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On the "Hot Spots" Conjecture of J. Rauch

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1. INTRODUCTION

Suppose that D is an open connected bounded subset of $\mathbf{R}^d$, $d \geqslant 1$. Let $u(t, x)$, $t \geqslant 0$, $x \in D$, be the solution of the heat equation $\partial u/\partial t = (1/2)\, \Delta_x u$ in D with the Neumann boundary conditions and the initial condition $u(0, x) = u_0(x)$. That is, $u(t, x)$ is a solution to the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = \dfrac{1}{2} \Delta_x u(t, x), & x \in D, \quad t > 0, \\ u(0, x) = u_0(x), & x \in D, \\ \dfrac{\partial u}{\partial n}(t, x) = 0, & x \in \partial D, \quad t > 0. \end{cases} \tag{1.1}$$

Informally speaking, the hot spots conjecture of J. Rauch is as follows. (This form of the conjecture is stated only for didactic reasons and was never proposed by J. Rauch.)

Conjecture R_1. For "most" initial conditions $u_0(x)$, if z_t is a point at which the function $x \to u(t, x)$ attains its maximum, then the distance from

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z_t to the boundary of D tends to zero as t tends to ∞. In other words, the "hot spots" move towards the boundary.

We will state several rigorous versions of this conjecture, review some known results, and prove the conjecture under some additional assumptions. Let us, however, first observe that the conclusion cannot hold for *all* initial conditions. Consider, for example, $D=(0, 2\pi)\times(0, 2\pi)\subset\mathbf{R}^2$ and $u_0(x) = u_0(x_1, x_2) = -(\cos x_1+\cos x_2)$. The function $u_0(x)$ is an eigenfunction corresponding to the 4th eigenvalue μ_4 and so we have $u(t, x)=u_0(x)\, e^{-\mu_4 t}$. It is not true that $\operatorname{dist}(z_t, \partial D)\to 0$ because $z_t=(\pi, \pi)$ for all t.

The long term behavior of the solution of the heat equation considered in the last example is determined by the 4th eigenfunction. However, the long term behavior of the "generic" solution is obtained from the long term behavior of the Neumann heat kernel which is determined by the second eigenfunction. In other words, under suitable conditions on the domain, such as convexity or Lipschitz boundary, and for a "typical" initial condition $u_0(x)$, we have

$$u(t, x)=c_1+c_2\varphi_2(x)\, e^{-\mu_2 t}+R(t, x), \tag{1.2}$$

where c_1 and $c_2\neq 0$ are constants depending on the initial condition, μ_2 is the second eigenvalue for the Neumann problem in D, $\varphi_2(x)$ is a corresponding eigenfunction, and $R(t, x)$ goes to 0 faster than $e^{-\mu_2 t}$, as $t\to\infty$. We will make this precise below in Proposition 2.1. The eignfunction expansion (1.2) leads to a version of the "hot spots" conjecture which involves the second eigenfunction. We will state several versions of the conjecture, with varying strength of the analytic condition and for various classes of domains. Consider the following statements for a domain D.

(HS1) For every eigenfunction $\varphi_2(x)$ corresponding to μ_2 which is not identically 0, and all $y\in D$, we have $\inf_{x\in\partial D}\varphi_2(x)<\varphi_2(y)<\sup_{x\in\partial D}\varphi_2(x)$.

(HS2) For every eigenfunction $\varphi_2(x)$ corresponding to μ_2 and all $y\in D$, we have $\inf_{x\in\partial D}\varphi_2(x)\leqslant\varphi_2(y)\leqslant\sup_{x\in\partial D}\varphi_2(x)$.

(HS3) There exists an eigenfunction $\varphi_2(x)$ corresponding to μ_2 which is not identically 0, and such that for all $y\in D$, we have $\inf_{x\in\partial D}\varphi_2(x)\leqslant\varphi_2(y)\leqslant\sup_{x\in\partial D}\varphi_2(x)$.

The strongest statement (HS1) asserts that the inequalities are strict, while the other two statements involve weaker assertions. Note that all statements (HS1)–(HS3) make assertions about both "hot spots" and "cold spots" of eigenfunctions. This is because if φ is an eigenfunction, so is $-\varphi$ and so maxima and minima are indistinguishable in the context of this problem.

Conjecture R_2 (Rauch). The statement (HS1) is true for every domain $D \subset \mathbf{R}^d$.

The "hot spots" conjecture was made, as we recently learned from Rauch, in 1974 in a lecture he gave at a Tulane University PDE conference. Despite the fact that the conjecture has been around for so many years and that it is very well known, it has never, according to Rauch, appeared in print under his name. According to Kawohl [12], Conjecture R_2 had been proved for parallelepipeds, balls, and annuli (see [12, p. 46]). It seems that the only published result which deals with less restrictive classes of domains is the following theorem of Kawohl [12].

THEOREM 1.1 (Kawohl [12]). *Suppose that* $D = D_1 \times (0, a)$ *where* $D_1 \subset \mathbf{R}^{d-1}$ *has boundary of class* $C^{0,1}$. *Then* (HS2) *holds for* D.

In Theorem 1.1, there is practically no restriction on the shape of D_1 while the second factor of the product has the simplest possible form. We will give an intuitive explanation for this product structure assumption in Remark 3.1. Proposition 2.6 below contains a generalization of Kawohl's result with a very simple proof.

Since there were some doubts that conjecture R_2 was true, Kawohl [12, p. 56] proposed the following.

Conjecture K (Kawohl). The statement (HS1) is true for convex domains $D \subset \mathbf{R}^d$.

Theorem 1.1 shows that (HS2) holds for some non-convex domains. In Example 3.2 below, we will show that (HS1) holds for some other non-convex domains.

The main purpose of the present paper is to present a method of proving theorems rather than a single result. These techniques are discussed in Sections 3 and 4 below. However, in order to give the reader some idea about the main results, we state here two such theorem. A triangle is called *obtuse* if one of its angles is obtuse, that is, greater than $\pi/2$.

THEOREM 1.2. *The hot spots conjecture in the form* (HS3) *is true for obtuse triangles. The "hot" and "cold" spots are located at the most distant vertices.*

THEOREM 1.3. *Suppose that* D *is a sufficiently long convex planar domain which has a line of symmetry* S *which intersects* ∂D *at* x *and* y. *Suppose that the ratio of the diameter of* D *to its width is greater than* 1.54. *The hot spots conjecture in the form* (HS1) *holds under either of the following additional assumptions.*

(A1) *D has another line of symmetry S_1 which is perpendicular to S;*

(A2) *For every $r > 0$, the intersection of the circle $\partial B(x, r)$ with D is either empty or it is a connected arc, and the same holds for $\partial B(y, r)$.*

See Theorem 3.3, Corollary 3.1, and Corollary 4.1 for a more precise statement of these results.

We briefly outline the idea of the proofs. They are based on various properties of eigenfunctions and eigenvalues, which may be of independent interest, and on coupling arguments which have been developed by Burdzy and Kendall [5]. Suppose that D is an obtuse triangle, $A \subset D$, $u_0(x) = 1$ for $x \in A$, and $u_0(x) = 0$ for $x \in D \setminus A$. Suppose that X_t and Y_t are reflected Brownian motions in D with $X_0 = x$ and $Y_0 = y$. We construct the two Brownian motions so that they are dependent in a very special way—their dependence is a crucial element of the proof. Our goal is to choose A and construct X_t and Y_t in such a way that for every t both the processes X_t and Y_t are in A or in $D \setminus A$, or $X_t \in A$ and $Y_t \in D \setminus A$, but we never have $X_t \in D \setminus A$ and $Y_t \in A$ for the same t. Then

$$u(t, x) - u(t, y) = P(X_t \in A, \ Y_t \in D \setminus A \mid X_0 = x, \ Y_0 = y) > 0.$$

By choosing appropriate x, y, A and a "coupling" (X_t, Y_t), we can prove that the function $x \to u(t, x)$ is monotone on every line segment in D parallel to the longest side of the triangle D, for every fixed t. This implies the monotonicity of $\varphi_2(x)$ on the same family of line segments. Hence, the maximum of $\varphi_2(x)$ cannot be attained inside D.

The proof of Theorem 1.3 follows along similar lines but is a bit more subtle. The choice of coupling in the proof of Theorem 1.2 is rather easy and to a certain extent arbitrary. The proof of Theorem 1.3 is based on the detailed analysis of the "mirror coupling" originally developed for a different project by Burdzy and Kendall [5].

Our method of proof works best in cases in which the function $u(t, x)$, for some initial conditions, is monotone in a particular direction in x-space. We are able to derive (HS3) from these results and even identify the location of the "hot" and "cold" spots. In order to prove (HS2) or (HS1) we need the following result which may have some independent interest. For a precise statement, see Proposition 2.4.

PROPOSITION 1.1. *If D is convex and the ratio of its diameter to its width is greater than 3.07 then the second eigenvalue corresponds to a 1-dimensional subspace of $L^2(D)$.*

A disc and a square are examples of domains which have two orthogonal eigenfunctions corresponding to μ_2. The diameter to width ratios for these domains are 1 and $\sqrt{2}$, respectively. The square seems to be the extreme case and we

Conjecture. If a convex domain has two orthogonal eigenfunctions corresponding to μ_2 then its diameter to width ratio is not greater than $\sqrt{2}$.

Remark 1.1. Several people have suggested the following approach to the "hot spots" conjecture for convex domains. Take two copies of a convex domain D and glue them together along their common boundary. With some minimal smoothing, we obtain a compact manifold with no boundary and with non-negative curvature. Then the "hot spots" conjecture essentially says that the maximum for the second eigenfunction of the Laplacian on this manifold cannot be attained at a point where the curvature is zero.

This line of attack does not seem to be plausible in view of the following example. Let D be the surface of the cylinder $\{(x_1, x_2, x_3): 0 < x_1 < 100, x_2^2 + x_3^2 < 1\}$. We smooth the edges $\{(x_1, x_2, x_3): x_1 = 0, x_2^2 + x_3^2 = 1\}$ and $\{(x_1, x_2, x_3): x_1 = 100, x_2^2 + x_3^2 = 1\}$ so that D is a smooth Riemannian manifold with non-negative curvature. It seems that the second eigenfunction for this domain should be antisymmetric with respect to the plane $\{x_1 = 50\}$ and it should attain its maximum and minimum at the points $(0, 0, 0)$ and $(100, 0, 0)$, where the curvature is zero.

Remark 1.2. R. Varadhan pointed out that one may be able to perturb the domain D without destroying the property that the "hot spots" lie on the boundary. This should be possible in the case when the second eigenvalue is non-degenerate, that is, when the second eigenvalue corresponds to a 1-dimensional subspace of $L^2(D)$. Using this method we should be able to relax the symmetry assumption of Theorem 1.3 but so far we have not been able to implement it.

The rest of the paper consists of three sections. Section 2 collects several preliminary results on eigenfunctions and eigenvalues for the Neumann Laplacian which we believe will be of independent interest. These results can be derived using either probabilistic or analytic methods (see Remark 2.1 below), and we indeed use both methods in our proofs. We find this approach both interesting and appropriate, particularly in light of the fact that historically eigenvalue estimates have been of interest to both analysts and probabilists. Section 3 contains a rigorous version of Theorem 1.2 with a proof and a sketch of several other results which can be proved using the same method. It also contains an estimate for the direction of the gradient of $u(t, x)$. A probabilistic proof of Kawohl's theorem is given in the same section (Remark 3.1). This proof only gives (HS3) but it has the advantage that it gives some estimates on the gradient of the function as well. Section 4 starts with a discussion of mirror couplings. A rigorous version of Theorem 1.3 and its proof follow.

We would like to point out some advantages of our method of proof. We are able to show that the conjecture holds in some non-convex domains, i.e., in the case which is usually considered harder than the case of convex domains. We give estimates on the direction of the gradient of eigenfunctions, and we identify the locations of the hot and cold spots in the domains considered in our theorems. Finally, there has been some further progress on the "hot spots" conjecture since the first draft of this paper appeared. Burdzy and Werner [6] have used the methods previously developed in this paper and an earlier idea of Werner to give a rigorous counterexample to Conjecture R_2.

