Convex Solutions of Certain Elliptic Equations Have Constant Rank Hessians

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1. Introduction

In this note we first consider solutions u of

$$\Delta u = f(u, \Delta u) > 0 \tag{1.1}$$

in a region Ω of Euclidean n space ($\mathbb{R}^n$). Here ∇u and Δu denote the gradient and Laplacian of u. We assume that f has Hölder continuous second partial derivatives on some open set containing the range of the function $x \mapsto (u(x), \nabla u(x))$, $x \in \Omega$. We also assume that f is strictly positive, with

$$2(f_u)^2 (\cdot, \nabla u) - f(\cdot, \nabla u) f_{uu}(\cdot, \nabla u) \geqq 0, \tag{1.2}$$

that is, $1/f(\cdot, \nabla u)$ is convex in u.

Let H denote the Hessian matrix of u. Our main result is

Theorem 1. *Let u, f satisfy* (1.1), (1.2), *and suppose that H is positive semidefinite on Ω. Then H has constant rank in Ω.*

Thus if H is positive definite in a neighborhood of the boundary of Ω, then H is positive definite in Ω. CAFFARELLI and FRIEDMAN [2] have proved Theorem 1 in $\mathbb{R}^2$ when f has the form

$$f(u, \nabla u) = h(u) + |\nabla u|^2 k(u).$$

Our method is a generalization to $\mathbb{R}^n$, $n \geqq 2$, of their proof.

The minimum principle in Theorem 1 can be compared with an important recent result of KENNINGTON [7, 8]. To state KENNINGTON's result, suppose that u is a solution of

$$\sum_{i,j} a^{ij}(\nabla u)\, u_{x_i x_j} = f(u, \nabla u) \tag{1.3}$$

in Ω, where the sum is taken over $1 \leqq i, j \leqq n$. Assume that each a^{ij}, $1 \leqq i$, $j \leqq n$, has Hölder continuous second partial derivatives. We also assume that

(a^{ij}) is symmetric and positive definite on some open set containing the range of the function $x \mapsto \nabla u(x)$. Define T on $\Omega \times \Omega \times (0, 1)$ by

$$T(x, y, \lambda) = \lambda u(x) + (1 - \lambda)\, u(y) - u(\lambda x + (1 - \lambda)\, y).$$

If u and f satisfy (1.2), (1.3) and

$$\frac{\partial f}{\partial u}(\cdot, \nabla u) > 0, \tag{1.4}$$

then KENNINGTON shows that T cannot have a negative relative minimum at an interior point of $\Omega \times \Omega \times (0, 1)$. Previously, KOREVAAR [9] had obtained a similar conclusion under the assumption that f be a concave function of u, which is a stronger restriction than (1.2). KAWOHL [6] has found results between those of KOREVAAR and KENNINGTON.

Theorem 1, combined with the method of continuity (see [2]), can often be used to establish that certain solutions of (1.1) are convex functions in Ω. To illustrate the method, let Ω be a convex region and suppose that

$$\Delta w = -1 \quad \text{in } \Omega, \tag{1.5}$$

while $w = 0$ on $\partial\Omega$ (boundary of Ω). Put $u = -w^{1/2}$. Then u satisfies the equation

$$\Delta u = -(|\nabla u|^2 + \tfrac{1}{2})/u = f(u, \nabla u) > 0 \tag{1.6}$$

in Ω. Note that $1/f(\cdot, \nabla u)$ is convex. Now, if Ω is the unit ball B, then

$$u(x) = -[(1 - |x|^2)/2n]^{1/2}, \quad x \in B,$$

so clearly u is a convex function. For an arbitrary convex region Ω, deform B continuously into Ω by a family (Ω_t), $0 \leqq t < 1$, of strictly convex regions in such a way that $\Omega_0 = B, \Omega_1 = \Omega$, and $\partial\Omega_t \to \partial\Omega_s$ as $t \to s$ in the sense of Hausdorff distance, whenver $0 \leqq s \leqq 1$. The deformation also is chosen so that $\partial\Omega_t$, $0 \leqq t < 1$, can be locally represented for some α, $0 < \alpha < 1$, by a function whose norm in the space $C_{2,\alpha}$ of functions with Hölder continuous second derivatives depends only on δ, whenever $0 < t \leqq \delta < 1$.

Let $u(\cdot, t)$ be the solution of (1.6) in Ω_t with boundary value zero on $\partial\Omega_t$. Let H_t be the corresponding Hessian matrix. Then from standard estimates and the choice of deformation, it follows that for each δ, $0 < \delta < 1$, there exists $\varepsilon = \varepsilon(\delta) > 0$ such that H_t is positive definite in an ε neighborhood of $\partial\Omega_t$, whenever $0 < t \leqq \delta$. This fact, Theorem 1, and convergence of u_t to u_s locally in the $C_{2,\alpha}$ norm as $t \to s$ imply that $H = H(1)$ is positive definite. Indeed, using the above observations and the explicit nature of $u(\cdot, 0)$, it is easily seen that H_t is positive definite in $\Omega(t)$ when $t > 0$ is sufficiently small. If H were not positive definite, then it would follow for some δ, $0 < \delta < 1$, that H_δ is positive semidefinite but not positive definite in $\Omega(\delta)$. From Theorem 1, H_δ has constant rank $< n$ in $\Omega(\delta)$, which is impossible since H_δ is positive definite in an ε neighborhood of $\partial\Omega(\delta)$. Hence, if w satisfies (1.5) in a convex region Ω and has boundary value zero on $\partial\Omega$, then $w^{1/2}$ is a strictly concave function in Ω. We note that KENNINGTON [7, 8] used his previously mentioned minimum principle to show that

$w^{1/2}$ is concave in Ω. His method, though, does not appear to imply the strict concavity of $w^{1/2}$.

