ASYMPTOTIC EXPANSIONS FOR A CLASS OF FOURIER INTEGRALS AND APPLICATIONS TO THE POMPEIU PROBLEM

By

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1. Introduction

In 1929 D. Pompeiu [P1] posed the following problem: *For which bounded sets D in $\mathbf{R}^2$ is it true that knowing that for a given $f \in C(\mathbf{R}^2)$*

$$\int_{\sigma(D)} f(x)\,dx = 0 \qquad \text{for every rigid motion } \sigma \text{ of } \mathbf{R}^2, \tag{1.1}$$

implies $f \equiv 0$?

This fascinating question, known as the Pompeiu problem, has challenged the efforts of many mathematicians since its original enunciation. A bounded set $D \subset \mathbf{R}^2$ for which it is true that if (1.1) holds, then $f \equiv 0$, is said to have the *Pompeiu property.* We emphasize that there is to the present date no explicit characterization, possibly geometric, of those sets in $\mathbf{R}^2$ which have the Pompeiu property. There has been in the last fifteen years a burst of interest in the above problem. The work of several people has progressively unraveled connections with problems in harmonic analysis, complex function theory, symmetry in partial differential equations, not to mention relevant applications such as computerized tomography. In order to provide the reader with some historical background, and better motivate our work, further in this section we outline the development of the subject and mention the existing results.

Although motivated by applications to the Pompeiu problem, in this paper we are primarily concerned with studying the asymptotic behavior along certain algebraic varieties of $\mathbf{C}^2$ of a class of Fourier integrals with a complex phase. To be more specific, we need to introduce some notation. For $\alpha > 0$ we let M_α denote the algebraic variety in $\mathbf{C}^2$

$$M_\alpha = \left\{ \zeta = (\zeta_1, \zeta_2) \in \mathbf{C}^2 \,\middle|\, \zeta_1^2 + \zeta_2^2 = \alpha \right\}. \tag{1.2}$$

Let V be an analytic surface in $\mathbf{C}^2$. If $\Omega \subset \mathbf{C}$ is a simply-connected open set and $x_i : \Omega \to \mathbf{C}$, $i = 1,2$, are two analytic functions, we suppose that V is parametrized by $x = (x_1, x_2) : \Omega \to \mathbf{C}^2$, i.e.,

$$V = \left\{ x(z) = (x_1(z), x_2(z)) \,\middle|\, z \in \Omega \right\}. \tag{1.3}$$

If Γ is a closed simple C^1 curve on V, and $\alpha > 0$ is fixed, we are interested in the asymptotic behavior when $\zeta \to \infty$ along M_α of the integral

$$I(\zeta) = \int_\Gamma e^{-i\langle x,\zeta\rangle}(dx_1 + i\,dx_2), \tag{1.4}$$

where $\langle\ ,\ \rangle$ denotes the usual inner product in $\mathbf{R}^2$. For the interpretation of (1.4), see (2.11) in Section 2. Our motivation to study integrals such as that in (1.4) stems from Brown, Schreiber and Taylor's Theorem 1.2 quoted below. We also acknowledge the strong influence of a lecture on inverse problems delivered by Carlos Berenstein at the University of Bologna and of his paper [B], which we discuss below. We stress that there are so far no general results on the asymptotic behavior of Fourier integrals with a complex phase whose imaginary part is allowed unrestricted sign. Our approach is based on Riemann's method of the steepest descent; see [Ri], and also [Ol], [Bl H]. For a class of curves Γ satisfying certain geometric assumptions we obtain an asymptotic expansion of $I(\zeta)$ in (1.4) which is completely characterized in terms of the geometry of V in (1.3) and does not depend on the particular parametrization $x = (x_1, x_2)$ of V. For the precise statement one should see Theorem 2.1 below. As a consequence, we deduce the Pompeiu property for the class of those domains in $\mathbf{R}^2$ whose boundary is the $\mathbf{R}^2$-*section* of an analytic surface V in $\mathbf{C}^2$ satisfying the geometric conditions of Theorem 2.1. We now give an outline of the existing literature on the Pompeiu problem. This will allow a better understanding of our results, whose discussion we resume at the end of this section, where we also formulate a conjecture and indicate some open problems. The rest of the paper is organized as follows. Section 2 is devoted to the study of the asymptotic behavior of $I(\zeta)$ in (1.4). Theorem 2.1 is the main result. In Section 3 we use Theorem 2.1 to characterize a class of subsets of $\mathbf{R}^2$ which have the Pompeiu property. The paper ends with some illustrative examples.

In [P1] Pompeiu showed that if D is any square in $\mathbf{R}^2$, then (1.1), along with the assumption that $f \to 0$ at ∞, implies that $f \equiv 0$. In [Ch] Christov removed that restriction, thus proving that every square in $\mathbf{R}^2$ has the Pompeiu property. In [P2] it was asserted, and even incorrectly proved, that every disk in $\mathbf{R}^2$ has the Pompeiu property. Fifteen years later Chakalov [C] showed this assertion to be false. In fact, if $x = (x_1, x_2) \in \mathbf{R}^2$ let us set $f(x) = \sin(ax_1)$, with $a \in \mathbf{R}$ to be suitably chosen, and

$$B_R = \left\{x \in \mathbf{R}^2 \,\middle|\, |x| = \sqrt{x_1^2 + x_2^2} < R\right\}.$$

Then if $x_0 = (\xi_0, \eta_0) \in \mathbf{R}^2$ is fixed, and τ_{x_0} denotes the translation in $\mathbf{R}^2$ defined by $x \mapsto \tau_{x_0}(x) = x_0 + x$, an easy computation yields

$$(1.5)\qquad \int_{\tau_{x_0}(B_R)} \sin(ax_1)\,dx = 4R^2 \sin(a\xi_0) \int_0^{\pi/2} \cos^2 u \cos(aR\sin u)\,du$$

$$= \frac{2\pi R}{a} \sin(a\xi_0) J_1(aR),$$

where J_1 is the Bessel function of the first kind and order one. In the last equality in (1.5) we have used formula (10) on p. 401 in [GR]. It is then enough to choose $a \in R$ such that aR is a zero of J_1 to conclude from (1.5) that for any $x_0 \in \mathbf{R}^2$ the integral of $\sin(ax_1)$ on $\tau_{x_0}(B_R)$ is zero. Since B_R is rotation invariant this is enough to prove that B_R does not have the Pompeiu property. One might have surmised that what makes the things go wrong for the disk is precisely the fact that this set is not affected by rotations.

The Pompeiu problem is closely related to another problem in complex function theory which generalizes the content of the classical theorem of Morera. A collection $\{\Gamma\}$ of closed rectifiable curves in $\mathbf{C}$ is said to have the *Morera property* if for each $f \in C(\mathbf{C})$ the condition

$$\int_{\sigma(\Gamma)} f(z)\,dz = 0 \qquad \text{for every rigid motion } \sigma \text{ of } \mathbf{R}^2 \text{ and every } \Gamma \in \{\Gamma\}$$

implies that f is entire. It was first realized by Zalcman [Z] in 1972 that if D has the Pompeiu property and ∂D is rectifiable, then ∂D has the Morera property. The converse of this is also true; see [BST]. Zalcman was also the first one to use the Fourier transform and a deep result of L. Schwartz, see e.g., [E], to prove the following sophisticated two-circle theorem (cf. [Z]).

Theorem 1.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbf{C})$ and assume that there exist two distinct positive real numbers r_1, r_2 such that for a.e. $z \in \mathbf{C}$*

$$\int_{C_r(z)} f(w)\,dw = 0, \qquad r \in \{r_1, r_2\},$$

where $C_r(z) = \{w \in C : |w - z| = r\}$. If r_1/r_2 is not a quotient of zeros of the Bessel function J_1, then f coincides a.e. with an entire function.

Using ideas closely related to those presented in [Z], in 1973 Brown, Schreiber and Taylor [BST] proved the following

Theorem 1.2. *A bounded set $D \subset \mathbf{R}^2$ has the Pompeiu property iff the complexified Fourier transform of the characteristic function of D does not vanish identically on M_α for any $\alpha \neq 0$, where M_α is defined by* (1.2).

If D is also simply-connected, one can replace "for any $\alpha \neq 0$" with "for any $\alpha > 0$" (see Berenstein [B]).

If χ_D is the characteristic function of D, the Fourier transform of χ_D is

$$\hat{\chi}_D(x) = \int_D e^{-i\langle x,\xi\rangle}\, d\xi. \tag{1.6}$$

The complexified Fourier transform is obtained from (1.6) by analytic continuation. We remark that it is an entire function on $\mathbf{C}^2$ of exponential type; see e.g., [E]. To prove Theorem 1.2, Brown, Schreiber and Taylor showed that every rotation and translation invariant closed subspace of $C^\infty(\mathbf{R}^2)$ is generated by the polynomial-exponential functions it contains. The latter result was proved in [BST] with the aid of the above-mentioned theorem of Schwartz.

Chakalov's example above now becomes apparent in the light of Theorem 1.2. If B_R is as in (1.5), a computation yields (cf. also [Z, (10) on p. 242])

$$\hat{\chi}_{B_R}(\zeta_1,\zeta_2) = 2\pi R\,\frac{J_1\big(R\sqrt{\zeta_1^2+\zeta_2^2}\big)}{\sqrt{\zeta_1^2+\zeta_2^2}}, \qquad \zeta_1,\zeta_2 \in \mathbf{C}. \tag{1.7}$$

From (1.7) it is obvious that if we choose $\alpha > 0$ such that $R\sqrt{\alpha}$ is a zero of J_1, then $\hat{\chi}_{B_R} \equiv 0$ on M_α, and therefore B_R does not have the Pompeiu property. On the other hand, *any elliptical region*

$$E_{ab} = \left\{(x_1,x_2)\in\mathbf{R}^2 \,\middle|\, \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} < 1\right\}, \qquad a,b > 0, \tag{1.8}$$

has the Pompeiu property. This was shown in [BST] as a consequence of Theorem 1.2 and of the formula

$$\hat{\chi}_{E_{ab}}(\zeta_1,\zeta_2) = 2\pi ab\,\frac{J_1\big(\sqrt{a^2\zeta_1^2+b^2\zeta_2^2}\big)}{\sqrt{a^2\zeta_1^2+b^2\zeta_2^2}}, \qquad \zeta_1,\zeta_2 \in \mathbf{C}, \tag{1.9}$$

from which it is clear that for no $\alpha > 0$ can $\hat{\chi}_{E_{ab}} \equiv 0$ on M_α. Using Theorem 1.2 Brown, Schreiber and Taylor also proved that every polygon in $\mathbf{R}^2$, and, more generally, every convex set with at least one true corner have the Pompeiu property. However, the case of domains with smooth boundaries was left open.