In this paper, we consider the solutions of the heat equation relative to the "one-half Laplacian" operator, $(1/2)\Delta$, which is a convenient normalization for arguments involving Brownian motion. The results hold for the usual Laplacian Δ, by scaling. We caution the reader that because of this normalization, some of the familiar formulas for eigenvalues change by a factor of $\frac{1}{2}$.

2. SOME RESULTS ON EIGENFUNCTIONS AND EIGENVALUES

In this section we derive some basic facts about eigenfunctions that we will need in the subsequent sections. As it has been pointed out to us by several people, some of these results seem to be well known to the experts in spectral geometry. However, we have not been able to find them in the literature, particularly in the form that we need, and hence we present them here. We begin by giving a precise meaning to (1.2). Let $P_t(x, y)$ denote the Neumann heat kernel for the domain D. Under some minimal smoothness assumptions on the domain (convex or Lipschitz boundary is more than enough by Bass and Hsu [3] or Davies [9, Theorems 1.7.9 and 3.2.9]), we have the following bound for the heat kernel,

$$P_t(x, y) \leqslant \frac{C_1}{t^{n/2}} \exp\left(-\frac{|x-y|^2}{c_2 t}\right),$$

for all $x, y \in D$ and all $0 < t \leqslant 1$. In particular

$$0 \leqslant P_1(x, y) \leqslant C_1. \tag{2.1}$$

Here C_1 and c_2 are constants depending on the domain. It follows from this that there are positive constants c_1' and c_2' such that

$$\sup_{x, y \in D} \left| P_t(x, y) - \frac{1}{\text{vol}(D)} \right| \leqslant c_1' e^{-c_2' t}, \tag{2.2}$$

for $t \geqslant 1$ (see Bass and Hsu [3], Davies [9, p. 112], or Smits [17]). It follows from (2.2) that $\int_D P_t(x, x)\, dx < \infty$. That is, the semigroup is of finite trace. By Theorems 1.7.9 and 1.7.12 of Davies [9], it also has a discrete spectrum on $L^2(D)$. Let $\varphi_1, \varphi_2, ...$ be an orthonormal basis for $L^2(D)$ of eigenfunctions with eigenvalues $0 = \mu_1 < \mu_2 \leqslant \mu_3 \cdots$ where we repeat the eigenvalues if needed to take into account their multiplicity. Recall that $\varphi_1 = 1/\text{vol}(D)$. The following proposition provides the extension of (2.2) needed in our paper.

Proposition 2.1. *Let D be a domain in $\mathbf{R}^d$ whose Neumann heat kernel satisfies* (2.1). *Let $u_0 \in L^\infty(D)$ and let $u(t, x)$ be the solution of* (1.1). *Suppose that $\mu_2 = \mu_3 = \cdots = \mu_{k-1} < \mu_k$. Then,*

$$u(t, x) = \sum_{j=1}^{k-1} a_j e^{-\mu_j t} \varphi_j(x) + R(t, x), \tag{2.3}$$

and there is a constant C depending on u_0 and k such that

$$|R(t, x)| \leqslant C e^{-\mu_k t},$$

for all $t \geqslant 2$ and all $x \in D$.

Proof. Let T_t be the semigroup generated by the kernel P_t. By (2.1), T_1 maps $L^2(D)$ into $L^\infty(D)$. Let $C_1 = \| T_1 \|_{2, \infty}$ be the operator norm. Since $T_t \varphi_j = e^{-\mu_j t} \varphi_j$, we have (by the normalization of our eigenfunctions) that $\| \varphi_j \|_\infty \leqslant C_1 e^{\mu_j}$. Writing $u_0(x) = \sum_{j=0}^\infty a_j \varphi_j$, where the coefficients a_j are square summable, we obtain for all $t \geqslant 2$

$$\begin{aligned}
|R(t, x)| &= \left| \sum_{j=k}^{\infty} a_j \varphi_j(x)\, e^{-\mu_j t} \right| \\
&\leqslant C_1 \sum_{j=k}^{\infty} e^{\mu_j} e^{-\mu_j t} \\
&= C_1 e^{-\mu_k (t-1)} \sum_{j=k}^{\infty} e^{(-\mu_j + \mu_k)(t-1)} \\
&\leqslant C_1 e^{-\mu_k (t-1)} \sum_{j=k}^{\infty} e^{(-\mu_j + \mu_k)} \\
&= \left(C_1 e^{2\mu_k} \sum_{j=k}^{\infty} e^{-\mu_j} \right) e^{-\mu_k t}.
\end{aligned}$$

The last inequality follows from the fact that $(-\mu_j+\mu_k)(t-2)$ is non-positive and takes its largest value at $t=2$. Finally, the last sum converges since

$$\sum_{j=0}^{\infty} e^{-\mu_j}=\int_D P_1(x,x)\,dx<\infty,$$

by our bounds above on the kernel $P_t(x,y)$. ∎

We proceed with a sequence of results which leads to a geometric criterion for a convex planar domain to have only one eigenfunction corresponding to μ_2.

The following result is a special case of "bracketing" and well known to specialists (Reed and Simon [16, pp. 270–271]). The proof of this lemma is almost exactly the same as the proof of the Courant Nodal Line Theorem. We supply it for the convenience of the reader.

LEMMA 2.1. *Suppose that a domain D in $\mathbf{R}^2$ is divided by a smooth curve Γ into two subdomains D_1 and D_2. Let λ_j be the first eigenvalue for the mixed Neumann–Dirichlet problem on D_j, with the Neumann boundary conditions on $\partial D\cap\partial D_j$ and the Dirichlet boundary conditions on Γ. If μ_2 is the second Neumann eigenvalue for D then $\mu_2\leqslant\max\{\lambda_1,\lambda_2\}$.*

Proof. First, let us recall that the second Neumann eigenvalue $\mu_2(D)$ for $(1/2)\Delta$ in any domain D of $\mathbf{R}^d$ is given by

$$\mu_2(D)=\min\left(\frac{(1/2)\int_D|\nabla u(x)|^2\,dx}{\int_D|u(x)|^2\,dx}\right), \tag{2.4}$$

where the minimum is taken over all functions in $W^{1,2}(D)$ (the gradient in $L^2(D)$) and with integral zero over the domain. (See Bandle [1, p. 101] or Kawohl [12, p. 45].) Let ψ_1 and ψ_2 be the first eigenfunctions for the mixed boundary value problem for the domain D_1 and D_2, respectively. Both of these eigenfunctions are positive [1, p. 112]. Choose constants α_1 and α_2 such that the function $\psi=\alpha_1\psi_1+\alpha_2\psi_2$ has integral zero over the domain. Using this function in the variational formula (2.4) for μ_2 and the similar characterization for the mixed boundary value problem proves the lemma. ∎

One may generalize Lemma 2.1 to certain higher eigenvalue problems using a similar argument. For example, suppose that Γ is the union of a finite number of smooth curves which divide a domain D into three disjoint subdomains D_1, D_2, and D_3. Let $\mu_3(D)$ be the third Neumann eigenvalue for D and let $\mu_1(D_j)$ be the first eigenvalue for the mixed problem in D_j, with the Dirichlet boundary conditions on Γ and the Neumann boundary conditions on $\partial D_j \backslash \Gamma$. Then $\mu_3(D) \leqslant \max\{\mu_1(D_1), \mu_1(D_2), \mu_1(D_3)\}$. Heuristic arguments and computer simulations of Burdzy *et al.* [4] suggest that for some related eigenvalue minimization problems, the higher Dirichlet eigenvalues are not optimal.

We remind the reader that the nodal set of an eigenfunction u is the set of all points in $\bar{D}$ where the function u vanishes. By the famous Courant Nodal Line Theorem (Chavel [7, p. 19]; or Bandle [1, p. 112]), the nodal set of a second eigenfunction for a domain in $\mathbf{R}^2$ is a smooth curve, called the nodal line, dividing the domain into two subdomains. In the case of a Neumann second eigenfunction, there are no closed nodal lines [1, p. 128]. Note that if Γ is the nodal line for a second Neumann eigenfunction in D, then $\mu_2 = \lambda_1 = \lambda_2$, in the notation of Lemma 2.1.

Let $B(x, r)$ denote the open ball with center x and radius r. We recall that the first Dirichlet eigenvalue for $B(x, r)$ is $j_0^2/2r^2$ where j_0 is the smallest positive zero of the first Bessel function [1, p. 92].

PROPOSITION 2.2. *Suppose that D is a planar domain with piecewise smooth bounday, $z_L, z_R \in \bar{D}$, $\rho > 0$, and a smooth curve Γ divides D into two subdomains D_1 and D_2 with $z_L \in \bar{D}_1$ and $z_R \in \bar{D}_2$. Assume that the distance from z_L to Γ is greate than or equal to ρ, and the same for z_R. Suppose that $B(z_L, \rho) \cap D_1$ and $B(z_R, \rho) \cap D_2$ are star-shaped domains with respect to z_L and z_R, respectively. If μ_2 is the second Neumann eigenvalue for D then $\mu_2 \leqslant j_0^2/2\rho^2$.*

Proof. By Lemma 2.1, it is enough to prove that $\max\{\lambda_1, \lambda_2\} \leqslant \lambda = j_0^2/2\rho^2$ where λ_i is the first eigenvalue for the D_i with Dirichlet boundary conditions on Γ and Neumann conditions elsewhere on the boundary. Let $D_3 = B(z_L, \rho) \cap D_1$ and let η_1 be the first eigenvalue for the domain D_3 with Dirichlet boundary conditions on $\partial_1 D_3 = \partial B_3 \cap \partial B(z_L, \rho)$ and Neumann on $\partial_2 D_3 = \partial D_3 \backslash \partial_1 D_3$. By domain monotonicity, $\lambda_1 \leqslant \eta_1$. We now prove that $\eta_1 \leqslant \lambda$. Towards this end, let X_t be a Brownian motion in D_3 starting from a point $y \in D_3$, killed on $\partial_1 D_3$ and reflected on $\partial_2 D_3$. Without loss of generality assume that z_L is the origin. The radial component of the inward normal vector at any point of $\partial_2 D_3$ points towards the origin (or vanishes) because D_3 is star-shaped with respect to z_L. It follows that $|X_t|$ is a 2-dimensional Bessel process plus a non-increasing process corresponding to the local time push on $\partial_2 D_3$. Hence, the time τ when the process $|X_t|$ reaches the level ρ and gets killed is not smaller than the analogous time for the 2-dimensional Bessel process. The probability that the 2-dimensional Bessel process does not hit ρ by the time t is the same as the probability that the exit time for disc of radius ρ is larger than t. Such probability, starting from y, is bounded below by $c(y)\, e^{-\lambda t}$, for large time. Here we may take $c(y)$ to be the first Dirichlet eigenfunction for the disc by verifying that the semigroup of the Dirichlet Laplacian is “intrinsically ultracontractive” and

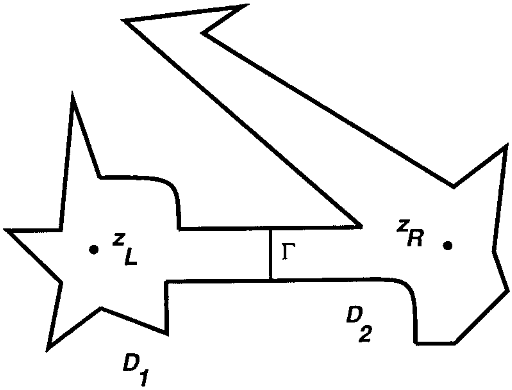

FIGURE 2.1

applying Theorem 4.2.5 in Davies [9]. The same estimate applies to τ and it follows that $\eta_1 \leqslant \lambda$. A similar bound holds for D_2 and the proposition follows. ▮

An example of a domain D, points z_L and z_R, and a curve Γ satisfying the assumptions of Proposition 2.2 is given in Fig. 2.1.