As another example, suppose Ω is a bounded convex ring. That is, $\mathbb{R}^n - \Omega$ consists of two components and if E denotes the bounded component of $\mathbb{R}^n - \Omega$, then E and $\Omega \cup E$ are convex. Let w be a solution of Laplace's equation in Ω, and suppose that w has boundary value zero on $\partial\Omega \cap E$, while w has boundary value 1 on the rest of $\partial\Omega$. Let $u = w^k$ and observe that

$$\Delta u = \left(1 - \frac{1}{k}\right) |\nabla u|^2/u = f(u, \nabla u) > 0$$

in Ω. Clearly f satisfies (1.2) but not (1.4). Again from standard estimates, it can be seen that if k is sufficiently large and $\partial\Omega$ is locally of class $C_{2,\alpha}$ for some α $(0 < \alpha < 1)$, then the Hessian matrix of u is positive definite in a neighborhood of $\partial\Omega$. Also, if $n > 2$ and $\Omega = \{x \in \mathbb{R}^n\colon 1 < |x| < 2]$, then

$$w(x) = [1 - |x|^{2-n}]/(1 - 2^{2-n}),$$

so clearly u is convex when k is large.

The method of continuity and Theorem 1 can now be applied to a strictly convex ring Ω of class $C_{2,\alpha}$, to deduce that u is strictly convex when $k = k(\Omega)$ is large enough. Thus in this case the level sets of $w = u^{1/k}$ are strictly convex. Approximating a general convex ring by strictly convex rings with smooth boundaries, it follows that the corresponding w has convex level sets. This method, however, does not seem to be strong enough to show that the level sets of w are strictly convex, a fact which was proved by GABRIEL in [3] from a rather involved computation. Also, we note that KENNINGTON's method does not appear to imply that w as above has convex level sets, since (1.4) is false. For further applications of Theorem 1 in $\mathbb{R}^2$, as well as more details in the above examples, see [2].

We next consider solutions u of (1.3), under the assumption that $1/f(\cdot, \nabla u)$ is strictly convex, that is

$$(1.7) \qquad 2(f_u)^2\,(\cdot, \nabla u) - f_{uu}(\cdot, \nabla u) f(\cdot, \nabla u) > 0.$$

We prove

Theorem 2. *Let u and f be as in* (1.3) *and* (1.7). *Then H has constant* rank r *on Ω. Moreover, u is constant in $n - r$ coordinate directions.*

The proof of Theorem 2 is somewhat complicated and, in fact, will be deduced from some inequalities we derive in proving Theorem 1. If in addition to the above assumptions we also assume that

$$(1.8) \qquad a^{ij}\ (1 \leqq i, j \leqq n) \text{ and } f \text{ are real analytic},$$

(on their respective domains), then a straightforward and relatively simple proof of Theorem 2 can be given. Moreover, the proof parallels in several respects the ideas of KENNINGTON's convexity minimum principle. This proof is given in Section 2. The proofs of Theorems 1 and 2 are given in Sections 3–4. In Section 5 we show that if H has rank r in Theorem 1, then through each point in Ω there is an $(n-r)$-dimensional plane on which u is linear. We also consider other implications of Theorems 1 and 2 in Section 5.

2. A Weak Form of Theorem 2

Let u and f satisfy (1.3), (1.7) and (1.8), and suppose that H has rank $r < n$ at x_0 in Ω. By performing a translation and rotation, we may assume that $x_0 = 0$ and $u_{y_iy_i}(0) = 0$, $r+1 \leqq i \leqq n$, where $\hat{y}_i$, $1 \leqq i \leqq n$, is an orthonormal coordinate system. Given a function F, in the sequel we shall write F_{ij} for $F_{y_iy_j}$.

Let $v = (v_1, v_2, \dots, v_n)$ be an arbitrary unit vector whose scalar projection on $\hat{y}_i$ is v_i, $1 \leqq i \leqq n$. We know that $H(\varepsilon v) \geqq 0$ for small ε. In particular, if $\omega = (\omega_1, \omega_2, \dots, \omega_{n-1})$ and $|\omega| \leqq 1$, we consider second derivatives of u in the directions $(\varepsilon\mu\omega, 1)$, $\mu \in \mathbb{R}$, whence

$$u_{nn}(\varepsilon v) + 2 \sum_{i<n} \varepsilon\mu u_{in}(\varepsilon v)\, \omega_i + \varepsilon^2\mu^2 \sum_{i,j<n} u_{ij}(\varepsilon v)\, \omega_i\omega_j \geqq 0. \tag{2.1}$$

Since $u_{nn}(0) = 0$ and u is convex, we have

$$u_{nn}(0) = u_{nnk}(0) = u_{nk}(0) = 0, \quad 1 \leqq k \leqq n. \tag{2.2}$$

Since u has continuous fourth partials derivatives, we get (using (2.2) to eliminate some terms)

$$\begin{aligned} u_{nn}(\varepsilon v) &= \tfrac{1}{2}\varepsilon^2 \sum_{k,l} u_{nnkl}(0)\, v_k v_l + o(\varepsilon^2), \\ u_{in} &= \varepsilon \sum_k u_{ink}(0)\, v_k + o(\varepsilon), \\ u_{ij}(\varepsilon v) &= u_{ij}(0) + o(1), \end{aligned} \tag{2.3}$$

where $o(\varepsilon^l)$ denotes a term which tends to 0 as $\varepsilon^l \to 0$.

