On the Hopf Lemma

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Dedicated to Vladimir Mazya on his 70th birthday

Abstract. The Hopf Lemma for second order elliptic operators is proved to hold in domains with $C^{1,\alpha}$, and even less regular, boundaries. It need not hold for C^1 boundaries. Corresponding results are proved for second order parabolic operators.

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1.

The Hopf Lemma, a purely local result, is a basic tool in the study of second order elliptic operators in $\mathbb{R}^n$, of the form

$$L := a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + b_i(x)\frac{\partial}{\partial x_i} + c(x) =: M + c. \tag{1.1}$$

It says essentially that if u is a positive function in a domain G, which vanishes at a boundary point P, and satisfies

$$Lu \le 0, \tag{1.2}$$

then the interior normal derivative of u at P is positive. A standard, precise form of the lemma, which applies also to degenerate elliptic operators is stated below in Proposition 1.

Up to now, one required that the boundary ∂G be of class C^2 near P. In this paper we relax that condition considerably. As stated in Corollary 1 in section 3, it suffices that the boundary be in $C^{1,\alpha}$ for some $0 < \alpha < 1$. In fact a much weaker condition suffices — almost that the boundary is in C^1. But in section 2 we give examples, Example 1 and 2, of C^1 boundaries where the Hopf Lemma does not hold. It is, of course, well known that it must not hold in Lipschitz domains. For example, in the plane, near the origin, in $x, y > 0$, the harmonic function

$$u = xy$$

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O.A. Oleinik independently discovered the Hopf Lemma, in [**5**], at the same time as Hopf. But we will continue to call it Hopf lemma, as it is usually known.

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is positive, vanishes at the origin, and grows quadratically as we go on the line $y = x > 0$.

Here is a standard form of the Hopf Lemma.

Concerning L, we always assume $a_{ij}\xi_i\xi_j \geq 0$ for $\xi \in \mathbb{R}^n$ (degenerate elliptic), all coefficients are bounded, in absolute value by C, and, near P, we assume, if ν is the interior unit normal to ∂G at P, that

$$a_{ij}\nu_i\nu_j \geq c_0, \text{ a positive constant.} \tag{1.3}$$

Proposition 1. *Assume that $u > 0$ near P in G is of class C^2 there, continuous in $G \cup \{P\}$, and vanishes at P, and that it satisfies (1.2) in G. If ∂G is of class C^2 then, if ν is the interior unit normal to ∂G at P,*

$$\liminf_{t \to 0} \frac{u(P + t\nu)}{t} > 0. \tag{1.4}$$

A few words about the well known proof: It is clear that if the Hopf Lemma holds at P in G then it holds at P on any domain D, containing G, having P also on the boundary.

The method of proof is to consider a slightly smaller domain Ω lying in G, with P on its boundary. In $\overline{\Omega}$, near P,

$$u > 0 \text{ except at } P. \tag{1.5}$$

Now one constructs a comparison function h in $\overline{\Omega}$ satisfying, near P,

$$h(P) = 0, \quad h \leq 0 \text{ on } \partial\Omega, \tag{1.6}$$

$$Lh \geq 0 \text{ in } \Omega, \tag{1.7}$$

and

$$\liminf_{t \to 0^+} \frac{h(P + t\nu)}{t} > 0. \tag{1.8}$$

Having such a function h, by (1.5), it is clear that for some $0 < R$ small, and $0 < \epsilon$ small,

$$w := u - \epsilon h \geq 0 \quad \text{on } \partial\{\Omega \cap B(P, R)\}.$$

Here $B(P, R)$ is the ball about P of radius R.

In $\Omega \cap B(P, R)$, we have

$$Lw \leq 0.$$

We may apply the maximum principle and conclude that

$$w \geq 0 \quad \text{in } \Omega \cap B(P, R).$$

Recall that, because R is small, the maximum principle holds in $\Omega \cap B(P, R)$.