For any domain D in $\mathbf{R}^d$ let d_D denote its diameter.

COROLLARY 2.1. *Let D be a convex domain in the plane and let μ_2 be its second Neumann eigenvalue. Then $\mu_2 \leqslant 2j_0^2/d_D^2$.*

Proof. Consider any points $z_L, z_R \in \partial D$ with $|z_L - z_R| = d_D$. Let Γ be the intersection of D with the line of symmetry for these points. Then we can apply Proposition 2.2 with $\rho = d_D/2$. ▮

The estimate of Corollary 2.1 follows from Cheng [8]. Indeed, using the construction described in Remark 1.1, we can translate the estimate in Theorem 2.1 in Cheng [8] into a statement about the second Neumann eigenvalue for a convex domain. Note that Cheng's result holds for compact Riemannian manifolds with Ricci curvature bounded below by $(n-1)k$, where n is the dimension of the manifold. If we take $k=0$, we obtain the bound in Corollary 2.1.

It is well known that the bound in Corollary 2.1 is the best possible estimate in the class of all convex planar domains. It is nearly sharp for isosceles triangles with vertices $(-1, 0)$, $(1, 0)$, and $(0, a)$, with very small $a>0$. For small a, the second eigenvalue μ_2 for this triangle is bounded below by $\tan^{-1}(a)\, j_0^2/(2a)$, by a simple reflection argument; see Bandle [1, p. 114]. A version of Corollary 2.1 without the sharp constant is proved in [17].

Let us define the width of the domain, which we will denote by width(D), to be the infimum of the widths of all strips containing D. We let w_D^V be the length of the projection of D on the vertical axis and w_D^H its projection on the horizontal axis. We assume, without loss of generality, that our domains are always oriented so that $w_D^V \leqslant w_D^H$ and we set $w_D = w_D^V$. Observe that $w_D \geqslant \text{width}(D)$ but in general w_D is not necessarily the same as width(D). We will say that a function $h(x^1, x^2)$ is antisymmetric with respect to the horizontal axis if $h(x^1, x^2) = -h(x^1, -x^2)$ for all (x^1, x^2).

PROPOSITION 2.3. *Suppose that D is a convex planar domain which is symmetric with respect to the horizontal axis. Let j_0 be the smallest positive zero of the first Bessel function. Suppose the ratio d_D/w_D is greater than* $2j_0/\pi \approx 1.53096$. *Then there is no eigenfunction $\varphi_2(x)$ corresponding to the second eigenvalue which is antisymmetric with respect to the horizontal axis.*

Proof. Let $z_L = (z_L^1, 0)$ and $z_R = (z_R^1, 0)$ be the points of intersection of ∂D with the horizontal axis. Without loss of generality,

$$D = \{(x^1, x^2) : z_L^1 < x^1 < z_R^1,\ -f(x^1) < x^2 < f(x^1)\},$$

where f is a positive concave function on $[z_L^1, z_R^1]$. Suppose we have an eigenfunction φ_2 corresponding to μ_2 such that

$$\int_{-f(x^1)}^{f(x^1)} \varphi_2(x^1, x^2)\, dx^2 = 0,$$

for every $x^1 \in (z_L^1, z_R^1)$. Since $\mu_2(a, b) = \pi^2/[2(b-a)^2]$, for any interval (a, b), applying (2.4) we get

$$\int_{-f(x^1)}^{f(x^1)} |\varphi_2(x^1, x^2)|^2\, dx^2 \leqslant \frac{(2f(x^1))^2}{\pi^2} \int_{-f(x^1)}^{f(x^1)} \left|\frac{\partial \varphi_2}{\partial x^2}(x^1, x^2)\right|^2 dx^2$$

$$\leqslant \frac{w_D^2}{\pi^2} \int_{-f(x^1)}^{f(x^1)} |\nabla \varphi_2(x^1, x^2)|^2\, dx^2.$$

Integrating this inequality from z_L^1 to z_R^1 with respect to x^1 gives that

$$\frac{\pi^2}{2w_D^2} \leqslant \mu_2.$$

However, by Corollary 2.1,

$$\mu_2 \leqslant \frac{2j_0^2}{d_D^2}.$$

Thus, if there is antisymmetric eigenfunction corresponding to μ_2 we must have $d_D^2/w_D^2 \leqslant 4j_0^2/\pi^2 \approx 2.3438$. Therefore if $d_D/w_D > 2j_0/\pi \approx 1.53096$, there is no antisymmetric eigenfunctions corresponding to μ_2, as stated in the lemma. ▮

PROPOSITION 2.4. (i) *Suppose that D is a convex domain with the ratio* d_D/width(D) *greater than* $4j_0/\pi \approx 3.06$. *Then there exists only one eigenfunction corresponding to* μ_2, *up to a multiplicative constant.*

(ii) *Suppose that D is a convex domain which is symmetric with respect to the horizontal axis. If* $d_D/w_D > 2j_0/\pi \approx 1.53$ *then the subspace of* $L^2(D)$ *corresponding to* μ_2 *is one-dimensional.*

Proof. (i) Assume without loss of generality that D is oriented in such a way that its projection on the vertical axis is equal to width(D). Choose points $z_L = (z_L^1, z_L^2)$ and $z_R = (z_R^1, z_R^2)$ in ∂D with the smallest z_L^1 and the largest z_R^1. The choice might not be unique and it is not necessarily true that $|z_L - z_R| = d_D$ or that z_L and z_R lie on the horizontal axis. Suppose that there exist two independent eigenfunctions $\hat{\varphi}_2(x)$ and $\tilde{\varphi}_2(x)$ corresponding to the second eigenvalue μ_2. First we will show that there exists an eigenfunction $\varphi_2(x)$ corresponding to μ_2 and such that $\varphi_2(z_L) = 0$. If $\hat{\varphi}_2(z_L) = 0$ or $\tilde{\varphi}_2(z_L) = 0$ then we are done. Otherwise we let

$$\varphi_2(x) = \hat{\varphi}_2(z_L)\, \tilde{\varphi}_2(x) - \tilde{\varphi}_2(z_L)\, \hat{\varphi}_2(x).$$

Recall that the nodal line Γ for $\varphi_2(x)$ divides D into two subdomains D_1 and D_2 and does not form a closed loop. One of the endpoints of Γ is z_L; let the other be called v. Without loss of generality we will assume that v lies on the lower part of the boundary, between z_L and z_R (we may have $v = z_R$). Let D_1 be the subdomain which lies "below" Γ. The function $\varphi_2(x)$, restricted to D_1, is the first eigenfunction for the mixed Neumann–Dirichlet problem in D_1, with the Neumann boundary conditions on $\Lambda = \partial D \cap \partial D_1$ and Dirichlet boundary conditions on Γ. We will estimate the first eigenvalue for this problem, which is the same as μ_2.

Let $a_1 = \inf\{ y^2 : (y^1, y^2) \in D\}$ and $a_2 = \sup\{ y^2 : (y^1, y^2) \in D\}$. Let $X_t = (X_t^1, X_t^2)$ be a Brownian motion in D_1, starting from $(a, b) \in D$, which is reflected on Λ and killed upon hitting Γ. Note that the vertical component of the inward normal vector points upward at every point of Λ. Thus, the process X_t^2 is the sum of a one-dimensional Brownian motion and a nondecreasing process, corresponding to the upward component of reflection when X_t is reflecting on Λ. The process X_t^2 cannot take values outside $[a_1, a_2]$; it is killed at the hitting time of a_2 or before hitting this value, it is pushed upward when it hits a_1 and, possibly, when it is strictly inside (a_1, a_2). A standard comparison argument for the solutions of stochastic differential equations shows now that the distribution of X^2 at time t is minorized by the distribution of the one-dimensional Brownian motion in $[a_1, a_2]$, starting from a, reflecting on a_1, and killed upon hitting a_2. The probability that such a process does not exit $[a_1, a_2]$ by the time t is bounded by $ce^{-\lambda t}$, where λ is the eigenvalue for the Laplacian on $[a_1, a_2]$ with the Neumann condition at a_1 and the Dirichlet condition at the other endpoint. Hence,

$$\lambda = \frac{\pi^2}{8(a_2 - a_1)^2} = \frac{\pi^2}{8(\text{width}(D))^2}.$$

For large t, the probability that X_t^2 has not hit Γ by the time t is also bounded by $ce^{-\lambda t}$, and so the first eigenvalue for the mixed problem in D_1 cannot be smaller than $\pi^2/8(\text{width}(D))^2$. Recall that this eigenvalue is the same as μ_2.

By Corollary 2.1, $\mu_2 \leqslant 2j_0^2/d_D^2$, so we have

$$\frac{\pi^2}{8(\text{width}(D))^2} \leqslant \frac{2j_0^2}{d_D^2},$$

which gives $d_D/\text{width}(D) \leqslant 4j_0/\pi$. If this inequality is not satisfied, we only one eigenfunction corresponding to μ_2.

(ii) In this part we let z_L and z_R be the points of intersection of ∂D with the horizontal axis. We assume, as in part (i), that there are two independent eigenfunctions corresponding to μ_2 and we construct an eigenfunction $\varphi_2^*(x)$ which vanishes at z_L. This means that the nodal line of $\varphi_2^*(x)$ has one of its endpoints at z_L. It follows that $\varphi_2^*(x)$ cannot be symmetric with respect to the horizontal axis. Hence, the function

$$\varphi_2(x^1, x^2) = \varphi_2^*(x^1, x^2) - \varphi_2^*(x^1, -x^2)$$

cannot be identically equal to zero. Note that $\varphi_2(x)$ is an antisymmemtric eigenfunction, i.e., $\varphi_2(x^1, x^2) = -\varphi_2(x^1, -x^2)$ for all $(x^1, x^2) \in D$. Now we use Proposition 2.3 to conclude that $d_D/w_D > 2j_0/\pi$. Alternatively, we can repeat the proof in part (i) with $[a_1, a_2]$ replaced by $[a_1, 0]$, since we know in the present case that the nodal line for $\varphi_2(x)$ lies on the horizontal axis. ▮

Remark 2.1. As we have just mentioned, the probabilistic proof of Proposition 2.4 can also be adapted to give a probabilistic proof of Proposition 2.3. On the other hand, one can give an analytic proof of Proposition 2.4, based on the following result from Sperb [18, Corollary 5.2]. Suppose D is a convex domain in $\mathbf{R}^d$. Let φ_2 be a Neumann eigenfunctions corresponding to μ_2. Then the function

$$P(z) = |\nabla\varphi_2(z)|^2 + 2\mu_2\, |\varphi_2(z)|^2$$

must have its maximum where φ_2 assumes its maximum M or its minimum m. (Again, recall that we are dealing with $(1/2)\Delta$.) Let $\tilde{M}=\max\{M, m\}$. Then

$$|\nabla \varphi_2| \leqslant \sqrt{2\mu_2}\sqrt{\tilde{M}^2-\varphi_2^2}. \tag{2.5}$$

Now, continuing with the notation of the proof of Proposition 2.4(i), let z_M be a point where φ_2 (the φ_2 of the proof of the proposition) reaches its maximum and z_m a point where it reaches its minimum. We may assume that $z_M \in \bar{D}_1$ (otherwise take $-\varphi_2$). Let z_0 be the point on Γ directly above z_M. Clearly, $|z_M - z_0| \leqslant \text{width}(D)$. Dividing (2.5) by $\sqrt{\tilde{M}^2-\varphi_2^2}$ and integrating on the segment from z_0 to z_M and doing the same for point z_m, leads to

$$\frac{\pi}{2} \leqslant \text{width}(D)\sqrt{2\mu_2},$$

which gives the crucial estimate for the Proposition 2.4.

Also, an analytic proof of Proposition 2.2 is possible. Once again, with the notation of the proposition we need to show that $\eta_1 \leqslant \lambda$. We again assume that the point z_L is the origin. We let $\varphi(z)$ be the Dirichlet eigenfunction for $B(0, \rho)$. Using this as a test function in the variational characterization for η_1, the result follows. (The fact that the intersection of D_1 with $B(0, \rho)$ is star-shaped with respect to 0 is used to write the integrals in the variational characterization for η_1 in polar coordinates and to reduce to a simple inequality about Bessel functions. We leave the details to the interested reader.)

