DOMAINS ON WHICH ANALYTIC FUNCTIONS SATISFY QUADRATURE IDENTITIES

By

DOV AHARONOV AND HAROLD S. SHAPIRO

1. Introduction

1.1. One of the most basic properties of an analytic (or harmonic) function f is its mean value property:

$$f(z_0) = A^{-1} \int_D f d\sigma. \tag{1.1}$$

Here D denotes a circular disc centered at z_0, and A its area. By $d\sigma$, here and throughout this paper, we denote the area element on $\mathbf{R}^2 = \mathbf{C}$. We can also interpret (1.1) as a "one point quadrature identity": to integrate an analytic function over a disc D, evaluate it at the center of the disc, then multiply by the area of the disc.

The papers [8, 9, 10] have established, under successively more general hypotheses, that circular discs are characterized by this property (a more detailed accounting will be given below). In [1] (see also [2, 3]) the present authors were led to the study of domains on which analytic functions satisfy exact quadrature identities (q. i.) in terms of function values *and a finite number of derivatives* at a fixed point; a natural generalization, moreover, is to allow function values and derivatives at *several* points; and a method was given for determining all *simply connected* domains admitting q.i. in this wider sense. Our motivation for investigating this problem is that it arose in connection with certain extremal problems for univalent functions, namely: among all functions univalent in the disc U and normalized by prescribing their values and derivatives up to a certain order at the origin, to find that function with least Dirichlet integral. The extremal functions can be shown to map U on domains which satisfy q.i. Because, however, the admissible functions f in these identities are subject to certain restrictions the results we have obtained so far concerning q.i. yield only limited applications to the extremal problems. It is our hope, of course, that further progress either in the general theory of q.i., or in relaxing the newly mentioned restrictions on the test functions (so as to make the present results fully applicable to the extremal problems) will

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 30 (1976)

lead to complete solutions of those problems. In the meantime we present here a self-contained account of our results on q.i., which we feel have intrinsic interest, and which may be read independently of [1].

1.2. The following notations will be employed. For a plane domain D, $L^p(D)$ denotes the usual Lebesgue class (with respect to the areal measure $d\sigma$). By $L^p_a(D)$ we denote the set of single-valued analytic functions in $L^p(D)$ and by $L^p_{a.s}(D)$ the subset of $L^p_a(D)$ consisting of functions with single-valued integrals (these classes coincide when D is simply connected). We shall say that D *admits a quadrature identity* (relative to some prescribed "test class" of integrable analytic functions on D) if there exist points $z_1, \cdots, z_n$ in D and complex numbers $a_{j,k}$ where

$$1 \leqq j \leqq n; \quad 0 \leqq k \leqq r_j - 1; \quad r_j \geqq 1; \quad a_{j,r_j-1} \neq 0$$

such that

$$\int_D f d\sigma = \sum_{j=1}^{m} \sum_{k=0}^{r_j-1} a_{j,k} f^{(k)}(z_j) \tag{1.2}$$

for every f in the test class.

It would perhaps be more natural, as the previous authors who studied the mean value property (one-point quadrature) have done, to use test classes of *harmonic*, rather than analytic, functions. Our preference for the latter is motivated by a search for results applicable to the above mentioned extremal problems.

1.3. Let us give a brief account of the known results about *one-point* q.i. Epstein [8] proved:

Theorem A. *Let D be a simply connected plane domain of finite area and z_0 a point of D such that* (1) *holds whenever f is harmonic and integrable on D. Then D is a disc centered at z_0.*

In his proof, Epstein in fact only required that (1.1) hold for $f \in L^2_a(D)$. In our work, we shall always take as the "test class" either $L^1_a(D)$ or $L^1_{a.s}(D)$. For D of finite area and finite connectivity, the distinction between L^1_a and L^2_a as test class is not significant, since $L^p_a(D)$ is, for every $1 < p \leqq \infty$, dense in $L^1_a(D)$ (a proof of this fact, using methods from [11], was kindly communicated to us by L. I. Hedberg. (See also [17, p. 112].) The restriction on D to have finite area is, as we shall see, not essential to Epstein's result.

Later, Epstein and Schiffer [9] proved

Theorem B. *Let D be any domain in $\mathbf{R}^n$ of finite measure whose complement has non-empty interior. Suppose there exists a point $x_0 \in D$ such that*

$$u(x_0) = V^{-1} \int_D u(x)\,dx \tag{1.3}$$

for every function u harmonic and integrable on D. Then D is a ball centered at x_0. (Here dx denotes Lebesgue measure on $\mathbf{R}^n$, $n \geqq 2$, and V is the measure of D.)

Kuran [13] then showed that the assumption that the complement of D has non-empty interior is superfluous, giving moreover a new, very short and elegant proof. The method of Epstein and Schiffer, when $n = 2$, can easily be modified so as to require only test functions u that are *analytic* and integrable on D, that is $u \in L^1_a(D)$.