Inspired by the results in [BST], in 1976 Williams [Wi 1] discovered a remarkable connection between the Pompeiu problem and a symmetry problem in pde known as *Schiffer's conjecture* (cf. [Y, Problem 80, p. 688]):

Let $D \subset \mathbf{R}^n$ be a connected, bounded open set with C^2 boundary. Does the existence of a non-trivial solution of the overdetermined boundary value problem

$$\begin{cases} \Delta u = -\lambda u & \text{in } D,\quad \lambda > 0, \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial D} = 0, & u\big|_{\partial D} = const., \end{cases} \tag{1.10}$$

imply that D is a ball?

It was shown in [Wi 1] that a non-trivial solution to (1.10) exists iff D fails to have the Pompeiu property. We remark that if there exists a non-trivial solution to (1.10), then the L^2 average of u on D cannot be arbitrary. This can be readily seen by using the following identity due to Rellich [R],

$$\int_{\partial D} |\nabla u|^2 \langle x, \nu \rangle \, d\sigma = (n-2) \int_D |\nabla u|^2 \, dx + 2 \int_{\partial D} \langle x, \nabla u \rangle \frac{\partial u}{\partial \nu} \, d\sigma - 2 \int_D \langle x, \nabla u \rangle \Delta u \, dx, \tag{1.11}$$

valid for any bounded domain $D \subset \mathbf{R}^n$ with C^2 boundary and any $u \in C^2(\bar{D})$. If in (1.10) we let $a = u\big|_{\partial D}$, observing that

$$\int_{\partial D} \langle x, \nu \rangle \, d\sigma = n|D|,$$

for a non-trivial solution u to (1.10), we obtain from (1.11), after an integration by parts in the last integral,

$$\begin{aligned} 0 &= (n-2) \int_D |\nabla u|^2 \, dx + \lambda \int_{\partial D} \langle x, \nu \rangle u^2 \, d\sigma - n\lambda \int_D u^2 \, dx \\ &= na^2 \lambda |D| - 2\lambda \int_D u^2 \, dx. \end{aligned} \tag{1.12}$$

In the last equality in (1.12) we have used the fact that if u solves (1.10) we have

$$\int_D |\nabla u|^2 \, dx = \lambda \int_D u^2 \, dx.$$

(1.12) yields

$$a^2 = \frac{2}{n} \frac{1}{|D|} \int_D u^2 \, dx,$$

which implies, in particular, that $a \neq 0$. Therefore, if we set

$$v = \frac{u}{\lambda a} - \frac{1}{\lambda}$$

it is immediate to verify that u solves (1.10) iff v solves

$$\begin{cases} \Delta v = -\lambda v - 1 & \text{in } D, \\ \dfrac{\partial v}{\partial \nu}\Big|_{\partial D} = 0, \quad v\big|_{\partial D} = 0. \end{cases} \tag{1.13}$$

(1.13) bears a resemblance to the symmetry problem considered by Serrin in [S]; see also [W] and the recent paper [GL], although the boundary conditions there are different, Serrin's result being concerned with positive solutions. We mention that Aviles [A] has recently obtained various partial results concerning (1.10). In particular, in the two-dimensional case he has proved that if there exists a nontrivial solution to (1.10) and if $\lambda \le \nu_7$, where ν_7 is the seventh Neumann eigenvalue of the Laplacian in D, then D is a disk. If we let $D = B_R = \{x \in \mathbf{R}^n \mid |x| < R\}$, then (1.10) becomes

$$\begin{cases} u''(r) + \dfrac{n-1}{r}\, u'(r) + \lambda u(r) = 0 \quad \text{in } [0,R], \\ u'(R) = u'(0) = 0, \quad u(R) = a. \end{cases} \tag{1.14}$$

(1.14) admits the general solution

$$u_\lambda(r) = C r^{-(n-2)/2} J_{(n-2)/2}(\sqrt{\lambda}\, r), \tag{1.15}$$

where C is a constant, and $J_{(n-2)/2}$ is the Bessel function of the first kind and order $(n-2)/2$. Using the identity

$$\frac{d}{dz}\,[z^{-\nu} J_\nu(z)] = -z^{-\nu} J_{\nu+1}(z)$$

(see, e.g., [L, (5.35) on p. 103]), we obtain from (1.15)

$$u_\lambda'(R) = -C\sqrt{\lambda}\, R^{-(n-2)/2} J_{n/2}(\sqrt{\lambda}\, R). \tag{1.16}$$

From (1.16) it is clear that in order to have $u_\lambda'(R) = 0$ it is enough to choose $\lambda > 0$ such that $\sqrt{\lambda} R$ is a zero of $J_{n/2}$. It follows that if $D = B_R$ there are infinitely many u_j's and infinitely many u_j's, $u_j = u_{\lambda_j}$, that solve (1.10).

In 1980 Berenstein [B] proved a two-dimensional converse to this result.

Theorem 1.3. *Let $D \subset \mathbf{R}^2$ be a simply-connected, bounded, open set with C^2 boundary. If there exist infinitely many solutions to* (1.10), *then D is a disk.*

Although Theorem 1.3 is not directly linked to the Pompeiu problem, the method of proof is quite ingenious. Berenstein's approach is based on the use of asymptotic expansions of the complexified Fourier transform of the characteristic function χ_D of D and ultimately relies on Theorem 1.2 above. In order to be more specific, we need to introduce some notation. For $\alpha > 0$ let M_α be as in (1.2). We observe that the points of M_α can be represented as follows:

$$\zeta = r\xi + it\eta, \qquad r > 0, \quad t \in \mathbf{R}, \tag{1.17}$$

where

$$\xi = (\cos\theta, \sin\theta), \quad \eta = (-\sin\theta, \cos\theta), \quad 0 \le \theta \le 2\pi. \tag{1.18}$$

If we set

$$\zeta_1 = r\cos\theta - it\sin\theta, \qquad \zeta_2 = r\sin\theta + it\cos\theta, \tag{1.19}$$

then

$$\zeta = (\zeta_1, \zeta_2) \in M_\alpha \qquad \text{iff } \zeta_1^2 + \zeta_2^2 = r^2 - t^2 = \alpha. \tag{1.20}$$

From (1.20) we see that for $\alpha > 0$ fixed

$$r = |t|\left(1 + O\left(\frac{1}{|t|^2}\right)\right), \qquad \text{as } |t| \to \infty. \tag{1.21}$$

The asymptotic analysis of the complexified Fourier transform $\hat{\chi}_D$ under the assumption (1.21) is very difficult. One has to deal with a Fourier integral with a complex phase whose imaginary part oscillates; as mentioned above, there is no general theory for such integrals. If there exist infinitely many eigenvalues of (1.10), then there exists a sequence $(\alpha_j)_{j\in\mathbf{N}}$, with $\alpha_j \nearrow +\infty$ as $j +\infty$, such that $\zeta \in \bigcup_{j=1}^{\infty} M_{\alpha_j}$. Under this assumption Berenstein can allow $r = r(t)$ to have a superlinear growth in $|t|$, as $|t| \to \infty$. Specifically,

$$r = e^{|t| g(|t|)} \quad \text{with } g(|t|) \to +\infty \quad \text{as } |t| \to \infty. \tag{1.22}$$

By means of a partition of unity he then writes $\hat{\chi}_{\partial D}(\zeta)$ as a sum of integrals of the type (cf. [B])

$$I = \int_{-\infty}^{+\infty} e^{-ir\langle \xi, x(s)\rangle} \alpha(s, t\eta)\, ds$$

where $\alpha(s, t\eta) = \beta(s) e^{t\langle \eta, x(s)\rangle}$, $s \mapsto x(s) = (x_1(s), x_2(s))$ is a parametrization of ∂D, and $\beta \in C_0^\infty(\mathbf{R})$ is such that either (a) there are no critical points of the phase $\langle \xi, x(s)\rangle$ in $\operatorname{supp}\beta$ or (b) there is only one critical point of $\langle \xi, x(s)\rangle$ in the interior of $\operatorname{supp}\beta$. (1.22) allows one to estimate I in the difficult case (a) as follows:

$$I = o\left(\frac{1}{\sqrt{r}}\right) \qquad \text{as } r \to +\infty. \tag{1.23}$$

If (b) occurs, the asymptotic behavior of I can be determined by Laplace's method, which yields

$$I \sim \frac{1}{\sqrt{r}} \qquad \text{as } r \to +\infty. \tag{1.24}$$

(See [B] for precise asymptotics.) From (1.23), (1.24) Berenstein concludes that under the assumption (1.22) the main contribution to the asymptotic expansion of I comes from the critical points of the phase $\langle \xi, x(s)\rangle$. The final statement is as follows:

Theorem 1.4 (see [B]). *Let $D \subset \mathbf{R}^2$ be a simply-connected bounded open set with real analytic boundary. Assume that at those points $x_1, x_2, \ldots, x_n \in \partial D$ in which the normal to ∂D is parallel to $\xi = (\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi)$, the corresponding curvatures of ∂D, $k_1, k_2, \ldots, k_n$ are different from zero. If $\zeta \in \mathbf{C}^2$ is as in* (1.17) *with r, t satisfying* (1.22), *one has for $r \to +\infty$*

$$\hat{\chi}_{\partial D}(\zeta) = \sqrt{\frac{2\pi}{r}} \left\{ \sum_{j=1}^{n} |k_j|^{-1/2} \sigma_j e^{-i\langle x_j, \zeta \rangle} + o(1) \right\}, \tag{1.25}$$

where $\sigma_j = \tau_j \exp((-i\pi/4) \operatorname{sign} k_j)$ and τ_j is the (complex) unit vector tangent to ∂D in x_j.

We mention that Berenstein's Theorem 1.3 above has been recently extended to any number of dimensions by Berenstein and Yang [BY 2]. There are also versions of Theorem 1.3 relative to the hyperbolic disc in dimension two [BY 1] and to the hyperbolic ball in any number of dimensions [BY 2].

In this paper we take up Berenstein's approach. The new fact here is that for a class of domains $D \subset \mathbf{R}^2$ we establish an asymptotic expansion of $\hat{\chi}_{\partial D}(\zeta)$ when $\zeta = r\xi + it\eta \in M_\alpha$, $\alpha > 0$, and r and t are related by (1.21). We consider bounded simply-connected domains $D \subset \mathbf{R}^2$, whose boundary ∂D is real analytic. In our framework it is natural to study the asymptotic behavior of integrals of the type (1.4), since, as mentioned above, we employ Riemann's method of the steepest descent. The latter requires a complexification of both phase and amplitude in the integral whose asymptotic behavior is to be determined. This fact makes it unsuitable to obtain an asymptotic expansion for $I(\zeta)$ in (1.4) in terms of the geometry of Γ, where $\Gamma = \partial D \subset \mathbf{R}^2$. For instance, a known conjecture claims that *every bounded, convex region with real analytic boundary has the Pompeiu property, unless it is a disk.* To the best of our knowledge the only existing result in this direction is Brown and Kahane's [BK], which states that if the minimum diameter of the region is less than or equal to half of the maximum diameter, then the conjecture is true. We should recall, however, that any elliptical region has the Pompeiu property. An important achievement would be the determination of the asymptotic behavior of those integrals that arise as complexified Fourier transforms of characteristic functions of convex sets with real analytic boundary. We point out that the requirement that the boundary be analytic is not excessive since, as Williams proved in [Wi 2], if ∂D is *a priori* known to be Lipschitz, and if D fails to have the Pompeiu property, then ∂D is real analytic; see also [C].

