AN INVERSE SPECTRAL THEOREM AND ITS RELATION TO THE POMPEIU PROBLEM

By

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1. The study of the relations between the eigenvalues of the Laplace operator and the geometry of the domain is a very old one and it remains an area of extremely active research. Beyond the beauty of many of the results obtained, a sustaining force has been its wide number of applications (see for instance [13, 23]). The problem we will consider here, though it has originated in harmonic analysis, is related to questions as diverse as nuclear reactors' construction and tomography.

The question is very simple. Let D be a simply-connected bounded region in the euclidean plane, with sufficiently smooth boundary ∂D. Assume that some of the eigenfunctions for the Neumann problem are also constant along ∂D. What can we say about D? We can prove in some cases that D is a disk (Propositions 1 and 2). There is a similar result for the Dirichlet problem (Proposition 3).

Part of the results presented here were the subject of lectures given at the Universidade Federal de Pernambuco [2]; further progress on this problem has been encouraged by discussions with several colleagues, L. Nirenberg, M. Schiffer, S. Wolpert and P. Yang foremost, to whom I express my appreciation.

2. The problem proposed above is both an eigenvalue problem and an overdetermined problem. Let us begin by recalling some similar problems. We will assume throughout that D is a bounded simply-connected open plane domain and ∂D is of class $C^{2+\varepsilon}$ for some $\varepsilon > 0$, though some of the statements below hold under more general conditions, both with respect to the number of dimensions as well as the smoothness and connectedness conditions.

In a situation arising from fluid dynamics, Serrin [16] and Weinberger [20] considered the equation

$$(1) \qquad \Delta u = -1 \quad \text{in } D,$$

together with the boundary conditions

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$$(2)\qquad u = 0, \qquad \frac{\partial u}{\partial n} = \text{constant on } \partial D \quad (\boldsymbol{n} = \text{exterior normal to } \partial D).$$

They showed that if there is a solution $u \in C^2(\bar{D})$ to this problem, then D is a disk of radius R and

$$(3)\qquad u = \frac{R^2 - r^2}{4}$$

where r denotes the distance from the center of D. The proof in [20] is elementary, based on the simple nature of the unique solution (3) to the problem (1)–(2). Regretfully, it doesn't extend to our problem below.

We denote by $0 < \lambda_1 < \lambda_2 \leqq \lambda_3 \leqq \cdots$ the eigenvalues of the Dirichlet problem:

$$(4)\qquad \Delta u + \lambda u = 0 \quad \text{in } D, \qquad u \not\equiv 0 \quad \text{in } D, \quad \text{and}$$

$$(5)\qquad u = 0 \quad \text{on } \partial D.$$

It is an open question from [14] what happens when an eigenfunction for the Dirichlet problem satisfies (2). The only case known to date was that of $\lambda = \lambda_1$. Then it is well known that $u > 0$ [6] and hence, e.g. from theorem 2 [16], it follows again that D is a disk and u is a radial function. We consider this question in Proposition 3.

The problem we want to consider here first, is the existence of $\alpha \neq 0$ and $u \not\equiv 0$ such that

$$(6)\qquad \Delta u + \alpha u = 0 \quad \text{in } D, \qquad \text{grad } u = 0 \quad \text{on } \partial D.$$

The standard argument shows that $\alpha > 0$, as it should be since u is an eigenfunction of the Neumann problem which is constant on the boundary, i.e.

$$(7)\qquad u = c = \text{constant}, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D.$$

In this form, our problem bears a resemblance to the above problem (4)–(2) and also to (1)–(2) since the function $v = u/\alpha c - 1/\alpha$ satisfies

$$(8)\qquad \Delta v + \alpha v = -1 \quad \text{in } D, \qquad \text{and}$$

$$(9)\qquad v = 0, \quad \frac{\partial v}{\partial n} = 0 \qquad \text{on } \partial D.$$

There are several ways of showing that v is well-defined, i.e. $c \neq 0$; once this is done $v \not\equiv 0$ is immediate since u cannot be constant by (6). One way of showing that $c \neq 0$ is by using the fundamental solution G of $\Delta G + \alpha G = \delta$ and Green's identities, another is via the curious identity

$$\int_D u^2 dxdy = 2c^2 A, \qquad A = \text{area of } D, \tag{10}$$

which shows that maybe one should use the classical isoperimetric inequality [11] to study (6).

