Lemma 2.3 there is a function h analytic in D, except for a double pole at $z=0$, that is

$$h(z)=\frac{b_1}{z}+\frac{b_2}{z^2}+H(z),\quad H \text{ analytic in } D \tag{4.7}$$

satisfying $h(z)=\bar{z}$, $z\in\partial D$. Observe first that *there is no loss of generality in assuming b_2 real and positive*. Indeed, if $b_2=0$ we are back to the Epstein–Schiffer case. If $b_2\neq 0$, make the change of variables $z=\lambda\zeta$ where λ is a complex number of modulus one, and write $D_0=\bar{\lambda}D$, $h_0(\zeta)=\lambda h(\lambda\zeta)$, $H_0(\zeta)=\lambda H(\lambda\zeta)$. Then $h_0(\zeta)=\bar{\zeta}$ for $\zeta\in\partial D_0$, and (4.7) becomes

$$h_0(\zeta)=\frac{b_1}{\zeta}+\frac{b_2}{\lambda\zeta^2}+H_0(\zeta),\quad H_0 \text{ analytic in } D_0. \tag{4.7'}$$

For suitable choice of λ, the coefficient of ζ^{-2} in (4.7') is real and positive. So, we assume henceforth $b_2>0$.

Now, by Theorem 3, D is bounded by smooth curves; denoting by Γ the positively oriented boundary of D we have, using (4.7)

$$\frac{1}{2\pi i}\int_\Gamma \bar{z}\,d\sigma=\frac{1}{2\pi i}\int_\Gamma h(z)dz=b_1.$$

The left-hand integral is π^{-1} times the area of D, hence $b_1>0$.

Using (4.7), we obtain, for $z\in\partial D$

$$b_1+\frac{b_2}{z}+zH(z)=z\bar{z}$$

$$\frac{b_1}{z}+\frac{b_2}{z^2}+(z+H(z))=z+\bar{z}$$

$$\frac{2b_1b_2}{z}+\frac{b_2^2}{z^2}+H_1(z)=(z\bar{z})^2,$$

where the third of these equations is gotten by squaring the first; H_1 stands for a certain function analytic in D. Multiplying these equations by $-b_1$, $-b_2$, 1 respectively and adding causes the z^{-1} and z^{-2} terms in the left hand members to cancel. We therefore get a function analytic in D, which, on ∂D is equal to the real-valued function $-b_1z\bar{z}-b_2(z+\bar{z})+(z\bar{z})^2$, hence both functions are constant; in particular,

$$(z\bar{z})^2 - b_1 z\bar{z} - b_2(z+\bar{z}) = \text{const.}, \quad z \in \partial D.$$

Denoting the (real) constant on the right by c and (for more uniform notation) writing $b_1 = a$, $b_2 = b$, we have therefore shown: *The boundary of D is a subset of the curve C whose equation is*

$$P(x,y) \equiv (x^2+y^2)^2 - a(x^2+y^2) - 2bx - c = 0. \tag{4.8}$$

Here a, b, c are positive constants (that $c > 0$ follows because P is positive in the unbounded component of the complement of D^-, hence negative on D, so that $-c = P(0,0) < 0$).

Since no straight line can intersect C in more than four points, ∂D cannot have more than two components. If it has one component, D is simply connected, which is what we wish to prove; so we shall assume that D has connectivity two, and derive a contradiction. The assumption implies that C consists of two closed loops, lying one within the other, and D is the domain between them. Let us denote the inner and outer loops by C_1, C_2 respectively. Writing (4.8) as

$$z^2\bar{z}^2 - (az+b)\bar{z} - (bz+c) = 0$$

and solving for $\bar{z}$, we get

$$2\bar{z} = z^{-2}(az + b + Q(z)^{1/2}), \quad z \in \partial D \tag{4.9}$$

where

$$Q(z) = 4bz^3 + (a^2+4c)z^2 + 2abz + b^2.$$

Now, by assumption, $\bar{z}$ coincides on ∂D with the function $h(z)$ which is analytic in D except for a double pole at $z = 0$. Let $x_1 < x_2 < x_3 < x_4$ denote the four intersections of C with the real axis. They are the four roots of

$$p(x) \equiv x^4 - ax^2 - 2bx - c = 0. \tag{4.10}$$

Since $0 \in D$, $x_4 > 0$, hence $Q(x_4) > 0$. If we take the *positive* square root in (4.9) the equation is valid at $z = x_4$. Indeed, in the contrary case we should have

$$2x_4 = x_4^{-2}(ax_4 + b - A)$$

where A is the positive square root of $Q(x_4)$.