Substituting (2.3) into (2.1), dividing by ε^2 and letting $\varepsilon \to 0$ yields

$$\tfrac{1}{2} \sum_{k,l} u_{nnkl} v_k v_l + 2\mu \sum_{i<n} \sum_k u_{ink}\omega_i v_k + \mu^2 \sum_{i,j<n} u_{ij}\omega_i\omega_j \geqq 0, \tag{2.4}$$

where all derivatives of u have been evaluated at the origin. Letting ω be the projection of v, $\omega = (v_1, \dots, v_{n-1})$, we find that

$$\tfrac{1}{2} \sum_{k,l} u_{nnkl} v_k v_l + 2\mu \sum_{i<n} \sum_k u_{ink} v_i v_k + \mu^2 \sum_{i,j<n} u_{ij} v_i v_j \geqq 0.$$

From (2.2), we observe that this expression is unchanged if i and j are allowed to vary from 1 to n. Using this fact, we see that the above inequality can be expressed in terms of directional derivatives by

$$\tfrac{1}{2}(u_{nn})_{vv} + 2\mu(u_n)_{vv} + \mu^2 u_{vv} \geqq 0, \tag{2.5}$$

where all expressions are evaluated at the origin.

Before proceeding further, we mention that the key expression (2.5) was derived by looking at second derivatives of $u(\varepsilon v)$ in directions $(\varepsilon\mu(v_1, \dots, v_{n-1}), 1)$. The same idea is used in KENNINGTON's proof of his convexity minimum principle. Near three colinear points $\{y, z, x = (1-\lambda)\, y + \lambda z\}$ of minimum convexity, he considers small perturbations and studies their effect on the function $T(y, z, \lambda)$ defined in Section 1. One can derive his inequalities by perturbing each of $\{y, z, x\}$ in the direction of a vector v, but different magnitudes determined by μ. Thus if $\{y, z, x\}$ are assumed to lie on a line in the $\hat{y}^n$ direction, then KENNINGTON studies the convexity of u along lines displaced by multiples of ε in the v direction from $\{y, z, x\}$ with direction vectors $(\varepsilon\mu\omega, 1)$, $\omega = (v_1, v_2, \dots, v_{n-1})$.

To continue the proof, choose an orthonormal system of vectors $\{v^1, \dots, v^n\}$ so that, with respect to these coordinates, $[a^{ij}(\nabla u(0))]$ is diagonal with eigenvalues $\lambda_1 \dots \lambda_n$. By adding multiples of (2.5) n times, we get

$$\frac{1}{2} \sum_k \lambda_k u_{nnv^k v^k} + 2\mu \sum_k \lambda_k u_{nv^k v^k} + \mu^2 \sum_k \lambda_k u_{v^k v^k} \geqq 0. \tag{2.6}$$

From (2.2) we find that

$$(a^{ij}(\nabla u)\, u_{ij})_n \Big|_{x=0} = \left(\sum_k a^{ij}_{u_k} u_{kn}\right) u_{ij} + a^{ij} u_{ijn} \Big|_{x=0} = a^{ij}(\nabla u(0))\, u_{ijn}(0). \tag{2.7}$$

Also,

$$\begin{aligned}(a^{ij}(\nabla u)\, u_{ij})_{nn} \Big|_{x=0} &= \left(\sum_{k,l} a^{ij}_{u_k u_l} u_{kn} u_{ln}\right) u_{ij} \\ &\quad + 2\left(\sum_k a^{ij}_{u_k} u_{kn}\right) u_{ijn} + a^{ij} u_{ijnn} \Big|_{x=0} \\ &= a^{ij}\,(\nabla u(0))\, u_{ijnn}(0).\end{aligned} \tag{2.8}$$

From (1.3), (2.7) and (2.8) it follows that (2.6) can be written as

$$\tfrac{1}{2} f_{nn} + 2\mu f_n + \mu^2 f \geqq 0 \quad \text{at } x = 0.$$

This inquality can hold for all $\mu \in \mathbb{R}$ if and only if the discriminant is $\leqq 0$, that is

$$2(f_n)^2 - f_{nn} \leqq 0 \quad \text{at } x = 0. \tag{2.9}$$

But at $x = 0$ we have

$$\begin{aligned} f_n &= \sum_k f_{u_k} u_{kn} + f_u u_n = f_u u_n, \\ f_{nn} &= \sum_{k,l} f_{u_k u_l} u_{kn} u_{ln} + \sum_k (f_{u_k} u_{knn} + 2 f_{u u_k} u_{kn} u_n) + f_u u_{nn} + f_{uu} u_n^2 = f_{uu} u_n^2. \end{aligned} \tag{2.10}$$

Hence (2.9) becomes

$$(2f_u^2 - f f_{uu})\, u_n^2 \leqq 0 \quad \text{at } x = 0.$$

By (1.7), we must have $u_n(0) = 0$.