So far as we are aware, no study of the mean value property has been published when the test class is $L^1_{a,s}$. If D is bounded and of finite connectivity, and (1.1) holds for all $f \in L^1_{a,s}(D)$ then it can be shown that D is a disc centered at z_0. A proof of this was kindly shown to one of the authors by M. Schiffer; it is based on properties of canonical slit mappings, and is considerably deeper than the corresponding proof for L^1_a. With Prof. Schiffer's kind permission, we include below (a modified version of) this proof.

1.4. The plan of the paper is as follows. Section 2 presents a reformulation of the q.i. problem, which is easier to work with. Section 3 treats the general q.i. in *simply connected* domains. Here a fairly complete result is obtained, apart from the question of uniqueness. The uniqueness question, i.e. whether two distinct domains may admit the same q.i., appears to be the most difficult in the whole present circle of ideas; some remarks concerning this will be made below.

Section 4 studies domains D (of arbitrary connectivity) which admit a q.i. relative to test functions in $L^1_a(D)$. Our main result is that the boundary of any such D is a subset of some irreducible algebraic curve. The proof of this is based upon a simple but apparently new general theorem on meromorphic functions which are real on the boundary of a domain, which also has other applications. As a concrete application of our method, we find the most general D admitting a q.i. of the special form

$$\int_D f\,d\sigma = af(z_0) + bf'(z_0). \tag{1.4}$$

A feature of our work is that D is not required *a priori* to have finite area, but only to satisfy (2.4) below.

Section 5 studies domains D of any finite connectivity which admit a q.i. relative to the narrower class of test functions $L^1_{a,s}$. This is the case needed for application to the above-mentioned extremal problems. The restriction to $L^1_{a,s}$, as already remarked in connection with the very simplest (i.e., one-point) q.i. causes serious difficulties, and even the complete description of domains D for which (1.4) holds seems in this case not attainable by our methods. We do however deduce some remarkable differential-geometric properties of the boundary of any domain D admitting a q.i., e.g. the slope and curvature must be algebraically related. Of course these results apply *a fortiori* to the domains studied in Section 4.

The final Section 6 deals with some open questions, digressions and variants of the main problems.

Acknowledgements. The authors wish to express their gratitude to Professor M. M. Schiffer for several very illuminating discussions relating to the present material. They also acknowledge with warm thanks their indebtedness to numerous colleagues for stimulating discussions, useful information, improvements or corrections, especially Bernard Epstein, Björn Gustafsson, Lars–Inge Hedberg, Alexei Levin, Bernt Øksendal, Allen Shields, Uri Srebro, and John Wermer. Finally, we apologize for the fact that certain overlapping prior work, especially of Philip Davis, came to our attention too late to permit incorporation of detailed references within the body of our article; instead, the appropriate credits, and comparison of results, are given at the end of our paper.

2. Preparatory results and a reformulation of the problem

In this Section we first establish some basic properties of Cauchy transforms. These are known in principle, but do not seem to have been enunciated elsewhere in just the form in which we shall require them. With the aid of these results we shall reformulate the q.i. problem in a form which is analytically more tractable, *an overdetermined Dirichlet problem for the Cauchy–Riemann equations, with a free boundary*. This technique was employed by Epstein and Schiffer [9], and is a modification of a method apparently first used by Schiffer [15].

Lemma 2.1. *Let $u \in L^\infty(\mathbf{R}^2)$ satisfy*

$$\int_{\mathbf{R}^2} \left| \frac{u(z)}{z} \right| d\sigma < \infty \quad (z = x + iy). \tag{2.1}$$

Then the Cauchy transform

$$U(\zeta)=\int \frac{u(z)}{z-\zeta}\,d\sigma \tag{2.2}$$

where the integration is over the whole z-plane, exists as an absolutely convergent integral for every complex ζ, is continuous, and satisfies

$$U(\zeta)=O(|\zeta|^{1/2}),\quad \zeta\to\infty. \tag{2.3}$$

Moreover, the derivative

$$\frac{\partial U}{\partial \bar z}=\frac{1}{2}\left(\frac{\partial U}{\partial x}+i\frac{\partial U}{\partial y}\right)$$

in the sense of distribution theory, equals $-\pi u$.

Proof. The existence of the integral (2.2) for each ζ is straightforward, and we omit this verification.

As to the continuity of U, observe first that U is certainly continuous if u has compact support, being the convolution of u with the function z^{-1} in $L^1_{\mathrm{loc}}(\mathbf{R}^2)$. For the general case, write

$$u_n(z)=\begin{cases} u(z), & |z|\leqq n\\ 0\ , & |z|>n,\end{cases}$$

and let U_n denote the Cauchy transform of u_n. It suffices to prove $U_n\to U$ uniformly on compact sets. Now,

$$|U(\zeta)-U_n(\zeta)|=\left|\cdot\int_{|z|>n}\frac{u(z)}{z-\zeta}\,d\sigma\right|\leqq\left[\int_{|z|>n}\frac{|u(z)|}{|z|}\,d\sigma\right]\cdot\sup_{|z|>n}\left|\frac{z}{z-\zeta}\right|.$$

(integrations with respect to z). For ζ in a fixed compact set and n large enough, the sup is less than 2, therefore, by (2.1), the right side is arbitrarily small for n large enough.