Since L is not assumed to be uniformly elliptic, we include the proof of that. Consider the positive function in $\Omega \cap B(P, R)$,

$$m := e^{\alpha R} - e^{\alpha\nu\cdot(x-P)}, \qquad \alpha > 0.$$

We have

$$Mm = (-\alpha^2 a_{ij}\nu_i\nu_j - \alpha b_i\nu_i)e^{\alpha\nu\cdot(x-P)}.$$

Because of (1.3) we may fix α large so that

$$Mm \leq -\frac{\alpha^2 c_0}{2} e^{\alpha\nu\cdot(x-P)} \quad \text{and} \quad \frac{\alpha^2 c_0}{2} > 5|c|.$$

Then

$$e^{-\alpha\nu\cdot(x-P)}Lm \le -\frac{\alpha^2 c_0}{2} + 2|c||e^{\alpha(R-\nu\cdot(x-P))} - 1|.$$

Finally, choosing R small we find that

$$Lm < 0 \qquad \text{in } \Omega \cap B(P,R).$$

But then the function $z = w/m$ satisfies

$$a_{ij}z_{ij} + b_i' z_i + \frac{Lm}{m} z < 0.$$

Since $Lm \le 0$, z cannot have a negative minimum in $\Omega \cap B(P,R)$. So $z \ge 0$, and hence

$$w \ge 0 \quad \text{in} \quad \Omega \cap B(P,R).$$

Finally, since $u \ge \epsilon h$, (1.4) follows from (1.8).

Thus the proof of the Hopf Lemma hinges on constructing a function h satisfying (1.6)-(1.8).

REMARK 1. *It is clear from the proof that the conclusion of the Hopf Lemma also holds if ν is any unit vector at P pointing inside the domain and not tangent to the boundary. This will be true for all the results of the paper, but we will just take ν =unit inner normal.*

In section 4 we treat parabolic operators: we wish to call attention to Theorem 4.

2.

From now, for convenience we let P be the origin, $\nu = (0, \cdots, 0, 1)$ and write the coordinates as (x, y), $x = (x_1, \cdots, x_{n-1})$, $y \in \mathbb{R}$, and assume that G contains a domain Ω of the form

$$\Omega = \{(x,y) \mid y > f(|x|)\}, \quad f(0) = 0. \tag{2.1}$$

We begin with a result yielding a Hopf Lemma for $L = \Delta$. For such L the conditions on f are very clean.

THEOREM 1. *Assume that*

$$f \in C^1[0,1] \cap C^2(0,1), \tag{2.2}$$

$$\int_0^1 \frac{f(s)}{s^2} ds < \infty, \tag{2.3}$$

$$f(0) = f'(0) = 0, \quad f'(r) \ge 0 \textit{ for } 0 < r < 1, \tag{2.4}$$

$$f''(r) + \frac{n-1}{r} f'(r) \textit{ is nonincreasing.} \tag{2.5}$$

Then, for $L = \Delta$, if $u > 0$ in Ω near the origin, $u \in C^2(\Omega)$, and u is continuous in $\Omega \cup (0,0)$, vanishes at $(0,0)$ and satisfies

$$\Delta u \le 0 \quad \textit{in } \Omega.$$

Then (1.4) holds:

$$\liminf_{y \to 0^+} \frac{u(0,y)}{y} > 0. \tag{2.6}$$

A consequence is, taking $f(r) = r^{1+\alpha}$, that

Corollary 1. *The Hopf Lemma holds in domains with $C^{1,\alpha}$ boundary, $0 < \alpha < 1$.*

Proof of Theorem 1. It is easy to see that there exists some small positive constant ϵ such that either $f > 0$ in $(0, \epsilon)$ or $f \equiv 0$ in $(0, \epsilon)$. Without loss of generality, we assume that $f > 0$ in $(0, 1)$. The conditions on f also hold for af, $a > 1$, so then in

$$\{(x, y) \mid y > af(|x|)\}, \quad y \text{ small}$$

$u > 0$ in the closure except at $(0, 0)$. We will consider that domain and we simply rewrite af as f. So in our new Ω, $u > 0$ on $\overline{\Omega}$ near the origin, except at the origin.