Proof of (10). The main component of the proof is a formula of Rellich [15] stating that if v is an eigenfunction for (4)–(5) and λ its corresponding eigenvalue,

$$\lambda \int_D v^2 dxdy = \int_{\partial D} \left(\frac{\partial v}{\partial n}\right)^2 r \cos(\boldsymbol{n}, \boldsymbol{r}) ds, \tag{11}$$

where r is the distance to any fixed point in D, $\boldsymbol{r}$ stands for the corresponding radius vector, $(\mathbf{n}, \mathbf{r})$ is the angle between $\boldsymbol{r}$ and the normal $\boldsymbol{n}$, and ∂D is positively oriented. In our case we note that if u is the solution to (6), then both u_x and u_y are eigenfunctions for the Dirichlet problem (4)–(5) also with eigenvalue α and we can apply to them (11). It is clear we will need to compute the values of the second derivatives of u at the boundary points of D, which we do presently. Let us parametrize ∂D by arc length and, as usual, denote $\dot{x} = dx/ds$, etc. The functions u_x, u_y satisfy

$$u_x(x(s), y(s)) \equiv 0, \qquad u_y(x(s), y(s)) \equiv 0,$$

and by differentiation,

$$\begin{cases} u_{xx}\dot{x} + u_{xy}\dot{y} = 0, \\ u_{yx}\dot{x} + u_{yy}\dot{y} = 0. \end{cases}$$

It follows that

$$u_{xy}(\dot{x}^2 + \dot{y}^2) + \dot{x}\dot{y}(u_{xx} + u_{yy}) = 0$$

or, using (6)–(7) and $\dot{x}^2 + \dot{y}^2 = 1$,

$$u_{xy} = \alpha c\, \dot{x}\dot{y} \qquad \text{on } \partial D.$$

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$$u_{xx} = -\alpha c\,\dot{y}^2, \quad u_{yy} = -\alpha c\,\dot{x}^2 \qquad \text{on } \partial D.$$

Hence,

$$\frac{\partial u_x}{\partial n} = u_{xx}\dot{y} - u_{xy}\dot{x} = -\alpha c\dot{y} \qquad \text{and} \qquad \frac{\partial u_y}{\partial n} = -\alpha c\dot{x}.$$

By Rellich's formula (11) we get

$$(12) \qquad \alpha \int_D (u_x^2 + u_y^2)dxdy = \alpha^2 c^2 \int_{\partial D} (\dot{x}^2 + \dot{y}^2) r \cos(\boldsymbol{n}, \boldsymbol{r}) ds = 2A\alpha^2 c^2.$$

On the other hand, by Green's formula, the left hand side of (12) can be computed as

$$\alpha \int_D u_x^2 + u_y^2 = \alpha \int_{\partial D} u \frac{\partial u}{\partial n} ds - \alpha \int_D u \Delta u = \alpha^2 \int_D u^2,$$

which completes the proof of (10).

Out of the equivalence of (6) and (8)–(9) one obtains the following lemma.

Lemma 1. *Let D be a simply-connected open bounded plane domain with $C^{2,\varepsilon}$ boundary $(0 < \varepsilon)$ for which there is an eigenvalue α for the problem* (6), *then the boundary ∂D is real analytic.*

Proof. From [1, 6] we conclude that the solution u to (6) is of class $C^2(\bar{D})$. Hence the corresponding solution v to (8)–(9) is also of class $C^2(\bar{D})$, and theorem 1′ from [9] applies, which concludes the proof that ∂D is a real analytic Jordan curve. □

Let us discuss briefly the case where D is a disk of radius R and center at the origin of coordinates. We can apply Holmgren's uniqueness theorem to the problem (6)–(7) and conclude that the solution u must be a radial function. Let us write $u = u(r)$ and $u' = du/dr$, then (6) becomes

$$(13) \qquad u'' + \frac{1}{r}u' + \alpha u = 0, \quad 0 \leqq r \leqq R, \qquad \text{and} \qquad u'(R) = 0,$$

where u is a smooth function up to $r = 0$. From here it follows that

$$u(r) = \text{const. } J_0(\sqrt{\alpha} r),$$

where J_n denotes the Bessel function of order n [6]. The boundary condition becomes

$$J_0'(\sqrt{\alpha} R) = -J_1(\sqrt{\alpha} R) = 0. \tag{14}$$

We see that there are in fact infinitely many eigenvalues for the problem (6), each of them simple and given by

$$\alpha_n = \beta_n^2/R^2, \qquad n = 1, 2 \cdots$$

where $0 < \beta_1 < \beta_2 < \cdots$ is the sequence of positive zeros of the function J_1. This explicit formula allows us to compare the smallest α, $\alpha_1 = \beta_1^2/R^2$, with the lowest eigenvalue for the Dirichlet problem, $\lambda_1 = \gamma_1^2/R^2$, $\gamma_1 =$ smallest positive zero of J_0. The intertwining of the zeros of J_0 and J_1 shows that $\lambda_1 < \alpha_1$. This holds for any domain D, since as we have seen in the proof of (10), both u_x and u_y are linearly independent eigenfunctions for the Dirichlet problem with eigenvalue α_1, hence in general $\alpha_1 \geqq \lambda_2$ [6]. In the case of a disk it is easy to see that actually $\alpha_1 = \lambda_2$. The converse is also true.