Then

$$2x_4^3 - ax_4 - b = -A.$$

But, the left hand side is $p'(x_4)/2$; thus $p'(x_4) < 0$. However, since p has four real roots, it is clear that $p'(x_4) > 0$. It follows that $2h(z)$ *coincides throughout D with the function*

$$F(z) = z^{-2}(az + b + Q(z)^{1/2}).$$

Here F is defined by analytic continuation of that branch of $Q(z)^{1/2}$ which is positive at $z = x_4$; continuation along all paths in D must lead to a single-valued function. From this we shall now derive a contradiction. Indeed, $Q(z)$ cannot have a zero of *odd* order on the interval (x_3, x_4) (otherwise F would have a branching singularity in D). There are now two alternatives: (a) Q does not vanish anywhere on (x_3, x_4), *or* (b) Q has at least one zero of *even* order on (x_3, x_4). Suppose first that (a) holds. This implies that, continuing analytically the original branch of $Q(z)^{1/2}$ from x_4 to x_3, we arrive at a non-negative value for $Q(x_3)^{1/2}$. Thus

$$2x_3 = 2h(x_3) = F(x_3) = x_3^{-2}(ax_3 + b + B)$$

where $B = Q(x_3)^{1/2} \geqq 0$. This implies, however, that $p'(x_3) \geqq 0$. But $p'(x_3) < 0$, and we have reached a contradiction.

Hence alternative (b) must hold, therefore Q has a zero of order at least 2 on (x_3, x_4). An identical reasoning also establishes that Q has a zero of order at least 2 on (x_1, x_2). Thus Q has at least four zeros, contradicting the fact that it is of degree 3. This concludes the proof of Theorem 4 for bounded D.

It remains only to prove *a priori* that a domain satisfying the hypotheses of the theorem is bounded. Boundedness entered in the proof that b_1 in (4.7) is positive; we now permit b_1, b_2 in (4.7) to be any complex numbers. Consider the 7 functions $zh(z), (zh(z))^2, (zh(z))^3, z + h(z), i(z - h(z)), zh(z)(z + h(z)), izh(z)(z - h(z))$. Each of these is analytic in D except for a pole at $z = 0$ of order at most 3, and is real on ∂D. Hence, by the reasoning employed in proving Theorem 3, some nontrivial linear combination of these functions with real coefficients is analytic in D and, being real on ∂D, is constant. Hence there are real constants c_i not all zero, such that

$$c_1(x^2+y^2)^3 + c_2(x^2+y^2)^2 + (x^2+y^2)(c_3x + c_4y) + c_5(x^2+y^2) + c_6x + c_7y + c_8 = 0$$

for all $x + iy \in \partial D$. A simple enumeration of cases, which we omit, shows that for every choice of the c_i the resulting (real) locus either is bounded, or else fails to bound a domain for which (4.2) holds. This completes the proof of Theorem 4.

We can complete Theorem 4 by showing: *Suppose D_1 and D_2 are plane open sets satisfying* (4.2), *and each satisfies* (4.6) (*with the same a_0, a_1*). *Then $D_1 = D_2$.*

Proof. We have already shown that D_1 and D_2 are images of U under conformal maps by quadratic polynomials. The hypotheses imply

$$\int_{D_1} w^n d\sigma = \int_{D_2} w^n d\sigma\,; \qquad n = 0, 1, 2, \cdots$$

or, if Q_1, Q_2 are the respective mapping functions

$$\int_U Q_1(z)^n\,|\,Q_1'(z)|^2\,d\sigma = \int_U Q_2(z)^n\,|\,Q_2'(z)|^2\,d\sigma\,; \qquad n = 0, 1, 2, \cdots. \tag{4.11}$$

Hence, we have only to prove: *if $Q_1(z) = b_1 z + c_1 z^2$ and $Q_2(z) = b_2 z + c_2 z^2$ are univalent in U and satisfy* (4.11), *and* $\arg b_1 = \arg b_2$, *then $Q_1 = Q_2$.*

Clearly there is no loss of generality if we suppose $b_1 = 1$. By simple computations, the cases $n = 0$, $n = 1$ of (4.11) yield

$$|\,b_1|^2 + 2\,|\,c_1|^2 = |\,b_2|^2 + 2\,|\,c_2|^2 \tag{4.12}$$

$$b_1^2 \bar{c}_1 = b_2^2 \bar{c}_2\,. \tag{4.13}$$

Writing $B_i = |\,b_i|^2$, $C_i = |\,c_i|^2$, $i = 1, 2$ we have

$$\begin{aligned} B_1 + 2C_1 &= B_2 + 2C_2 \\ B_1^2 C_1 &= B_2^2 C_2\,. \end{aligned} \tag{4.14}$$