We now construct a comparison function h satisfying (1.6)-(1.8). Namely, with $r = \sqrt{|x|^2 + y^2}$,

$$h = y + g(y) - 2f(r)$$

with g defined by, using (2.3),

$$g = 2f(y) + 2(n-1)y \int_0^y \frac{f(s)}{s^2} ds. \tag{2.7}$$

By (2.2) and (2.4), $g \in C^1[0, 1) \cap C^2(0, 1)$. We have

$$g''(y) = 2\left(f''(y) + (n-1)\frac{f'(y)}{y}\right). \tag{2.8}$$

Hence, by (2.5),

$$\frac{1}{2}\Delta h = \left(f''(y) + (n-1)\frac{f'(y)}{y}\right) - \left(f''(r) + (n-1)\frac{f'(r)}{r}\right) \geq 0, \tag{2.9}$$

thus (1.7) holds.

Since $g(0) = g'(0) = 0$ and $f' \geq 0$, it follows from the fact that $y = f(|x|)$ on $\partial\Omega$, that (1.6) holds:

$$h \leq 0 \quad \text{on } \partial\Omega.$$

Finally we have $h_y(0, 0) = 1$, so (1.8) holds. h satisfies the conditions (1.6)-(1.8).

□

Before treating the general operator L we first describe a class of C^1 function f such that for Δ, the Hopf Lemma does *not* hold in $\widetilde{\Omega}$, at the origin. Here

$$\widetilde{\Omega} = \{(x, y) \mid y > f(|x|)\}. \tag{2.1$'$}$$

Theorem 2. *Assume that f satisfies conditions (2.2) and (2.4). Instead of (2.3) assume*

$$\int_0^1 \frac{f(s)}{s^2} ds = \infty. \tag{2.10}$$

Assume in addition, for some positive constant C, that

$$\frac{f}{r} \leq Cf' \qquad \text{in } (0, 1) \tag{2.11}$$

and

$$f'' + \frac{n - \frac{3}{2}}{r} f' \geq 0 \qquad \text{in } (0, 1). \tag{2.12}$$

Then the Hopf Lemma does not hold for Δ in $\widetilde{\Omega}$, at the origin.

Observe first that the domain $\widetilde{\Omega} = \{y > f(|x|)\}$ lies in the domain $\Omega = \{y > \frac{1}{2}f(r)\}$, hence, near the origin,

$$f(|x|) > \frac{1}{2}f\left(\sqrt{|x|^2 + f(|x|)^2}\right) = \frac{1}{2}f(r), \quad \text{if } y = f(|x|).$$

Indeed, since $f' \geq 0$, the right hand side is

$$\leq \frac{1}{2}f(|x| + f(|x|)) =: J.$$

Now

$$J = \frac{1}{2}\left(f(|x|) + f'(\xi)f(|x|)\right)$$

for some ξ in $|x| < \xi < |x| + f(|x|)$. Since $f'(s) \to 0$ as $s \to 0$, it follows that

$$J \leq f(|x|).$$

We will show that the Hopf Lemma does not hold for Δ at the origin in $\{y > \frac{1}{2}f(r)\}$.

For convenience we replace f by $2f$, and work in the domain

$$\Omega = \{(x, y) \mid y > f(r)\}.$$

Proof. Let g be the function

$$g = exp\left(-\beta \int_r^1 \frac{f(s)}{s^2}ds\right), \qquad \beta > 0. \tag{2.13}$$

Because of (2.10), $g(0) = 0$. Consider

$$u = (y - f(r))g(r).$$

Clearly $u > 0$ in Ω and $u = 0$ on $\partial\Omega$. The constant β will be chosen so that

$$\Delta u \leq 0 \ \text{ in } \Omega.$$

Then, since $u_y(0,0) = 0$, the Hopf Lemma does not hold at the origin.