Concerning our contribution we mention that we have the following

Conjecture. *Every bounded, simply-connected region with real analytic boundary, except the disk, has the Pompeiu property.*

The results in this paper seem to partially confirm this conjecture since the only fact that is crucial to us is the analyticity of the boundary. Among the illustrative examples we give at the end of Section 3 there are two nonconvex, bounded, simply-connected domains having the Pompeiu property. The results of Section 2 provide a tool to construct plenty of such examples. We emphasize, however, that in order to apply Theorem 2.1 we must know that the analytic extension of the parametrization of the boundary of the domain, $s \mapsto (x_1(s), x_2(s))$, $s \in [0,2\pi]$, has at least one critical point. Otherwise, we can say nothing concerning the Pompeiu property. An example of this unsatisfactory aspect of our method is provided by

$$x(s) = (x_1(s), x_2(s)) = (e^{\cos s}\cos(\sin s), e^{\cos s}\sin(\sin s)), \qquad s \in [0,2\pi], \tag{1.26}$$

which corresponds to

$$\varphi(s) = x_1(s) + ix_2(s) = e^{e^{is}}.$$

The curve parametrized by (1.26) bounds a convex region in the plane. We are not able to decide whether this region has the Pompeiu property or not.

A weaker conjecture than the one above stems from the following considerations. Let E_{ab} be as in (1.8). The boundary of E_{ab} can be parametrized by

$$x(s) = (x_1(s), x_2(s)) = (a\cos s, b\sin s), \qquad s \in [0,2\pi]. \tag{1.27}$$

If we set $\varphi(s) = x_1(s) + ix_2(s)$, then (1.27) corresponds to

$$\varphi(s) = \frac{a+b}{2}\, e^{is} + \frac{a-b}{2}\, e^{-is} = p(e^{is}) + q(e^{-is}),$$

where we have set

$$p(\sigma) = \frac{a+b}{2}\,\sigma, \quad q(\sigma) = \frac{a-b}{2}\,\sigma, \qquad \sigma \in \mathbf{R}.$$

More generally, we can consider polynomials with real coefficients

$$p(\sigma) = \sum_{k=0}^{m} a_h \sigma^h, \quad q(\sigma) = \sum_{k=0}^{n} b_k \sigma^k, \qquad m > 1, \tag{1.28}$$

and form the complex-valued function

$$\varphi(s) = p(e^{is}) + q(e^{-is}), \qquad s \in [0,2\pi]. \tag{1.29}$$

The assumption $m > 1$ in (1.28) rules out the possibility that φ in (1.29) yields a circle in $\mathbf{C}$. We have some strong evidence that every region in $\mathbf{C}$ bounded by a curve parameterized by (1.29), with p and q as in (1.28), has the Pompeiu property. We hope to come back to this in a future study.

Finally, we would like to thank the referees, whose comments have improved the presentation of the paper.

2. Asymptotic expansions for a class of Fourier integrals with a complex phase

In what follows Ω denotes a simply-connected open set in $\mathbf{C}$. Points in Ω are denoted by $z = u + iv$. We let $\tilde{\Omega} = \{(u,v) \in \mathbf{R}^2 \,|\, z = u + iv \in \Omega\}$, and assume that $\tilde{\Omega} \supset \{(u,0) \,|\, u \in \mathbf{R}\}$. Furthermore, we assume that if $(u,v) \in \tilde{\Omega}$, then also $(u + 2\pi, v) \in \tilde{\Omega}$ or, what is the same, that $z \in \Omega$ implies $z + 2\pi \in \Omega$. We are given two analytic functions $x_i : \Omega \to \mathbf{C}$, $i = 1,2$, such that

$$x_i(z + 2\pi) = x_i(z), \qquad i = 1,2, \quad \text{for any } z \in \Omega. \tag{2.1}$$

Then we consider the subset of $\mathbf{C}^2$

$$V = \{x(z) = (x_1(z), x_2(z)) \,|\, z \in \Omega\}. \tag{2.2}$$

Letting $\lambda_i = \operatorname{Re} x_i$, $\mu_i = \operatorname{Im} x_i$, $i = 1,2$, we can identify V in a natural way with a subset of $\mathbf{R}^4$ by setting

$$\tilde{V} = \{(\lambda_1(u,v), \lambda_2(u,v), \mu_1(u,v), \mu_2(u,v)) \,|\, (u,v) \in \tilde{\Omega}\}. \tag{2.3}$$

Henceforth, to abbreviate the notation we let

$$\lambda(u,v) = (\lambda_1(u,v), \lambda_2(u,v)), \mu(u,v) = (\mu_1(u,v), \mu_2(u,v)), \quad (u,v) \in \tilde{\Omega}. \tag{2.4}$$

Using (2.4), and identifying $(u,v) \in \tilde{\Omega}$ with $z = u + iv \in \Omega$, we simply denote by

$$\tilde{x}(u,v) = \tilde{x}(z) = (\lambda(z), \mu(z)) = (\lambda(u,v), \mu(u,v)) \tag{2.5}$$

the vector $(\lambda_1(z), \lambda_2(z), \mu_1(z), \mu_2(z)) = (\lambda_1(u,v), \lambda_2(u,v), \mu_1(u,v), \mu_2(u,v))$ in $\mathbf{R}^4$. Because of (2.1) we see that if for $\epsilon \in \mathbf{R}$ we set

$$\Omega_{\epsilon,2\pi} = \{z \in \Omega \,|\, \epsilon \le \operatorname{Re} z < \epsilon + 2\pi\}, \quad \tilde{\Omega}_{\epsilon,2\pi} = \{(u,v) \in \tilde{\Omega} \,|\, \epsilon \le u < \epsilon + 2\pi\}, \tag{2.6}$$

then letting

$$V_\epsilon = \{x(z) \,|\, z \in \Omega_{\epsilon,2\pi}\}, \qquad \tilde{V}_\epsilon = \{\tilde{x}(u,v) \,|\, (u,v) \in \tilde{\Omega}_{\epsilon,2\pi}\}, \tag{2.7}$$

we have in fact

$$V = V_\epsilon, \qquad \tilde{V} = \tilde{V}_\epsilon.$$

If $\tilde{x}$ is as in (2.5) we let $\tilde{\mathbf{x}}_u$, $\tilde{\mathbf{x}}_v$ respectively denote the vectors $(\partial\lambda_1/\partial u, \partial\lambda_2/\partial u, \partial\mu_1/\partial u, \partial\mu_2/\partial u)$, $(\partial\lambda_1/\partial v, \partial\lambda_2/\partial v, \partial\mu_1/\partial v, \partial\mu_2/\partial v)$. We observe that requiring that $\tilde{\mathbf{x}}_u$, $\tilde{\mathbf{x}}_v$ be linearly independent is equivalent to

$$x'(z) = (x_1'(z), x_2'(z)) \neq 0, \tag{2.8}$$

where $x_i' = dx_i/dz$, $i = 1,2$. More precisely, it follows from (2.8) and the Cauchy–Riemann equations (see (2.25) below) that

$$\|\tilde{\mathbf{x}}_u\| = \|\tilde{\mathbf{x}}_v\| \neq 0, \qquad \langle \tilde{\mathbf{x}}_u, \tilde{\mathbf{x}}_v \rangle_{\mathbf{R}^4} = 0,$$

where we have denoted by $\| \ \|$, $\langle \ \rangle_{\mathbf{R}^4}$ respectively the Euclidean length and the inner product in $\mathbf{R}^4$.

Throughout this section we make the following

(2.9) Assumption. *For every* $\epsilon \in \mathbf{R}$ *the set* $\tilde{V}_\epsilon$ *is a two-dimensional manifold immersed in* $\mathbf{R}^4$ (cf., e.g., [Bo]). *We remark that this is true if* (2.8) *holds.*

We denote by $\tilde{\Sigma}$ the collection of all closed, simple, C^1 curves $\tilde{\Gamma} : [a,b] \to \tilde{V}_\epsilon$ for which there exists a simple, C^1 curve $\tilde{\gamma} : [a,b] \to \bar{\tilde{\Omega}}_{\epsilon,2\pi} = \left\{ (u,v) \in \tilde{\Omega} \,\middle|\, \epsilon \leq u \leq \epsilon + 2\pi \right\}$ such that $\tilde{\gamma}(b) - \tilde{\gamma}(a) = 2\pi$ and

$$\tilde{\Gamma}(s) = \tilde{x}(\tilde{\gamma}(s)) = \tilde{x} \circ \tilde{\gamma}(s), \qquad s \in [a,b].$$

Via the identification of V_ϵ with $\tilde{V}_\epsilon$ the collection $\tilde{\Sigma}$ identifies a corresponding family of curves, Σ, on V_ϵ. Therefore, if $\Gamma \in \Sigma$ there exists a simple, C^1 curve $\gamma : [a,b] \to \bar{\Omega}_{\epsilon,2\pi} = \left\{ z \in \Omega \,\middle|\, \epsilon \leq \operatorname{Re} z \leq \epsilon + 2\pi \right\}$, such that $\gamma(b) - \gamma(a) = 2\pi$ and for which

$$\Gamma(s) = x(\gamma(s)) = x \circ \gamma(s), \qquad s \in [a,b]. \tag{2.10}$$

Throughout the forthcoming discussion we will tacitly assume that an $\epsilon \in \mathbf{R}$ *has been fixed so that* V *and* $\tilde{V}$ *are given by* (2.7).

The object of this section is to study the asymptotic behavior of integrals of the type (1.4)

$$I(\zeta) = \int_\Gamma e^{-i\langle x,\zeta \rangle}(dx_1 + i\,dx_2)$$

when $\zeta \to \infty$ along a set M_α in $\mathbf{C}^2$ given by (1.2), and $\Gamma \in \Sigma$.