Now, the requirement of univalence implies that $Q_1'(z)$ and $Q_2'(z)$ have no zeroes in U, which implies

$$|\,b_j| \geqq 2\,|\,c_j|\,, \quad j = 1, 2$$

or

$$B_j \geqq 4C_j, \quad j = 1, 2. \tag{4.15}$$

It is enough to show, under these assumptions, that $B_1 = B_2$, for then $b_1 = b_2$, and (from (4.13)) $c_1 = c_2$, whence $Q_1 = Q_2$ follows. Thus, we have to show that for given positive S, T the equations

$$B+2C=S, \quad B^2C=T, \quad B \geqq 4C$$

have at most one positive solution, i.e. $B^2(S-B)=2T$ has at most one root B satisfying $B \geqq 2(S-B)$, that is $B \geqq (2/3)S$. Putting $x = B/S$ this comes down to showing that $x^2(1-x)=2T/S^3$ has at most one root with $2/3 \leqq x \leqq 1$, which is evident since $x^2(1-x)$ is monotone on this interval. This completes the proof.

It should be emphasized that the univalence was used in an essential way: purely algebraically the system (4.11) (which is equivalent to (4.12) and (4.13)) does *not* imply $|b_1|=|b_2|$. This is why the general uniqueness problem discussed at the end of Section 3 appears difficult, and may well have a negative answer in general. For example, the polynomials

$$Q_1(z)=z+\frac{z^2}{12}, \quad Q_2(z)=\frac{\sqrt{2}}{4}z+\frac{2}{3}z^2$$

satisfy (4.11). Q_1 is univalent in U, while Q_2 is not. This may be interpreted so, that the *doubly-sheeted* domain (Riemann surface) $Q_2(U)$ admits the same q.i. as does the Jordan domain $Q_1(U)$.

It follows from Theorem 3 that if D satisfies a q.i. it has finite connectivity, say n. As is well known, there is a canonical conformal map $w=\varphi(z)$ of D onto an n-times covered unit disc; φ can be chosen so that an arbitrary point of D maps into $w=0$, and then it is uniquely determined apart from a constant factor (see [4]). This function is sometimes called the *Ahlfors function* of D.

Theorem 5. *If D satisfies the hypotheses of Theorem* 3, *the Ahlfors function of D is algebraic.*

Proof. The function occurring in the proof of Theorem 3 satisfies $h(z)=\bar{z}$ on ∂D. As we saw, there exists a nonconstant polynomial $P(X, Y)$ such that $P(h(z)+z, h(z)-z)=0$, *hence h is an algebraic function.*

Now, since $|\varphi(z)|=1$ for $z \in \partial U$ the function $\Psi(z)=i(\varphi(z)+1)(\varphi(z)-1)^{-1}$ is analytic in D and has real boundary values (it has poles at points of ∂D where $\varphi(z)=1$). By Theorem 3 there is a non-constant polynomial $Q(X, Y)$ such that $Q(\Psi(z), h(z)+z)=0$. Hence Ψ, and finally also φ, is algebraic.

Theorem 6. *Under the hypotheses of Theorem* 3, *the leading form* (*i.e. homogeneous terms of highest degree*) *of P is divisible by* X^2+Y^2.

Proof. We know that P has real coefficients and

$$P\left(\frac{z+h(z)}{2}, \frac{z-h(z)}{2i}\right) = 0. \tag{4.16}$$

Let P_0 denote the leading form of P, say

$$P_0(X, Y) = \sum_{j=0}^{n} c_j X^j Y^{n-j}$$

where at least one c_j is non-zero. Now, (4.16) can be rearranged in the form $Q(z, h(z)) = 0$ where Q is a certain polynomial of degree n. The coefficient of h^n in this expression must vanish, indeed, all other terms involve h^j with $j < n$, and this would lead to a contradiction when $z \to z_0$, a pole of h, unless the h^n term is absent. The coefficient of h^n is clearly $P_0\left(\frac{1}{2}, \frac{i}{2}\right) = 2^{-n} P_0(1, i)$, hence $P_0(1, i) = 0$. Writing

$$P_0(X, Y) = X^n p(Y/X)$$

where p is a polynomial with real coefficients (in one variable) of degree n, we see that $p(i) = 0$, hence $p(t) = (t^2 + 1) q(t)$ for some polynomial q of degree $n-2$, and $P_0(X, Y) = (X^2 + Y^2) \cdot X^{n-2} q(Y/X)$ proving the assertion.