To prove (2.3), we write

$$|U(\zeta)|\leqq\int_{\mathbf{R}^2}\frac{|u(z)|}{|z-\zeta|}\,d\sigma_z=\left(\int_{E_1}+\int_{E_2}\right)\frac{|u(z)|}{|z-\zeta|}\,d\sigma_z$$

where $E_1=\{z:|z-\zeta|\leqq|\zeta|^{1/2}\}$ and $E_2=\{z:|z-\zeta|>|\zeta|^{1/2}\}$.

Now,

$$\int_{E_1} \leqq \| u \|_\infty \int_{E_1} \frac{d\sigma_z}{|z-\zeta|} = \| u \|_\infty \cdot \int_{|w| \leqq |\zeta|^{1/2}} \frac{d\sigma}{|w|} = 2\pi \| u \|_\infty |\zeta|^{1/2}$$

and

$$\int_{E_2} \leqq \int_{E_2} \frac{|z|}{|z-\zeta|} \cdot \left| \frac{u(z)}{z} \right| d\sigma_z \leqq \sup_{z \in E_2} \left| \frac{z}{z-\zeta} \right| \cdot \int_{\mathbf{R}^2} \left| \frac{u(z)}{z} \right| d\sigma.$$

Now, the second factor on the right is finite (hypothesis (2.1)), and moreover, for $z \in E_2$ we have $\left| \frac{z}{z-\zeta} \right| \leqq 1 + \left| \frac{\zeta}{z-\zeta} \right| \leqq 1 + |\zeta|^{1/2}$. Thus, $\int_{E_2}$ is $O(|\zeta|^{1/2})$ as $\zeta \to \infty$, and combining this with the above estimate for $\int_{E_1}$ yields (2.3).

We can note in passing that *the exponent* 1/2 *in* (2.3) *is sharp*. Indeed, let n_j be a rapidly increasing sequence of positive numbers, and let D_j denote the intersection with the upper half-plane of the disc $\{z : |z - n_j| < j^{-1} n_j^{1/2}\}$. Then, if D is defined as the union of the D_j (we could also make D connected if desired by joining up the D_j with sufficiently thin channels) and u is the indicator function of D, it is easily verified that (2.1) holds, whereas $|U(n_j)| >$ const. $j^{-1} n_j^{1/2}$, which is not $O(n_j^\alpha)$ for any $\alpha < 1/2$ if n_j increases rapidly enough, e.g. $n_j > 2^j$.

Finally, to verify the distributional identity $\frac{\partial U}{\partial \bar{z}} = -\pi u$, we have, for each $\varphi \in C^\infty(\mathbf{R}^2)$ with compact support (all integrations are over $\mathbf{R}^2$):

$$\left\langle U, \frac{\partial \varphi}{\partial \bar{\zeta}} \right\rangle = \int U \frac{\partial \varphi}{\partial \bar{\zeta}} d\sigma_\zeta = \int \frac{\partial \varphi}{\partial \bar{\zeta}} \left(\int \frac{u(z)}{z-\zeta} d\sigma_z \right) d\sigma_\zeta$$

$$= \int u(z) \left(\int \frac{1}{z-\zeta} \frac{\partial \varphi}{\partial \bar{\zeta}} d\sigma_\zeta \right) d\sigma_z$$

and, by a standard variant of Green's formula, the inner integral in the last expression equals $\pi \varphi(z)$.

Hence

$$\left\langle \frac{\partial U}{\partial \bar{\zeta}}, \varphi \right\rangle = -\left\langle U, \frac{\partial \varphi}{\partial \bar{\zeta}} \right\rangle = \langle -\pi u, \varphi \rangle,$$

which completes the proof of the lemma.

Lemma 2.2. *Let D be any open subset of* $\mathbf{C}$ *such that*

$$\int_D \frac{d\sigma}{|z|} < \infty. \tag{2.4}$$

Let

$$S(\zeta) = \int_D \frac{d\sigma}{z - \zeta} \quad (\textit{integration with respect to } z). \tag{2.5}$$

Then S is continuous on the whole complex plane, is $0(|\zeta|^{1/2})$ *as* $\zeta \to \infty$, *and for* $\zeta \in D$ *satisfies*

$$S(\zeta) = -\pi\bar{\zeta} + g(\zeta) \tag{2.6}$$

where g is a function continuous on D^- *(the closure of D) and analytic in D.*

Proof. The first two assertions follow by applying the previous lemma to $u = 1_D$, the indicator function of D. Also, by that lemma, $\partial S/\partial\bar{\zeta} = -\pi$ on D, and so $g = S + \pi\bar{\zeta}$ has a vanishing (distributional) $\bar{\zeta}$ derivative on D. By Weyl's lemma (see e.g. [18, p. 42]) g is an analytic function in D, and finally, since both $S(\zeta)$ and $\bar{\zeta}$ are everywhere continuous, g is continuously extendible to D^-.