Finally, observe from (1.8) and a theorem of HOPF [5] that u is real analytic in Ω. We shall use the real analyticity of u to show that if $u_n(0) = 0$, then u is constant on lines on the $\hat{y}_n$ direction. Differentiating (1.3) and evaluating at a point x near 0, we obtain

$$\text{(2.11)} \quad \sum_{i,j} a^{ij}(\nabla u(0))\, u_{nnij} = \sum_{i,j} [a^{ij}(\nabla u(0)) - a^{ij}(\nabla u)]\, u_{nnij} - \sum_{i,j,k} [2a^{ij}_{u_k} u_{kn} u_{ijn}] - \sum_{i,j,k,l} [a^{ij}_{u_k u_l} u_{kn} u_{ln} u_{ij}] + f_{nn}.$$

Now, in a neighborhood of the origin it follows from the real analyticity of u that either $u_{nn} \equiv 0$ or there exists a positive integer m such that

$$u_{nn}(x) = P(x) + Q(x),$$

where P is a homogeneous polynomial of degree m and Q is the remainder in the power series expansion for u_{nn}, starting with terms of degree $m+1$. Since u_{nn} has a minimum at 0, we see that m is even. From the positive semi-definiteness of H, we deduce that

$$(u_{in})^2 \leqq u_{nn} u_{ii}, \quad 1 \leqq i \leqq n,$$

so there exist $c > 0$ and $\varrho > 0$ small enough so that

$$|u_{in}| \leqq c\,|x|^{m/2}, \quad |u_{ijn}| \leqq c\,|x|^{[(m/2)-1]}, \quad |u_n| \leqq c\,|x|^{[(m/2)+1]},$$
$$|u_{nnk}| \leqq c\,|x|^{m-1}, \quad |u_{nnij}| \leqq c\,|x|^{m-2},$$

when $|x| \leqq \varrho$. Representing f_{nn} as in (2.10) and using the above inequalities in (2.11), we find that

$$\sum_{i,j} a^{ij}(Du(0))\, P_{ij}(x) \leqq c\,|x|^{m-1}$$

for $|x| \leqq \varrho$. The left-hand side of this inequality must be identically zero, since it is a homogeneous polynomial of degree $m-2$. Thus

$$\Sigma\, a^{ij}(\nabla u(0))\, P_{ij} \equiv 0.$$

Observe that $P(0) = 0$ and $P \geqq 0$, as follows from $u_{nn} \geqq 0$. Using the strong minimum principle for uniformly elliptic equations, we conclude that $P \equiv 0$. Hence $u_{nn} \equiv 0$ in a neighborhood of 0. Consequently $u_{nn} \equiv 0$ in Ω. Moreover, since u is convex, we also have $u_{in} \equiv 0$ in Ω, $1 \leqq i \leqq n$. Hence u_n is constant. Since $u_n(0) = 0$, it follows that $u_n \equiv 0$. Thus u is constant in Ω on lines parallel to $\hat{y}_n$.

Since $u_{ii}(0) = 0$ when $r + 1 \leqq i \leqq n$, we can repeat the above argument with u_{nn} replaced by u_{ii}, $1 + r \leqq i \leqq n$. This completes the proof of Theorem 2 under assumption (1.8).

3. Proof of Theorem 1

Let u and f be as in (1.1) and (1.2). As in Section 2, we assume that $H(0)$ has rank $r < n$. Observe from (1.1) that $r \geqq 1$. Let ϕ be the sum of all $r+1$ by

$r+1$ principal minors of H. We shall show that $\phi \equiv 0$. Following CAFFARELLI and FRIEDMAN, we say that $h(y) \lesssim k(y)$ provided there exist positive constants c_1 and c_2 such that

$$(h-k)(y) \leqq (c_1 |\nabla \phi| + c_2 \phi)(y).$$

We also write $h(y) \sim k(y)$ if $h(y) \lesssim k(y)$ and $k(y) \lesssim h(y)$. Next , we write $h \lesssim k$ if the above inequality holds in a neighborhood of the origin, with the constants, c_1 and c_2 independent of y in this neighborhood. Finally, $h \sim k$ if $h \lesssim k$ and $k \lesssim h$. We shall show that

$$\Delta \phi \lesssim 0. \tag{3.1}$$

Since $\phi \geqq 0$ in Ω and $\phi(0) = 0$, it then follows from the strong minimum principle (see [4], p. 34) that $\phi \equiv 0$ in a neighborhood of the origin.

Hereafter, if we say that a condition is satisfied "locally," we mean that there exists $\varrho > 0$ such that the condition is satisfied for all $|z| < \varrho$. To begin the proof, pick $c > 0$ so that the r non-zero eigenvalues of $H(0)$ are bounded below by $2c$. Thus $H(z)$ has r eigenvalues $\geqq c$, locally. For such a z, choose a coordinate system $\{\hat{y}_1, \ldots, \hat{y}_n\}$ as in Section 2 so that $H(z)$ is a diagonal matrix. Then

$$u_{jj}(z) \geqq c, \quad 1 \leqq j \leqq r; \qquad u_{ij}(z) = 0, \quad 1 \leqq i \neq j \leqq n. \tag{3.2}$$

Let $G = \{1, \ldots, r\}$ and $B = \{r+1, \ldots, n\}$ be the "good" and "bad" sets of indices, and define

$$Q = \prod_{j \in G} u_{jj}(z),$$

$$Q_j = Q/u_{jj}(z), \quad j \in G; \qquad R = \sum_{j \in G} Q_j.$$

Let α and β be two unit vectors. We compute ϕ and its first and second derivatives in the directions α and β. (In this section only $\alpha, \beta \in \{\hat{y}_1, \ldots, \hat{y}_n\}$.) We find, for ϕ and ϕ_α,

$$0 \sim \phi(z) \sim \left(\sum_{i \in B} u_{ii}(z)\right) Q \sim \sum_{i \in B} u_{ii}(z) \quad (\text{so } u_{ii}(z) \sim 0, i \in B), \tag{3.3}$$

$$0 \sim \phi_\alpha(z) \sim Q \sum_{i \in B} u_{\alpha ii}(z) \sim \sum_{i \in B} u_{\alpha ii}(z). \tag{3.4}$$