As a corollary, for instance, *the domain bounded by an ellipse satisfies no q.i.*

Combining Theorems 1, 3 and 6 we get the following corollary about rational conformal maps, independent of the notion of q.i.:

Suppose D is the image of U under a conformal map by a rational function without poles in U^-. Then there is a non-constant polynomial $P(X, Y)$ with real coefficients irreducible over **C**, *such that $P(x, y) = 0$ for all $x + iy \in \partial D$. Moreover, the leading form of P is divisible by $X^2 + Y^2$.*

This proposition, apart possibly from the last sentence, applies to any curve (not necessarily simple) which is the image of a circle under the mapping by a rational function, regardless whether it is univalent or where its poles lie; this can easily be shown directly.

After our completion of this manuscript, Björn Gustafsson, using a Riemann surface argument (some details of which are mentioned below in Section 6.2) showed us a considerable improvement of Theorem 6, namely

Theorem 6' (Gustafsson). *Under the hypotheses of Theorem 3, the leading form of P is a non-zero constant times $(X^2 + Y^2)^n$. Here n is the order of the q.i., i.e. the sum of the numbers r_j appearing in* (1.2), *or what is the same thing, the number of poles of the associated function h.*

Quite likely Gustafsson's line of reasoning could culminate in a complete and explicit description of all domains D for which (4.2) holds, which satisfy q.i. relative to $L_a^1(D)$.

5. Domains admitting q.i. in $L_{a,s}^1$

We begin with a special theorem for one-point q.i., communicated to us by M. Schiffer; here a result of greater finality is attained than in the case of general q.i.

Theorem 7. *Let D be a bounded domain of finite connectivity, and $z_0 \in D$. If* (1.1) *holds for all $f \in L_{a,s}^1(D)$, then D is a circular disc centered at z_0.*

Proof. Let n denote the connectivity, and $\partial D = \bigcup_{j=1}^n \Gamma_j$ (we suppose none of the Γ_j is a point, the exceptional cases being left to the reader). By Lemma 2.4, there is a function h analytic in D except for a simple pole at z_0 and continuously extendible to D^-, such that (2.12) holds. Let

$$\varphi(z) = z + h(z), \quad \Psi(z) = -z + h(z).$$

Then, φ, Ψ are analytic in D except for simple poles at 0, and continuously extendible to D^-. Because of (2.12), the boundary values of φ lie on a union of n horizontal segments, those of Ψ lie on a union of n vertical segments. By a standard application of the argument principle, φ and Ψ are univalent in D. Hence the n horizontal segments are pairwise disjoint, likewise for the vertical segments, so that φ and Ψ are (apart from normalization) the canonical slit mappings of D. Each of them has, at their common pole z_0, the same residue which we may assume is 1. Now, $\varphi' - \Psi'$ has $2n-2$ zeroes in D (see Nehari [14, p. 340]). However, $\varphi' - \Psi' = 2$. so we have a contradiction unless $n = 1$. Thus, D is simply connected and hence, by Epstein's theorem, a disc centered at z_0.

We turn next to some differential-geometric aspects of the domains D admitting q.i. *Henceforth in this chapter, unless otherwise specified, we shall always assume D to be of finite connectivity, bounded by continua $\Gamma_1, \cdots, \Gamma_n$.*

Lemma 5.1. *Let D, bounded by the continua $\Gamma_1, \cdots, \Gamma_n$ satisfy the q.i.* (2.10). *Then each Γ_j is an analytic arc or Jordan curve. Moreover, the function h of Lemma* 2.4 *is analytic on D^-, and satisfies*

$$h'(z) = \overline{T(z)}^2, \quad z \in \partial D \tag{5.1}$$

$$h''(z) = -2ik(z)\overline{T(z)}^3, \quad z \in \partial D. \tag{5.2}$$

Here $T(z)$ denotes a unit tangent vector to the (oriented) boundary Γ of D at the point z (represented as a complex number) and $k(z)$ is the curvature of Γ at z.

Proof. The analyticity of the Γ_i follows from a local result to be proved in Section 6. Since the boundary values of $h(z)+z$ on each Γ_i lie on a horizontal segment, $h(z)+z$, and hence h, is analytically continuable across the Γ_j. Fix a point z_0 on the boundary component Γ_j, and let us represent Γ_j by the equation $z = z(s)$, when s is arc length along Γ_j measured from $z_0 = z(0)$. Then, from (2.12) $h(z(s)) = \overline{z(s)} + C_j$.