We now compute. First

$$\begin{aligned}
\Delta\left(yg(r)\right) &= y\Delta g + 2g'\frac{y}{r} = y\left(g'' + \frac{n+1}{r}g'\right) && (2.14)\\
&= yg\left(\frac{\beta^2 f^2}{r^4} + \frac{\beta f'}{r^2} - \frac{2\beta f}{r^3} + \frac{n+1}{r}\frac{\beta f}{r^2}\right) \\
&= yg\left(\frac{\beta^2 f^2}{r^4} + \frac{(n-1)\beta f}{r^3} + \frac{\beta f'}{r^2}\right) && (2.15)\\
&> 0. && (2.16)
\end{aligned}$$

On the other hand,

$$\Delta(fg) = g(f'' + \frac{n-1}{r}f') + 2f'g' + f(g'' + \frac{n-1}{r}g').$$

By (2.14), $\Delta u \leq 0$ is thus equivalent to:

$$(y - f)(g'' + \frac{n+1}{r}g') \leq g(f'' + \frac{n-1}{r}f') + 2f'g' - \frac{2fg'}{r}.$$

In view of (2.16) it suffices to show that

$$r(g'' + \frac{n+1}{r}g') \leq g(f'' + \frac{n-1}{r}f') + 2f'g' - \frac{2fg'}{r}$$

and, because of (2.12) and (2.15), this will follow, provided

$$\frac{(\beta^2+2\beta)f^2}{r^3}+\frac{(n-1)\beta f}{r^2}+\frac{\beta f'}{r}\le\frac{1}{2r}f'+\frac{2\beta f f'}{r^2}. \tag{2.17}$$

Using (2.11) we see that for β small, the LHS of (2.17) is

$$\le\frac{3C^2\beta}{r}(f')^2+\left(C(n-1)\beta+\beta\right)\frac{f'}{r}\le\left(C(n-1)+2\right)\beta\frac{f'}{r},$$

since f' is small for r small.

But the RHS of (2.17) is $\ge\frac{1}{2r}f'$. Hence for β small we see that (2.17) holds. Indeed

$$\Delta\left((y-f(r))g(r)\right)\le-\frac{g}{4r}f'.$$

An example of an f satisfying the conditions of Theorem 2 is

Example 1. $f=\dfrac{r}{\log\frac{1}{r}}$.

Another, which is rather sharp in view of Example 3 below is the following: Set

$$E_0=\log\frac{1}{r},$$

and define recursively E_k by

$$E_k=\log E_{k-1}.$$

Set

$$G_k=E_0\cdots E_k.$$

Note that

$$E'_{k+1}=-\frac{1}{rG_k}. \tag{2.18}$$

Example 2. For every positive integer k, the function

$$f_k=\frac{r}{G_k(r)} \tag{2.19}$$

satisfies the conditions of Theorem 2. Hence for this f_k, the Hopf Lemma does not hold.

On the other hand we have

Example 3. For every positive integer k, and $a>0$, the function

$$f_k=\frac{r}{G_k(r)E_k(r)^a} \tag{2.20}$$

satisfies the conditions of Theorem 1. Thus the Hopf Lemma holds in $\{y>f_k(|x|)\}$. This is stronger than Corollary 1.

The reader may easily check Examples 2 and 3.

3.

We now take up the general operator L and prove that the Hopf Lemma holds at the origin, for L in the domain

$$\Omega = \{(x, y) \mid y > f(|x|)\}, \tag{3.1}$$

assuming some mild conditions on f in addition to those in Theorem 1.

THEOREM 3. *Assume that L satisfies the conditions described earlier, in particular that (1.3) holds, i.e.*

$$a_{nn} \geq c_0 > 0. \tag{3.2}$$

About f we assume (2.2)-(2.5) and, in addition, for some positive constants C_1 and c_2,

$$f(r) = \circ(f'(r)) \quad \textit{as } r \to 0, \tag{3.3}$$

$$\frac{f'(r)}{r} \leq C_1 \left(f''(r) + \frac{n-1}{r} f'(r) \right) \quad \textit{in } (0,1) \tag{3.4}$$

$$\int_0^r \frac{f(s)}{s^2} ds = \circ(\frac{f'(r)}{r}) \quad \textit{as } r \to 0 \quad \textit{and } \frac{f'(r)}{r} \geq c_2 > 0. \tag{3.5}$$

Then the Hopf Lemma holds for L in Ω at the origin.