The argument in Remark 2.1, together with Cheng's estimate for μ_2, has the following corollary which gives some information on the location of the "hot spots" relative to the nodal line. We leave the formal proof to the reader.

Corollary 2.2. *Let D be a planar convex domain of diameter d_D. Let φ_2 be any Neumann eigenfunction corresponding to μ_2 and let z_M and z_m be points in $\bar{D}$ where φ_2 has a maximum and a minimum, respectively. If $d(z_M, \Gamma)$ and $d(z_m, \Gamma)$ denotes the distance for these points to the nodal line Γ, then*

$$\max\{d(z_M, \Gamma), d(z_m, \Gamma)\} \geqslant \left(\frac{\pi}{4j_0}\right) d_D \approx 0.327 d_D.$$

The following questional naturally arises from the above results. For an arbitrary convex domain D in the plane, what is the dimension of the eigenspace corresponding to μ_2? In the case of the Dirichlet problem, it is known [13] that the number of linearly independent eigenfunctions corresponding to the second eigenvalue is at most two. A similar result has been proved by Nadirashvili [14, 15] for the Neumann problem. We give a new and, perhaps, a little easier proof of Nadirashvili's theorem.

PROPOSITION 2.5 (Nadirashvili). *Let D be a simply connected planar domain with smooth boundary. The multiplicity of the second Neumann eigenvalue μ_2 is at most* 2.

Proof. Suppose we have three independent eigenfunctions φ_1, φ_2, and φ_3 corresponding to μ_2. Let z_∞ be a point on the boundary of D with $\varphi_1(z_\infty)=0$. Such a point exists since the nodal line does not form a closed loop. As in the proof of Proposition 2.4(i) we find a linear combination φ_4 of φ_2 and φ_3 with the property that $\varphi_4(z_\infty)=0$.

Let $z_k \in \partial D$ be a point at the distance $1/k$ from z_∞. We will show that for every k there exists an eigenfunction ψ_k vanishing at both z_k and z_∞. If φ_1 or φ_4 has this property then we are done. Otherwise we take

$$\psi_k(x)=\varphi_1(z_k)\,\varphi_4(x)-\varphi_1(x)\,\varphi_4(z_k).$$

The function ψ_k is not identically equal to 0 because φ_1 and φ_4 are independent. By multiplying ψ_k by a constant, if necessary, we may assume that

$$\psi_k=C_{1,k}\varphi_1+C_{2,k}\varphi_2+C_{3,k}\varphi_3,$$

where

$$C_{1,k}^2+C_{2,k}^2+C_{3,k}^2=1.$$

By passing to a subsequence, if necessary, we may assume that the sequence $\{C_{1,k}\}$ converges as $k\to\infty$, and so do the sequences $\{C_{2,k}\}$ and $\{C_{3,k}\}$. Hence, the functions ψ_k converge to an eigenfunction ψ_∞. Let Γ_k be the nodal line for ψ_k and let D_k be the component of $D\setminus\Gamma_k$ which touches the smaller component of $\partial D\setminus\{z_k, z_\infty\}$. We can assume without loss of generality that ψ_k is positive on D_k. Let D_∞ be the set where ψ_∞ is positive and $D_\infty(\varepsilon)=\{x\in D_\infty : \psi_\infty(x)>\varepsilon\}$. Fix some small $\varepsilon>0$ such that $D_\infty(\varepsilon)\neq\emptyset$. Since the ψ_k converge to ψ_∞, we must have $D_\infty(\varepsilon)\subset D_k$, for large k. Recall that $z_k\to z_\infty$. This implies that the cluster set for Γ_k's is contained in the nodal line for ψ_∞ and it contains a closed loop around $D_\infty(\varepsilon)$. We obtain a contradiction, since the nodal line for the second eigenfunction cannot contain a closed loop. ∎

The following proposition extends Kawohl's result (Theorem 1.1). The proof is the same as the one for $D\times(0,a)$ given in [12, Remark 2.37, p. 56].

PROPOSITION 2.6. *Consider domains* $D_1 \subset \mathbf{R}^{d_1}$ *and* $D_2 \subset \mathbf{R}^{d_2}$. *If* (HS2) *holds for* D_1 *then* $D_1 \times D_2$ *also satisfies* (HS2).

Proof. Let $\{\varphi_k^j\}_{k \geqslant 1}$ be a complete orthonormal system of Neumann eigenfunctions for the domain D_j. Let μ_k^j be the eigenvalue corresponding to φ_k^j. In this notation, $0 = \mu_1^j < \mu_2^j = \mu_3^j = \cdots = \mu_m^j < \mu_{m+1}^j \leqslant \cdots$, i.e., the functions φ_k^j, $k = 2, 3, ..., m$, correspond to the second lowest eigenvalue. We will use x and y to denote generic elements of D_1 and D_2.

The family of functions $\{\varphi_k^1(x)\, \varphi_n^2(y)\}_{k, n \geqslant 1}$ is a complete orthonormal system of Neumann eigenfunctgions for $D_1 \times D_2$. The eigenfunction $\varphi_k^1(x)$ $\varphi_n^2(y)$ corresponds to the eigenvalue $\mu_k^1 + \mu_n^2$. The lowest eigenvalue is $\mu_1^1 + \mu_1^2 = 0 + 0 = 0$, as expected. The only candidates for the second lowest eigenvalue for this system are $\mu_1^1 + \mu_2^2 = \mu_2^2$ and $\mu_2^1 + \mu_1^2 = \mu_2^1$. If $\mu_2^1 < \mu_2^2$ then every eigenfunction corresponding to μ_2^1 has a form $\tilde{\varphi}(x)$, i.e., it is a function of the x variable only, $x \in D_1$. Such a function satisfies the condition in (HS2). The proof is similar when $\mu_2^1 > \mu_2^2$.

Now suppose that $\mu_2^1 = \mu_2^2$. Then a second eigenfunction for $D_1 \times D_2$ may have the form $\tilde{\varphi}(x) + \hat{\varphi}(y)$, where $\tilde{\varphi}(x)$ is a second eigenfunction for D_1 and $\hat{\varphi}(y)$ is a second eigenfunction for D_2. Fix arbitrary $x_1 \in D_1$ and $y_1 \in D_2$. We have assumed that D_1 satisfies (HS2) so there exists $x_2 \in \partial D_1$ with $\tilde{\varphi}(x_2) \geqslant \tilde{\varphi}(x_1)$. Then $\tilde{\varphi}(x_1) + \hat{\varphi}(y_1) \leqslant \tilde{\varphi}(x_2) + \hat{\varphi}(y_1)$, which proves (HS2) for $D_1 \times D_2$ since $(x_2, y_1) \in \partial(D_1 \times D_2)$. ∎

3. RESULTS BASED ON "SYNCHRONOUS COUPLINGS"

We start this section with a review of basic properties of the "synchronous coupling" of reflected Brownian motions.

In this section, we will choose the value of the angle $\angle K$ formed by a straight line K with the horizontal axis so that $\angle K \in (-\pi/2, \pi/2]$.

Let $X_t = (X_t^1, X_t^2)$ be a 2-dimensional Brownian motion with $X_0 = (x^1, x^2)$ where $x^2 > 0$. Let $W_t^X = 0 \wedge \min_{s \leqslant t} X_s^2$. Then the Skorohod Lemma [11, Lemma 3.6.14] implies that $\tilde{X}_t = (X_t^1, X_t^2 - W_t^X)$ is a reflected Brownian motion in the upper half-plane. The process $\tilde{X}_t$ has the same distribution as $(X_t^1, |X_t^2|)$, which is one of the most popular definitions of reflected Brownian motion in the half space. It is essential for our coupling arguments that we use a representation derived from Skorohod's lemma. Let $Y_t = (Y_t^1, Y_t^2) = (X_t^1 + (y^1 - x^1), X_t^2 + (y^2 - x^2))$, where $y_2 > 0$. Then Y_t is a Brownian motion starting from (y^1, y^2). If we let $W_t^Y = 0 \wedge \min_{s \leqslant t} Y_s^2$, we obtain a reflected Brownian motion $\tilde{Y}_t$ by the means of the formula $\tilde{Y}_t = (Y_t^1, Y_t^2 - W_t^Y)$. Let K_t be the straight line passing through $\tilde{X}_t$ and $\tilde{Y}_t$ and recall that $\angle K_t$ denotes the angle between K_t and the horizontal axis.

It is elementary to check that $\angle K_t$ converges in a monotone way to 0 as $t \to \infty$ and, moreover, $\angle K_t$ is constantly equal to 0 after X_t and Y_t hit the horizontal axis simultaneously. The pair $(\tilde{X}_t, \tilde{Y}_t)$ of reflected Brownian motions is called a *synchronous coupling*. A straightforward generalization of the above construction gives for every polygonal domain D and every pair of starting points $x, y \in \bar{D}$, a pair of reflected Brownian motions $(\tilde{X}_t, \tilde{Y}_t)$ with $(\tilde{X}_0, \tilde{Y}_0) = (x, y)$, and such that $\tilde{X}_t - \tilde{Y}_t$ remains constant on every interval during which both process stay in the interior of the domain. It should be noted that none of the processes $\tilde{X}_t$ and $\tilde{Y}_t$ can hit any vertices of ∂D, by the results of Varadhan and Williams [19]. It is not hard to prove that with probability 1 there will be u such that $X_u = Y_u$ if and only if ∂D contains perpendicular line segments. If such a u exists then $X_t = Y_t$ for all $t \geqslant u$.

Our first result is concerned with the direction of the gradient of $u(t, x)$ in obtuse triangles. Consider an obtuse triangle D, i.e., a triangle whose one angle is greater than $\pi/2$. We will assume that the longest side of the triangle lies on the horizontal axis, the triangle lies in the first quadrant, and one of its vertices is at the origin. The smaller sides of the triangle form angles a and b with the horizontal axis, with $a \in (-\pi/2, 0)$ and $b \in (0, \pi/2)$ (see Fig. 3.1).

Let $\angle \nabla_x u(t, x)$ be the angle formed by the gradient $\nabla_x u(t, x)$ with the horizontal axis.

THEOREM 3.1. *Suppose that $u(0, x)$ is C^1 and $c < \angle \nabla_x u(0, x) < d$ for all $x \in D$, where $c > b - \pi/2$ and $d < \pi/2 + a$. Then for every t and x we have*

$$\min(a, c) \leqslant \angle \nabla_x u(t, x) \leqslant \max(b, d).$$

Proof. Consider a line K with

$$\max(b, d) - \pi/2 \leqslant \angle K \leqslant \min(a, c) + \pi/2.$$

Suppose that $x, y \in K \cap D$, $x = (x^1, x^2)$, $y = (y^1, y^2)$, and $x^1 < y^1$. Let X_t and Y_t be a pair of reflected Brownian motions in D with $X_0 = x$ and $Y_0 = y$. We assume that (X_t, Y_t) is a synchronous coupling as explained at the beginning of this section. Let K_t be the line passing through X_t and Y_t. Since the sides of the obtuse triangle are not perpendicular to each other, we will never have $X_t = Y_t$ and so K_t is defined in a unique way for all t, a.s. Recall that the direction of K_t either remains constant or approaches the direction of the side on whch one of the process is currently reflecting. This implies that $\angle K_t$ can never leave the interval $[\max(b, d) - \pi/2, \min(a, c) + \pi/2]$ and in fact it will be confined to the subinterval $[a, b]$ for

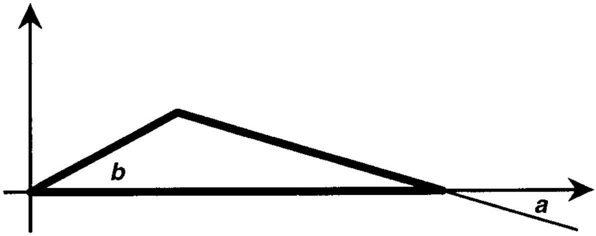

FIGURE 3.1

large t. Moreover, we will always have $X_t^1 < Y_t^1$. The last two observations and the assumption that $c < \angle \nabla_x u(0, x) < d$ imply that $u(0, X_t) < u(0, Y_t)$ for all t, a.s.