Proposition 1. *Let D be a domain for which (6) has a non-trivial solution. If $\alpha = \lambda_2 =$ second eigenvalue for the Dirichlet problem in D, then D is a disk of radius $R = \beta_1/\sqrt{\alpha}$, $\beta_1 =$ smallest positive zero of the Bessel function J_1.*

Proof. It is an immediate consequence of the isoperimetric inequality of Payne and Weinberger [12]. They proved that if Λ_1 is the smallest eigenvalue of the clamped plate problem

$$\begin{cases} \Delta(\Delta w) + \Lambda \Delta w = 0 & \text{in } D, \\ w = \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial D \end{cases} \tag{15}$$

then $\Lambda_1 \geqq \lambda_2$ and, furthermore, equality is attained only for D a disk. In our case, let $\alpha = \lambda_2$ and take $w = u - c$; we see that w satisfies (15) with $\Lambda = \alpha$, hence $\Lambda_1 \leqq \alpha = \lambda_2$. □

To end this section, let us remark that ring domains always have solutions to (6). On the other hand, using elliptical coordinates one can see that a true ellipse cannot have any eigenvalues for (6), but we shall see this more easily in §3, where we indicate the problem in harmonic analysis from which (6) arose. Furthermore, any attempt to show that a domain D for which there exist eigenvalues for the problem (6) must be a disk should be of a global nature. In fact, ∂D must be real analytic by Lemma 1 and hence the Cauchy–Kowalewski theorem [6] guarantees the existence of a solution u near ∂D to the problem (6), for *arbitrary* α.

3. While (6) can be directly linked to problems in plasma physics [19] and nuclear reactors [10], it can also be related to tomography [17, 18]. In fact, suppose we had an "X-ray machine" that computes the total density of a tissue above and below a certain plane domain D as is usually done, that is by assigning a plus sign to those portions above it and a minus to those below. We would be free to move this machine around, and the question arises as to whether we can reconstruct the "true" picture of the tissue; in particular, is it possible that two different tissues produce the same set of X-rays? The surprising answer is that in some sense if D is a disk this could happen. The problem we are looking at in this construction is the following: let f be a C^∞ function (continuous or just locally integrable would do as well), could it be that

$$\int_{\sigma(D)} f(x,y)\,dx\,dy = 0, \quad \text{for all possible rigid motions } \sigma, \tag{16}$$

and nevertheless $f \not\equiv 0$? For the purpose of the construction of this machine we could just assume that D is a bounded measurable subset of the plane with positive measure. If we impose on f the restriction that it has compact support (e.g. the density of tissue in the brain) then it is easy to conclude from (16) that $f \equiv 0$. The difficulty that arises is that in the "real world" we cannot distinguish between a function of compact support and one that remains "very small" in an annulus of "sufficiently large" outer radius. (We call them functions of *almost compact support*.) Hence, the apparently unrealistic consideration of functions of arbitrary support seems justified. The existence of non-trivial solution to (16) is usually known as the failure of the Pompeiu property for D [3, 22], and as it is shown in [3] it is equivalent to the existence of a common zero to all the functions $\hat{\chi}_{\sigma(D)}$, where $\hat{\chi}_A$ stands for the Fourier transform of the characteristic function of the set A. The common value of $\hat{\chi}_{\sigma(D)}(0)$ is the measure of D, hence if there is a common zero $\zeta \in \mathbf{C}^2$, it must be different from zero. Using well-known properties of the Fourier transform, it can be seen that this means that $\hat{\chi}_D$ vanishes identically on some "circle" C_α, $\alpha > 0$,

$$C_\alpha = \{\zeta = (\zeta_1, \zeta_2) \in \mathbf{C}^2 : \zeta_1^2 + \zeta_2^2 = \alpha\}. \tag{17}$$

Since $\alpha \neq 0$, the polynomial $\zeta_1^2 + \zeta_2^2 - \alpha$ is irreducible and hence the function

$$T(\zeta) = \hat{\chi}_D(\zeta)/(\zeta_1^2 + \zeta_2^2 - \alpha)$$

is entire; furthermore, by the Ehrenpreis–Malgrange lemma [8], it follows that T is the Fourier transform of a distribution of compact support ψ. Therefore

$$(\zeta_1^2 + \zeta_2^2 - \alpha)\hat{\psi}(\zeta) = \hat{\chi}_D(\zeta)$$

or, equivalently,

$$\Delta\psi + \alpha\psi = -\chi_D, \tag{18}$$

where this equation is understood in the sense of distributions. Suppose now that the boundary ∂D of D is a C^2-Jordan curve, then ψ being of compact support and real analytic outside D, by (18), vanishes identically outside D. Furthermore ψ and $\partial\psi/\partial n$ are continuous across ∂D, hence $v = \psi \,|\, D$ provides a solution to the problem (8)–(9). This shows that for such domains the failure of the Pompeiu property is equivalent to our original problem (6), and (18) is a version of (6) that is meaningful even when ∂D is not sufficiently regular. The spectral synthesis theorem [3] shows that functions f of almost compact support satisfying (16) can only occur if we have infinitely many eigenvalues α for (18).