Remark. For D bounded, a simple elementary proof of (2.6) can be given, as was done in [9]. Namely, fix R so large that $\Delta = \{z : |z| < R\}$ contains D, and write (for $\zeta \in D$)

$$S(\zeta) = \int_\Delta \frac{d\sigma}{z - \zeta} - \int_{\Delta \setminus D} \frac{d\sigma}{z - \zeta}$$

The first integral is easily evaluated and equals $-\pi\bar{\zeta}$. The second defines a function, $-g(\zeta)$, continuous for all complex ζ and analytic in D. This proves (2.6).

Let us now come back to q.i. It is notationally convenient to designate the right hand member of (1.2) by $\langle \alpha, f \rangle$ where α is a *distribution with finite support in D*, or what is the same thing, a finite linear combination of δ-functions $\delta_{(z_j)}$ and their partial derivatives; we will for brevity call such a distribution an *elementary distribution*. Of course, distinct elementary distributions may determine the same q.i., that is, have the same action on analytic functions; for example the distributions α, β defined by

$$\langle \alpha, \varphi \rangle = \frac{\partial \varphi}{\partial x}\bigg|_{x=y=0}$$

$$\langle \beta, \varphi \rangle = -i\frac{\partial \varphi}{\partial y}\bigg|_{x=y=0} \qquad (\varphi \in C_0^\infty)$$

satisfy $\langle \alpha, f \rangle = \langle \beta, f \rangle = f'(0)$ for f analytic in the neighbourhood of 0, because of the Cauchy–Riemann equations.

If α denotes any distribution with compact support K, it is meaningful to evaluate

$$A(\zeta) = \langle \alpha, k_\zeta \rangle, \quad \zeta \in \mathbf{C} \backslash K$$

where $k_\zeta(z) = (z - \zeta)^{-1}$. The function $A(\zeta)$, which is analytic and single-valued off K, is called the *Cauchy transform* of α. Obviously the Cauchy transform of an elementary distribution with support in D is a *rational function, with poles outside* D^- and vanishing at ∞. The converse is easily established, too. We now have:

Lemma 2.3. *Let D be any open subset of* $\mathbf{C}$ *satisfying* (2.4). *The following are equivalent*:

i) *D satisfies a q.i.* (1.2) *for all* $f \in L_a^1(D)$.

ii) *There is an elementary distribution* α *supported in D such that*

$$\int_D f d\sigma = \langle \alpha, f \rangle, \quad f \in L_a^1(D). \tag{2.7}$$

(We call a formula of this type a *quadrature identity*.)

iii) *The Cauchy transform S of* 1_D *is a rational function. More precisely: there is a rational function R such that* $S(\zeta) = R(\zeta)$ *for all* $\zeta \in \mathbf{C} \backslash D$. (*R has then necessarily all its poles in D.*)

iv) *There is a function h, analytic in D except for finitely many poles, continuously extendible to* D^-, *and satisfying*

$$h(\zeta) = \bar{\zeta}, \quad \zeta \in \partial D \tag{2.8}$$

$$|h(z) - \bar{z}| \leqq M_1 + M_2 |z|^{1/2}, \quad z \in D_0, \tag{2.9}$$

where D_0 *is a domain obtained from D by omitting a small disc around each pole of* h (M_i *are constants*).

Proof. We have already discussed the equivalence of (i) and (ii), and (iii) follows from (ii) by taking $f(z) = (z - \zeta)^{-1}$ in (2.7). Conversely, (iii) implies that (2.7) holds, initially for $f(z) = (z - \zeta)^{-1}$ with $\zeta \notin D$, and hence for all $f \in L^1_a(D)$ since finite linear combinations of the former functions are dense in $L^1_a(D)$ (see [12]). Hence (i), (ii) and (iii) are equivalent.

(iii) $\Rightarrow$ (iv). Suppose $S(\zeta) = R(\zeta)$ on $\mathbf{C}\backslash D$. Define

$$h(z) = \pi^{-1}(g(z) - R(z)), \quad z \in D.$$

This function has the analytic behaviour required in (iv). From (2.6) we see that (2.8) holds. Moreover, (2.6) shows that $g(z) - \pi\bar{z} = O(|z|^{1/2})$ on D, and since $R(z)$ is bounded on D_0, (2.9) holds.

(iv) $\Rightarrow$ (iii). Let V denote the open set $\mathbf{C}\backslash\partial D$. The function F defined by

$$F(z) = \begin{cases} S(z), & z \in \mathbf{C}\backslash D^- \\ g(z) - \pi h(z), & z \in D \end{cases}$$

(where g is the function defined in Lemma 2.2) is analytic on V except for a finite number of poles in D, and continuously extendible to the whole complex plane by virtue of (2.6) and (2.8). Suppose now, for the moment, that ∂D consists of smooth arcs. Then, by Painlevé's theorem, F is analytically continuable across all points of ∂V. Since it is bounded in a neighbourhood of ∞, and has only polar singularities, it is a rational function.

The fact that ∂D consists of smooth arcs, in fact that it is a subset of an algebraic curve, is nontrivial and will be proved later, in Section 4. (There is no danger of circularity in our reasoning, since in fact the implication (iv) $\Rightarrow$ (iii) is never used in the sequel, and was included here only for the sake of completeness.)