The function $u(t, x)$ may be probabilistically represented as $u(t, x) = Eu(0, X_t)$ and, by analogy, $u(t, y) = Eu(0, Y_t)$. This and the inequality $u(0, X_t) < u(0, Y_t)$ imply that $u(t, x) = u(t, (x^1, x^2))$ is a strictly increasing function of x^1 for $(x^1, x^2) \in K \cap D$. Since this is true for every line K with

$$\max(b, d) - \pi/2 \leqslant \angle K \leqslant \min(a, c) + \pi/2,$$

the gradient $\nabla_x u(t, x)$ must satisfy

$$\min(a, c) \leqslant \angle \nabla_x u(t, x) \leqslant \max(b, d). \quad \blacksquare$$

The assumption that $u(0, x)$ is C^1 is needed in Theorem 3.1 only so that we can define $\angle \nabla_x u(0, x)$. The same method of proof gives the following result.

THEOREM 3.2. *Suppose that Γ is a piecewise smooth curve such that for any tangent line K_1 at any point of Γ we have $\angle K_1 \in (c + \pi/2, \pi/2] \cup (-\pi/2, d - \pi/2)$ for all $x \in D$, where $c > b - \pi/2$ and $d < a + \pi/2$. Let A be the set to the right of Γ and let $u(0, x) = \mathbf{1}_A(x)$ for $x \in D$. Then for every fixed t we have*

$$\min(a, c) \leqslant \angle \nabla_x u(t, x) \leqslant \max(b, d).$$

Proof. The proof is the same as for Theorem 3.1. We will indicate some minor adjustments to the proof needed in the present case. We have shown that $\angle K_t$ can never leave the interval $[\max(b, d) - \pi/2, \min(a, c) + \pi/2]$ and we always have $X_t^1 < Y_t^1$. This and the assumption that for any line K_1 tangent to Γ we have $\angle K_1 \in (c + \pi/2, \pi/2] \cup (-\pi/2, d - \pi/2)$ imply that $u(0, X_t) < u(0, Y_t)$ for all t, a.s. The rest of the proof is the same as in the case of Theorem 3.1. $\blacksquare$

THEOREM 3.3. *For every obtuse triangle D there exists an eigenfunction $\varphi_2(x)$ corresponding to the second eigenvalue μ_2 for the Neumann problem in D such that for every $y \in D$ we have*

$$\inf_{x \in \partial D} \varphi_2(x) < \varphi_2(y) < \sup_{x \in \partial D} \varphi_2(x).$$

Proof. Fix some eigenfunction $\varphi_2^*(x)$ corresponding to the second eigenvalue μ_2 and fix $x_* \in D$ such that $\varphi_2^*(x_*) > 0$. We will consider two curves Γ and Γ_1 and the corresponding regions A and A_1 to the right of Γ and Γ_1, as in Theorem 3.2. We let Γ be the vertical line passing through x_*. We choose Γ_1 in such a way that $A \subset A_1$ and $B = A_1 \backslash A$ is a small triangle which contains x_* in its boundary. We make B so small that, by continuity of φ_2^*, we have $\varphi_2^*(x) > 0$ for $x \in B$. We also require that the sides of B are close to vertical; more precisely, we require that for every line K_1 tangent to Γ_1 we have $\angle K_1 \in (a + \pi/2, \pi/2) \cup (-\pi/2, b - \pi/2)$.

We set $u_0(x) = \mathbf{1}_A(x)$ and $u_0^1(x) = \mathbf{1}_{A_1}(x)$ for $x \in D$. Let $u(t, x)$ and $u^1(t, x)$ be the solutions of the Neumann problem in D with initial conditions $u_0(x)$ and $u_0^1(x)$, respectively. By Proposition 2.1,

$$u(t, x) = \alpha_1 + \alpha_2^* \varphi_2^*(x)\, e^{-\mu_2 t} + \tilde{\alpha}_2 \tilde{\varphi}_2(x)\, e^{-\mu_2 t} + R(t, x),$$

where $R(t, x)$ converges to 0 faster than $e^{-\mu_2 t}$ as $t \to \infty$. Here $\tilde{\varphi}_2(x)$ is an eigenfunction corresponding to μ_2 which is orthogonal to $\varphi_2^*(x)$. The analogous formula for $u^1(t, x)$ is

$$u^1(t, x) = \beta_1 + \beta_2^* \varphi_2^*(x)\, e^{-\mu_2 t} + \tilde{\beta}_2 \tilde{\varphi}_2(x)\, e^{-\mu_2 t} + R^1(t, x).$$

We have

$$\beta_2^* - \alpha_2^* = \int_{A_1 \cap D} \varphi_2^*(x)\, dx - \int_{A \cap D} \varphi_2^*(x)\, dx = \int_B \varphi_2^*(x)\, dx > 0,$$

so at least one of the coefficients α_2^* or β_2^* is non-zero. Let us assume that $\beta_2^* \neq 0$, the other case being analogous. Then

$$u^1(t, x) = \beta_1 + \beta_2 \varphi_2(x)\, e^{-\mu_2 t} + R^1(t, x), \tag{3.1}$$

where $\varphi_2(x)$ is a second eigenfunction and $\beta_2 \neq 0$.

Suppose that $\beta_2 > 0$; the other case can be dealt with in a similar way. Theorem 3.2 implies that $u^1(t, x)$ is monotone on every horizontal line passing through D, for every fixed t. Without loss of generality, let us assume that $u^1(t, (x^1, x^2))$ is an increasing function of x^1. For every $x \in D$, let $V(x)$ be the set of $y \in D$ such that the angle between the vector $\overrightarrow{x, y}$ and the horizontal axis lies within $(b - \pi/2, a + \pi/2)$. By Theorem 3.2 and our choice of Γ and Γ_1, we have $u^1(t, y) \geqslant u^1(t, x)$, for all $x \in D$, $y \in V(x)$, and

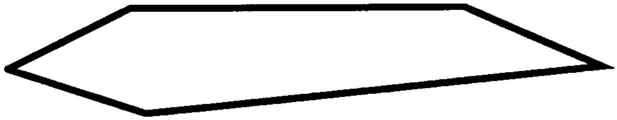

FIGURE 3.2

$t > 0$. Since $R^1(t, x)$ converges to 0 faster than $e^{-\mu_2 t}$, the last fact and (3.1) imply that $\varphi_2(y) \geqslant \varphi_2(x)$ for $y \in V(x)$. The remark following Corollary (6.31) in Folland [10] may be applied to the operator $\Delta + \mu_2$ to conclude that the eigenfunctions are real analytic and therefore they cannot be constant on an open set unless they are constant on the whole domain D. It follows that the maximum of $\varphi_2(x)$ cannot be attained on an open subset of D and thus it can be attained only at the right vertex. The proof that the minimum is attained at the left vertex is completeley analogous. ∎

COROLLARY 3.1. *If D is an obtuse triangle with $d_D/\text{width}(D) > 3.07$ then* (HS1), *the strongest form of the "hot spots" conjecture, holds for D.*

Proof. The result follows from Theorem 3.3 and Proposition 2.4(i). ∎

The assumption that D is a triangle plays no role in the arguments—all we need is a bound on the angles formed by the sides of D. We will present a few examples of domains to which Theorems 3.1–3.3 can be easily extended. We leave it to the reader to formulate the corresponding theorems in a rigorous way.

EXAMPLE 3.1. Theorems 3.1–3.3 apply to any convex polygonal domain which has an "upper" and "lower" sides such that there is an interval of length less than $\pi/2$ which contains all angles formed by edges of ∂D with the horizontal axis. See Fig. 3.2.

EXAMPLE 3.2. The assumption of convexity does not play an essential role in Theorems 3.1–3.3. It is elementary to check that if two reflected Brownian motions in the domain in Fig. 3.3 are related by a "synchronous coupling" then the "left" particle will always stay to the left of the other one. This is the main property of the coupling used in the proofs of Theorems 3.1–3.3.

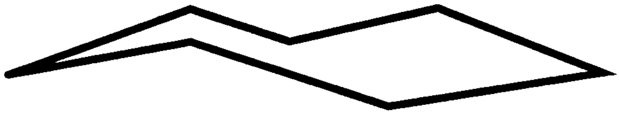

FIGURE 3.3

FIGURE 3.4

EXAMPLE 3.3. The results can be further extended from polygonal domains to domains with piecewise smooth boundaries. Figure 3.4 shows an example of a domain with piecewise smooth boundaries, similar to that in Fig. 3.3. Note that the leftmost and rightmost vertices must stay sharp. Our proofs are based on the fact that the "left" and "right" particles have to preserve these relative positions forever. The reflected Brownian motion can be thought of as Brownian motion with a "push" at the boundary which has the direction of the inward pointing normal vector at the current position of the particle. This observation can be used to show that the reflected Brownian motions coupled in a synchronous way will not switch from the left to the right side and vice versa in the domain in Fig. 3.4. We leave the details of the proof to the reader.

EXAMPLE 3.4. Wendelin Werner pointed out to us that our method can be applied to reflected Brownian motion with oblique angle of reflection. This corresponds to the heat equation with oblique boundary conditions. We first indicate how oblique reflection changes the properties of the synchronous coupling.

Suppose that $N=\{(x^1, x^2): x^2=ax^1\}$, and let N^+ be the region above N. Let $X_t=(X_t^1, X_t^2)$ be a 2-dimensional Brownian motion with $X_0=(x_0^1, x_0^2)\in N^+$. Let $W_t^X=0\wedge \min_{s\leqslant t} X_s^2-aX_s^1$. Then $\tilde{X}_t=(X_t^1, X_t^2-W_t^X)$ is a reflected Brownian motion in N^+ with the vector of reflection pointing upward. The angle of reflection θ is equal to $\pi/2-\angle N$. Let $Y_t=(Y_t^1, Y_t^2)=(X_t^1+(y_0^1-x_0^1),\ X_t^2+(y_0^2-x_0^2))$, where $(y_0^1, y_0^2)\in N^+$. Then Y_t is a Brownian motion starting from (y_0^1, y_0^2). If we let $W_t^Y=0\wedge \min_{s\leqslant t} Y_s^2-aY_s^1$, we obtain a reflected Brownian motion $\tilde{Y}_t$ by the means of the formula $\tilde{Y}_t=(Y_t^1, Y_t^2-W_t^Y)$. The vector of reflection for $\tilde{Y}$ is the same as for $\tilde{X}$. Assume that $y_0^1>x_0^1$. Let K_t be the straigth line passing through $\tilde{X}_t$ and $\tilde{Y}_t$. Then $\angle K_t$ converges in a monotone way to $\angle N$ as $t\to\infty$. By rotating the above picture we can obtain similar statements for arbitrary directions of the line N and vector of reflection.

The proof of the following statement is left to the reader. It is completely analogous to that of Theorem 3.1 except that we have to apply synchronous coupling for reflected Brownian motions with oblique reflection, as discussed above.