Because of (3.2), the positive constants in the definition of $\sim$ can be chosen locally, here and in what follows. To compute $\phi_{\alpha\beta}$, we use the second part of (3.2), and then use (3.3) and (3.4) to discard terms uneffected by the $\sim$ relation. We obtain

$$\phi_{\alpha\beta} \sim Q \sum_{i \in B} u_{ii\alpha\beta}(z) - 2 \sum_{i \in B} \sum_{j \in G} Q_j u_{ij\alpha} u_{ij\beta} + R \sum_{i,j \in B}{}' [u_{ii\alpha} u_{jj\beta} - u_{ij\alpha} u_{ij\beta}],$$

where Σ' means the sum is taken over $i \neq j$. Using (3.4), we may replace $\Sigma'_{i \in B} u_{ii\alpha}$ with $-u_{jj\alpha}$, thus

$$\phi_{\alpha\beta} \sim Q \sum_{i \in B} u_{ii\alpha\beta}(z) - 2 \sum_{i \in B} \sum_{j \in G} Q_j u_{ij\alpha} u_{ij\beta} - R \sum_{i,j \in B} u_{ij\alpha} u_{ij\beta}. \tag{3.5}$$

If we pick $\alpha = \beta = \hat{y}_k$ and sum over k, (3.5) yields

$$\Delta\phi \sim Q \sum_{i\in B} (\Delta u)_{ii} - 2 \sum_{i\in B} \sum_{j\in G} |\nabla u_{ij}|^2 Q_j - R \sum_{i,j\in B} |\nabla u_{ij}|^2. \tag{3.6}$$

We relate the terms in (3.6) to derivatives of f as follows. If $i \in B$, then from (3.2), (3.3) and (3.4), we have

$$f_i = \sum_k f_{u_k} u_{ki} + f_u u_i \sim f_u u_i, \tag{3.7}$$

$$f_{ii} = \sum_{k,l} (f_{u_k u_l}) u_{ki} u_{il} + \sum_k (f_{u_k} u_{kii} + 2f_{uu_k} u_{ki} u_i) + f_u u_{ii} + f_{uu} u_i^2,$$

$$\sum_{i\in B} f_{ii} \sim f_{uu} \sum_{i\in B} u_i^2. \tag{3.8}$$

Finally we show that

$$Q \sum_{i\in B} f_i^2 \lesssim \Big(\sum_{j\in G} \sum_{i\in B} |\nabla u_{ij}|^2 Q_j \Big) f. \tag{3.9}$$

Indeed from (1.1), (3.4), the Schwarz inequality and (3.3), we find

$$f_i^2 = \Big(\sum_j u_{jji} \Big)^2 \sim \Big(\sum_{j\in G} u_{jji} \Big)^2 \leqq \Big[\sum_{j\in G} (u_{jji})^2 / u_{jj} \Big] \Big(\sum_{j\in G} u_{jj} \Big)$$

$$\sim (1/Q) \Big[\sum_{j\in G} (u_{jji})^2 Q_j \Big] f,$$

and (3.9) follows. Substituting (3.8) and (3.9) into (3.6), and using (3.7), we obtain

$$\Delta\phi \lesssim Q(f_{uu} - 2f_u^2/f) \sum_{i\in B} u_i^2$$

at z. It then follows from (1.2) that $\Delta\phi(z) \lesssim 0$. Since the constants can be chosen locally, we conclude that (3.1) is valid. From the remark following (3.1), we conclude that $\phi \equiv 0$ locally. Thus, the set $F = \{x: \text{rank } H(x) = r\}$ is open. If $x_0 \in \Omega$ is a boundary point of F, then by continuity of ϕ we see that rank $H(x_0) \leqq r$. Applying the above argument, we get

$$\text{rank } H = \text{constant} = r$$

in a neighborhood of x_0. Thus F is also closed in Ω and so, by connectivity, $F = \Omega$.

4. Proof of Theorem 2

Let u and f be as in (1.3) and (1.7). We again assume that $0 \in \Omega$ and that $H(0)$ has rank $r < n$. Define ϕ as in Section 3. We first show as in Theorem 1 that $\phi \equiv 0$ in a neighborhood of the origin. We shall assume that

$$(a^{ij}(\nabla u(0))) = (\delta_{ij}), \tag{4.1}$$

where δ_{ij} is the Kronecker δ, that is, (1.3) reduces to (1.1) at the origin. To see that this assumption is permissible, let u satisfy (1.3), where we have picked a fixed x coordinate system near the origin so that $a^{ij}(\nabla u(0))$ is diagonal, with positive eigenvalues $\lambda_1, \dots, \lambda_n$. Define

$$w(z) = u(\sqrt{\lambda_1} z_1, \dots, \sqrt{\lambda_n} z_n) = u(\bar{z}).$$

Then

$$w_{x_i}(z) = \sqrt{\lambda_i} u_{x_i}(\bar{z}), \qquad w_{x_i x_j}(z) = \sqrt{\lambda_i \lambda_j}\, u_{x_i x_j}(\bar{z}).$$

Hence, if we define

$$b^{ij}(p_1, \dots, p_n) = \frac{1}{\sqrt{\lambda_i \lambda_j}} a^{ij}\left(\frac{p_1}{\sqrt{\lambda_i}}, \dots, \frac{p_n}{\sqrt{\lambda_n}}\right),$$

then

$$\sum_{i,j} b^{ij}(\nabla w)\, w_{x_i x_j}(z) = \sum_{i,j} a^{ij}(\nabla u)\, u_{x_i x_j}(\bar{z}) = f\left(w, \frac{w_{x_1}}{\sqrt{\lambda_1}}, \dots, \frac{w_{x_n}}{\sqrt{\lambda_n}}\right).$$

Since the Hessians of w and u are "conjugates", their positivity and rank are the same at corresponding points. By inspection, w satisfies an equation of the type desired. We now re-label w again as u.