Let us point out that a function f satisfying (16) can be easily found using (17). It is enough to take $f(x, y) = \exp(-i(\zeta_1 x + \zeta_2 y))$, with $\zeta \in C_\alpha$, or just the real or imaginary part of that exponential. If D is an ellipse,

$$D = \left\{(x, y) \in \mathbf{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\right\}$$

then

$$\hat{\chi}_D(z) = \frac{2\pi ab J_1(\sqrt{a^2\zeta_1^2 + b^2\zeta_2^2})}{\sqrt{a^2\zeta_1^2 + b^2\zeta_2^2}}$$

and hence, $\hat{\chi}_D$ cannot vanish identically on a circle C_α unless $\sqrt{a^2\zeta_1^2 + b^2\zeta_2^2}$ remains real valued there (since all the zeros of J_1 are real). This can only happen if $a = b$, which shows clearly that the Pompeiu property does not fail for a true ellipse, and it also shows again the role of the zeros of J_1 for the case of a disk ($a = b = R$). A similar computation shows that the Pompeiu property does not fail for convex sets with corners, i.e. a boundary point with two distinct lines of support, see [3]. Using Holmgren's uniqueness theorem one can also show that ∂D cannot contain line segments, even when ∂D is not smooth [7, 21].

4. We have seen that the failure of the Pompeiu property is equivalent to the existence of a single eigenvalue for (6); on the other hand, the disk has infinitely many eigenvalues for that overdetermined boundary value problem. In this section we will show that the converse is also true.

Proposition 2. *Let D be a simply-connected bounded open subset of $\mathbf{R}^2$ with $C^{2+\varepsilon}$ boundary $(\varepsilon > 0)$. Assume that the eigenvalue problem* (6) *has infinitely many solutions α, then D is a disk.*

Proof. The idea of the proof is very simple; one tries to show by asymptotic methods that $\hat{\chi}_D$ can't vanish on a circle (17) unless a piece of ∂D is an arc of circle. Since, by Lemma 1, ∂D is real analytic, it follows that D is a disk. The difficulty is that we will be dealing with the asymptotic behavior of a Fourier integral with a complex phase function whose imaginary part changes sign and hence its behavior cannot be obtained from previously known results [4].

Let $\chi_{\partial D}$ denote the distribution defined by

$$\langle \chi_{\partial D}, \psi \rangle = \int_{\partial D} \psi(x, y)(dx + i dy), \qquad \psi \in C_0^\infty(\mathbf{R}^2). \tag{19}$$

An application of Green's formula in the complex plane $\mathbf{C}$ (identified to $\mathbf{R}^2$) gives

$$\iint_D \frac{\partial \psi}{\partial \bar{z}}\, dx dy = \frac{1}{2i} \int_{\partial D} \psi dz, \qquad z = x + iy, \tag{20}$$

for arbitrary C_0^∞ functions ψ. From the point of view of distributions, (20) means

$$\frac{\partial}{\partial \bar{z}} \chi_D = -\frac{1}{2i} \chi_{\partial D}, \tag{21}$$

with $\chi_{\partial D}$ defined by (19). Taking Fourier transforms of both sides of (21), we get

$$(\zeta_1 + i\zeta_2)\hat{\chi}_D(\zeta) = \hat{\chi}_{\partial D}(\zeta), \tag{22}$$

hence $\hat{\chi}_D$ vanishes identically on a circle C_α if and only if $\hat{\chi}_{\partial D}$ does.

Since ∂D is a real analytic Jordan curve there are only finitely many points at which the curvature k vanishes. Let $\theta \in \mathbf{R}$ and $\xi = \xi(\theta) = (\cos\theta, \sin\theta)$ be such that the finitely many points $p_j \in \partial D$, $j = 1, \cdots, N$ where the normal to ∂D is parallel to ξ are such that the corresponding curvatures $k_j \neq 0$. Only finitely many θ fail to have this property. We denote by $\eta = d\xi/d\theta = (-\sin\theta, \cos\theta)$, then the unit tangent vector T_j at those points satisfy $T_j = \varepsilon_j\eta$, $\varepsilon_j = \pm 1$. We denote by τ_j the vector T_j identified to a complex number, i.e. $T_j = (a, b)$, $\tau_j = a + ib$. With θ as above we choose the origin of coordinates (and the prospective center for D) once for all as follows. Choose Σ as the narrowest strip of axis parallel to η which contains D; the positive y-axis will be chosen in the direction of η dividing Σ into two equal parts, Σ^+ and Σ^-, the right and left portions. Let p_1 be a point in $(\partial\Sigma)^+ \cap \partial D$, hence $T_1 = \eta$, and the exterior normal at p_1 coincides with ξ. We choose the ray through p_1 in the direction of ξ as the positive x-axis. This determines the origin of coordinates, which in principle might not be a point in D; it also makes $\theta = 0$.