PROPOSITION 3.1. *Assume that* $\theta < \pi/2-(b-a)$. *Suppose that* D *is an obtuse triangle as in Theorem* 3.1 *and at each point* $z \in \partial D$ *we have a vector of reflection* v_z *such that its angle with the inward normal vector at* z *is less than* θ (*the angle does not have to be the same for all* z). *Let* $u(t, x)$ *be the solution to the heat equation with the oblique reflection defined by* v_z. *Suppose that* $u(0, x)$ *is* C^1 *and* $c < \angle \nabla_x u(0, x) < d$ *for all* $x \in D$, *where* $c > b-\pi/2$ *and* $d < \pi/2+a$. *Then for every fixed* t *we have*

$$\min(a, c)-\theta \leqslant \angle \nabla_x u(t, x) \leqslant \max(b, d)+\theta.$$

Remark 3.1. We sketch an argument which can be used to prove a weak version (HS3) of Kawohl's theorem (Theorem 1.1) but it has an advantage over the proof of Proposition 2.6 in that it yields some estimates for the gradient of $u(t, x)$. Suppose that $D=D_1 \times (0, a)$ where $D_1 \subset \mathbf{R}^{d-1}$. Take any points $z=(z^1, z^2, ..., z^d)$ and $y=(y^1, y^2, ..., y^d)$ in D with the property that $z^k=y^k$ for $k<d$ and $z^d<y^d$. Let X_t be a Brownian motion in $\mathbf{R}^d$ starting from z and let $Y_t=X_t+(y-z)$. For simplicity, we may assume that D_1 is a polygonal domain. Then we can construct reflected Brownian motions $\tilde{X}_t$ and $\tilde{Y}_t$ in D from X_t and Y_t by the means of a multi-dimensional analogue of the Skorohod lemma (see the beginning of the section). It is evident that the two processes will hit $(\partial D_1) \times (0, a)$ at the same time. The reflection on this part of the boundary will preserve (locally) the vector $X_t - Y_t$. When one of the processes hits $D_1 \times \{0\}$ or $D_1 \times \{1\}$, the vector $X_t - Y_t$ will decrease in length but it will remain vertical. Eventually, we will have $X_t = Y_t$ but before this happens the particle Y_t will be always directly above X_t. Now the same argument as in the proof of Theorem 3.1 shows that if the initial temperature $u_0(x)=u_0(x^1, x^2, ..., x^d)$ is an increasing function of x^d for fixed $x^1, x^2, ..., x^{d-1}$, then the same is true of $u(t, x^1, x^2, ..., x^d)$ for all fixed $t, x^1, x^2, ..., x^{d-1}$. This implies the weak version (HS3) of Theorem 1.1 via an argument similar to that in the proof of Theorem 3.3.

4. RESULTS BASED ON "MIRROR COUPLINGS"

First we will review some properties of "mirror couplings" for reflected Brownian motions. These results have been obtained by Burdzy and Kendall [5] in the course of study of "efficient Markovian couplings," i.e., couplings for which the probability of non-coupling by the time t is of order $e^{-\mu t}$, where μ is the spectral gap for a given Markov process.

Let us start by defining the mirror coupling for free Brownian motions in $\mathbf{R}^2$. Suppose that $x, y \in \mathbf{R}^2$, $x \neq y$, and that x and y are symmetric with respect to a line M. Let X_t be a Brownian motion starting from x and let

τ be the first time t with $X_t \in M$. Then we let Y_t be the mirror image of X_t with respect to M for $t \leqslant \tau$, and we let $Y_t = X_t$ for $t > \tau$. The process Y_t is a Brownian motion starting from y. The pair (X_t, Y_t) is a "mirror coupling" of Brownian motions in $\mathbf{R}^2$.

Next we turn to the mirror coupling of reflected Brownian motions in a half-plane $\mathscr{H}$, starting from $x, y \in \mathscr{H}$. Let M be the line of symmetry for x and y. The case when M is parallel to $\partial\mathscr{H}$ can be easily handled using Skorohod's lemma, so we focus on the case when M intersects $\partial\mathscr{H}$. By performing rotation and translation, if necessary, we may suppose that $\mathscr{H}$ is the upper half-plane and M passes through the origin. We will write $x = (r^x, \theta^x)$ and $y = (r^y, \theta^y)$ in the polar coordinates. The points x and y are at the same distance from the origin so $r^x = r^y$. Suppose without loss of generality that $\theta^x < \theta^y$. We first generate a 2-dimensional Bessel process R_t starting from r^x. Then we generate two coupled one-dimensional processes on the "half-circle" as follows. Let $\tilde{\Theta}^x_t$ be a 1-dimensional Brownian motion starting from θ^x. Let $\tilde{\Theta}^y_t = -\tilde{\Theta}^x_t + \theta^x + \theta^y$. Let Θ^x_t be the reflected Brownian motion on $[0, \pi]$, constructed from $\tilde{\Theta}^x_t$ by the means of the Skorohod lemma, using "local time" push on both sides of the interval $[0, \pi]$. The analogous reflected process obtained from $\tilde{\Theta}^y_t = \hat{\Theta}^y_t$. Let τ^Θ be the smallest t with $\Theta^x_t = \hat{\Theta}^y_t$. Then we let $\Theta^y_t = \hat{\Theta}^y_t$ for $t \leqslant \tau^\Theta$ and $\Theta^y_t = \Theta^x_t$ for $t > \tau^\Theta$. We define a "clock" by $\sigma(t) = \int_0^t R_s^{-2}\, ds$. Then $X_t = (R_t, \Theta^x_{\sigma(t)})$ and $Y_t = (R_t, \Theta^y_{\sigma(t)})$ are reflected Brownian motions in $\mathscr{H}$ with the normal vector of reflection. Moreover, X_t and Y_t behave like free Brownian motions coupled by the mirror coupling as long as they are both strictly inside $\mathscr{H}$. The processes will stay together after the first time they meet. This property is crucial in this section but was hardly relevant for the synchronous coupling. For definiteness, we let M_t be the horizontal line passing through X_t if $X_t = Y_t$.

The most important property of the above coupling is that the two processes X_t and Y_t remain at the same distance from a fixed point (the origin). We will describe how this property manifests itself in more general settings. First of all, suppose that $\mathscr{H}$ is again an arbitrary half-plane, and x and y belong to $\mathscr{H}$. Let M be the line of symmetry for x and y. Then our construction generates a pair of reflecting Brownian motions starting from x and y such that the distance from X_t to $M \cap \partial\mathscr{H}$ is the same as for Y_t, for every t. Let M_t be the line of symmetry for X_t and Y_t. Note that M_t may move, but only in a continuous way, while the point $M_t \cap \partial\mathscr{H}$ will never move. We will call M_t the *mirror* and the point $H = M_t \cap \partial\mathscr{H}$ will be called the *hinge*. The absolute value of the angle between the mirror and the normal vector to $\partial\mathscr{H}$ at H can only decrease.

The mirror coupling of reflected Brownian motions in a convex polygonal domain D can be described as follows. Suppose that X_t and Y_t start from x and y inside the domain D. As soon as one of the particles hits a side I of ∂D, the processes will evolve according to the coupling described in the previous paragraph. To be more precise, let K be the straight line containing I. Since the process which hits I does not "feel" the shape of D except for the direction of I, it follows that the two processes will remain at the same distance from the hinge $H_t = M_t \cap K$. The mirror M_t can move but the hinge H_t will remain constant as long as I remains the side of ∂D where the reflection takes place. The hinge H_t will jump when the reflection location moves from I to another side of ∂D. Since D is convex, H_t will be always on ∂D or outside D.

We will say that X_t is *active* if it is currently reflecting from a side of ∂D and similarly for Y_t. U_t^1 and U_t^2 be the intersection points of the mirror M_t with ∂D. Let ∂D^a be the "active" part of ∂D, i.e., this connected component of $\partial D \setminus \{U_t^1, U_t^2\}$ which contains the active particle. We note that the active part ∂D^a can only increase with time as a subset of the boundary. However, the active part will switch from one side of M_t to the other from time to time. It will later turn out that this is a convenient way to describe all possible movements of the mirror M_t.

THEOREM 4.1. *Suppose that D is a convex planar polygonal domain which is symmetric with respect to the horizontal axis. Let z_L and z_R be the intersection points of ∂D with the horizontal axis. Assume that for every $r > 0$, the intersection of the circle $\partial B(z_L, r)$ with D is either empty or it is a connected arc, and the same holds for $\partial B(z_R, r)$. Let Γ be any vertical line and let A be the half-plane to the right of Γ. Consider the solution $u(t, x)$ to the heat equation with Neumann boundary conditions in D and with the initial condition $u(0, x) = \mathbf{1}_A(x)$ for $x \in D$. Suppose that $x_0 \in D$ lies above the horizontal axis. Then for every t, the line containing $\nabla_x u(t, x_0)$ and passing through x_0 passes above or through each of the points z_L and z_R. An analogous statement holds for points on the other side of the horizontal axis, by symmetry. The horizontal component of $\nabla_x u(t, x_0)$ points to the right, for every $x_0 \in D$ and t.*

Note that every ellipse D can be oriented in such a way that it satisfies the condition that for every $r > 0$, the intersection of the circle $\partial B(z_L, r)$ with D is either empty or it is a connected arc, and the same holds for $\partial B(z_R, r)$.

Proof. Consider any straight line M which intersects the upper part of ∂D at a point U^1 and the lower part of ∂D at U^2. Take any points $x, y \in D$ which are symmetric with respect to M and let X_t and Y_t be reflected Brownian motions in D, starting from x and y, respectively, and related by the mirror coupling. Recall that the mirror for X_t and Y_t is denoted M_t. As long as M_t intersects both the upper and the lower parts of ∂D, we will denote the intersection points U_t^1 and U_t^2. This is true for $t = 0$, by assumption, and we intend to prove that this will remain true for all t, a.s.

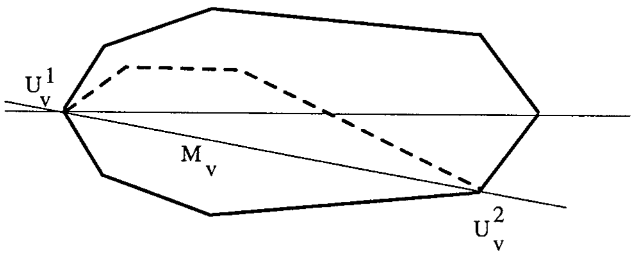

FIG. 4.1. The dotted line is the mirror image, with respect to M_v, of the non-active side of the boundary just before time v.

The points U_t^1 and U_t^2 move in a continuous fashion because M_t does. Recall that the active part ∂D^a of the boundary can only increase. This means that both U_t^1 and U_t^2 move towards z_L or both points move towards z_R. Both points can reach either one of these points at the same time only if X_t and Y_t hit one of these points. This event has probability zero by the results of Varadhan and Williams [19]. Suppose that U_t^1 reaches z_L at time v, before U_t^2 does. Then M_v forms a negative angle with the horizontal axis. Since the points U_t^1 and U_t^2 must have been moving towards z_L just before the time v, it follows that the active side ∂D^a of the boundary was above and to the right of M_t. Figure 4.1 illustrates the fact that the mirror image of the non-active side of the boundary with respect to M_v lies strictly inside D—this is due to the assumption that for every $r > 0$, the intersection of the circle $\partial B(z_L, r)$ with D is either empty or it is a connected arc, and the same holds for $\partial B(z_R, r)$. We obtain a contradiction as both processes X_t and Y_t must always stay within the set $\bar{D}$ and they are always mirror images of each other with respect to M_t (or $X_t = Y_t$).

The same argument shows that none of the points U_t^1 or U_t^2 can hit z_L or z_R before the coupling time for X_t and Y_t. This implies that the mirror M_t cannot attain the horizontal direction before the coupling time and so we conclude that $X_t^1 - Y_t^1$ does not change the sign. This implies, in the same way as in the proof of Theorem 3.1, that for every t, the function $u(t, x) = u(t, (x^1, x^2))$ is increasing in x^1 on every straight line M_1 which is perpendicular to any line M which crosses the upper and lower parts of ∂D. This easily implies the claim about the direction of the gradient $\nabla_x u(t, x_0)$. ∎

Recall that we say that a function $h(x^1, x^2)$ is antisymmetric with respect to the horizontal axis if $h(x^1, x^2) = -h(x^1, -x^2)$ for all (x^1, x^2).