Remark. It is formulation (iv) which we actually work with below, that is, we ask: *for which domains D is the function* $\bar{z}|_{\partial D}$ *the boundary value of a function analytic in D, except for a finite number of polar singularities*? This is equivalent to the boundary value problem: *Find a function analytic in D, which on* ∂D *coincides with* $\bar{z} + R(z)$ (R a given rational function). For a *given* domain D this problem is in general senseless (overdetermined); we have to find those special domains D for which it is solvable.

To put the problem in perspective, we may also make this observation: the function $\bar{z}|_{\partial D}$ is analytically extendible to *some* neighbourhood of ∂D, intersected with D, if and only if ∂D is an analytic curve (see Section 6). We are here interested in those special analytic curves for which the extension is to *all* of D, apart from finitely many polar singularities.

Next, we give a reformulation of the q.i. problem in the case when the test class is $L^1_{a,s}$. To avoid certain approximation problems, we content ourselves with *necessary* conditions for q.i., which will suffice for our purposes.

Lemma 2.4. *Let D be any open subset of $\mathbf{C}$ satisfying* (2.4), *and suppose that D satisfies a q.i.* (1.2) *for all $f \in L^1_{a,s}(D)$; or (what is the same thing), there is an elementary distribution α supported in D such that*

$$\int_D f d\sigma = \langle \alpha, f \rangle, \quad f \in L^1_{a,s}(D). \tag{2.10}$$

Then, on any connected subset K of ∂D, $S(\zeta)$ (the Cauchy transform of 1_D) and $A(\zeta)$, the (rational) Cauchy transform of α, differ by a constant. Moreover, there exists a function $C(\zeta)$ continuous on ∂D and constant on each connected subset of ∂D, and a function h with all the properties enunciated in Lemma 2.3. (iv) *except that* (2.8) *is to be replaced by*

$$h(\zeta) = \bar{\zeta} + C(\zeta), \quad \zeta \in \partial D. \tag{2.11}$$

Proof. Let ζ_1, ζ_2 belong to the connected subset K of ∂D. Then, since the function

$$F(z) = \log \frac{z - \zeta_1}{z - \zeta_2}$$

is single-valued on D, its derivative $f(z) = (z - \zeta_1)^{-1} - (z - \zeta_2)^{-1}$ belongs to $L^1_{a,s}(D)$. Substituting this f into (2.10) gives

$$S(\zeta_1) - S(\zeta_2) = A(\zeta_1) - A(\zeta_2),$$

or $S(\zeta_1) - A(\zeta_1) = S(\zeta_2) - A(\zeta_2)$, showing that $S - A$ is constant on K. Write $B = S - A$. Then, from (2.6),

$$A(\zeta) + B(\zeta) = -\pi\bar{\zeta} + g(\zeta)$$

showing that $h = \pi^{-1}(g - A)$ and $C = \pi^{-1}B$ have the asserted properties and satisfy (2.11).

Remark. The special difficulties of the $L^1_{a,s}$ problem arise from the presence of the term $C(\zeta)$ in (2.11). In a pathological case, say if some portion of ∂D is totally disconnected, we know only that $C(\zeta)$ is some continuous function there. Usually we shall assume that *∂D consists of a finite union of continua*, say $\Gamma_1, \cdots, \Gamma_n$. In that case $C(\zeta)$ is a constant C_j on each Γ_j and (2.11) becomes

$$h(\zeta) = \bar{\zeta} + C_j, \quad \zeta \in \Gamma_j \qquad (j = 1, 2, \cdots, n). \tag{2.12}$$

Here we arrive at a boundary value problem like that discussed in the previous paragraph, but with the difference that on each boundary component the boundary values $\bar{\zeta}$ may be adjusted by an arbitrary additive constant. Thus, *a priori* there is more chance to solve this boundary value problem than (2.8), corresponding to the fact that D is now required to satisfy a q.i. for a smaller class of test functions. Despite the similarity of the two problems, it will turn out that there is a profound difference in the character of the possible domains D; for instance, in case (2.8) ∂D is part of some algebraic curve, whereas for (2.12), unless all the C_j are equal, no sub-arc of ∂D can be algebraic.

3. The simply connected case

Theorem 1. *Let D be a simply connected domain such that*

$$\int_D \frac{d\sigma}{|z|} < \infty. \tag{3.1}$$

The necessary and sufficient condition that D satisfy a quadrature identity (1.2) *for all $f \in L^1_a(D)$ is that some (and hence every) conformal map of the unit disc U on D be a rational function with all poles outside U^-. More precisely: let φ map U conformally on D, and let t_j be the preimage of z_j $(j = 1, 2, \cdots, n)$. Then φ is a rational function whose poles lie at the points $t_j^* = 1/\bar{t}_j$. The order of the pole at t_j^* is r_j. Conversely, if φ is any rational function univalent in U, with poles of order r_j at points $t_j^*, |t_j^*| > 1$ and no other poles, then $D = \varphi(U)$ admits a q.i. of the form* (1.2).