Let ϱ, z and $\{\hat{y}_i\}$ be chosen as in Section 3. We use the notation of Section 3, and recall formulas (3.3), (3.4) and (3.5) with α, β chosen to be $\hat{x}_k$, $\hat{x}_l$. Multiplying (3.5) by a^{kl} and summing yields

$$\begin{aligned}(4.2)\qquad & \sum_{k,l} a^{kl}(\nabla u)\, \phi_{x_k x_l} - Q \sum_{i \in B} \sum_{k,l} a^{kl}(\nabla u)\, u_{x_k x_l ii} \\ & \sim -2 \sum_{k,l} \sum_{i \in B} \sum_{j \in G} Q_j a^{kl}(\nabla u)\, u_{x_k ij} u_{x_l ij} - R \sum_{k,l} \sum_{i,j \in B} a^{kl}(\nabla u)\, u_{x_k ij} u_{x_l ij} \\ & = -2\Sigma_1 - \Sigma_2, \qquad \text{say}.\end{aligned}$$

Differentiating $a^{kl} u_{x_k x_l}$ as in (2.7) and (2.8), and using (3.2) and (3.3), we see that for $i \in B$

$$(4.3)\qquad (a^{kl}(\nabla u)\, u_{x_k x_l})_i = \left(\sum_j a^{kl}_{u_{x_j}} u_{x_j i}\right) u_{x_k x_l} + a^{kl}(\nabla u)\, u_{x_k x_l i} \sim a^{kl}(\nabla u)\, u_{x_k x_l i}$$

$$\begin{aligned}(4.4)\qquad (a^{kl}(\nabla u)\, u_{x_k x_l})_{ii} = & \left(\sum_{j,m} a^{kl}_{u_{x_j} u_{x_m}} u_{x_j i} u_{x_m i}\right) u_{x_k x_l} \\ & + 2\left(\sum_j a^{kl}_{u_{x_j}} u_{x_j i}\right) u_{x_k x_l i} + a^{kl} u_{ii x_k x_l} \\ & \sim a^{kl} u_{ii x_k x_l},\end{aligned}$$

where all expressions have been evaluated at z. Using (1.3) and summing (4.4) over k, l and $i \in B$ improves (4.2) to

$$\sum_{k,l} a^{kl}(\nabla u)\,\phi_{x_k x_l} \sim Q \sum_{i\in B} f_{ii} - 2\Sigma_1 - \Sigma_2. \tag{4.5}$$

We seek to estimate $\sum_{i\in B} f_i^2$ in terms of Σ_1 as in Section 3, using the fact that $a^{ij} \approx \delta_{ij}$ near 0. Write

$$\begin{aligned} \sum_{i\in B} f_i^2 &= \sum_{i\in B} (\Delta u_i)^2 + E_0, \\ \Sigma_1 &= \sum_{j\in G}\sum_{i\in B} (|\nabla u_{ij}|^2 Q_j) + E_1, \\ f &= \Delta u + E_2. \end{aligned} \tag{4.6}$$

Here, the first term on the right-hand side of each equation was obtained by replacing a^{kl} with δ_{kl} in the definition of the corresponding term on the left-hand side of the equation.

The error terms can be estimated using (4.2), (4.3) and (4.1). Given $\delta > 0$, we obtain locally

$$|E_0| + |E_1| \lesssim \delta\left(\sum_j \sum_{i\in B} |\nabla u_{ij}|^2\right), \quad |E_2| \lesssim \delta. \tag{4.7}$$

Now using (4.6) to modify the inequality following (3.9), we see that

$$\begin{aligned} Q/f \sum_{i\in B} f_i^2 &= Q/f\left(\sum_{i\in B}(\Delta u_i)^2 + E_0\right) \sim Q/f\left(\sum_{i\in B}\left(\sum_{j\in G} u_{ijj}\right)^2 + E_0\right) \\ &\leqq Q/f\left[\left(\sum_{i\in B}\sum_{j\in G} u_{ijj}^2/u_{jj}\right)\left(\sum_{j\in G} u_{jj}\right) + E_0\right] \\ &\lesssim 1/f[(\Sigma_1 - E_1)(f - E_2) + QE_0] \\ &\leqq \Sigma_1 + |E_1|\left(1 + \frac{|E_2|}{f}\right) + \frac{|E_2|}{f}\Sigma_1 + \frac{Q|E_0|}{f} \\ &= \Sigma_1 + E. \end{aligned} \tag{4.8}$$

E can be estimated using (4.7). Indeed given $\delta > 0$, we have locally

$$E \lesssim \delta\left(\sum_j \sum_{i\in B} |\nabla u_{ij}|^2\right). \tag{4.9}$$

Using (4.8) judiciously in (4.5), along with (3.7) and (3.8), we have

$$\sum_{k,l} a^{kl}(\nabla u)\,\phi_{x_k x_l} \lesssim Q\left(f_{uu} - \frac{2\eta f_u^2}{f}\right)\sum_{i\in B} u_i^2 + 2\eta E - 2(1-\eta)\,\Sigma_1 - 2\Sigma_2, \tag{4.10}$$

for any $0 \leqq \eta \leqq 1$. But by inspection of (4.2), we see that there exists a constant $c > 0$ such that locally

$$\sum_j \sum_{i\in B} |\nabla u_{ij}|^2 \leqq c(\Sigma_1 + \Sigma_2). \tag{4.11}$$