THEOREM 4.2. *Suppose that D is a convex polygonal planar domain satisfying the hypotheses of Theorem* 4.1 *and there is no eigenfunction $\varphi_2(x)$ corresponding to the second eigenvalue which is antisymmetric with respect to* *the horizontal axis. Then there exists an eigenfunction* $\varphi_2(x)$ *corresponding to the second eigenvalue such that for every* $y \in D$ *we have* $\inf_{x \in \partial D} \varphi_2(x) < \varphi_2(y) < \sup_{x \in \partial D} \varphi_2(x)$.

Proof. Let z_L and z_R be intersection points of ∂D with the horizontal axis. Take any eigenfunction $\varphi_2^*(x)$ corresponding to the second eigenvalue and let $\varphi_2(x^1, x^2) = \varphi_2^*(x^1, x^2) + \varphi_2^*(x^1, -x^2)$ for all $(x^1, x^2) \in D$. Note that $\varphi_2(x)$ is an eigenfunction corresponding to the second eigenvalue. By assumption, $\varphi_2(x)$ is not identically equal to zero. By Courant's nodal domain theorem [7, p. 19; 1, p. 112], $\varphi_2(x)$ divides D into only 2 nodal domains. This and the fact that $\varphi_2(x)$ is symmetric with respect to the horizontal axis imply that the nodal line must lie at a positive distance from the points z_L and z_R. Hence, $\varphi_2(x)$ does not vanish at either point and this is also true for some neighborhoods of both points, by the continuity of $\varphi_2(x)$. We will suppose that $\varphi_2(z_R) > 0$; the proof is analogous when we have the opposite inequality.

The rest of the proof is very similar to the proof of Thoerem 3.3. Consider two distinct vertical lines Γ and Γ_1 and the corresponding regions A and A_1 to the right of Γ and Γ_1. We assume the Γ and Γ_1 are so close to z_R that $\varphi_2(x) > 0$ for all $x \in (A \cup A_1) \cap D$.

We set $u_0(x) = \mathbf{1}_A(x)$ and $u_0^1(x) = \mathbf{1}_{A_1}(x)$ for $x \in D$. Let $u(t, x)$ and $u^1(t, x)$ be the solutions of the Neumann problem in D with initial conditions $u_0(x)$ and $u_0^1(x)$, respectively. We have, by Proposition 2.1,

$$u(t, x) = \alpha_1 + \alpha_2 \varphi_2(x)\, e^{-\mu_2 t} + \tilde{\alpha}_2 \varphi_2(x)\, e^{-\mu_2 t} + RR(t, x),$$

where $R(t, x)$ converges to 0 faster than $e^{-\mu_2 t}$ as $t \to \infty$. Here $\tilde{\varphi}_2(x)$ is an eigenfunction corresponding to μ_2 which is orthogonal to $\varphi_2(x)$. The analogous formula for $u^1(t, x)$ is

$$u^1(t, x) = \beta_{12} + \beta_2 \varphi_2(x)\, e^{-\mu_2 t} + \tilde{\beta}_2 \tilde{\varphi}_2(x)\, e^{-\mu_2 t} + R^1(t, x).$$

We have

$$\beta_2 - \alpha_2 = \int_{A_1 \cap D} \varphi_2(x)\, dx - \int_{A \cap D} \varphi_2(x)\, dx = \int_{(A_1 \backslash A) \cap D} \varphi_2(x)\, dx \neq 0,$$

so at least one of the coefficients α_2 or β_2 is non-zero. Let us assume that $\beta_2 \neq 0$, the other case being analogous. Then

$$u^1(t, x) = \beta_1 + \hat{\beta}_2 \hat{\varphi}_2(x)\, e^{-\mu_2 t} + R^1(t, x), \tag{4.1}$$

where $\hat{\varphi}_2(x)$ is a second eigenfunction and $\hat{\beta}_2 \neq 0$.

Without loss of generality we assume that $\hat{\beta}_2 > 0$. Theorem 4.1 implies that $u^1(t, (x^1, x^2))$ is an increasing function of x^1, for every fixed t. For every $x \in D$, let $V(x)$ be the intersection of D with the ball $B(z_R, |x - z_R|)$. Using the information about the direction of the gradient $\nabla_x u(t, x)$ provided by Theorem 4.1, it is elementary to see that $u^1(t, y) \geqslant u^1(t, x)$, for all $x \in D$, $y \in V(x)$, and $t > 0$. Since $R^1(t, x)$ converges to 0 faster than $e^{-\mu_2 t}$, the last fact and (4.1) imply that $\hat{\varphi}_2(y) \geqslant \hat{\varphi}_2(x)$ for $y \in V(x)$. Since the maximum of $\hat{\varphi}_2(x)$ cannot be attained on a non-empty open subset of D (recall the argument from the proof of Theorem 3.3), it can be attained only at z_R. The location of the minimum is z_L, by the same argument. ∎

THEOREM 4.3. *Suppose that D is a convex polygonal planar domain which is symetric with respect to the horizontal and vertical axes. Then there exists an eigenfunction $\varphi_2(x)$ corresponding to the second eigenvalue such that for every $y \in D$ we have $\inf_{x \in \partial D} \varphi_2(x) < \varphi_2(y) < \sup_{x \in \partial D} \varphi_2(x)$.*

Proof. First we are going to show that there exists an eigenfunction $\varphi_2(x)$ which is antisymmetric with respect to one of the axes, i.e., we either have $\varphi_2(x^1, x^2) = -\varphi_2(x^1, -x^2)$ for all $(x^1, x^2) \in D$ or we have $\varphi_2(x^1, x^2) = -\varphi_2(-x^1, x^2)$ for all $(x^1, x^2) \in D$. Take any eigenfunction $\varphi_2^*(x)$ corresponding to the second eigenvalue. Let $\tilde{\varphi}_2(x^1, x^2) = \varphi_2^*(x^1, x^2) + \varphi_2^*(-x^1, x^2)$. If $\tilde{\varphi}_2(x)$ is identically 0 then we can take $\varphi_2(x) = \varphi_2^*(x)$. Otherwise we let $\hat{\varphi}_2(x^1, {}^2) = \tilde{\varphi}_2({}^1, {}^2) + \tilde{\varphi}_2(x^1, -x^2)$.

We will prove that $\hat{\varphi}_2(x)$ is identically equal to 0. Suppose that $\hat{\varphi}_2(x)$ is not identically equal to 0. The function $\hat{\varphi}_2(x)$ is symmetric with respect to both axes. Let $\tilde{D}$ be the part of D in the first quadrant. Since $\hat{\varphi}_2(x)$ must take both positive and negative values and is symmetric with respect to both axes, the nodal line must intersect the interior of $\tilde{D}$. The nodal line cannot form a closed loop inside $\tilde{D}$, see, e.g., [1, p. 128]. (The fact that the nodal line cannot form a closed loop follows easily from the fact that $\lambda_1 > \mu_2$ [1, p. 155], where λ_1 is the first Dirichlet eigenvalue for D.) The part of the nodal line inside $\tilde{D}$ cannot touch both axes because, by symmetry, we would have a closed loop formed by the nodal line inside D. If the nodal line inside $\tilde{D}$ touches ∂D, then D must be divided into more than 2 nodal domains. This is ruled out by the Courant nodal domain theorem quoted in the proof of Theorem 4.2. This completes the proof that $\hat{\varphi}_2(x)$ is identically equal to 0 and so we can take $\varphi_2(x) = \tilde{\varphi}_2(x)$.

The nodal line for $\varphi_2(x)$ must lie on one of the axes. Without loss of generality, suppose that it lies on the vertical axis. Let $\mathscr{H}^+$ denote teh right half plane. Then $\varphi_2(x)$ is the first eigenfunction for the mixed Dirichlet–Neumann problem in $D' = D \cap \mathscr{H}^+$, with the Dirichlet boundary conditions on $\partial_1 D' = \partial D \cap \partial \mathscr{H}^+$ and the Neumann boundary conditions elesewhere on the boundary. We will prove that $\varphi_2(x)$ is monotone on all horizontal lines.

The probabilistic representation of the solutions $u(t, x)$ to the mixed Dirichlet–Neumann heat problem involves Brownian motion X_t reflected on $\partial_2 D' = \partial D \cap \mathscr{H}^+$ and killed on $\partial_1 D'$. Let $u_0(x) = 1$ for all $x \in D'$. If $u(0, x) = u_0(x)$ for all $x \in D'$ and $X_0 = x$ then $u(s, x)$ is equal to the probability that X_t is not killed on $\partial_1 D'$ before time s. Suppose that $x = (x^1, x^2)$ and $y = (y^1, y^2)$ are any points in D' with $x^2 = y^2$, and $x^1 < y^1$. In order to prove monotonicity of $\varphi_2(x)$ on horizontal lines it will suffice to construct Brownian motions X_t and Y_t, starting from x and y, and such that X_t exists D' through $\partial_1 D'$ no later than Y_t does.

Our proof will use the mirror coupling except that if any of the processes X_t or Y_t hits $\partial_1 D'$, it will be killed, and the other process, if it survives beyond this point, will continuoe on its own. he other process may be killed later.

Since the points x and y lie on a horizontal line, the initial direction of the mirror M_0 for X_0 and Y_0 is vertical. Let U_0^1 and U_0^2 be the upper and lower points of intersection of M_0 with $\partial D'$. Since M_t moves in a continuous way, we can choose the labels U_t^1 and U_t^2 for the intersection points of M_t with $\partial D'$ in such a way that U_t^1 and U_t^2 are continuous functions of t. In this proof we change the conventions concerning the angles and we choose the angle $\angle M_t$ between M_t and the horizontal axis so that $t \to \angle M_t$ is a continuous function. We set $\angle M_0 = \pi/2$.

Let z_R be the intersection point of $\partial_2 D'$ and the horizontal axis. We will argue that neither U_t^1 nor U_t^2 can ever touch z_R and $\angle M_t$ always stays in $[0, \pi]$. Suppose that this is not always true and let v be the infimum of t such that $U_t^1 = z_R$ or $U_t^2 = z_R$ or $U_t^2 = z_R$ or $\angle M_t \notin [0, \pi]$. First we consider the case when $U_v^1 = z_R$ (the case $U_v^2 = z_R$ is analogous). One can prove that M_v cannot be horizontal in this case but we do not need to do this—if M_v is horizontal and $U_v^1 = z_R$ then M_v lies on the horizontal axis, so, by symmetry, the line M_t will stop moving at time v and X_t and Y_t will hit $\partial_1 D'$ at the same time.

Next suppose that $U_v^1 = z_R$ and M_v is not horizontal, and so $\angle M_v \in (0, \pi/2)$. In this case, the argument is very similar to that in the proof of Theorem 4.1. The main difference is that we do not assume any more that for every $r > 0$, the intersection of the circle $\partial B(z_L, r)$ with D is either empty or it is a connected arc, and the same holds for $\partial B(z_R, r)$. However, we use the assumption of D having two lines of symmetry as follows. Note that the active part of the boundary must have been the part of $\partial_2 D'$ above U_t^1, just before time v, because U_t^1 was pushed down to z_R. The mirror image, with respect to M_v, of the part of $\partial_2 D'$ below M_v lies strictly inside D', or on $\partial_1 D'$, or outside D'. This contradicts the fact that X_t and Y_t must always stay inside the domain D', and that the active side of the boundary just before time v was above U_t^1.

Now suppose that $\angle M_v = 0$; as usual, the symmetric case $\angle M_v = \pi$, is left to the reader. We have already discussed the case when M_v lies on the horizontal axis, so let us assume that it does not. Suppose U_v^1 lies in the

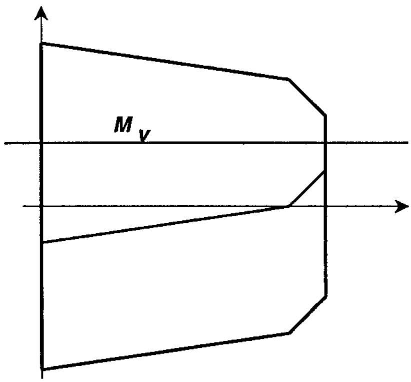

FIGURE 4.2

first quadrant. Then the mirror image, with respect to M_v, of the part of D' above M_v lies inside the part of D' below M_v. We have two possibilities for what may happen after time v. The first one is illustrated by Fig. 4.2. The point z_R may lie on a vertical segment of the boundary of D'. Then both X_t and Y_t may be reflecting at the same time from this line segment for some time after v. In this case M_t will not be moving.