Proof. In the proof we shall use the fact, already referred to (and proved in the following Section) that *the hypotheses imply ∂D is a subset of an algebraic curve.* It would not be hard to avoid this "loan" at the expense either of a more complicated proof, or weakening the theorem slightly by assuming *a priori* that D is bounded.

Suppose then that D admits the q.i. (1.2), which we may for brevity write in the form (2.7). By Lemma 2.3 there is a function h analytic in D except for poles of order r_j at the points z_j, continuously extendible to $\bar{D}$, and satisfying

$$h(\zeta) = \bar{\zeta}, \quad \zeta \in \partial D. \tag{3.2}$$

Let φ map U conformally on D. Then the regularity of ∂D ensures that φ is continuously extendible to ∂U except perhaps for a finite subset E of ∂U (consisting of points τ such that $\lim_{t\to\tau}|\varphi(t)|=\infty$); it will turn out that E is, in fact, empty but we do not assume *a priori* that D is bounded. From (3.2) we have

$$h(\varphi(t))=\overline{\varphi(t)},\quad t\in\partial U\backslash E. \tag{3.3}$$

Now, the function ψ defined by

$$\psi(s)=\overline{\varphi(1/\bar{s})},\quad |s|>1$$

is analytic in $|s|>1$, continuously extendible to $|s|\geqq 1$ except for points $s\in E$, and satisfies $\psi(t)=\overline{\varphi(t)}$ for $t\in\partial U\backslash E$. From (3.3) we see therefore that the function

$$F(t)=\begin{cases} h(\varphi(t)), & |t|<1 \\ \psi(t), & |t|>1 \end{cases} \tag{3.4}$$

is analytically continuable across all points of $\partial U\backslash E$ and is thus analytic and single-valued on the Riemann sphere except for

a) poles of order r_j at the points $t_j\in U$ where $\varphi(t_j)=z_j$

b) singularities at points of E.

We now show that E must be empty. Indeed, suppose $\tau\in E$, and let V be a neighbourhood of τ containing no other point of E, nor any t_j. Observe that (3.1) is equivalent to

$$\int_0^{2\pi}\int_0^1 \frac{|\varphi'(re^{i\theta})|^2}{|\varphi(re^{i\theta})|}\,r\,dr\,d\theta<\infty. \tag{3.5}$$

Since φ is univalent, the distortion theorem [5, p. 394] implies $|\varphi(re^{i\theta})|^{-1}>c(1-r)^2$ for $\frac{1}{2}<r<1$, and hence

$$\int_0^{2\pi}\int_0^1 (1-r)^2|\varphi'(re^{i\theta})|^2\,r\,dr\,d\theta<\infty.$$

By simple estimates, the details of which are left to the reader, the last inequality implies $|\varphi'(re^{i\theta})|=O(1-r)^{-2}$, hence $|\varphi(re^{i\theta})|=O(1-r)^{-1}$, and feeding this improved estimate back into (3.5) and repeating the reasoning yields $|\varphi(re^{i\theta})|=O(1-r)^{1/2}$. But this, together with (2.9) and (3.4) imply that $F(t)=O(1-|t|)^{-1/2}$

on $V \cap U$, so τ cannot be a pole of F. Since moreover ψ is univalent in $|t| > 1$, $F(t) = O(|t - \tau|^{-2})$ in a full neighborhood V, hence τ cannot be an essential singularity either. We conclude that E is empty, and consequently ***F is a rational function.*** From the relation

$$\varphi(t) = \overline{F(1/\bar{t})}, \quad t \in U$$

we see that φ too is a rational function and its poles are at the points $t_j^* = 1/\bar{t}_j$, with multiplicities r_j.

To complete the proof we have only to show that if φ is a univalent function in U of this form, then $D = \varphi(U)$ satisfies a q.i. of the form (1.2). In view of Lemma 2.3 we have only to show that $\bar{z}|_{\partial D}$ is the (continuous) boundary value of a function analytic in D except for poles of order r_j at the points z_j. This is the same as saying there is a function analytic in U except for poles of order r_j at the points t_j, and attaining (continuously) the boundary values $\overline{\varphi(t)}$ on $|t| = 1$. The function $\Psi(t) = \overline{\varphi(1/\bar{t})}$ does just this. The proof is finished.

The question of uniqueness. Consider a specific q.i. of the form (1.2). Is it possible for each of two distinct domains D_1, D_2 to admit this q.i. for all $f \in L^1_a(D_1 \cup D_2)$? Even assuming D_1 and D_2 are simply connected, in which case Theorem 1 tells us much about the nature of these domains, we could not prove uniqueness. Possibly uniqueness does not hold in general. If we restrict attention to the even more special case when D_1 and D_2 are Jordan domains, the problem is equivalent to the following: *Let D_1 and D_2 be smoothly bounded Jordan domains with non-empty intersection, and suppose there exists an elementary distribution α with support in $D_1 \cap D_2$ such that*

$$\int_{D_1} f d\sigma = \int_{D_2} f d\sigma = (\alpha, f) \tag{3.6}$$

holds for all polynomials f. Must we have $D_1 = D_2$?