If γ is as in (2.10) the previous integral must be interpreted as follows:

$$\int_\Gamma e^{-i\langle x,\zeta \rangle}(dx_1 + i\,dx_2) = \int_\gamma e^{-i\langle x(z),\zeta \rangle}(x_1'(z) + ix_2'(z))\,dz \tag{2.11}$$

where $x_i' = dx_i/dz$, $i = 1,2$. We recall that $\langle \ , \ \rangle$ in (2.11) denotes the inner product defined by

$$\langle (x_1,x_2),(\zeta_1,\zeta_2) \rangle = x_1\zeta_1 + x_2\zeta_2 \qquad \text{for}\,(x_1,x_2),(\zeta_1,\zeta_2) \in \mathbf{C}^2.$$

The l.h.s. of (2.11) is well-defined since if γ_1 is another curve in $\bar{\Omega}_{\epsilon,2\pi}$ such that $\Gamma = x \circ \gamma_1$, then by Cauchy's theorem and (2.1) we have

$$\int_\gamma e^{-i\langle x(z),\zeta \rangle}(x_1'(z) + ix_2'(z))\,dz = \int_{\gamma_1} e^{-i\langle x(z),\zeta \rangle}(x_1'(z) + ix_2'(z))\,dz.$$

If $\alpha > 0$ we use the representation (1.17)–(1.19) for points $\zeta \in M_\alpha$. If ξ, η are as in (1.17) we denote by $(\xi,-\eta)$ the vector in $\mathbf{R}^4$ $(\cos\theta, \sin\theta, \sin\theta, -\cos\theta)$, $\theta \in [0,2\pi]$. We note that if $\theta = 0$, then $(\xi,-\eta) = (1,0,0,-1)$. For a given $\theta \in [0,2\pi]$ and ξ, η as in (1.18) we define

$$\psi(z) = \langle x(z), \xi + i\eta\rangle = e^{-i\theta}(x_1(z) + ix_2(z)), \tag{2.12}$$

where we recall $x(z) = (x_1(z), x_2(z))$. If for $\alpha > 0$ we let

$$M_{-\alpha} = \left\{\zeta = (\zeta_1,\zeta_2) \in \mathbf{C}^2 \,\middle|\, \zeta_1^2 + \zeta_2^2 = -\alpha\right\},$$

then $\zeta \in M_\alpha$ iff $i\zeta \in M_{-\alpha}$. We have found it convenient, for both notational and computational reasons, to study the integral

$$\int_\Gamma e^{-\langle x,\zeta\rangle}(dx_1 + i\,dx_2) = \int_\gamma e^{-\langle x(z),\zeta\rangle}(x_1'(z) + ix_2'(z))\,dx$$

when $\zeta \to \infty$ along $M_{-\alpha}$, rather than the one in (2.11). This makes no difference from the point of view of the applications.

The main result of this section is given by the following

Theorem 2.1. *Let V be as in* (2.2), Γ *be a closed, simple, C^1 curve on V, with $\Gamma \in \Sigma$, and let $\tilde\Gamma$ be the closed, simple, C^1 curve on $\tilde V$ corresponding to* Γ. *We assume that*

(i) *The set $\tilde X$ of the points of $\tilde V$ in which a tangent plane to $\tilde V$ is normal to* $(1,0,0,-1)$ *(and therefore to $(\xi,-\eta)$ for any $\theta \in [0,2\pi]$) is not empty.*

(ii) *$\tilde\Gamma$ meets n points $\tilde x_1,\ldots,\tilde x_n$ of $\tilde X$, and for each of the hyperplanes in* $\mathbf{R}^4$ *normal to $(\xi,-\eta)$ and passing through $\tilde x_j$, $j = 1,\ldots,n$, $\tilde\Gamma$ lies all on the side containing* $(\xi,-\eta)$.

(By "$\tilde\Gamma$ meets $\tilde x$", we mean that $\tilde x \in \tilde\Gamma(\,]a,b[\,)$.)

(iii) *The Gauss curvature of $\tilde V$ at $\tilde x_1,\ldots,\tilde x_n$ is negative.*

We denote by $x_1,\ldots,x_n$ the points on V corresponding to $\tilde x_1,\ldots,\tilde x_n$. Then we have for $\zeta = r\xi + it\eta \in M_{-\alpha}$, $t > 0$ and $r \to +\infty$

$$\int_\Gamma e^{-\langle x,\zeta\rangle}(dx_1 + i\,dx_2) = -\frac{i\alpha e^{i\theta}}{2r^{5/2}}\sum_{j=1}^n e^{-\langle x_j,\zeta\rangle}\{a_j + o(1)\}, \tag{2.13}$$

with

$$a_j \neq 0, \qquad j = 1,\ldots,n. \tag{2.14}$$

Remark 2.1. We mention explicitly that the points $\tilde x_j \in \tilde V$, $j = 1,\ldots,n$, in (ii), and therefore the points $x_j \in V$, $j = 1,\ldots,n$, in (2.13), depend on the particular $\theta \in [0,2\pi]$ for which $(\xi,-\eta) = (\cos\theta,\sin\theta,\sin\theta,-\cos\theta)$ and such that (ii) holds. In (2.13) $o(1)$ denotes a function whose absolute value tends to zero uniformly for $r \to +\infty$.

Remark 2.2. We obtain (2.13) under the assumption that $r \to +\infty$ by letting $t \to +\infty$ in (1.21). Following the proof of (2.13) below we can easily obtain the asymptotic expansion of the l.h.s. of (2.13) for $\zeta \in M_{-\alpha}$, and $r \to +\infty$ by letting $t \to -\infty$ in (1.21). We must only replace $(\xi, -\eta)$ with (ξ, η) in the assumptions (i) and (ii).

We now turn to the

Proof of Theorem 2.1. Let $\zeta \in M_{-\alpha}$. Using (1.17) and (2.11) we can rewrite the l.h.s. of (2.13) as follows:

$$\int_\Gamma e^{-\langle x,\zeta\rangle}(dx_1 + i\,dx_2) = \int_\gamma e^{-[r\langle x(z),\xi\rangle + it\langle x(z),\eta\rangle]}(x_1'(z) + ix_2'(z))\,dz, \tag{2.15}$$

where now (see (1.20))

$$r^2 - t^2 = -\alpha. \tag{2.16}$$

By means of a Taylor expansion we have

$$\begin{aligned}
&\int_\gamma e^{-[r\langle x(z),\xi\rangle + it\langle x(z),\eta\rangle]}(x_1'(z) + ix_2'(z))\,dz \\
&= \int_\gamma e^{-r\langle x(z),\xi+i\eta\rangle}(x_1'(z) + ix_2'(z))\,dz \\
&\quad - i(t-r)\int_\gamma e^{-r\langle x(z),\xi+i\eta\rangle}\langle x(z),\eta\rangle\,(x_1'(z) + ix_2'(z))\,dz \\
&\quad - \frac{(t-r)^2}{2}\int_\gamma e^{-r\langle x(z),\xi+i\eta\rangle}\langle x(z),\eta\rangle^2(x_1'(z) + ix_2'(z))\,dz \\
&\quad + i\,\frac{(t-r)^3}{6}\int_\gamma e^{-r\langle x(z),\xi\rangle}k(t,r,z)(x_1'(z) + ix_2'(z))\,dz.
\end{aligned} \tag{2.17}$$

The remainder in the last integral in the r.h.s. of (2.17) is estimated as follows:

$$|k(t,r,z)| \le C|e^{-ir\langle x(z),\eta\rangle}|e^{C|t-r|} \qquad \text{on } \gamma, \tag{2.18}$$

where $C > 0$ is a constant depending only on V and Γ. Using (2.12) and (2.1) we recognize that

$$\int_\gamma e^{-r\langle x(z),\xi+i\eta\rangle}(x_1'(z) + ix_2'(z))\,dz = e^{i\theta}\int_\gamma e^{-r\psi(z)}\psi'(z)\,dz = 0. \tag{2.19}$$

Now we look at the *critical points* of the phase $\langle x, \xi + i\eta\rangle$ in the second and third integral in the r.h.s. of (2.17). These are the points $z \in \Omega_{\epsilon,2\pi}$ at which

$$\langle x'(z), \xi + i\eta\rangle = 0, \tag{2.20}$$

where $x'(z) = (x_1'(z), x_2'(z))$. We recall the notation (2.4). Letting $\partial\lambda/\partial u = (\partial\lambda_1/\partial u, \partial\lambda_2/\partial u)$, $\partial\mu/\partial u = (\partial\mu_1/\partial u, \partial\mu_2/\partial u)$ and an analogous meaning for $\partial\lambda/\partial v$, $\partial\mu/\partial v$, we see that (2.20) is equivalent to

$$\begin{cases} \left\langle \dfrac{\partial\lambda}{\partial u}, \xi \right\rangle - \left\langle \dfrac{\partial\mu}{\partial u}, \eta \right\rangle = 0, \\ \left\langle \dfrac{\partial\lambda}{\partial u}, \eta \right\rangle + \left\langle \dfrac{\partial\mu}{\partial u}, \xi \right\rangle = 0. \end{cases} \tag{2.21}$$

By the Cauchy–Riemann equations we have

$$\frac{\partial\lambda}{\partial u} = \frac{\partial\mu}{\partial v}, \qquad \frac{\partial\lambda}{\partial v} = -\frac{\partial\mu}{\partial u}. \tag{2.22}$$

Using (2.22) in (2.21) we conclude that at a point $z_0 \in \Omega_{\epsilon,2\pi}$ (2.20) holds iff the vector $(\xi, -\eta)$ is normal to the surface $\tilde V$ in $\tilde x(z_0)$ (see (2.5)). Also, (2.13) implies that the critical points of $\langle x, \xi + i\eta\rangle$ do not depend on θ in (1.18). In conclusion

(2.23) *$z_0 \in \Omega_{\epsilon,2\pi}$ is a critical point of $z \mapsto \langle x(z), \xi + i\eta\rangle$ iff the vector* $(1,0,0,-1)$ *is normal to the surface $\tilde V$ in $\tilde x(z_0)$. Moreover,* $(1,0,0,-1)$ *is normal to $\tilde V$ in $\tilde x(z_0)$ iff $(\xi, -\eta)$ is, for every $\theta \in [0,2\pi]$.*

If z_0 is a critical point for $z \to \langle x(z), \xi + i\eta\rangle$, we set

$$b_{11} = \left\langle \left(\frac{\partial^2\lambda}{\partial u^2}(z_0), \frac{\partial^2\mu}{\partial u^2}(z_0) \right), (\xi, -\eta) \right\rangle_{\mathbf{R}^4},$$

$$b_{12} = b_{21} = \left\langle \left(\frac{\partial^2\lambda}{\partial u \partial v}(z_0), \frac{\partial^2\mu}{\partial u \partial v}(z_0) \right), (\xi, -\eta) \right\rangle_{\mathbf{R}^4},$$

$$b_{22} = \left\langle \left(\frac{\partial^2\lambda}{\partial v^2}(z_0), \frac{\partial^2\mu}{\partial v^2}(z_0) \right), (\xi, -\eta) \right\rangle_{\mathbf{R}^4},$$

where $\langle\ ,\ \rangle_{\mathbf{R}^4}$ denotes the inner product in $\mathbf{R}^4$.