The only other possibility is that the upper side of the boundary of D' will be active. If a part of the boundary of D' is horizontal and one of the processes is reflecting from this part, the mirror will move but it will not change its direction. Otherwise, since D is symmetric with respect to the vertical axis, the hinge for the mirror, if it exists must lie to the right of $M_v \cap \partial_2 D'$ and so the mirror will be turning counterclockwise; i.e, the angle $\angle M_t$ will move from 0 to inside the interval $(0, \pi/2)$.

It is routine to restart the argument at the next time when $\angle M_v = 0$ or $\angle M_v = \pi$ and complete the proof of the claim that U_t^1 and U_t^2 never hit z_R and $\angle M_t$ always stays in $[0, \pi]$.

Since $\angle M_t \in [0, \pi]$ for all t, we have $X_t^1 \leqslant Y_t^1$ for all t, until one or both the processes are killed. This proves X_t must hit $\partial_1 D'$ before or at the same time when Y_t hits $\partial_1 D'$. This in turn proves the monotonicity of $u(t, x)$ along horizontal lines within D', for every fixed t.

Next we extend our argument to points $x = (x^1, x^2)$ and $y = (y^1, y^2)$ such that the line of symmetry for these points crosses both the upper and the lower sides of ∂D. The same reasoning as for x and y lying on a horizontal line proves that if $x^1 < y^1$ then the process starting from x will hit $\partial_1 D'$ no later than the process starting from y. It follows that $u(t, x) \leqslant u(t, y)$ for all t. We recall from the proof of Theorem 4.1 that this implies that given any $x_0 \in D$ which lies above the horizontal axis and any t, the line containing $\nabla_x u(t, x_0)$ and passing through x_0 passes above or through each of the points z_L and z_R.

Then we can argue as in the proof of Theorem 4.2 that if $V(x)$ is the intersection of D with the ball $B(z_R, |x-z_R|)$ then $u(t, x) \leqslant u(t, y)$, for all $y \in V(x)$ and all t.

We have $u(t, x) = \alpha\varphi_2(x)\, e^{-\mu_2 t} + R(t, x)$, where $\alpha \neq 0$ and $R(t, x)$ goes to 0 faster than $e^{-\mu_2 t}$. Without loss of generality suppose that $\alpha > 0$. It follows that $\varphi_2(x) \leqslant \varphi_2(y)$ for $y \in V(x)$. This implies that $\varphi_2(x)$ attains its maximum only at z_R. For the same reason, the minimum is attained at z_L. ▮

Recall that d_D and w_D denote the diameter of D and the length of the projection of D on the vertical axis.

COROLLARY 4.1. *Suppose that a convex polygonal domain D is symmetric with respect to the horizontal axis S and the ratio d_D/w_D is greater than* 1.54. *Let x and y be the intersection points of S with ∂D. Make at least one of the following additional assumptions.*

(A1) *D has another line of symmetry S_1 which is perpendicular to S*;

(A2) *For every $r > 0$, the intersection of the circle $\partial B(x, r)$ with D is either empty or it is a connected arc, and the same holds for $\partial B(y, r)$.*

Then (HS1), *the strongest version of "hot spots" conjecture, holds for D.*

Proof. In the proof of Proposition 2.4(ii) it is shown that if there are two independent eigenfunctions corresponding to μ_2 then a linear combination of them is antisymmetric. An assumption of Theorem 4.2 asserts that there is no such eigenfunction. This, Theorem 4.3, and Proposition 2.4(ii) imply the lemma. ▮

Theorems 4.1–4.3 and Corollary 4.1 hold for domains with smooth boundaries. We stated them only for polygonal domains in order to avoid the discussion of the mirror coupling to smooth domains. The generalization of the mirror coupling to smooth domains is not too hard and does not involve any fundamentally different ideas.

EXAMPLE 4.1. One may ask whether the counterexample to "Chavel's conjecture" about domain monotonicity for the Neumann heat kernel given in Bass and Burdzy [2] can be adapted to give a counterexample to the "hot spots conjecture" of Rauch. The question is rather vague but the answer seems to be negative in view of the following example. We will only sketch the argument and only for the three dimensional space.

Let D consist of two-cream cones,

$$D = \{(x^1, x^2, x^3) \in \mathbf{R}^3 : (x^2)^2 + (x^3)^2 < a \min[(x^1+1)^2, (-x^1+1)^2]\},$$

where $a \in (0, \infty)$. Let $D_1 = \{(x^1, x^2, x^3) \in D : x^1 < 0\}$ and let $\varphi(x)$ be the first eigenfunction for the mixed Neumann–Dirichlet problem in D_1 with the Neumann boundary conditions on $\partial_1 D_1 = \partial D_1 \cap \partial D$ and the Dirichlet conditions on $\partial_2 D_1 = \partial D \backslash \partial_1 D_1$. We extend $\varphi(x)$ to the whole domain D so that it is antisymmetric in x^1 variable. Then $\varphi(x)$ is a Neumann eigenfunction in D but we expect it to correspond to μ_2 only for sufficiently small a. We will argue that $\varphi(x)$ attains its maximum on the boundary of D.

Consider any points $y = (y^1, y^2, y^3)$ and $z = (z^1, z^2, z^3)$ in D_1 with the same distance ρ from $(-1, 0, 0)$. We will construct a Brownian motion X_t in D_1 with the normal reflection on $\partial_1 D_1$ and killed on $\partial_2 D_1$. It will be convenient to consider $(-1, 0, 0)$ as the origin as we will use spherical coordinates. First we generate the radial part of X_t, i.e., a 3-dimensional Bessel process R_t starting from ρ. Then we generate the angle Θ_t^X between X_t and the x^1-axis. Finally, we generate the third component Σ_t^X of the spherical coordinates. We choose the starting points for these processes so that X_t starts from z. We construct another process Y_t in the same way, starting from y, with the important provision that the radial part of Y_t is R_t, i.e., the same as for X_t. Moreover, we couple the processes Θ_t^X and Θ_t^Y after the first time τ^Θ when they meet, i.e., we have $\Theta_t^X = \Theta_t^Y$ for $t \geqslant \tau^\Theta$. We conclude that $\Theta_t^X - \Theta_t^Y$ cannot change the sign. Hence, the process which started closer to the x^1-axis will hit $\partial_2 D_1$ no later than the other one. Our usual argument now shows that the function $\varphi(x)$ is a non-decreasing function of the distance from the x^1-axis on every fixed sphere centered at $(-1, 0, 0)$. We conclude that the maximum of $\varphi(x)$ must be attained on ∂D.

Exercise 4.1. We end with an exercise for readers who stayed with us until now and mastered the coupling arguments. Let Γ_a^1 be a curve defined by parametric equations as $\{(x^1(s), x^2(s)), 0 \leqslant s \leqslant 2\pi\}$, with

$$x^1(s) = -1 + \left(1 + \frac{a}{2\pi} s\right) \frac{s + 2\pi}{4\pi} \cos s,$$

$$x^2(s) = \left(1 + \frac{a}{2\pi} s\right) \frac{s + 2\pi}{4\pi} \sin s.$$

Let Γ_a be the union of Γ_a^1 and the curve symmetric to Γ_{-a}^1 with respect to the point $(0, 0)$. The union of $\Gamma_{0.05}$ and $\Gamma_{-0.05}$ forms the boundary of a domain D depicted in Fig. 4.3. Suppose that the function $u(0, x)$ (i.e., the initial condition in (1.1)) is equal to 1 for all $x = (x^1, x^2) \in D$ with $x^1 > -1/4$, $x^2 > 0$, and also for all points with $x^1 > 1/4$. The initial condition is zero elsewhere in D. Prove that for every fixed $t > 0$, the solution $u(x, t)$ of (1, 1) is monotone along every line Γ_a, for every $a \in (-0.05, 0.05)$. Hint: the synchronous coupling does not work in this case but the mirror coupling does.

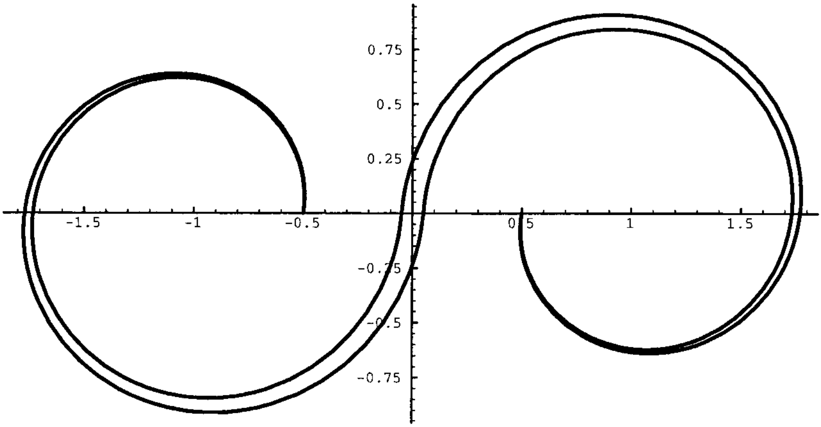

FIGURE 4.3

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REFERENCES

1. C. Bundle, "Isoperimetric Inequalities and Applications," Pitman, London, 1980.
2. R. Bass and K. Burdzy, On domain monotonicity of the Neumann heat kernel, *J. Funct. Anal.* **116** (1993), 215–224.
3. R. Bass and P. Hsu, Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains, *Ann. Probab.* **19** (1991), 486–508.
4. K. Burdzy, R. Hołyst, D. Ingerman, and P. March, Configurational transition in a Fleming–Viot-type model and probabilistic interpretation of Laplacian eigenfunctions, *J. Phys. A* **29** (1996), 2633–2642.
5. K. Burdzy and W. Kendall, Efficient Markovian couplings: Examples and counterexamples, preprint, 1998.
6. K. Burdzy and W. Werner, A counterexample to the "hot spots" conjecture, *Ann. of Math.*, in press.
7. I. Chavel, "Eigenvalues in Riemannian geometry," Academic Press, Orlando, 1984.

8. S. Y. Cheng, Eigenvalue comparison theorems and its geometric applications, *Math. Z.* **143** (1975), 289–297.
9. E. B. Davies, "Heat Kernels and Spectral Theory," Cambridge Univ. Press, Cambridge, UK, 1989.
10. G. B. Folland, "Introduction to Partial Differential Equations," Princeton Univ. Press, Princeton, NJ, 1976.
11. I. Karatzas and S. E. Shreve, "Brownian Motion and Stochastic Calculus," Springer-Verlag, New York, 1988.
12. B. Kawohl, "Rearrangements and Convexity of Level Sets in PDE," Lecture Notes in Mathematics, Vol. 1150, Springer-Verlag, Berlin, 1985.
13. C. S. Lin, On the second eigenfunction of the Laplacian in $\mathbf{R}^2$, *Comm. Math. Phys.* **111** (1987), 161–166.
14. N. S. Nadirashvili, On the multiplicity of the eigenvalues of the Neumann problem, *Soviet Math. Dokl.* **33** (1986), 281–282.
15. N. S. Nadirashvili, Multiple eigenvalues of the Laplace operator, *Math. USSR Sb.* **133–134** (1988), 225–238.
16. M. Reed and B. Simon, "Methods of Modern Operators," Academic Press, New York/London, 1978.
17. R. Smits, Spectral gaps and rates to equilibrium for diffusions in convex domains, *Michigan Math. J.* **43** (1996), 141–157.
18. R. Sperb, "Maximum Principles and Their Applications," Academic Press, San Diego, 1981.
19. S. R. S. Varadhan and R. J. Williams, Brownian motion in a wedge with oblique reflection, *Comm. Pure Appl. Math.* **38** (1985), 405–443.