Differentiating with respect to s, and denoting z-derivatives by dashes

$$h'(z(s))\frac{dz}{ds} = \overline{\frac{dz}{ds}}. \tag{5.3}$$

Since dz/ds is a unit tangent vector at the point $z(s)$, this is just (5.1). Another differentiation in (5.3) yields

$$n''(z(s))\frac{dz}{ds} = 2\overline{\frac{dz}{ds}}\cdot\overline{\frac{d^2z}{ds^2}} \tag{5.4}$$

and, since $(d^2z/ds^2) = (d/ds)(T(z(s)))$ is the curvature at $z(s)$ times a unit normal vector, we have $(d^2z/ds^2) = ik(z)T(z)$. Substituting this into (5.4) gives (5.2).

Corollary. *Under the stated hypotheses, the functions (defined on ∂D) $\overline{T(z)}^2$ (T: unit tangent vector) and $k(z)^2$ (curvature) are the boundary values of functions analytic in D except for polar singularities (whose location moreover can be specified for each concrete q.i.).*

Theorem 8. *Let D be a domain of finite connectivity satisfying*

$$\int_D \frac{d\sigma}{|z|} < \infty \tag{5.5}$$

and admitting a q.i. for all $f \in L^1_{a.s}(D)$. Let $\mu(z)$, $z \in \Gamma$ denote the slope of the tangent line to Γ at the point z, and $k(z)$ the curvature there (since, as we know, Γ consists of analytic arcs, these quantities are well-defined). Then there is a non-constant polynomial $P(X, Y)$ with real coefficients, irreducible over the complex field, such that $P(\mu(z), k(z)^2) = 0$ for all $z \in \Gamma$.

Remark. By the slope we mean, as is customary, the tangent of the angle made with the x-axis; replacing the x-axis by another reference line alters $\mu(z)$ rationally, and we would of course then get a different polynomial $P(X, Y)$.

Proof. Suppose first that D is bounded. Let α denote the angle, which the tangent line to Γ at z_0 makes with the x-axis. Then

$$\mu(z_0) = \tan\alpha = i\frac{e^{-2i\alpha}-1}{e^{-2i\alpha}+1} = i\frac{\overline{T(z_0)}^2-1}{\overline{T(z_0)}^2+1}.$$

As we see from (5.1), the last expression is the boundary value, at $z = z_0 \in \partial D$, of a certain function analytic in D except for poles, and continuously extendible to ∂D except for finitely many poles (where $T(z) = -1$; their number is finite because of the analyticity of the boundary). Since $k(z)^2$ is the boundary value of a function in D of similar nature, the conclusion now follows from Theorem 2. The case when D is unbounded but merely satisfies (5.5) follows by our standard Phragmén–Lindelöf argument.

Remark 1. All "familiar" domains, except the circular disc, seem to be excluded as carriers of q.i. by the preceding results. In particular, no such D can have a line segment on its boundary, nor a circular arc except in the case of a circular disc.

Remark 2. We can add to Theorem 8 the observation: *the leading form of $P(X, Y)$ is divisible by Y*, i.e. the X^n term (n = degree P) vanishes. For otherwise $P(\mu(z), k(z)^2)$ would have a term $c \cdot \mu(z)^n$ standing alone, $c \neq 0$, and so could not vanish at points where $\mu(z) \to \infty$.

Theorem 9. *If D is of finite connectivity, satisfies (5.5) and satisfies a q.i. for all $f \in L^1_{a,s}(D)$ its Ahlfors function φ satisfies a differential equation of the form $Q(\varphi', \varphi) = 0$ where Q is a non-constant polynomial.*

Proof. By Lemma 5.1,

$$\frac{h''(z)^2}{h'(z)^3} = 4k(z)^2, \quad z \in \partial D$$

i.e. the function on the left has real boundary values on ∂D. Since also h' and φ have absolute value one on ∂D we deduce by our usual argument that any two of the three functions $(h'')^2/(h')^3$, h', φ satisfy a non-trivial algebraic relation. In particular, there exists an algebraic function F such that $\varphi = F(h')$, hence $\varphi' = F'(h')h''$. Since h'' is, in turn an algebraic function of h', we see that both φ

and φ' are algebraic functions of h', and hence of one another, which implies the assertion. (Of course, the theorem applies not only to the Ahlfors function, but to any function analytic except for finitely many poles in D^-, and with boundary values of modulus 1, or with real boundary values.)