Furthermore by the strict convexity of $1/f(\cdot, \nabla u)$ at the origin, there is a neighborhood of 0 and an $\eta < 1$ such that, in this neighborhood,

$$\left(f_{uu} - \frac{2\eta f_u^2}{f}\right) < 0. \tag{4.12}$$

For this η, we see from (4.9), (4.10) and (4.11) that there exists $c > 0$ so that locally

$$\sum_{k,l} a^{kl}(\nabla u)\phi_{x_k x_l} \lesssim Q(f_{uu} - 2\eta f_u^2/f) \sum_{i \in B} u_i^2 - c \sum_j \sum_{i \in B} |\nabla u_{ij}|^2. \tag{4.13}$$

In particular

$$\sum_{k,l} a^{kl}(\nabla u)\phi_{x_k x_l} \lesssim 0,$$

and it follows from the strong maximum principle used in Section 3 that $\phi \equiv 0$ in a neighborhood of the origin. As in Section 3, we conclude that $\operatorname{rank} H \equiv r$ in Ω.

We now show that u is constant in $n - r$ directions. Since $\phi \equiv 0$ we can replace $\lesssim$ by $\leqq$ in each of our inequalities. It follows from (4.13) that

$$\sum_j \sum_{i \in B} |\nabla u_{ij}|^2 (z) = 0. \tag{4.14}$$

Equation (4.14) implies that the coordinates $\hat{y}_i$, $r + 1 \leq i \leq n$, can be chosen independently of z locally. To see this, note from (4.14) that there exists $c > 0$ such that

$$|u_{ij}(w) - u_{ij}(z)| = |u_{ij}(w)| \leqq c\,|z - w|^2 \tag{4.15}$$

locally, when $i \in B$. Also, if $i \neq j$,

$$|u_{ij}(w)| \leqq c\,|z - w|, \tag{4.16}$$

for c large, since $u_{ij}(z) = 0$ for $i \neq j$. Let v be a null vector of $H(w)$. If $v = \sum_i b_i \hat{y}_i$, then from (4.15),

$$0 = \sum_i b_i u_{ij}(w) = \sum_{i \in B} b_i\, u_{ij}(w) + \sum_{i \in G} b_i u_{ij}(w), \quad j \in G.$$

Choose j so that $|b_j| = \max_{k \in G} |b_k|$. Using (4.15), (4.16), the above inequality and the fact that $u_{jj}(z) \geqq c$ (see (3.2),) we get locally

$$|b_j| \leqq c_1\,|w - z|^2$$

for $c_1 > 0$ large enough. Note that $q = \sum_{i \in B} b_i \hat{y}_i$ is a null vector for $H(z)$ since H has rank r and (3.2) holds. From the last inequality it follows that

$$|q/|q| - v| \leqq c_2\,|w - z|^2 \tag{4.17}$$

locally, for an appropriate choice of $c_2 > 0$. Note that c_2 is independent of w and z locally. From (4.17) we conclude that each unit null vector of $H(w)$ lies within $c_2\,|z - w|^2$ of a null vector of $H(z)$. Since w and z are arbitrary (subject to $|z|, |w| \leqq \varrho$), it follows from a simple argument that actually the null-spaces of $H(w)$ and $H(z)$ are the same. (Divide the line segment from z to w into N equal parts, N large, and use (4.15) in each subsegment.) From (4.14) and our choice of coordinate system, it follows that $u_{ij} \equiv 0$ in a neighborhood of the origin whenever $i \in B$. Thus u_i is constant in a neighborhood of the origin, and so must equal zero by (4.13). Repeating the argument, it follows that the set

$$\{x \in \Omega: u_i(x) = u_{ii}(x) = 0 \text{ for all } i \in B\}$$

is open. Since this set is clearly closed in Ω, we conclude from the connectivity of Ω that u is constant in $n - r$ coordinate directions.

5. Remarks

Remark 1. Let u and f be as in Theorem 1 and suppose that H has constant rank $r < n$. If Ω is convex, then it follows as in [2] that through each $x_0 \in \Omega$ there is an $(n - r)$-dimensional plane on which u is linear in Ω. To give the proof (following [2]), suppose $x_0 = 0$ and put

$$v(x) = u(x) - u(0) - \nabla u(0) \cdot x, \qquad x \in \Omega.$$

Since $v(0) = 0$, $\nabla v(0) = 0$, and v is convex, we have $v \geqq 0$. Also, the set

$$E = \{x: v(x) = 0\} = \{x: v(x) \leqq 0\}$$

is convex in Ω. Next choose an orthogonal coordinate system $(\hat{y}_j)$ so that, in this system, $u_{ij}(0) = 0$ when $i \neq j$ or $r + 1 \leqq i \leqq n$. We claim that E is contained in the $(n - r)$-dimensional vector space L generated by $\hat{y}_{r+1}, \ldots, \hat{y}_n$. Indeed, suppose $v(x) = 0$, where $x \cdot \hat{y}_j \neq 0$ for some j, $1 \leqq j \leqq r$. Then $v(tx) > 0$ for small $t > 0$, as follows from the fact that $u_{jj}(0) \geqq 0$. Since E is convex, we have reached a contradiction. Thus $E \subseteq L$. If $E = L \cap \Omega$, then u is linear on E and we are done. Otherwise, since E considered as a convex subset of L is the intersection of $(n - r)$-dimensional half-spaces, there is a point z in E and a ray l emanating from z with $l - \{z\} \subseteq L - E$. By making a rotation if necessary, we may assume that $l = \{z - t\hat{y}_n: 0 \leqq t < \infty\}$. Let π denote the $(r + 1)$-dimensional plane through z generated by $\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_r, \hat{y}_n$, and let v_1 be the restriction of v to this plane. If $B_\varrho = \{x: |x - z| < \varrho\} \subseteq \Omega$, we claim that for some $\varepsilon_0 > 0$,