Before proceeding, let us take a closer look at the circles C_α where $\hat{\chi}_{\partial D}$ vanishes identically. If $\zeta \in C_\alpha$, we write $\zeta = \xi + i\eta$, $\xi, \eta \in \mathbf{R}^2$, then

$$\zeta_1^2 + \zeta_2^2 = |\xi|^2 - |\eta|^2 + 2i\langle \xi, \eta\rangle = \alpha,$$

hence, since $\alpha > 0$,

$$\langle \xi, \eta \rangle = \xi_1\eta_1 + \xi_2\eta_2 = 0, \qquad \text{and}$$
$$|\xi|^2 = \xi_1^2 + \xi_2^2 = \alpha + |\eta|^2.$$

Hence with $\xi = \xi(\theta) = (\cos\theta,\ \sin\theta)$, $\eta = \eta(\theta) = (-\sin\theta,\ \cos\theta)$, we can write $\zeta \in C_\alpha$ as

$$\zeta = r\xi + it\eta, \qquad r > 0, \quad t \in \mathbf{R}, \tag{23}$$

related by

$$r^2 = \alpha + t^2. \tag{24}$$

Since we have infinitely many α, $\alpha \in \{\alpha_k : k \in \mathbf{N}\}$ at our disposal we can choose α, r, t so that

$$|t| \to \infty \qquad \text{and} \qquad |t|/\log r \to 0. \tag{25}$$

We are now ready to state and prove the main asymptotic formula we need. The notation remains as above.

Lemma 2. *Let Ω be an open arc of S^1 containing $\xi_0 = \xi(0)$ and such that for $\xi \in \Omega$ the corresponding points $p_1 \cdots p_N$ in ∂D have non-vanishing curvature. Let the points ζ be chosen according to the conditions* (23) *through* (25), *then*

$$\hat{\chi}_{\partial D}(\zeta) = \left(\frac{2\pi}{r}\right)^{1/2}\left\{\sum_{j=1}^{N} |k_j|^{-1/2}\sigma_j \exp(-i\langle p_j, \zeta\rangle) + o(1)\right\}, \tag{26}$$

where $\sigma_j = \tau_j \exp((-i\pi/4)\operatorname{sign} k_j)$. Furthermore, the $o(1)$ is uniform in any compact subset of Ω.

Remark. This lemma holds when ∂D is just of class C^5.

Proof. It is necessary to go through the proof of (26) because, as we said above,

$$\hat{\chi}_{\partial D}(\zeta) = \int_{\partial D} \exp(-i(\zeta_1 x + \zeta_2 y))(dx + idy) \tag{27}$$

is a Fourier integral with complex phase whose imaginary part changes sign. Hence the classical theory of asymptotics [4] does not apply. It is precisely for this reason that we cannot get by with the existence of a single eigenvalue α, since we can prove (26) only for those ζ for which (25) holds. Let us denote by s, arc length in ∂D. Then we have to look at the critical points of the phase function $\langle \xi, p(s)\rangle$, $p = (x, y) \in \partial D$, which occur precisely when

$$\frac{d}{ds}\langle \xi, p(s)\rangle = \langle \xi, p'(s)\rangle = 0. \tag{28}$$

That is, precisely at the points where the tangent line is perpendicular to ξ, $p_1, \cdots, p_N$. By the choice of ξ, we can use a partition of unity and reduce the computation of (27) to a finite number of integrals of the form

$$\int_{-\infty}^{\infty} \exp(-ir\langle \xi, p(s)\rangle)\alpha(s, t\eta)\,ds, \tag{29}$$

where $\alpha(s, \eta) = \beta(s)\exp\langle \eta, p(s)\rangle$, $\beta \in C_0^\infty$ and such that either

(i) there are no critical points in the support of β, or

(ii) there is only one critical point and the support of β is contained in a small neighborhood of that point.

Note that in the second case the critical point is non-degenerate, since

$$\frac{d^2}{ds^2}\langle \xi, p(s)\rangle = \pm k(s)$$

at those points where (28) holds.