Since $b_{11} + b_{22} = 0$, the Gauss curvature of $\tilde V$ at $\tilde x(z_0)$ is given by

$$-(b_{11}^2 + b_{12}^2) = \left| \left(\frac{\partial^2\lambda_1}{\partial u^2}(z_0) - \frac{\partial^2\mu_2}{\partial u^2}(z_0) \right) + i \left(\frac{\partial^2\lambda_2}{\partial u^2}(z_0) + \frac{\partial^2\mu_1}{\partial u^2}(z_0) \right) \right|^2.$$

Hence we can conclude that

$$\textit{Gauss curvature at } \tilde x(z_0) = -|\psi''(z_0)|^2. \tag{2.24}$$

We remark that if z_0 is a simple critical point, then $\tilde x(z_0)$ is a saddle-point for $\tilde V$. In virtue of (2.23) the assumption (i) in the statement of Theorem 2.1 guarantees that there exists at least one critical point $z_1 \in \Omega_{\epsilon,2\pi}$ of $\langle x, \xi + i\eta\rangle$: If $\tilde x_1 = \tilde x(z_1) \in \tilde V$ we set $x_1 = x(z_1) \in V$. Clearly, there may be other points $z_2, \dots, z_r$ in

$\Omega_{\epsilon,2\pi}$ at which (2.20) holds. Among them we single out those that are mapped by $\tilde{x}$ into $\tilde{x}_2, \ldots, \tilde{x}_n \in \tilde{V}$, with $\tilde{x}_j$, $j = 1, \ldots, n$, being as in (ii). We then set

$$\tilde{x}(z_j) = \tilde{x}_j \in \tilde{V}, \qquad x(z_j) = x_j \in V, \qquad j = 1, \ldots, n. \tag{2.25}$$

We go back to (2.17), which we now rewrite using (2.15), (2.12) and (2.19):

$$\begin{aligned}\int_\Gamma e^{-\langle x,\zeta\rangle}(dx_1 + i\,dx_2) &= -i(t-r)e^{i\theta}\int_\gamma e^{-r\psi(z)}\langle x(z),\eta\rangle\psi'(z)\,dz \\ &\quad - \frac{(t-r)^2}{2}\,e^{i\theta}\int_\gamma e^{-r\psi(z)}\langle x(z),\eta\rangle^2\psi'(z)\,dz \\ &\quad + i\,\frac{(t-r)^3}{6}\,e^{i\theta}\int_\gamma e^{-r\langle x(z),\xi\rangle}k(t,r,z)\psi'(z)\,dz.\end{aligned} \tag{2.26}$$

Now a problem arises due to the fact that the amplitudes in the integrals in the r.h.s. of (2.26) vanish at the critical points of the phase ψ. To overcome this difficulty we perform an integration by parts using (2.1). This yields

$$\int_\gamma e^{-r\psi(z)}\langle x(z),\eta\rangle\psi'(z)\,dz = \frac{1}{r}\int_\gamma e^{-r\psi(z)}\langle x'(z),\eta\rangle\,dz, \tag{2.27}$$

$$\int_\gamma e^{-r\psi(z)}\langle x(z),\eta\rangle^2\psi'(z)\,dz = \frac{2}{r}\int_\gamma e^{-r\psi(z)}\langle x(z),\eta\rangle\langle x'(z),\eta\rangle\,dz. \tag{2.28}$$

Substituting (2.27), (2.28) in (2.26) gives

$$\begin{aligned}\int_\Gamma e^{-\langle x,\zeta\rangle}(dx_1 + i\,dx_2) &= -ie^{i\theta}\,\frac{(t-r)}{r}\int_\gamma e^{-r\psi(z)}\langle x'(z),\eta\rangle\,dz \\ &\quad - e^{i\theta}\,\frac{(t-r)^2}{r}\int_\gamma e^{-r\psi(z)}\langle x(z),\eta\rangle\langle x'(z),\eta\rangle\,dz \\ &\quad + ie^{i\theta}\,\frac{(t-r)^3}{6}\int_\gamma e^{-r\langle x(z),\xi\rangle}k(t,r,z)\psi'(z)\,dz\end{aligned} \tag{2.29}$$

After these reductions we turn to the asymptotic analysis of the l.h.s. of (2.29) as $\zeta \to \infty$ along $M_{-\alpha}$. We consider the points $z_1, \ldots, z_n \in \Omega_{\epsilon,2\pi}$ in (2.25). We fix our attention on z_1 and consider the region (recall (2.12))

$$\begin{aligned}\operatorname{Re}\psi(z) &= \operatorname{Re}[e^{-i\theta}(x_1(z) + ix_2(z))] \\ &\geq \operatorname{Re}[e^{-i\theta}(x_1(z_1) + ix_2(z_1))] \\ &= \operatorname{Re}\psi(z_1).\end{aligned} \tag{2.30}$$

We want to show that because of (i) and (ii) the trace of the curve γ is entirely contained in the region defined by (2.30). To this purpose we observe that if

$\gamma = \gamma_1 + i\gamma_2$, and we set $\tilde{\gamma} = (\gamma_1, \gamma_2)$, then recalling (2.4) a computation shows that

$$\operatorname{Re}\psi(\gamma(s)) \geq \operatorname{Re}\psi(z_1), \qquad s \in [a,b], \tag{2.31}$$

iff for every $s \in [a,b]$

$$\langle(\lambda(\tilde{\gamma}(s)) - \lambda(z_1), \mu(\tilde{\gamma}(s)) - \mu(z_1)), (\xi, -\eta)\rangle_{\mathbf{R}^4} \geq 0. \tag{2.32}$$

But (2.32) is equivalent to requiring that $\tilde{\Gamma} = \tilde{x} \circ \tilde{\gamma}$ lies all on one side of the hyperplane in $\mathbf{R}^4$ normal to $(\xi, -\eta)$ and passing through $\tilde{x}_1 = \tilde{x}(z_1)$ – precisely, the side that contains the vector $(\xi, -\eta)$. The assumption (ii) guarantees that this is actually the case. Moreover, (i) implies that in fact

$$\operatorname{Re}\psi(z_j) = \operatorname{Re}\psi(z_1), \qquad j = 2, \ldots, n. \tag{2.33}$$

Finally, the assumption that γ be C^1 prevents γ leaving z_1 from the same sector of the region $\operatorname{Re}\psi(z) \geq \operatorname{Re}\psi(z_1)$ from which it arrives to z_1. In conclusion, γ is a C^1, simple curve in $\bar{\Omega}_{\epsilon, 2\pi} = \{z \in \Omega \mid \epsilon \leq \operatorname{Re} z \leq \epsilon + 2\pi\}$ with $\gamma(b) - \gamma(a) = 2\pi$, whose trace entirely lies in the region $\operatorname{Re}\psi(z) > \operatorname{Re}\psi(z_1)$, with the exception of the points $z_1, \ldots, z_n$ at which (2.33) holds.

Moreover, from (2.24) and (iii) it follows that

$$\psi''(z_j) \neq 0, \qquad j = 1, \ldots, n \tag{2.34}$$

We are now in a position to apply Riemann's method of the steepest descent to the first integrals in the r.h.s. of (2.29).

Then by [Ol, Theorem 7.3 on p. 127] we have as $\zeta \in M_{-\alpha}$ and $r \to +\infty$

$$\int_\gamma e^{-r\psi(z)} \langle x'(z), \eta\rangle \, dz = \frac{1}{\sqrt{r}} \sum_{j=1}^n e^{-r\psi(z_j)} \left(a_j + O\left(\frac{1}{r}\right)\right), \tag{2.35}$$

$$\int_\gamma e^{-r\psi(z)} \langle x(z), \eta\rangle \langle x'(z), \eta\rangle \, dz = \frac{1}{\sqrt{r}} \sum_{j=1}^n e^{-r\psi(z_j)} \left(b_j + O\left(\frac{1}{r}\right)\right), \tag{2.36}$$

where the a_j's in (2.35) are given by

$$a_j = \left(\frac{\pi}{2\psi''(z_j)}\right)^{1/2} \langle x'(z_j), \eta\rangle, \qquad j = 1, \ldots, n, \tag{2.37}$$

whereas

$$b_j = \left(\frac{\pi}{2\psi''(z_j)}\right)^{1/2} \langle x(z_j), \eta\rangle \langle x'(z_j)\eta\rangle, \qquad j = 1, \ldots, n. \tag{2.38}$$

The branch of $\psi''(z_j)^{1/2}$ must be suitably chosen.

We have used the symbol $O(1/r)$ to denote functions whose absolute value is bounded by C/r, uniformly for large r, with a constant $C > 0$ which solely depends on V and Γ.

Before inserting (2.35), (2.36) in (2.29), we proceed to estimate the third integral in the r.h.s. of (2.29). By (2.18) we have

$$\begin{aligned}\left|\int_\gamma e^{-r\langle x(z),\xi\rangle}k(t,r,z)\psi'(z)\,dz\right| &\le Ce^{C|t-r|}\int_\gamma |e^{-r\langle x(\gamma(s)),\xi\rangle}e^{-ir\langle x(\gamma(s)),\eta\rangle}|\,ds \\ &= Ce^{C|t-r|}\int_\gamma |e^{-r\psi(\gamma(s))}|\,ds,\end{aligned} \tag{2.39}$$

where in the last equality we have used (2.12). By (2.31) we have for $s \in [a,b]$

$$|e^{-r\psi(\gamma(s))}| = e^{-r\operatorname{Re}\psi(\gamma(s))} \le e^{-r\operatorname{Re}\psi(z_1)}. \tag{2.40}$$

By (2.39), (2.40) we can write

$$\int_\gamma e^{-r\langle x(z),\xi\rangle}k(t,r,z)\psi'(z)\,dz = q(t,r) \tag{2.41}$$

where

$$|q(t,r)| \le Ce^{-r\operatorname{Re}\psi(z_1)}e^{C|t-r|}. \tag{2.42}$$

We now insert (2.35), (2.36), (2.41) in (2.29). Using (2.33) we obtain

$$\begin{aligned}&\int_\Gamma e^{-\langle x,\zeta\rangle}(dx_1 + i\,dx_2) \\ &= \frac{ie^{i\theta}(t-r)}{r^{3/2}}\left\{-\sum_{j=1}^n e^{-r\psi(z_j)}\left(a_j + O\left(\frac{1}{r}\right)\right)\right. \\ &\qquad\left. + i(t-r)\sum_{j=1}^n e^{-r\psi(z_j)}\left(b_j + O\left(\frac{1}{r}\right)\right) + \frac{(t-r)^2}{6}\,r^{3/2}q(t,r)\right\} \\ &= ie^{[i\theta-\operatorname{Re}\psi(z_1)]}\,\frac{(t-r)}{r^{3/2}}\sum_{j=1}^n e^{-ir\operatorname{Im}\psi(z_j)}\left\{-a_j + i(t-r)b_j + O\left(\frac{1}{r}\right) + i(t-r)O\left(\frac{1}{r}\right)\right. \\ &\qquad\left. + \frac{(t-r)^2r^{3/2}}{6n}\,q(t,r)e^{[ir\operatorname{Im}\psi(z_j)+r\operatorname{Re}\psi(z_1)]}\right\}.\end{aligned} \tag{2.43}$$