Proof. As usual by restricting u to a slightly smaller domain, which we still call Ω, $\{y > \alpha f(|x|)\}$ for $0 < \alpha - 1$ small we may suppose that near the origin, $u > 0$ in $\overline{\Omega}$ except at the origin where it is zero.

We have only to construct a comparison function h satisfying conditions (1.6)-(1.8) in the new Ω, i.e.

$$h(0,0) = 0, \quad h \leq 0 \text{ on } \partial\Omega, \tag{3.6}$$

$$Lh \geq 0 \text{ in } \Omega, \tag{3.7}$$

$$\liminf_{y \to 0} \frac{h(0,y)}{y} > 0. \tag{3.8}$$

We take

$$h = y + g(y) - 2f(r)$$

with, as usual, $r = \sqrt{|x|^2 + y^2}$ and g defined by

$$g(y) = K \left(f(y) + (n-1)y \int_0^y \frac{f(s)}{s^2} ds \right). \tag{3.9}$$

K will be chosen large. For r small, h satisfies (3.6). We have

$$g''(y) = K(f''(y) + \frac{n-1}{y} f'(y)),$$

and hence

$$\begin{aligned} L(y+g) &\geq Ka_{nn}(f'' + \frac{n-1}{y} f') - C(g' + g) - 2C \\ &\geq Kc_0(f'' + \frac{n-1}{y} f') - CK(f' + \frac{f}{y} + \int_0^y \frac{f(s)}{s^2} ds) - 2C. \end{aligned}$$

Hence, using (3.3)-(3.5) we find that for y small, K large,

$$L(y+g) \geq \frac{Kc_0}{2}(f'' + \frac{n-1}{y}f') + \frac{Kc_0}{4}\frac{f'}{y} - 2C \geq \frac{Kc_0}{2}(f'' + \frac{n-1}{y}f'). \tag{3.10}$$

Next we compute $Lf(r)$. Here α, β run from 1 to n (recall that $y = x_n$).

$$\partial_\alpha f = f'\frac{x_\alpha}{r}, \quad \partial_{\alpha\beta} f = f''\frac{x_\alpha x_\beta}{r^2} + f'\frac{\delta_{\alpha\beta}}{r} - f'\frac{x_\alpha x_\beta}{r^3}.$$

Hence

$$\begin{aligned} Lf &= f''\frac{a_{\alpha\beta}x_\alpha x_\beta}{r^2} + \frac{f'}{r}(\sum a_{\alpha\alpha} - \frac{a_{\alpha\beta}x_\alpha x_\beta}{r^2}) + f'b_\alpha\frac{x_\alpha}{r} + cf \\ &\leq C(f'' + \frac{n-1}{r}f') + C\frac{f'}{r} + cf. \end{aligned}$$

By (2.5), (3.3) and (3.4),

$$Lf \leq \overline{C}(f''(y) + \frac{n-1}{y}f'(y))$$

with some $\overline{C} > 0$.

Combining the last inequality and (3.10) we see that for K large, and y small, (3.7) holds:

$$Lh \geq \frac{Kc_0}{4}\left(f''(y) + \frac{n-1}{y}f'(y)\right) \geq \frac{Kc_0}{4C_1}\frac{f'(y)}{y} \geq 0, \tag{3.11}$$

by (3.4) and (3.5). In section 4, we will use (3.11).

Finally, (3.8) holds because

$$h_y(0,0) = 1.$$

□

Remark 2. *The function f_k given in (2.20) satisfies all the conditions of Theorem 3, so does $f = r^{1+\alpha}$, with some $\alpha \in (0,1)$.*

4.