We assume for simplicity that $\xi = \xi(0) = (1,0)$ and $\eta = (0,1)$. Let us examine first those integrals for which (i) holds. In that case $x'(s) \neq 0$ in the support of β and we can introduce a new variable u defined by $u = x(s)$, hence (29) becomes an integral of the form

$$I_1 = \int_{-\infty}^{\infty} e^{-iur}\gamma(u)\exp(t\psi(u))du, \tag{30}$$

with γ of compact support, integrating by parts once we obtain

$$I_1 = \frac{1}{2r}\int_{-\infty}^{\infty} e^{-iur}(\gamma'(u) + t\gamma(u)\psi'(u))\exp(t\psi(u))du,$$

and the best possible estimate for this integral is the following. For some $A > 0$

$$|I_1| \leqq C_1 \frac{(1+|t|)}{r} e^{A|t|}, \tag{31}$$

since $\psi(u)$ could change sign. In fact,

$$A = \begin{cases} \max\{\psi(u): \quad u \in \operatorname{supp}\gamma\} & \text{if} \quad t \geqq 0, \\ -\min\{\psi(u): \quad u \in \operatorname{supp}\gamma\} & \text{if} \quad t < 0. \end{cases} \tag{32}$$

Since (25) holds we have

$$I_1 = o(r^{-\frac{1}{2}}).$$

It is hard to see how to obtain (32) under different assumptions, since when there are finitely many eigenvalues α one has $|t| \sim r$.

For integrals of the second kind, we can assume the critical point occurs at $s = 0$, and then we make the change of variables

$$u^2 = \operatorname{sign}(x''(0))[x(s) - x(0)].$$

Hence we can get an integral of the form, $\delta = -\operatorname{sign}(x''(0))$,

$$I_2 = \exp(-i\langle\zeta, p(0)\rangle)\int_{-\infty}^{\infty} e^{\delta i u^2 r}\gamma(u)\exp(t\psi(u))du, \tag{33}$$

where γ, ψ are not the same functions as in (30) but $\gamma(u)$ still has compact support. For some convenient $a > 0$, this integral can be rewritten as

$$\gamma(0) e^{i\psi(0)} \int_{-a}^{a} e^{\delta i u^2 r} du + \int_{-\infty}^{\infty} e^{\delta i u^2 r} u \Phi(u, t) du,$$

where Φ is a function of compact support in u. The second integral can be integrated by parts and contributes an error term of the form $o(r^{-\frac{1}{2}})$. The first integral has the value

$$e^{\delta i\pi/4} \left(\frac{\pi}{r}\right)^{1/2} + O\left(\frac{1}{r}\right).$$

If we trace back all the changes of variables we find

$$I_2 = \left(\frac{2\pi}{r}\right)^{1/2} e^{\delta i\pi/4} | k(0)|^{-1/2} \tau(0) \exp(-i\langle \zeta, p(0)\rangle) + o(r^{-1/2}),$$

where $k(0)$ stands for the curvature at the critical point $s = 0$, $\tau(0) = x'(0) + iy'(0)$, $(x'(0), y'(0)) = \pm\eta$. This concludes proof of the lemma, since the last assertion follows from the method of proof. □

We will prove Proposition 2 first under the assumption that D is a convex set since the main ideas of the proof appear in this case. We have then $N = 2$ in the expansion (26) and the assumption that $\hat{\chi}_{\partial D} \equiv 0$ on $\bigcup_{k=1}^{\infty} C_{\alpha_k}$ implies that for each fixed $\xi \in \Omega$,

$$k_1^{-\frac{1}{2}} \exp(-i\langle p_1, \zeta\rangle) - k_2^{-\frac{1}{2}} \exp(-i\langle p_2, \zeta\rangle) = o(1) \tag{34}$$

since $k_i > 0$ and we have $\tau_1 = -\tau_2$, i.e. the unit tangent vectors point in opposite directions at the two points where the normal direction coincide. For (34) to hold we must have

$$\langle p_1, \eta\rangle = \langle p_2, \eta\rangle, \tag{35}$$

otherwise letting $t \to \infty$ or $t \to -\infty$ we get that the absolute value of the left hand side of (34) goes to ∞. This relation must persist over the whole arc Ω otherwise we get a contradiction with (34). We need to explore further the consequences of (35), but before doing so let us point out that we can also obtain

$$k_1 = k_2 \tag{36}$$

from (34). In fact, we can assume $\langle p_1, \eta\rangle > 0$ and let $t \to +\infty$, hence

$$k_1^{-\frac{1}{2}}\exp(-ir\langle p_1, \xi\rangle) - k_2^{-\frac{1}{2}}\exp(-ir\langle p_2, \xi\rangle) = o(1)$$

which yields (36) by comparison of the absolute values since k_1, k_2 are independent of t and r. We will show now that (35) implies that the radii of curvature R_1, R_2 satisfy the relation

$$R_1 + R_2 = C = \text{constant} \tag{37}$$

and, furthermore, D is strictly convex, i.e. Ω can be taken as the whole unit circle. These two relations (36)–(37) show that D is a circle of radius $C/2$.