Theorem 10. *Let D be as in the preceding theorem, and φ, Ψ the canonical conformal maps onto a horizontal resp. vertical slit domain. Then, there is a nonconstant polynomial Q such that $Q(\varphi', \Psi') = 0$.*

Proof. Writing $M = (\varphi - \Psi)/2$, $N = (\varphi + \Psi)/2$ it is known (see [14, p. 340]) that

a) M'/N' has modulus one on ∂D

b) $M'(z)N'(z)T(z)^2$ is real and positive on ∂D ($T(z)$ denotes, as before, a unit tangent vector at z). Now, by (5.1), $T(z)^2 = (h'(z))^{-1}$ on ∂D. Hence, any two of the three functions M'/N', $M'N'/h'$ and h' are related by a nontrivial polynomial identity. It is easy to see h' can be eliminated between these relations so as to obtain a nontrivial identity between M'/N' and $M'N'$, which implies the stated result.

Theorem 11. *Suppose D is of finite connectivity, satisfies* (5.5) *and satisfies a q.i. for all f in $L^1_{a,s}(D)$. If there is any sub-arc of ∂D which is algebraic then, in fact, the q.i. holds for all $f \in L^1_a(D)$, and consequently the conclusion of Theorem* 3 *is valid.*

Remark. This theorem shows a striking difference between the L^1_a and $L^1_{a,s}$ problems. Whereas in the former ∂D is *always* an algebraic curve, in the latter *no* portion of ∂D can be algebraic unless the q.i. actually holds in the wider class L^1_a, in other words, the constants C_j in (2.12) are all equal.

The theorem will follow easily from the following two lemmas.

Lemma 5.2. *Let G be a bounded domain and Γ_1, Γ_2 two continua lying in ∂G. Suppose h is an analytic function in G, continuously extendible to G^- and satisfying*

$$h(z) = \bar{z} + C_j; \quad z \in \Gamma_j, \quad j = 1, 2. \tag{5.6}$$

Then, if $F(X, Y)$ is any entire function of two complex variables such that

$$F(x, y) = 0, \quad x + iy \in \Gamma_1 \tag{5.7}$$

we also have

(5.8) $$F(x+ib,\ y+ia)=0, \quad x+iy \in \Gamma_1$$

where $C_2 - C_1 = a + ib$; *a and b real.*

Remark. It suffices, as the proof will show, if F is meromorphic, and h analytic and single-valued outside a compact subset of G.

Proof. By assumption

$$F\left(\frac{z+\bar{z}}{2},\ \frac{z-\bar{z}}{2i}\right)=0, \quad z \in \Gamma_1.$$

Substituting from (5.6)

$$F\left(\frac{z+h(z)-C_1}{2},\ \frac{z-h(z)+C_1}{2i}\right)=0, \quad z \in \Gamma_1.$$

Since the function on the left side is analytic throughout G, it vanishes identically. In particular, it vanishes for $z \in \Gamma_2$, where $h(z)=\bar{z}+C_2$, hence

$$F\left(\frac{z+\bar{z}+C_2-C_1}{2},\ \frac{z-\bar{z}-C_2+C_1}{2i}\right)=0, \quad z \in \Gamma_2,$$

or

(5.8) $$F(x+\lambda,\ y+i\lambda)=0, \quad x+iy \in \Gamma_2,$$

where $\lambda=(C_2-C_1)/2$. Hence

(5.9) $$F^*(x+\bar{\lambda}, y-i\bar{\lambda})=0, \quad x+iy \in \Gamma_2,$$

where $F^*(X, Y)$ denotes the entire function $\overline{F(\bar{X}, \bar{Y})}$.

Now, in exactly the same way as we derived (5.8) from (5.7), we can take (5.9) as a starting relation and obtain from it a new equation which holds on Γ_1. The result is

(5.10) $$F^*(x-\lambda+\bar{\lambda},\quad y-i\lambda-i\bar{\lambda})=0, \quad x+iy \in \Gamma_1.$$

Finally, writing $2\lambda = a+ib$, and taking complex conjugates in (5.10) we get

$$F(x+ib,\ y+ia)=0, \quad x+iy \in \Gamma_1$$

as was to be proved.

Lemma 5.3. *Let $P(x, y)$ be a polynomial with complex coefficients which satisfies the identity*

$$P(x+\alpha,\ y+\beta) = P(x, y) \tag{5.11}$$

where α, β are complex numbers not both zero. Then there exists a polynomial $p(t)$ (in one variable) such that

$$P(x, y) = p(\beta x - \alpha y). \tag{5.12}$$

(The converse is trivially true.)