Let now $\zeta = r\xi + it\eta \in M_{-\alpha}$. If we allow $r \to +\infty$ by letting $t \to +\infty$, then (2.16) gives

$$t = r\left(1 + \frac{\alpha}{2r^2} + O\left(\frac{1}{r^4}\right)\right) \qquad \text{as } r \to +\infty,$$

and therefore

$$t - r = \frac{\alpha}{2r}\left(1 + O\left(\frac{1}{r^3}\right)\right) \qquad \text{as } r \to +\infty, \qquad t > 0. \tag{2.44}$$

By (2.44) and (2.42) we obtain from (2.43)

$$\int_\Gamma e^{-\langle x,\zeta\rangle}(dx_1 + i\,dx_2) = -ie^{[i\theta - r\operatorname{Re}\psi(z_1)]}\,\frac{(t-r)}{r^{3/2}}\sum_{j=1}^{n} e^{-ir\operatorname{Im}\psi(z_j)}\left\{a_j + O\left(\frac{1}{\sqrt{r}}\right)\right\}.$$

(2.45)

Next, we observe that (2.12), (1.17) and (2.44) yield

$$e^{-r\psi(z_j)} = e^{-\langle x(z_j),\zeta\rangle}(1 + o(1)), \qquad j = 1, \ldots, n, \tag{2.46}$$

uniformly as $\zeta \in M_{-\alpha}$, $t > 0$, and $r \to +\infty$. Using (2.46) in (2.45), and (2.25), we have

$$\int_\Gamma e^{-\langle x,\zeta\rangle}(dx_1 + i\,dx_2) = -\frac{ie^{i\theta}(t-r)}{r^{3/2}}\sum_{j=1}^{n} e^{-\langle x_j,\zeta\rangle}\{a_j + o(1)\}, \tag{2.47}$$

uniformly as $\zeta \in M_{-\alpha}$, $t > 0$, and $r \to +\infty$. Inserting (2.44) in (2.47) we finally obtain

$$\int_\Gamma e^{-\langle x,\zeta\rangle}(dx_1 + i\,dx_2) = -\frac{i\alpha e^{i\theta}}{2r^{3/2}}\sum_{j=1}^{n} e^{-\langle x_j,\zeta\rangle}\{a_j + o(1)\}$$

as $\zeta = r\xi + it\eta \in M_{-\alpha}$, $t > 0$ and $r \to +\infty$. This completes the proof of (2.13). Finally, we observe that (2.14) holds. In fact, since at z_j, $j = 1, \ldots, n$, (2.20) holds, from the latter and (2.12) we have

$$x_1'(z_j) = -ix_2'(z_j), \qquad j = 1, \ldots, n. \tag{2.48}$$

(2.48) and (2.8) imply for every $\theta \in [0, 2\pi]$

$$\langle x'(z_j), \eta\rangle = x_2'(z_j)e^{i\theta} \neq 0. \tag{2.49}$$

(2.14) now follows form (2.49) and (2.37).

Remark 2.3. We observe the different decay at ∞ in (2.13) and (1.25) of Berenstein's Theorem 1.4 in Section 1. We have a factor $r^{-5/2}$ whereas $\hat{\chi}_{\partial D}(\zeta)$ in (1.25) decays like $r^{-1/2}$. This difference is due to the fact that Berenstein assumes (1.22) whereas we only use (1.21).

Remark 2.4. It is interesting to compare (2.13) with the formula that gives $\hat{\chi}_{\partial E_{ab}}(\zeta)$, where E_{ab} is the elliptical region in (1.8). By Green's theorem and (1.9) we obtain

$$\begin{aligned} \hat{\chi}_{\partial E_{ab}}(\zeta) &= (\zeta_1 + i\zeta_2)\hat{\chi}_{E_{ab}}(\zeta) \\ &= 2\pi ab(\zeta_1 + i\zeta_2)\,\frac{J_1\left(\sqrt{a^2\zeta_1^2 + b^2\zeta_2^2}\right)}{\sqrt{a^2\zeta_1^2 + b^2\zeta_2^2}} \end{aligned} \tag{2.50}$$

for $\zeta = (\zeta_1, \zeta_2) \in \mathbf{C}^2$. Let us assume that $a > b > 0$. If we take $\zeta \in M_\alpha$ with $\xi = (1,0)$, $\eta = (0,1)$ in (1.18), then (1.19) gives $\zeta_1 = r$, $\zeta_2 = it$. With this choice of ζ_1 and ζ_2 (2.50) becomes, using (1.20),

$$(2.51) \qquad \hat{\chi}_{\partial E_{ab}}(\zeta) = -\frac{2\pi ab(t-r)}{r\sqrt{(a^2-b^2)+\alpha b^2/r^2}} J_1\left(r\sqrt{(a^2-b^2)+\frac{\alpha b^2}{r^2}}\right)$$

where $\zeta = (r, it) \in M_\alpha$. Let us now recall the asymptotic behavior at ∞ of J_1 (see, e.g., [L, 5.16.1 on p. 134]):

$$(2.52) \qquad J_1(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4}\right) \qquad \text{as } x \to +\infty.$$

Letting $r \to +\infty$ in (2.51), by (2.52) and (1.20) we obtain for an $A \neq 0$

$$\hat{\chi}_{\partial E_{ab}}(\zeta) = \frac{1}{r^{3/2}} \sin\left(r\sqrt{a^2 - b^2} - \frac{\pi}{4}\right)(A + o(1))$$

$$\text{as } \zeta \in M_\alpha, \quad t > 0, \quad \text{and } r \to +\infty.$$

Remark 2.5. With Γ as in the statement of Theorem 2.1 let $\Gamma_1 \in \Sigma$ be a closed, simple, C^1 curve on V homotopic to Γ. Then if γ_1 is the curve in $\bar{\Omega}_{\epsilon,2\pi} = \{z \in \Omega \mid \epsilon \le \operatorname{Re} z \le \epsilon + 2\pi\}$ such that $\Gamma_1 = x \circ \gamma_1$, by (2.11), Cauchy's theorem and the periodicity we have for $\zeta \in M_\alpha$

$$\int_\Gamma e^{-i\langle x,\zeta\rangle}(dx_1 + i\,dx_2) = \int_{\Gamma_1} e^{-i\langle x,\zeta\rangle}(dx_1 + i\,dx_2).$$

Remark 2.6. For $\omega \in [0, 2\pi]$ let

$$A_\omega = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}$$

and consider the surface in $\mathbf{C}^2$

$$V_\omega = \{x_\omega(z) = A_\omega x(z) \mid z \in \Omega\}.$$

We claim that if there exists a curve Γ which satisfies (ii) with respect to the direction θ, then the curve $A_\omega \Gamma$ satisfies (ii) with respect to the direction $\omega + \theta$. In fact, letting $\xi_\theta = (\cos\theta, \sin\theta)$, $\eta_\theta = (-\sin\theta, \cos\theta)$ we have

$$\langle x_\omega(z), \xi_{\omega+\theta} + i\eta_{\omega+\theta}\rangle = \langle x(z), \xi_\theta + i\eta_\theta\rangle$$

and the critical points of $\langle x_\omega, (1,i)\rangle$ are the same as those of $\langle x, (1,i)\rangle$. We conclude that (ii) of Theorem 2.1 is invariant under rotations. The invariance under translations is obvious.

We close this section with two examples which shed some light on the assumptions of Theorem 2.1.

Example 2.1. The Circle of $\mathbf{C}^2$. Let $R > 0$ and consider the set in $\mathbf{C}^2$

$$V = \left\{ (R\cos z, R\sin z) \,\middle|\, z \in \mathbf{C} \right\}. \tag{2.53}$$

Following (2.3) we can identify V with the set in $\mathbf{R}^4$

$$\tilde{V} = \left\{ (R\cos u \cosh v, R\sin u \cosh v, -R\sin u \sinh v, R\cos u \sinh v) \,\middle|\, (u,v) \in \mathbf{R}^2 \right\}. \tag{2.54}$$

The condition that $(\xi, -\eta)$ be normal to the tangent plane to $\tilde{V}$ at the point $\tilde{x} = \tilde{x}(u,v)$ becomes in this case

$$\begin{cases} \sin(u-\theta)(\cosh v - \sinh v) = 0, \\ \cos(u-\theta)(\cosh v - \sinh v) = 0. \end{cases} \tag{2.55}$$

From (2.55) we see that for no $\theta \in [0.2\pi]$ and no point $\tilde{x} \in \tilde{V}$ is the tangent plane to $\tilde{V}$ in $\tilde{x}$ normal to $(\xi, -\eta)$. We remark that the $\mathbf{R}^2$-section of $\tilde{V}$ is the circle of radius R in $\mathbf{R}^2$

$$C_R = \left\{ (R\cos u, R\sin u, 0, 0) \,\middle|\, u \in \mathbf{R} \right\}$$

whose interior, the disk, does not have the Pompeiu property.

Example 2.2. The Ellipse of $\mathbf{C}^2$. Let $a > b > 0$ and consider the set in $\mathbf{C}^2$

$$V = \left\{ (a\cos z, b\sin z) \,\middle|\, z \in \mathbf{C} \right\}, \tag{2.56}$$

which is identified with the set in $\mathbf{R}^4$

$$\tilde{V} = \left\{ (a\cos u \cosh v, b\sin u \cosh v, -a\sin u \sinh v, b\cos u \sinh v) \,\middle|\, (u,v) \in \mathbf{R}^2 \right\}. \tag{2.57}$$

Let us take for simplicity $\theta = 0$, so that $(\xi, -\eta) = (1,0,0,-1)$. The condition that $(\xi, -\eta)$ be normal to the tangent plane to $\tilde{V}$ at $\tilde{x} = \tilde{x}(u,v)$ becomes now

$$\begin{cases} \sin u(a\cosh v - b\sinh v) = 0, \\ \cos u(a\sinh v - b\cosh v) = 0. \end{cases} \tag{2.58}$$

Solving (2.58) we find the solutions

$$(u_k, v_k) = \left(k\pi, \ln\sqrt{\frac{a+b}{a-b}} \right), \qquad k \in \mathbf{Z}. \tag{2.59}$$

Putting (2.59) into (2.57) we conclude that (modulo periodicity) there are two points on $\tilde{V}$ at which the tangent plane is normal to $(1,0,0,-1)$ (and therefore to any $(\xi, -\eta)$). They are given by

$$\left(\mp\frac{a^2}{\sqrt{a^2-b^2}},0,0,\mp\frac{b^2}{\sqrt{a^2-b^2}}\right). \tag{2.60}$$