If we only consider the first equality in (3.6), we are led to the following moment problem, an affirmative answer to which would imply an affirmative answer in the preceding problem:

Let D_1 and D_2 be smoothly bounded Jordan domains such that

$$\int_{D_1} z^n d\sigma = \int_{D_2} z^n d\sigma \qquad n = 0, 1, 2, \cdots.$$

Must we have $D_1 = D_2$?

As a moment problem in its own right, there is no need to assume *a priori* that $D_1 \cap D_2$ is non-empty. It is easy to prove the answer is "yes" if D_1^- and D_2^- are disjoint, or intersect in just one point.

4. Domains admitting quadrature identities in L_a^1

We begin with a general proposition which is the cornerstone of our method.

Theorem 2. *Let D be a bounded plane domain. Suppose f and g are holomorphic in D except for finitely many polar singularities. Suppose moreover that f and g are continuously extendible to ∂D and there take only real values. Then there is a non-trivial polynomial $P(X, Y)$ such that $P(f(z), g(z)) = 0$. Moreover, P can be taken to have real coefficients, and be irreducible over the complex field.*

Proof. The proof is very similar to one of the known proofs of the classical theorem [5] that, on a compact Riemann surface, any two meromorphic functions satisfy a polynomial relation. Let m denote the total number of poles (counting multiplicities) of f and g, and denote by P the set of these poles. Choose a positive integer n, and consider the functions

$$f^j g^k; \quad j \geqq 0, \quad k \geqq 0, \quad 1 \leqq j + k \leqq n. \tag{4.1}$$

Their number is $n(n+3)/2$. Each is analytic in D except for poles which lie in the set P, their total number (counting multiplicities) not exceeding mn. If each of these functions is expressed in the canonical way as the sum of a rational function (its "principal part") and a function analytic in D, all these principal parts lie in a certain vector space whose dimension is at most mn over the complex scalars, at most $2mn$ over the real scalars. Now, fix n so large that $n(n+3)/2$ exceeds $2mn$. Then, there exists a nontrivial linear combination of the functions (4.1) with real coefficients whose principal part vanishes, i.e. without poles in D. Being real on ∂D, this function must be a (real) constant. In other words, there is a nontrivial polynomial Q with real coefficients such that $Q(f, g) = 0$. Among all such polynomials choose one, which we denote by P_0, that has least degree d. To complete the proof we have only to show that P_0 is irreducible over the complex field. Suppose not, then $P_0 = P_1 P_2$ where P_1 and P_2 are polynomials (with complex coefficients) of degree less than d. Since $P_1(f, g)$ and $P_2(f, g)$ are meromorphic in D and their product is zero, one of them, say $P_1(f, g)$, is 0. Write $P_1 = P_3 + iP_4$ where P_3 and P_4 are polynomials with real coefficients, and each has degree less than d. Then

$$P_3(f(z), g(z)) + iP_4(f(z), g(z)) = 0, \quad z \in \partial D.$$

Because $f(z)$ and $g(z)$ are real on ∂D, so are $P_3(f(z), g(z))$ and $P_4(f(z), g(z))$, therefore both of these functions vanish on ∂D, and hence identically. Thus $P_3(f, g)$ and $P_4(f, g)$ are zero, and since at least one of them is non-constant, we have contradicted the supposed minimality of d. The proof is complete.

Remark. For later purposes, observe that the proof goes through unchanged even if f, g have a finite number of polar singularities on ∂D.

We are now in a position to prove

Theorem 3. *Let D be any plane open set such that*

$$\int_D \frac{d\sigma}{|z|} < \infty \tag{4.2}$$

and suppose that D admits a q.i. with respect to all $f \in L_a^1(D)$. Then there exists a non-constant polynomial $P(X, Y)$ with real coefficients, irreducible over the complex field, such that $P(x, y) = 0$ for all $x + iy \in \partial D$.

Before giving the proof, let us point out some simple corollaries (the first of which is unfortunate from the standpoint of possible utility of these q.i. in numerical analysis):

Corollary 1. *A triangle admits no q.i.*

Corollary 2. *An annulus admits no q.i.*

Indeed, in each of these cases there is no *irreducible* polynomial which vanishes on the whole boundary.

Proof of Theorem 3. By Lemma 2.3 there is a function h analytic in D except for poles, continuously extendible to ∂D and satisfying $h(z) = \bar{z}$ for $z \in \partial D$. Hence the functions defined by

$$f(z) = \frac{1}{2}(z + h(z))$$

and

$$g(z) = \frac{1}{2i}(z - h(z))$$

are analytic in D except for poles, continuously extendible to D^-, and moreover

$$f(z) = x, \quad g(z) = y \quad \text{for all} \quad x + iy \in \partial D. \tag{4.3}$$

Now, *suppose first that D is bounded.* Applying Theorem 2, we get a nonconstant polynomial P with real coefficients, irreducible over C, and satisfying $P(f, g) = 0$ whence by (4.3), $P(x, y) = 0$ for all $x + iy \in \partial D$, and the theorem is proved in this case.

