -\log u$ and show:

Theorem 2.5. *Let Ω be a smooth, bounded, strongly convex domain. Let $u \in C^2(\bar\Omega)$ be a nonnegative solution to the generalized eigenfunction equation*

$$a^{ij}\left(\frac{Du}{u}\right)u_{ij} = -u\,b\left(x, -\log u, \frac{Du}{u}\right), \tag{34}$$

where the function $b(x,\mathrm{v},p)$ satisfies (4) and where u satisfies the boundary conditions

(35) $$u > 0 \quad in\ \Omega, \qquad u = 0 \quad on\ \partial\Omega$$

(36) $$u \cdot \nu > 0 \qquad on\ \partial\Omega.$$

Then $\mathrm{v} = -\log u$ *is a convex function.*

Proof. u satisfies (34) if and only if v satisfies

$$a^{ij}(-D\mathrm{v})\mathrm{v}_{ij} = a^{ij}(-D\mathrm{v})\mathrm{v}_i\mathrm{v}_j + b(x,\mathrm{v},-D\mathrm{v})$$

which is of the form (3), (4). From (35), (36) Lemma 2.4 applies, so that for small enough δ and the (convex) domain Ω_δ, no positive maximum of concavity can occur on the boundary. Hence v is convex in Ω_δ. Hence v is convex in Ω.

Q.E.D.

Remark 2.6. Equation (34) is in general form. It applies in particular to equations like

(37) $$\Delta u = -u(\lambda + V(x)) \qquad \lambda > 0,\ V \text{ concave}$$

(38) $$\Delta u = -ug(u), \qquad g'(u) \leq 0 \quad \text{and} \quad g''(u)u + g'(u) \leq 0.$$

$$(\text{e.g.} \quad g(u) = \lambda - \mu u^p \quad p,\mu > 0).$$

Note too that condition (36) is implied by (35) and the Höpf boundary point lemma [9] provided that $b(x,-\log u,Du/u)$ is nonnegative for the solution u. This is true for (37) if $V \geq -\lambda$ and is also true for (38) as one can see by noting that $\Delta u \leq 0$ at the maximum value of u.

The answer to when solutions of (34), (35), (36) exist is not known in general, however. If the solutions arise from variational problems in which the candidate functions can be taken to be nonnegative, positive results are known (e.g. the first eigenfunction of the Laplacian).

Theorem 2.5 was first shown for (37) by H. J. Brascamp and E. H. Lieb [1] using completely different techniques. Extending their method, P. L. Lions has recently (concurrent to this work) shown it for (38).

Remark 2.7. One might at first suspect that the first (positive) eigenfunction of the Laplacian is itself concave. This is false for any domain: If $\partial\Omega$ is strongly convex at x, then $u = 0$ and $u \cdot \nu > 0$ on $\partial\Omega$ imply that D^2u is negative in all tangential directions. Since $\Delta u = 0$ at x, it follows that D^2u must be positive in the normal direction. Thus u is not concave.

The method of Brascamp-Lieb and Lions employs the parabolic equation corresponding to (34). They use the following theorem in the special case of the heat equation (proving it by an unrelated method):

Theorem 2.8. *If Ω is a smooth, bounded, strongly convex domain and if $u(x,t) \in C(\Omega \times [0,T])$ is such that*

$$u(x,t) \in C^2(\bar{\Omega}) \qquad \forall 0 < t \le T$$

$$u(x,t) \in C^1[(0,T]] \qquad \forall x \in \Omega$$

and such that (35), (36) are satisfied by the functions $u(x,t)$ $\forall 0 < t \le T$. If u satisfies $u_t = Lu$ with L as in (34) and if $u_0 = u(x,0)$ is Log *concave, the $u(x,t)$ is* Log *concave* $\forall 0 \le t \le T$.

Proof. Argue as in Theorem 2.5, using Theorem 1.6. Q.E.D.

Remark 2.9. I do not know of any other transformations of the form (31) that can be applied to any other natural boundary value problems. It would be interesting to find such problems. Unfortunately, once a function u satisfies an equation of the form (3), (4), a transformation $v = f(-u)$ of the form (31) will no longer satisfy such an equation. (The eigenfunction equation (34) does not satisfy (4) but its transformed equation does.)

Section 3. Counterexamples, related questions. We briefly describe a counterexample for constant Dirichlet data analogous to the one discussed in [7] for constant contact angle. In both cases convex domains with sharply rounded corners breed nonconvex solutions to reasonable elliptic equations.

Specifically, consider a convex domain $\Omega \subset \mathbf{R}^2$, symmetric about the x-axis, lying to the right of the y-axis, so that $\partial\Omega$ is smooth except at (0,0), where it has a corner. Let Ω_ε, also symmetric about the x-axis, be the same as Ω except with the sharp corner rounded off inside $B_{2\varepsilon}(0)$, and with the left-most point of $\partial\Omega_\varepsilon$ at $(\varepsilon,0)$. Let Λ_ε denote the (positive) curvature of $\partial\Omega_\varepsilon$ at $(\varepsilon,0)$. We round off Ω_ε in such a way that the $\Lambda_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Let $u^\varepsilon \in C^2(\bar{\Omega}_\varepsilon)$ be zero on $\partial\Omega_\varepsilon$ and nonpositive inside Ω_ε. A straightforward computation yields

$$u^\varepsilon_{yy}(\varepsilon,0) = -\Lambda_\varepsilon u^\varepsilon_x(\varepsilon,0). \tag{39}$$

Assume now that u^ε has positive bounded Laplacian in Ω_ε. (This example would also work for mean curvature and practically any solution of (3) with b positive in Ω_ε.)

As $\varepsilon \to 0$, equation (39) implies that either some of the $u^\varepsilon_{yy}(\varepsilon,0)$ approach infinity, or else the u^ε_x approach zero (since the $\Lambda_\varepsilon \to \infty$). In the first case some of the $u^\varepsilon_{xx}(\varepsilon,0)$ must approach $-\infty$ since Δu^ε is bounded. In particular they would become negative and u^ε would not be convex. In the second case the tangent plane to the graph of u^ε at $(\varepsilon,0)$ becomes horizontal which, if the u^ε were convex, would imply that $u^\varepsilon \to 0$ uniformly on (the bounded) Ω_ε. But one can easily construct barriers (e.g. a paraboloid with less Laplacian than u^ε whose zero height level is a circle contained in all the Ω_ε's) that lie above u^ε to show that the u^ε do not approach zero. Hence some of the u^ε are not convex.

This example indicates that what is needed to find convex solutions to elliptic boundary value problems in general is some relation between the boundary curvature of Ω, the boundary data of u and the elliptic operator L. This appears to be a difficult question, especially for nonlinear operators L.

Another interesting problem is to find natural conditions which force the level sets of u to be convex even though the function itself may not be.

Acknowledgements. I am indebted to R. Caflisch and H. Sexton for their proficiency in the perturbation techniques needed for Lemma 1.5, and to L. C. Evans for his parabolic proof.

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This work was partially supported by the U.S. Army under Contract No. DAAG29-80-C-0041 and by the National Science Foundation under Grant No. MCS-7927062, Mod. 1. Much of the research for this article was completed while the author was at Stanford University.

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Received November 20, 1981