The condition $\langle \xi, p'(s)\rangle = 0$ together with $k(s) \neq 0$ states precisely that the Gauss map has a local inverse, $s = s(\theta)$ (recall $\xi = (\cos\theta, \sin\theta)$ and $k(s) = d\theta/ds$). Indicate by $R(s) = 1/k(s) = ds/d\theta$ the radius of curvature at that point and $\rho(\theta) = R(s(\theta))$. Then

$$\frac{d}{d\theta}\langle \eta(\theta), p(s(\theta))\rangle = R(s)\left\langle \frac{dp}{ds}, \eta\right\rangle - \langle p, \xi\rangle = \rho(\theta)\varepsilon - \langle p, \xi\rangle. \tag{38}$$

The quantity $\langle dp/ds, \eta\rangle = \varepsilon = \pm 1$ and remains constant in the branch of $s = s(\theta)$ we are looking at in (28). In the case D convex, that we are presently considering, we have only two solutions which we can denote by $s(\theta)$ and $s(\theta + \pi)$ according to whether the exterior normal coincides with $\xi(\theta)$ or with $-\xi(\theta) = \xi(\theta + \pi)$. The corresponding values of ε are $+1$ and -1 respectively. In any case, i.e. even if D is not convex, from (28) and (38) it follows

$$\frac{d^2}{d\theta^2}\langle \eta, p\rangle = \varepsilon\frac{d\rho}{d\theta} - \rho\left\langle \frac{dp}{ds}, \xi\right\rangle + \langle p, \eta\rangle = \varepsilon\frac{d\rho}{d\theta} + \langle p, \eta\rangle. \tag{39}$$

Hence, (35) implies

$$\varepsilon_1\frac{d\rho_1}{d\theta} = \varepsilon_2\frac{d\rho_2}{d\theta} \tag{40}$$

in general, and in the convex case

$$\frac{d\rho}{d\theta}(\theta) = -\frac{d\rho}{d\theta}(\theta + \pi)$$

which is precisely (37). Note that (37) shows that D is also strictly convex because at the boundary points of Ω we should have either $\rho(\theta) \to +\infty$ or $\rho(\theta + \pi) \to +\infty$, while the other remains non-negative, this clearly contradicts (37) since $C < \infty$. This concludes the proof of Proposition 2 when D is convex.

We now go back to the general case. We want to show first that a statement analogous to (35) must hold. That is, if we group the points p_j according to the value $\langle p_j, \eta \rangle$, we must have at least two points in each group. The reason is the following, since the function $\hat{\chi}_{\partial D}$ is identically zero on each C_α, the same must be true for $d\hat{\chi}_{\partial D}/dt$, where the notation is still that of (23)–(24). By taking derivatives inside the integral (27) which defines $\hat{\chi}_{\partial D}$, we see that we get the same integrand as before except for the factor

$$- i\langle p, \xi \rangle \frac{dr}{dt} + \langle p, \eta \rangle = - i\langle p, \xi \rangle \frac{t}{r} + \langle p, \eta \rangle .$$

Hence, the proof of Lemma 2 shows that the leading term in the asymptotic expansion of $d\hat{\chi}_{\partial D}/dt$, when (23)–(25) holds, is given by

$$\sum_{j=1}^{N} |k_j|^{-\frac{1}{2}} \langle p_j, \eta \rangle \sigma_j \exp(-i\langle p_j, \zeta \rangle),$$

up to the irrelevant factor $(2\pi/r)^{\frac{1}{2}}$.

Let us denote by $b_1, \cdots, b_l$ the different values of $\langle p_j, \eta \rangle$. Then, the standard argument shows that by taking linear combinations of $\hat{\chi}_{\partial D}$ and its derivatives with respect to t, we can obtain the leading term

$$(41) \qquad (b_2 - b_1) \cdots (b_l - b_1) \exp(b_1 t) \sum |k_j|^{-\frac{1}{2}} \sigma_j \exp(-ir\langle p_j, \xi \rangle),$$

where the summation takes place over those indices j for which $\langle p_j, \eta \rangle = b_1$, say $j \in J_1$. It is clear then that for (41) to be $o(1)$ when $j \to \infty$ along the allowed set of values, one must have at least two distinct indices in J_1.

Let p_1 denote the points in the portion of the curve ∂D singled out by the inverse of the Gauss map, which passes for $\theta = 0$, through the point in the positive x-axis of largest abscissa. By our choice of coordinates $\theta = 0$ is a regular value of the Gauss map and this choice is possible. By the above, there is at least one other branch, say p_2, of the inverse of the Gauss map such that $\langle p_2, \eta \rangle = \langle p_1, \eta \rangle$ holds for an open set of θ, hence by analyticity it holds throughout the interval of definition of these two branches. Similarly, there is a $\delta > 0$ such that for $0 < \theta < \delta$ the index set J_1 mentioned above (which was of course dependent on θ) remains unchanged. As we have seen, the identity $\langle p_1, \eta \rangle = \langle p_2, \eta \rangle$ implies (40) and hence

$$(42) \qquad \varepsilon_1 \rho_1(\theta) = \varepsilon_2 \rho_2(\theta) + C.$$

From (38) we also obtain

$$\varepsilon_1\rho_1(\theta)-\langle p_1,\xi\rangle=\varepsilon_2\rho_2(\theta)-\langle p_2,\xi\rangle,$$

and

$$\langle p_2,\xi\rangle=\langle p_1,\xi\rangle-C.$$

Since ξ and η form an orthogonal basis, we get the identity

$$p_2=p_1-C\xi. \tag{43}$$

This last identity can be restated in the following form: the analytic curve obtained from ∂D by drawing it "parallel" to ∂D at a distance C along the inner normal, coincides with ∂D along an arc. Hence, it coincides with ∂D throughout. (It is clear from the choice of p_1 that $C>0$.)