Proof. From (5.11) we get, inductively

$$P(n\alpha, n\beta) = P(0,0), \quad n = 1, 2, \cdots$$

and since P is a polynomial this implies that $P(\alpha t, \beta t) - P(0,0)$ vanishes identically in t. Thus, the polynomial $P(x, y) - P(0,0)$ vanishes whenever $\beta x - \alpha y = 0$, and so is a multiple of $\beta x - \alpha y$. Thus

$$P(x, y) = P(0,0) + (\beta x - \alpha y) P_1(x, y)$$

for some polynomial P_1 of degree one less than the degree of P. It is readily verified that P_1 in turn satisfies (5.11), and now an obvious induction yields (5.12).

Proof of Theorem 11. Suppose there is an algebraic arc γ_1 lying on ∂D; we may suppose $\gamma_1 \subset \Gamma_1$. We have to prove (referring to (2.12)) that $C_j = C_1$ for $j = 2, \cdots, n$. Consider (say) $j = 2$. We suppose $C_2 \neq C_1$ and shall derive a contradiction.

Let $G \subset D$ be a bounded domain in which h is analytic, and whose boundary contains both γ_1 and some non-trivial subcontinuum γ_2 of Γ_2. By hypothesis there is some nonconstant polynomial P such that $P(x, y) = 0$ for $x + iy \in \gamma_1$. There is no loss of generality if we assume P is irreducible over **C** (since we can achieve this by passing to a sub-arc of γ_1). We now apply Lemma 5.2 with $F = P$, and obtain

$$P(x+\alpha,\ y+\beta) = 0, \quad x + iy \in \gamma_1$$

where $\alpha = ib$, $\beta = ia$ are by hypothesis not both zero ($C_2 \neq C_1$). Since $P(x, y)$ and $P(x+\alpha, y+\beta)$ both vanish on γ_1 they have a common factor; but since P is irreducible, and the two polynomials have the same leading form, they are indentical,

$$P(x+\alpha,\ y+\beta)=P(x,y).$$

But now, by Lemma 5.3, (5.12) holds. However, a polynomial of the form given by (5.12) is reducible over C, unless it is of degree one. Therefore P is linear and γ_1 is a line segment. But we have already seen that a q.i. implies D cannot have any line segment in its boundary. We have now reached a contradiction, and the theorem is proved.

Lemma 5.2 can also be used to yield other information on the structure of the boundary of domains satisfying q.i. We give only one example:

Theorem 12. *Suppose D is of finite connectivity, satisfies* (5.5) *and satisfies a q.i. for all f in $L^1_{a,s}(D)$. Suppose there is a nonconstant meromorphic function φ, real-valued on the real axis, such that $y=\varphi(x)$ for all $x+iy$ on some sub-arc γ_1 of Γ_1. Then if, in (2.12), $C_j \neq C_1$ for some j, then $b=\operatorname{Im}(C_j-C_1)\neq 0$, and φ' has period ib. Consequently, all the numbers $\{\operatorname{Im}(C_j-C_1)\}_{j=2}^n$ are integer multiples of some fixed number.*

Proof. Applying Lemma (5.2) to $F(x,y)=y-\varphi(x)$ yields

$$y+ia-\varphi(x+ib)=0,\quad x+iy\in\gamma_1 \tag{5.13}$$

where $a+ib=C_2-C_1$, which we shall suppose is not 0. Then $b\neq 0$ for $b=0$ and (5.13) imply $a=0$, a contradiction. Replacing y by $\varphi(x)$ in (5.13) and separating into real and imaginary parts gives

$$\varphi(x)=(\varphi(x+ib)+\varphi(x-ib))/2 \tag{5.14}$$

$$2ia=\varphi(x+ib)-\varphi(x-ib). \tag{5.15}$$

Since φ is entire, and (5.14) and (5.15) hold for some interval of x values (since γ_1 cannot be a vertical segment) they hold for all complex z and differentiating gives

$$\varphi'(z+ib)-2\varphi'(z)+\varphi(z-ib)=0$$

$$\varphi'(z+ib)-\varphi'(z-ib)=0.$$

These equations imply that φ' has period ib. Since a meromorphic function cannot have two pure imaginary incommensurable periods, the proof is complete.

6. Further remarks

6.1. Analyticity of the Cauchy transform

If D is a domain satisfying (2.4) we can define its Cauchy transform $S(\zeta)$ by (2.5). Then S is an analytic function outside of D^-, and continuously extendible to all of C. A basic question that has arisen in Section 5 is: *when is $S(\zeta)$ analytically continuable across a boundary arc γ of D?* The answer is: *when and only when γ is an analytic arc.* Indeed, by virtue of (2.6) this follows from the following well known lemma, whose proof we include for completeness (cf. Davis [7, p. 21]).