In this section we extend our results on the Hopf Lemma to parabolic operators of the form

$$\partial_t - L = \partial_t - \left(a_{\alpha\beta}(x,t)\frac{\partial^2}{\partial x_\alpha \partial x_\beta} + b_\alpha\frac{\partial}{\partial x_\alpha} + c\right). \tag{4.1}$$

We consider a domain G in (x,t) space, lying in $\{t < 0\}$ and whose boundary includes an open domain D on $\{t = 0\}$. We are interested in the Hopf Lemma at the origin, which lies on ∂D. $\overline{G} \setminus D =: P\partial G$, is called the parabolic boundary of G. For convenience we suppose that $(0,1)$ is the inner normal to ∂D at $(0,0)$ and denote x_n by y. Sometimes we use (x,y,t) to denote a point, with $x = (x_1, \cdots, x_{n-1})$.

We consider a positive function u in $G \cup D \cup \{(0,0)\}$,

$$u \in C^2(G) \cap C^0(G \cup D \cup \{(0,0)\}), \tag{4.2}$$

$$u(0,0) = 0 \tag{4.3}$$

and u satisfies

$$(\partial_t - L)u \geq 0 \qquad \text{in } G. \tag{4.4}$$

The coefficients of the operator L are assumed to satisfy the same conditions as those on L in section 3. In particular

$$a_{nn} \geq c_0 > 0$$

near the origin.

Concerning G we assume that the interior unit normal to $P\partial G$ *is not* $(0, 0, 1)$. The parabolic Hopf Lemma is said to hold for $\partial_t - L$ in G at the origin provided for every function $u > 0$ satisfying (4.2)-(4.4), one has

$$\liminf_{s \to 0^+} \frac{u(s\nu)}{s} > 0. \tag{4.5}$$

Here ν is the vector in D which is the inner normal to ∂D at the origin.

Remark 3. *It will be clear from the proofs that (4.5) will then also hold for some unit vectors $\nu = (\nu_1, \cdots, \nu_{n+1})$ at $(0,0)$ which points into $G \cup D$, and are not tangent to $P\partial G$ at the origin, so $\nu_{n+1} \leq 0$.*

In case $P\partial G$ is C^2, it is well known that the parabolic Hopf Lemma holds. See, for example [**6**]. A. Friedman [**2**] first proved a weaker form: he assumed $u > 0$ in a whole ball B in $\mathbb{R}^{n+1}$ tangent to $P\partial G$ at the origin, and with $B \cap \{t < 0\}$ lying in G. We study the case that $P\partial G$ is not C^2.

We will assume that near $(0,0)$, ∂D is given by

$$\{y > f(|x|)\}, \quad t = 0,$$

here $x = (x_1, \cdots, x_{n-1})$. We also assume that for some positive constant b, the domain

$$\Omega = \{(y,t) \mid t < 0, y > f(|x|) - bt\}, \tag{4.6}$$

near the origin, lies in G, and ν points into $\Omega \cup D$ and is not tangent to $P\partial\Omega$.

Here is an extension of Theorem 3.

Theorem 3′ Assume that f satisfies the conditions of Theorem 3, i.e. conditions (2.2)-(2.5), and (3.3)-(3.5) hold. Then the parabolic Hopf Lemma holds for L in Ω at the origin.

Proof. As before by replacing $f(|x|)$ by $\alpha f(|x|)$, $\alpha - 1 > 0$ small, and increasing b slightly, we may suppose

$$u > 0 \quad \text{on } P\partial\Omega \quad \text{except } u(0,0) = 0. \tag{4.7}$$

As in the proof of Theorem 3 it suffice to construct a comparison function in $B(R) \cap \Omega$, $B(R)$ is the ball centered at origin, of radius R, R small, which satisfies

$$(\partial_t - L)h \leq 0 \qquad \text{in } B(R) \cap \Omega, \tag{4.8}$$

$$h(0,0) = 0, \qquad h \leq 0 \quad \text{on } P\partial\Omega, \tag{4.9}$$

$$\nu \cdot \nabla h(0,0) > 0, \tag{4.10}$$

for ν pointing inside $\Omega \cup D$ at the origin.