3. Applications to the Pompeiu problem

In this section we apply Theorem 2.1 to characterize a large class of domains in $\mathbf{R}^2$ that have the Pompeiu property. Let D be a simply-connected, bounded domain in $\mathbf{R}^2$ with real analytic boundary. Suppose there exists an analytic manifold in $\mathbf{C}^2$, $V=\{x(z)=(x_1(z),x_2(z))\,|\,z\in\Omega\}$ as in Section 2, and suppose that the boundary of D, ∂D, is realized as the $\mathbf{R}^2$-*section* of V, i.e.,

$$\partial D=\{x(z)\,|\,z\in\Omega \text{ and } \operatorname{Im}x_1(z)=\operatorname{Im}x_2(z)=0\}. \tag{3.1}$$

For instance, if $s\mapsto(x_1(s),x_2(s))$, $s\in[0,2\pi]$ is a real analytic parametrization of ∂D with $x_i(0)=x_i(2\pi)$, $i=1,2$, then by Cauchy's theorem and the periodicity there exist a simply-connected, open set $\Omega\subset\mathbf{C}$ as in the opening of Section 2, and two analytic functions on Ω that coincide with x_i, $i=1,2$, on $\mathbf{R}$. If we continue to denote by x_i, $i=1,2$, those two functions and we form $x=(x_1,x_2):\Omega\to\mathbf{C}$, and V correspondingly, then (3.1) holds. Theorem 2.1 can then be used to provide an affirmative answer to the Pompeiu problem for D provided that ∂D is homotopic to a closed curve $\tilde{\Gamma}$ on $\tilde{V}$ (defined by (2.3)) satisfying the geometric assumptions (i) and (ii) of the theorem. In fact, if ∂D is homotopic to such a curve $\tilde{\Gamma}$, and Γ is the curve on V corresponding to $\tilde{\Gamma}$, we have from Remark 2.6 for $\zeta\in M_{-\alpha}$ and any $\alpha>0$

$$\begin{aligned}\hat{\chi}_{\partial D}(-i\zeta)&=\int_{\partial D}e^{-\langle x,\zeta\rangle}(dx_1+i\,dx_2)\\&=\int_{\Gamma}e^{-\langle x,\zeta\rangle}(dx_1+i\,dx_2).\end{aligned} \tag{3.2}$$

Suppose that D fails to have the Pompeiu property. Then by Theorem 1.2 in Section 1 there exists $\alpha>0$ such that $\hat{\chi}_{\partial D}(\zeta)=(\zeta_1+i\zeta_2)\hat{\chi}_D(\zeta)=0$ for every $\zeta\in M_\alpha$. Therefore, (3.2) and (2.13) in Theorem 2.1 yield for $\zeta=r\xi+it\eta\in M_{-\alpha}$

$$\sum_{j=1}^{n}e^{-\langle x_j,\zeta\rangle}a_j=o(1), \tag{3.3}$$

uniformly as $t>0$ and $r\to+\infty$, provided that for the function ψ, defined from $x=(x_1,x_2)$ via (2.12), (2.15) hold. Let us denote by $\langle\ ,\ \rangle_{\mathbf{R}^4}$ the inner product in $\mathbf{R}^4$ in order to distinguish it from the inner product in $\mathbf{R}^2$ in (3.3). In the notation of Section 2, and using (2.47), we can write for $j=1,\dots,n$

$$\langle x_j,\zeta\rangle=ir\langle\tilde{x}_j,(\eta,\xi)\rangle_{\mathbf{R}^4}+r\langle\tilde{x}_j,(\xi,-\eta)\rangle_{\mathbf{R}^4}+O(1/r), \tag{3.4}$$

where $|O(1/r)| \leq C/r$ uniformly as $r \to +\infty$, with $C > 0$ depending only on V and ∂D. Since by (i) in Theorem 2.1 $\langle \tilde{x}_j, (\xi, -\eta)\rangle_{\mathbf{R}^4} = 0$ for $j = 1, \ldots, n$, substituting (3.4) in (3.3) we conclude

$$\sum_{j=1}^{n} a_j e^{-ir\langle \tilde{x}_j, (\eta, \xi)\rangle_{\mathbf{R}^4}}(1 + o(1)) = o(1) \tag{3.5}$$

uniformly as $r \to +\infty$.

We summarize the previous considerations in the following

Proposition 3.1. *Let $D \subset \mathbf{R}^2$ be a simply-connected, bounded set with real analytic boundary ∂D, and suppose that ∂D lies on an analytic surface V in $\mathbf{C}^2$. If there exists a closed curve Γ on V satisfying the assumptions of Theorem* 2.1, *then D has the Pompeiu property provided that as $r \to +\infty$*

$$\sum_{j=1}^{n} a_j e^{-ir\langle \tilde{x}_j, (\eta, \xi)\rangle_{\mathbf{R}^4}} \not\to 0.$$

We conclude this section with some examples which illustrate the way Theorem 2.1 is applied.

Example 3.1. In Section 1 we mentioned that it was proved in [BST] that every elliptical region E_{ab} (see (1.8)) has the Pompeiu property as a consequence of (1.9) and Theorem 1.2. We give here a different proof of this fact that, instead of using (1.9), is based on Theorem 2.1. The value of such a longer approach is obviously demonstrative. Let $a > b > 0$ and consider the ellipse ∂E_{ab} parametrized by $s \mapsto (a\cos s, b\sin s)$, $s \in [0, 2\pi]$. Let V be the "ellipse" in $\mathbf{C}^2$ given by (2.56) and let

$$\psi(z) = e^{i\theta}(a\cos z + ib\sin z); \tag{3.6}$$

see (2.12). The critical points of ψ are given by

$$z_k = k\pi + i\ln\sqrt{\frac{a+b}{a-b}}, \qquad k \in \mathbf{Z}; \tag{3.7}$$

see (2.59). We emphasize that

$$\psi''(z_k) = -\psi(z_k) = (-1)^{|k|}e^{-i\theta}\sqrt{a^2 - b^2} \neq 0. \tag{3.8}$$

We let $\theta = 0$ in (3.6) and consider the region (see (2.30))

$$\operatorname{Re}\psi(z) \geq \operatorname{Re}\psi(z_1) \tag{3.9}$$

with z_1 given by (3.7). If $z = u + iv$ we see that (3.9) is equivalent to

$$[(a+b)e^{-v} + (a-b)e^{v}]\cos u \geq -2\sqrt{a^2 - b^2}. \tag{3.10}$$

The set of the couples (u, v) satisfying (3.10) is earmarked by a + sign in Fig. 1, its complementary by a − sign.

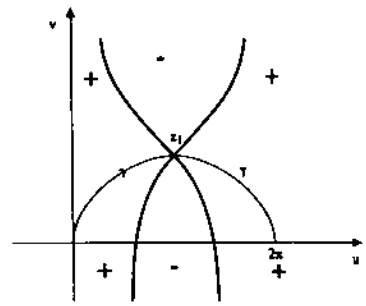

Fig. 1.

If we let γ be the curve whose trace in $\{z \in \mathbf{C} \,|\, 0 \leq \operatorname{Re} z \leq 2\pi\}$ is as in Fig. 1, and set $\Gamma(s) = (a\cos\gamma(s), b\sin\gamma(s))$, it is obvious that Γ is homotopic to ∂E_{ab}. Letting

$$x_1 = (a\cos z_1, b\sin z_1) = \left(-\frac{a^2}{\sqrt{a^2-b^2}}, -i\,\frac{b^2}{\sqrt{a^2-b^2}}\right) \in V$$

we obtain from Theorem 2.1 for $\zeta = (r, it) \in M_{-\alpha}$, $t > 0$, and $r \to +\infty$

$$\hat{\chi}_{\partial E_{ab}}(-i\zeta) = \int_\Gamma e^{-\langle x, \zeta\rangle}(dx_1 + i\,dx_2)$$

$$(3.11) \qquad = -\frac{i\alpha}{2r^{5/2}}\, e^{-\langle x_1, \zeta\rangle}(a_1 + o(1))$$

$$= -\frac{i\alpha}{2r^{5/2}}\, e^{r\sqrt{a^2-b^2}}(a_1 + o(1)).$$

Since $a_1 \neq 0$ (see (2.14)), we infer from (3.11) that for no $\alpha > 0$ can $\hat{\chi}_{\partial E_{ab}} \equiv 0$ on M_α and therefore E_{ab} has the Pompeiu property. It is interesting to compare (3.11) with the expansion obtained by (2.50) in Remark 2.4. As stressed in Remark 2.1, the points $x_j \in V$ corresponding to the critical points $z_j \in \Omega_{\epsilon,2\pi}$ do depend on the particular $\theta \in [0, 2\pi]$ for which (ii) in Theorem 2.1 holds. In fact, if we let $\theta = \pi/2$ in (3.6) and again consider the region (3.9), the latter is given by the sectors earmarked by a + sign in Fig. 2.

In this case we take a curve γ whose trace lies in the strip $\{z \in \mathbf{C} \,|\, -\pi/2 \leq \operatorname{Re} z \leq 3/2\pi\}$ and passes through the two critical points z_0, z_1 defined by (3.7). If we now set

$$x_0 = (a\cos z_0, b\sin z_0) = \left(\frac{a^2}{\sqrt{a^2-b^2}}, i\,\frac{b^2}{\sqrt{a^2-b^2}}\right) \in V,$$

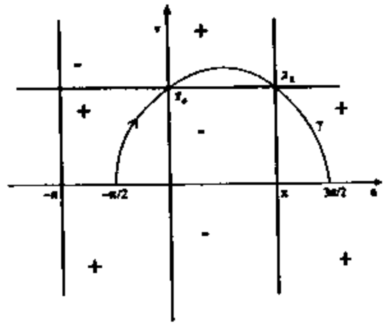

Fig. 2.

and $x_1 \in V$ is as above, then the curve $\tilde{\Gamma}$ on $\tilde{V}$ corresponding to γ in Fig. 2 passes through the two points $\tilde{x}_0$, $\tilde{x}_1$ on $\tilde{V}$ identified by (2.60). Again we can apply Theorem 2.1 and obtain an asymptotic expansion for $\hat{\chi}_{\partial E_{ab}}(\zeta)$. The latter is different, however, from (3.11) because we now have two contributions $e^{-\langle x_0, \zeta\rangle}$ and $e^{-\langle x_1, \zeta\rangle}$. We omit the details.