If D is unbounded, a more elaborate argument of Phragmén–Lindelöf type is needed, since Theorem 2 is not generally applicable to unbounded domains, and we proceed as follows. As in the proof of Theorem 2, we can in any case find a nonconstant polynomial P with real coefficients, irreducible over C, such that $P(f, g)$ is analytic in all of D. Thus the function $v(z) = \operatorname{Im} P(f(z), g(z))$ is harmonic in D, continuously extendible to D^-, and vanishes on ∂D. We have to show $v = 0$.

Observe first that (because of (2.9)) v is of polynomial growth, that is, for some $N > 0$,

$$v(z) = O(|z|^N); \quad |z| \to \infty, \quad z \in D. \tag{4.4}$$

Now, let D_0 be the domain obtained from D by omitting a small disc around each pole of h. By (2.9) we have for $z \in D_0$

$$h(z) = \bar{z} + O(|z|^{1/2})$$

$$zh(z) = |z|^2 + O(|z|^{3/2}), \quad z \to \infty,$$

and hence for each positive integer m

$$(zh(z))^m = |z|^{2m} + O(|z|^{2m-\frac{1}{2}}), \quad z \in D_0.$$

Now, fix $m > N/2$. The function $U(z) = \operatorname{Re}(zh(z))^m$ is harmonic in D_0, nonnegative on ∂D and satisfies

$$U(z) = |z|^{2m} + O(|z|^{2m-\frac{1}{2}}); \quad |z| \to \infty, \quad z \in D_0. \tag{4.5}$$

Fix a number ε, $0 < \varepsilon < 1$. By virtue of (4.4) and (4.5), the function $v - \varepsilon U$, which is harmonic in D_0, tends to $-\infty$ as $|z| \to \infty$, $z \in D_0$, hence it is bounded above in D_0. By the (extended) maximum principle it is therefore bounded there by its supremum on ∂D_0. Now, on ∂D it is less than or equal to zero, and therefore for all $z \in D_0$,

$$v(z) - \varepsilon U(z) \leqq \max_{z \in K} [|v(z)| + |U(z)|] = M$$

where K denotes the compact set $\partial D_0 \backslash \partial D$ (i.e. the union of the little circles). Letting $\varepsilon \to 0$ we obtain

$$\sup_{z \in D_0} v(z) \leqq M.$$

Thus we have proved v is bounded above on D_0, and hence on D. A similar argument applied to $-v$ shows v is bounded below on D. Since v is bounded on D and vanishes on ∂D, $v = 0$ and the proof is complete.

Strictly speaking, Theorem 3 does not contain the Epstein–Schiffer result (in two dimensions), however it is easy to deduce the latter by adapting the reasoning to the special situation. Suppose, indeed, that $0 \in D$ and $\int_D f d\sigma = af(0)$ for all $f \in L_a^1(D)$. Then h is analytic in D except for a simple pole at $z = 0$, and $h(z) = \bar{z}$ on ∂D. Therefore $zh(z)$ is analytic in D and real on ∂D. Therefore it is constant, i.e. $z\bar{z}$ is equal to some constant R^2 on ∂D. Since ∂D is a subset of the circle $\{|z| = R\}$ it follows easily that D is the disc $|z| < R$, and substituting $f = 1$ in the q.i. gives $\pi R^2 = a$. This is exactly the Epstein–Schiffer proof, except that the "Phragmén–Lindelöf" part of the argument, which is new, enables us to dispense with their hypothesis that D has finite area, in favour of the weaker hypothesis (4.2). The ultimate generalization, no doubt, is that the same conclusion holds under the weakest hypothesis for which the theorem is meaningful, i.e. whenever D is such that $L_a^1(D)$ contains at least one function not identically zero. Our method, however, essentially requires (4.2) in order to be able to define the Cauchy transform of 1_D.

If we wish to make a *detailed* analysis of a *specific* q.i. by the method employed in Theorem 3, we must carry out the "elimination of poles" step explicitly. This we shall now do in a concrete case.

Theorem 4. *Let D be a plane open set containing* 0 *and satisfying* (4.2). *Suppose for some constants a_0, a_1*

$$\int_D f d\sigma = a_0 f(0) + a_1 f'(0), \quad \textit{all} \quad f \in L_a^1(D). \tag{4.6}$$

Then, there is a quadratic polynomial $Q(z)$ satisfying $Q(0) = 0$, and univalent in U, such that $D = Q(U)$. Conversely, if Q is any such polynomial, $D = Q(U)$ admits the q.i. (4.6).

Proof. Suppose first D is simply connected. Then the theorem follows at once from Theorem 1, since it tells us that any (simply connected) domain for which (4.6) holds is the image of U under a univalent rational function whose only singularities are a double pole at infinity, i.e. a quadratic polynomial; and conversely. Thus, to complete the proof we have to establish that *D is simply connected.* We shall first prove this *under the additional assumption that D is bounded.* Now, by virtue of