Set

$$B(R) \cap \Omega = U. \tag{4.11}$$

Once we have constructed our h we infer from (4.2) and (4.7) that for some $\epsilon > 0$,

$$w := u - \epsilon h \geq 0 \qquad \text{on } P\partial U. \tag{4.12}$$

Then, since

$$(\partial_t - L)w \geq 0,$$

it follows from the parabolic maximum principle that

$$w \geq 0 \qquad \text{in } U. \tag{4.13}$$

This is easily seen: Take $\lambda > |c|$, c is the coefficients of L of the zero-order term. Then

$$0 \leq e^{-\lambda t}(\partial_t - L)w \geq (\partial_t - L + \lambda)e^{-\lambda t}w.$$

If $e^{-\lambda t}w$ had a negative minimum in $\overline{U}$, then at that point $\partial_t(e^{-\lambda t}w) \leq 0$ and so

$$0 \leq (\partial_t - L + \lambda)(e^{-\lambda t}w) \leq (-c + \lambda)e^{-\lambda t}w < 0.$$

Impossible.

Thus (4.13) holds and hence

$$\liminf_{s \to 0^+} \frac{u(s\nu)}{s} \geq \nu \cdot \nabla h(0) > 0.$$

Since $\nu = (\nu_1, \cdots, \nu_{n+1})$ points into $\Omega \cup D$ and is not tangent to $P\partial\Omega$, we have

$$\nu_n + b\nu_{n+1} > 0.$$

To construct h we just proceed as in section 3. Set, with $r = \sqrt{|x|^2 + y^2}$,

$$h = y + g(y) - 2f(r) + (b + \epsilon)t \tag{4.14}$$

with g as given in (3.9), and $\epsilon > 0$ small to be chosen later.

In the proof of Theorem 3 we showed, see (3.5) and (3.11), for K large, that

$$L(y + g(y) - 2f(r)) \geq \frac{Kc_0}{4C_1} \frac{f'(y)}{y} \geq \frac{Kc_0}{4C_1} c_2.$$

Hence, for K large,

$$(\partial_t - L)h \leq -\frac{Kc_0}{4C_1} c_2 + (b + \epsilon) < 0.$$

(4.8) is verified.

On the parabolic boundary $P\partial\Omega$ and near the boundary, $y = f(|x|) - bt$ and

$$h = [1 + \circ(1)]y - 2f(r) + (b+\epsilon)t = [1+\circ(1)]f(|x|) - 2f(r) - [1+\circ(1)]bt + (b+\epsilon)t \leq 0.$$

So (4.9) holds.

Finally, (4.10) holds, for $\nu_n + b\nu_{n+1} > 0$, so

$$\nu \cdot \nabla h(0, 0) = \nu_n + (b + \epsilon)\nu_{n+1} > 0,$$

if we take $\epsilon > 0$ small (depending on ν).

□

Next we have the analogue of Theorem 2: The parabolic Hopf Lemma does not hold. We only treat the heat operator $(\partial_t - \Delta)$.

As in Theorem 2, consider the domain

$$\Omega = \{(x, y, t) \mid t < 0, y > f(|x|) - bt\}, \qquad b > 0.$$

Ω is contained in the domain

$$\widetilde{\Omega} = \{(x, y, t) \mid t < 0, y > f(|x|)\}.$$

If the Hopf Lemma does not hold for $\partial_t - \Delta$ in $\widetilde{\Omega}$ then it does not hold in Ω. We will treat $\widetilde{\Omega}$.

Theorem 2′ Assume that f satisfies all the conditions of Theorem 2. Then the parabolic Hopf Lemma does not hold for $\partial_t - \Delta$ in $\widetilde{\Omega}$ at the origin.

Proof. We simply take the function u as given in the proof of Theorem 2; it is independent of t. There we showed that $Lu \le 0$, so $(\partial_t - L)u \ge 0$. The same holds for our L depending on t, since the dependence of the coefficients on t is irrelevant. $\nabla u(0,0) = 0$, the parabolic Hopf Lemma does not hold at $(0,0)$.