Recall that from the definition of the quantities ε_1, ε_2 we have $\varepsilon_1=1$ and $\varepsilon_2=\pm 1$ according to whether the outer normal at the point p_2 coincides with ξ or with $-\xi$. We can show now that $\varepsilon_2=-1$. In fact, if $\varepsilon_2=1$, then $p_2-C\xi$ also describes an arc of ∂D and hence the parallel to ∂D at a distance $2C$ also coincides with ∂D, but since $\varepsilon_2=1$ we can go on and ∂D would be unbounded.

Note that $\varepsilon_2=-1$ leads to p_2+C (inner normal) $=p_2+C\xi=p_1$ as expected. Furthermore, the cardinal of the set J_1 is exactly two. If we had a third arc p_3, it will satisfy

$$p_3=p_1-C'\xi,$$

and we can assume $0<C<C'$. By the same reasoning as above, we would have $p_2+C'\xi\in\partial D$, but for $\theta=0$ we obtain the point

$$p_1(0)+(C'-C)\xi(0),$$

which lies further to the right than $p_1(0)$, which is a contradiction.

The reasoning that leads to (36) can now be applied and we obtain

$$|\rho_1|=|\rho_2|.$$

Since $\rho_1>0$, if we can show that $\rho_2>0$ we would obtain from (42) that the arc described by p_1 is actually an arc of a circle of radius $C/2$ and we would have finished the proof of the proposition. If $\rho_2<0$, then $\rho_2=-\rho_1$, and (42) becomes

$$\rho_1=-\rho_1\cdot\varepsilon_2+C=\rho_1+C.$$

Hence $C = 0$, which is impossible. □

In the same way can prove the following:

Proposition 3. *Let D be a simply-connected plane domain with boundary ∂D of class $C^{2+\epsilon}$. Assume there are infinitely many eigenfunctions for the Dirichlet problem which have constant normal derivative along ∂D. Then D is a disk.*

Proof. This time we have infinitely many λ's and corresponding eigenfunctions such that

$$\Delta v + \lambda v = 0 \qquad \text{in } D,$$

$$v = 0, \qquad \frac{\partial v}{\partial n} = c = \text{constant} \quad \text{on } \partial D.$$

Since ∂D is of class $C^{2+\epsilon}$, we have that $v \in C^2(\bar{D})$ and hence we can apply theorem 2 of [9] (with the function $g(p) = |p|^2 - c^2$ in their statement), and conclude that ∂D is actually a real analytic Jordan curve. The other ingredient missing is the Fourier transform, but applying Green's third identity to the functions v and $u(x, y) = \exp(-i(\zeta_1 x + \zeta_2 y))$, we obtain

$$\int_{\partial D} \exp(-i(\zeta_1 x + \zeta_2 y))ds = 0 \qquad \text{if } \zeta_1^2 + \zeta_2^2 = \lambda.$$

Now, the same proof of Proposition 2 applies, the only difference arises in the expansion formula (26), where the numbers τ_j will not appear. The reason for this difference is that while in the previous proposition the eigenvalues α were related to the zeros of the Bessel function J_1, here they are related to J_0. □

We remark that both propositions leave open the problem of what happens when we have only one eigenvalue, except in the extreme cases already discussed $\alpha = \lambda_2$ or $\lambda = \lambda_1$. With respect to the regularity of the boundary, it is clear that in Proposition 1, 2 and 3 we only need to assume that ∂D is of class C^1 and the eigenfunctions are of class C^2 up to the boundary, since those are the hypotheses in [9]. The work of Cafarelli [5] shows that these conditions are automatically satisfied in the situation of Propositions 1 and 2 when we only assume that ∂D is a Lipschitz curve; on the other hand this does not seem to be the case for Proposition 3. Furthermore, the failure of the Pompeiu property is equivalent to the study of problem (6) when ∂D is at least a C^2-curve, but not otherwise, hence it would be interesting to study it when ∂D is not regular. For instance, we would like to know that if D has the Pompeiu property, then every other domain D' which is "sufficiently close" to D has the same property. Here, "sufficiently close" should be defined in a way that allows the consideration of very irregular boundaries.

Finally, it is clear that the method developed here works also in $\mathbf{R}^n$, $n \geqq 3$, and when the Laplace operator is replaced by the Laplace–Beltrami operator for different metrics; we plan to discuss these questions in the future.

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