Example 3.2. Let $D \subset \mathbf{R}^2$ be the domain whose boundary is parametrized by the curve

$$s \to e^{is} + ae^{-iNs}, \qquad \text{where } \mathbf{N} \text{ is natural and } a \in \mathbf{C}\setminus\{0\}.$$

For suitable values of $a = |a|e^{i\varphi}$, this curve is simple. Since the Pompeiu property is invariant under rigid motion, the domain D has the Pompeiu property iff the domain D', whose boundary is parametrized by

$$s \to e^{i\psi}[e^{is} + |a|e^{i\varphi}e^{-iNs}] = e^{i(s+\psi)} + |a|e^{-iN(s+\psi-\varphi/N-\psi/N-\psi)}, \qquad \psi \in \mathbf{R},$$

has the Pompeiu property. By choosing ψ in such a way that $-\varphi/N - \psi/N - \psi = 0$, we can reduce to prove the Pompeiu property for D, with $a > 0$. Finally, take $\psi(z) = (e^{iz} + ae^{-iNz})e^{i\pi}$ and consider the simple critical point

$$z_1 = -\frac{i}{N+1}\log(Na)$$

for ψ. Obviously, the path

$$u \mapsto u - \frac{i}{N+1}\log(Na)$$

is contained in the region $\operatorname{Re}\psi(z) \geq \operatorname{Re}\psi(z_1)$ and, hence, D has the Pompeiu property.

Example 3.3. Let $D \subset \mathbf{R}^2$ be the domain whose boundary ∂D is parametrized by

$$s \mapsto \left(1 + \frac{\cos 2s}{2}\right)(\cos s, \sin s), \qquad s \in [0.2\pi]$$

represented in Fig. 3.

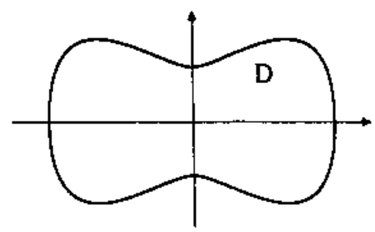

Fig. 3.

We let

$$V = \left\{ \left(\left(1 + \frac{\cos 2z}{2}\right) \cos z, \left(1 + \frac{\cos 2z}{2}\right) \sin z \right) \Big| z \in \mathbf{C} \right\},$$

and consider the function

$$\psi(z) = \left(1 + \frac{\cos 2z}{2}\right) e^{i(z-\theta)}, \qquad \theta \in [0, 2\pi]. \tag{3.12}$$

We consider the simple critical point

$$z_1 = -\frac{i}{2} \ln\left(\frac{\sqrt{7} - 2}{3}\right)$$

of ψ, let $\theta = \pi$ in (3.12), and look at the region

$$\operatorname{Re} \psi(z) \geq \operatorname{Re} \psi(z_1).$$

This is earmarked by a + sign in Fig. 4.

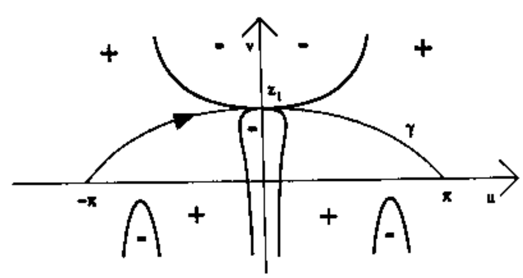

Fig. 4.

Letting γ be the curve whose trace is represented in Fig. 4, and Γ the curve on V defined by

$$\Gamma(s) = \left(\left(1 + \frac{\cos 2\gamma(s)}{2}\right)\cos\gamma(s), \left(1 + \frac{\cos 2\gamma(s)}{2}\right)\sin\gamma(s)\right),$$

we obtain from Theorem 2.1

$$\begin{aligned}\hat{\chi}_{\partial D}(-i\zeta) &= \int_\Gamma e^{-\langle x,\zeta\rangle}(dx_1 + i\,dx_2)\\ &= \frac{i\alpha}{2r^{5/2}}\, e^{-\langle x_1,\zeta\rangle}(a_1 + o(1)) \qquad (3.13)\\ &= \frac{i\alpha}{2r^{5/2}}\,(a_1 + o(1))e^{r(2\sqrt{7}-1)r/3\sqrt{3}(\sqrt{7}-2)^{1/2}},\end{aligned}$$

uniformly for $\zeta = (-r, -it) \in M_{-\alpha}$, $t > 0$ and $r \to +\infty$. From (3.13) we conclude that for no $\alpha > 0$ can $\hat{\chi}_{\partial D}$ vanish identically on M_α, therefore D has the Pompeiu property.

Example 3.4. Let $D \subset \mathbf{R}^2$ be the domain whose boundary ∂D, parametrized by

$$s \mapsto \left(1 - \frac{\sin^2 2s}{4}\right)(\cos s, \sin s), \qquad s \in [0, 2\pi],$$

is represented in Fig. 5.

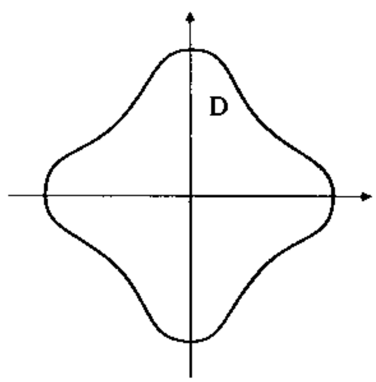

Fig. 5.

We let

$$V = \left\{\left(\left(1 - \frac{\sin^2 2z}{4}\right)\cos z, \left(1 - \frac{\sin^2 2z}{4}\right)\sin z\right) \,\middle|\, z \in \mathbf{C}\right\}$$

and consider the function

$$\psi(z) = \left(1 - \frac{\sin^2 2z}{4}\right)e^{i(z-\theta)}, \qquad \theta \in [0,2\pi]. \tag{3.14}$$

We let $\theta = \pi$ in (3.14) and look at the solutions of the equation $\psi'(z) = 0$. We consider the critical point $z_1 = \frac{1}{4}\ln 5$. We have $\psi(z_1) = -6/5^{5/4}$. Also, it is easy to check that z_1 is a simple critical point. The region

$$\operatorname{Re}\psi(z) \geq \operatorname{Re}\psi(z_1)$$

is earmarked by a + sign in Fig. 6, its complementary by a − sign.

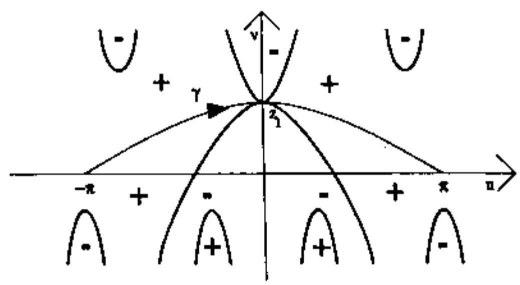

Fig. 6.

We let γ be the curve whose trace is represented in Fig. 6 and define a smooth curve on V by

$$\Gamma(s) = \left(\left(1 - \frac{\sin^2\gamma(s)}{4}\right)\cos\gamma(s), \left(1 - \frac{\sin^2\gamma(s)}{4}\right)\sin\gamma(s)\right).$$

Applying Theorem 2.1 we obtain

$$\begin{aligned} \hat{\chi}_{\partial D}(-i\zeta) &= \int_\Gamma e^{-\langle x,\zeta\rangle}(dx_1 + i\,dx_2) \\ &= \frac{i\alpha}{2r^{5/2}}\, e^{6r/5^{5/4}}(a_1 + o(1)), \end{aligned} \tag{3.15}$$

with $a_1 \neq 0$, uniformly for $\zeta = (-r,-it) \in M_{-\alpha}$, $t > 0$ and $r \to +\infty$. As in the previous examples, from (3.15) we conclude that D has the Pompeiu property.

References

[A] P. Aviles, *Symmetry theorems related to Pompeiu's problem*, Am. J. Math. **108** (1986), 1023–1036.

[B] C. A. Berenstein, *An inverse spectral theorem and its relation to the Pompeiu problem*, J. Analyse Math. **37** (1980), 128–144.

[BY 1] C. A. Berenstein and P. C. Yang, *An overdetermined Neumann problem in the unit disk*, Adv. Math. **44** (1982), 1–17.

[BY 2] C. A. Berenstein and P. C. Yang, *An inverse Neumann problem*, J. Reine Angew. Math. **382** (1987), 1–21.

[Bl H] N. Bleistein and R. A. Handelsman, *Asymptotic Expansions of Integrals*, Dover, New York, 1986.

[Bo] W. M. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*, Academic Press, New York 1986.

[BK] L. Brown and J. P. Kahane, *A note on the Pompeiu problem for convex domains*, Math. Ann. **259** (1982), 107–110.

[BST] L. Brown, B. M. Schreiber and A. B. Taylor, *Spectral synthesis and the Pompeiu problem*, Ann. Inst. Fourier **23** (1973), 125–154.

[Ca] L. A. Caffarelli, *The regularity of free boundaries in higher dimensions*, Acta Math. **139** (1977), 155–184.

[C] L. Chakalov, *Sur un problème de D. Pompeiu*, Annaire Univ. Sofia Fac. Phys. Math., Livre 1, **40** (1944), 1–44.

[Ch] C. Christov, *Sur un problème de M. Pompeiu*, Mathematica (Timosoara) **23** (1947–48), 103–107.

[E] L. Ehrenpreis, *Fourier Analysis in Several Complex Variables*, Wiley-Interscience, New York, 1970.

[GL] N. Garofalo and J. L. Lewis, *A symmetry result related to some overdetemrined boundary value problems*, Am. J. Math., to appear.

[GR] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, New York, 1980.

[L] N. N. Lebedev, *Special Functions and Their Applications*, Dover, New York, 1972.

[Ol] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974.

[P 1] D. Pompeiu, *Sur certains systèmes d'équations linéaires et sur une propriété intégrale des fonctions de plusieurs variables*, C. R. Acad. Sci. Paris **188** (1929), 1138–1139.

[P 2] D. Pompeiu, *Sur une propriété intégrale des fonctions de deux variables réelles*, Bull. Sci. Acad. Royale Belgique **15** (5) (1929), 265–269.

[R] F. Rellich, *Darstellung der Eigenwerte* $\Delta u + \lambda u$ *durch ein Randintegral*, Math. Z. **46** (1940), 635–646.

[Ri] B. Riemann, *Sullo svolgimento del quoziente di due serie ipergeometriche in frazione continua infinita*, Complete works, Dover, New York, 1953.

[S] J. Serrin, *A symmetry problem in potential theory*, Arch. Ration. Mech. Anal. **43** (1971), 304–318.

[W] H. Weinberger, *Remark on the preceding paper of Serrin*, Arch. Ration. Mech. Anal. **43** (1971), 319–320.

[Wi 1] S. A. Williams, *A partial solution of the Pompeiu problem*, Math. Ann. **223** (1976), 183–190.

[Wi 2] S. A. Williams, *Analyticity of the boundary for Lipschitz domains without the Pompeiu property*, Indiana Univ. Math. J. **30** (1981), 357–369.

[Y] S. T. Yau, *Seminar on Differential Geometry*, Annals of Math. Studies **102**, Princeton Univ. Press, 1982.

[Z] L. Zalcman, *Analyticity and the Pompeiu Problem*, Arch. Ration. Mech. Anal. **47** (1972), 237–254.

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