Lemma 6.1. *Let γ be an open Jordan arc, and $z_0 \in \gamma$. The following three assertions are equivalent:*

i) *There is a function f analytic in a neighborhood W of z_0 such that*

$$f(z) = \bar{z}, \quad z \in W \cap \gamma.$$

ii) *There is a function f analytic in a one-sided neighborhood V of z_0 (that is, the part of a full neighborhood which lies on one side of γ) and continuously extendible to γ such that*

$$f(z) = \bar{z}, \quad z \in V^- \cap \gamma.$$

iii) *There is a neighborhood V' of z_0 such that $V' \cap \gamma$ is an analytic arc.*

Proof. That (i) implies (ii) is trivial. We prove now (ii) implies (iii). Assuming (ii) holds, let z_1 and z_2 be points of $V^- \cap \gamma$ including z_0 between them, and join z_1 to z_2 by a smooth arc lying in V and having no other point in common with γ. This arc, together with γ, bounds a Jordan domain D. Let φ map the unit disc U upon D, and let $t_0 \in \partial U$ be the point which maps into z_0. We have to show φ is analytically continuable across t_0. Now, the hypothesis implies $f(z) = \bar{z}$ on some open arc of ∂D containing z_0, and hence

$$f(\varphi(t)) = \overline{\varphi(t)}, \quad t \in c \tag{6.1}$$

where c is some open subarc of ∂U containing t_0.

Now,

$$\psi(t) = \overline{\varphi(1/\bar{t})}, \quad |t| > 1 \tag{6.2}$$

is analytic for $|t| > 1$ and tends to $\overline{\varphi(s)}$ as $t \to s \in \partial U$.

mention that A. Levin has constructed an explicit example (unpublished) of a domain D of connectivity 2 such that

$$(*) \qquad \int_D f d\sigma = af(z_0) + bf'(z_0)$$

holds for all $f \in L^1_{a,s}(D)$; here z_0 is a suitable point in D. Because of Theorem 4, (*) *cannot* be true here for all f in L^1_a. Domains of arbitrary connectivity which satisfy q.i. in $L^1_{a,s}$ but not in all of L^1_a have been constructed by Gustafsson.

6.3. The Schwarz function

After the completion of our manuscript we became aware of the book [7] by Philip J. Davis. There is some overlap between this book and our paper, which we shall now discuss briefly. Inasmuch as our principal results seem still to be new, this retrospective accounting seems to us adequate; a complete rewriting of the paper would have enabled us to refer to [7] for a few computations and lemmas.

Given an analytic Jordan arc γ, the unique analytic function $S(z)$ equal to $\bar{z}$ on γ is called by Davis *the Schwarz function of* γ (this is the function we have usually called $h(z)$). Extending an earlier investigation of Davis and Pollak [6], Davis develops a series of interesting connections between conformal maps of D and the Schwarz function of ∂D. For the most part these complement, rather than anticipate, our results, but the following priorities should be especially noted.

a) Our Lemma 2.3 and Davis' theorem on p. 154 of [7] are quite similar; Davis however assumes an analytic boundary. Our method of proof is different and, as in most other places where our results overlap those in [7], we are able to get by with much weaker regularity assumptions.

b) Our Theorem 1 follows from combining the theorems on pages 154 and 158 of [7], assuming however that the domain is bounded and has an analytic boundary.

c) Various examples mentioned by us, e.g. that no triangle, annulus, or ellipse admits a q.i. are also given, or implicit, in [7].

d) The computations made by us in Sections 5, expressing the tangent vector and curvature in terms of the Schwarz function, are contained in chapter 7 of [7].

e) The elegant result proved on p. 107 of [7], that the Schwarz function of an arc extends meromorphically to the entire complex plane if and only if it is a linear fractional function, was rediscovered by us, and had been our Theorem 13; since our method of proof was exactly that of Davis, Theorem 13 has now been suppressed.

In conclusion, we remark (inspired by another paper of Davis, cf. also [7, p. 128]) that there might be some interest in studying by our methods q.i. of a more general type than those considered herein; for example, there is an identity due to Motzkin and Schoenberg and (independently) Grunsky, which states that

$$\int_D f'' d\sigma = \sum_{n=1}^{3} c_n f(z_n) \qquad (*)$$

holds for all f with $f'' \in L^1_a(D)$, whenever D is a triangle with vertices at z_1, z_2, z_3. Here the c_n depend only on D and can easily be given explicitly. The identity (*) is reflected in the fact that the Cauchy transform of a triangle has logarithmic singularities of a particular kind at the vertices, and no other singularities. It suggests a general study of domains whose Cauchy transforms (or, what comes to the same thing, whose Schwarz functions) have only finitely many singularities of prescribed (not necessarily polar) type, some of which may be on the boundary.

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