□

In Theorem $3'$ we have treated domains given by (4.6), with $b > 0$. Suppose we look at a larger domain $\widetilde{\Omega}$, given by (4.6), but with $b < 0$. Of course, under the same conditions on f, the parabolic Hopf Lemma also holds at the origin, in $\widetilde{\Omega}$, since $\widetilde{\Omega}$ contains Ω, where $b > 0$. But something more is true:

Here u is as in Theorem $3'$.

Theorem 3′′ Consider $\widetilde{\Omega}$ given by (4.6) but with $b < 0$. Assume f satisfies all the conditions of Theorem 3. Then we also have

$$\liminf_{t \to 0^+} \frac{u(0,t)}{(-t)} > 0. \tag{4.15}$$

Proof. We use the comparison function of (4.14)

$$h = y + g(y) - 2f(r) + \frac{b}{2}t,$$

but now $b < 0$. We have

$$(\partial_t - L)h = \frac{b}{2} - L(y + g(y) - 2f(r)) < 0$$

by (3.11).

On the parabolic boundary $P\partial\Omega$ and near the origin, $y = f(|x|) - bt$,

$$h = [1 + \circ(1)]y - 2f(r) + \frac{b}{2}t = [1 + \circ(1)]f(|x|) - 2f(r) - [\frac{b}{2} + \circ(1)]t \le 0.$$

We now argue as in the proof of Theorem $3'$ and conclude that for some $\epsilon > 0$, in $B(R) \cap \overline{\Omega}$,

$$u \ge \epsilon h.$$

Since $h_t = b < 0$, we find that (4.15) holds.

□

What happens if we permit the interior unit normal to $P\partial\Omega$ at the origin to be $(0, \cdots, 0, -1)$?

Consider the following simple example. Suppose Ω is given by

$$\Omega = \{(x,y,t) \mid t < 0, y - |x|^2 + \sqrt{-t} > 0\} \tag{4.16}$$

(so $P\partial\Omega$ is analytic, namely, $\{-t = (y - |x|^2)^2)\}$.

If we follow the preceding argument, using

$$h = y - |x|^2 + \sqrt{-t},$$

we have

$$(\partial_t - \Delta_x)h = -\frac{1}{\sqrt{-t}} + 2n < 0 \quad \text{for } |t| \text{ small.}$$

As a consequence we obtain

Theorem 4. *If u is a C^2 function near the origin in Ω given by (4.16), u is continuous and positive in $\Omega \cup D \cup \{(0,0)\}$ except that $u(0,0) = 0$, and u satisfies*

$$(\partial_t - \Delta)u \geq 0 \qquad \text{in } \Omega.$$

Then

$$\liminf_{t\to 0^+} \frac{u(0,t)}{\sqrt{-t}} > 0. \tag{4.17}$$

From this we see that u can not be C^1 at the origin. This is consistent with known facts about loss of regularity when the exterior normal to $P\partial\Omega$ is $(0, \cdots, 0, 1)$ at some point. See Kohn and Nirenberg [**4**] and Dong [**1**]. The cases treated there are just for one space variable. Our example is for higher dimension and seems to exhibit a phenomena not previously observed. The loss of regularity is not due to the value of u on the boundary, near $t = 0$, for there u could be $\equiv 0$, but with still $u > 0$ in Ω.

Proof of Theorem 4. With h as above we find, as before, that

$$u \geq \epsilon h$$

for some $\epsilon > 0$. Consequently

$$u(0,t) \geq \epsilon h(0,t) = \epsilon\sqrt{-t},$$

and (4.17) holds.

□

Remark 4. *Extension of the Hopf Lemma in some forms have been made in certain domains with corners; see Serrin* [**8**], *Gidas, Ni and Nirenberg* [**3**] *for elliptic operators, and by Rubinstein, Sternberg and Keller* [**7**] *for parabolic ones in one space dimension.*

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