$$w(y) \equiv -v_1(y) - \varepsilon(y - z) \cdot \hat{y}_n < 0, \tag{5.1}$$

whenever $y \in \pi \cap \{x: |x - z| = \varrho\}$ and $0 \leqq \varepsilon \leqq \varepsilon_0$. Observe that (5.1) holds for y in a neighborhood of $z + \varrho\hat{y}_n$, since $v_1 \geqq 0$. It also holds in a neighborhood of $z - \varrho\hat{y}_n$ for ε_0 sufficiently small, since $v(z - \varrho\hat{y}_n) > 0$. Thus we need only consider the set of $y \in \pi \cap \{x: |x - z| = \varrho\}$ such that

$$\sum_{i=1}^{r} [(y - z) \cdot \hat{y}_j]^2 \geqq \delta > 0.$$

Since $E \subseteq L$, we see by continuity that (5.1) holds on this set also if $\varepsilon_0 > 0$ is small. Hence (5.1) is true. We now apply a maximum principle of ALEXANDROV and BAKELMAN [1] to w in $\Omega \cap \pi$. If $A = (w_{ij})$ denotes the $r+1$ by $r+1$ matrix with $i, j \in \{1, \ldots, r, n\}$, we obtain

$$w(y) \leqq c \left(\int_{B_\varrho \cap \pi} |\det A| \, dx \right)^{1/(r+1)} = 0, \qquad y \in \pi \cap B_\varrho,$$

by Theorem 1. However, this inequality is clearly false when $|y - z|$ is small and $y \in l$, since $v(z) = 0$ and $\nabla v(z) = 0$ (v has a minimum at z). We have reached a contradiction. Hence $E = L \cap \Omega$, and u is linear on $L \cap \Omega$.

From Theorem 1 and the preceding discussion, we see that either

(i) *the Hessian of u is positive definite in Ω, or*
(ii) *through each point in Ω there is at least one line on which u is linear.*

Moreover, from Theorem 2 we see that if $1/f(\cdot, \nabla u)$ is strictly convex in Ω, then case (ii) may be improved to

(iii) *u is constant in at least one direction.*

We now show there are functions which satisfy (ii) but not (iii). We construct in $\mathbb{R}^2$ a solution u of (1.1), with $f(u) = -1/u$, which is convex in a suitable domain and whose graph is part of a cone. To do so, we use polar coordinates and write $u(r, \theta) = rg(\theta)$. If

$$\Delta u = u_{rr} + r^{-1} u_r + r^{-2} u_{\theta\theta} = -1/u,$$

then

$$g(\theta)/r + g''(\theta)/r = -1/(rg(\theta)),$$

which simplifies to

$$g''(\theta) = [-1 - g^2(\theta)]/g(\theta). \tag{5.2}$$

Now it is easy to see that (5.1) can be solved locally with, say, initial conditions $g(0) = -g_0 < 0$, $g'(0) = 0$. As long as $g < 0$, we see that $g'' > 0$. It can be shown from this inequality that $u = rg(\theta)$ is a convex function in a suitable domain. Clearly u is linear on rays through the origin.

If $1/f(\cdot, \nabla u)$ is strictly convex at some point in Ω, then from the proof of Theorem 2 it follows that u is constant along an entire line in Ω. In applications this possibility can often be eliminated. In these cases, if (1.2) holds and u is not strictly convex, it must be the case that $(1/f)_{uu} \equiv 0$ in Ω. Thus

$$\frac{1}{f(u, \nabla u)} = uA(\nabla u) + B(\nabla u).$$

In a future paper we hope to characterize those values of A and B for which the lines in (ii) intersect in a cone. In $\mathbb{R}^2$ we believe that a necessary and sufficient condition for this to happen is that B be a constant multiple of A.

Remark 2. We conjecture that Theorem 1 remains valid when (1.1) is replaced by (1.3). The conjecture is true in $\mathbb{R}^2$, as can be deduced from the proofs in [2] or in this paper.

There are also sharp results when f is allowed to have a suitable x dependence, just as KENNINGTON's method works for $f = f(x, u, \nabla u)$ when $1/f(\cdot, \cdot, \nabla u)$ is a convex function of (x, u) for fixed ∇u. For example, Theorem 1 generalizes to

Theorem 1. *Let the hypotheses of Theorem 1 hold, except that now* $f = f(x, u, \nabla u)$ *and* $1/f$ *is a convex function of* (x, u) *for fixed* ∇u. *Then* H *has constant rank in* Ω.

The proof is essentially unchanged. Indeed, the proof in Section 3 showed, without considering any dependence on f, that

$$\Delta\phi \lesssim Q \sum_{i \in B} (f_{ii} - 2f_i^2/f) = Qf^2 \sum_{i \in B} (1/f)_{ii}.$$

Using the explicit dependence of f, it is easily checked that $\Delta\phi \lesssim 0$.

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