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UNIVALENT FUNCTIONS AND THE POMPEIU PROBLEM

NICOLA GAROFALO AND FAUSTO SEGALA

ABSTRACT. In this paper we prove a result on the Pompeiu problem. If the Schwarz function Φ of the boundary of a simply-connected domain $\Omega \subset \mathbb{R}^2$ extends meromorphically into a certain portion E of Ω with a pole at some point $z_0 \in E$, then Ω has the Pompeiu property unless Φ is a Möbius transformation, in which case Ω is a disk.

1. INTRODUCTION

In 1929 the Rumanian mathematician D. Pompeiu formulated the following problem: "To characterize those bounded domains $\Omega \subset \mathbb{R}^2$ for which $f \equiv 0$ is the only continuous function such that

(1.1)
$$\int_{\sigma(D)} f \, dx = 0,$$

for every rigid motion σ of \mathbb{R}^2 ".

One says that Ω has the Pompeiu property if $f \equiv 0$ is the only continuous function for which (1.1) holds. For a historical introduction to the problem we refer the reader to [GS1]. In that paper we conjectured that (modulo sets of measure zero) the disk is the only simply-connected domain that does not have the Pompeiu property. Chakalov [C] was the first one to realize that the disk fails to have the Pompeiu property. In fact, if one considers the function $f(x_1, x_2) = \sin(ax_1)$, then one has

$$\int_{B_r(x_0)} f(x) \, dx = \frac{2\pi r}{a} \sin(ax_{0,1}) J_1(ar) \, ,$$

where $x_0 = (x_{0,1}, x_{0,2})$ is fixed, $B_r(x_0) = \{x | |x - x_0| < r\}$, and J_1 is the Bessel function of order one. It is therefore enough to choose a > 0, such that $J_1(ar) = 0$, for (1.1) to hold.

This paper contains some progress toward the above conjecture. Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain whose boundary $\partial \Omega$ is a piecewise C^1 Jordan curve. By the Riemann mapping theorem there exists a univalent function $h: D \to \Omega$, where $D = \{z \in \mathbb{C} | |z| < 1\}$. Moreover, h can be extended in a one-to-one fashion to a continuous map of \overline{D} onto $\overline{\Omega}$. In order

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to state the main result in this paper we need to introduce some definitions. We consider the Schwarz function of $\partial \Omega$ given by

$$\Phi(w) = \overline{h\left(\frac{1}{\overline{h^{-1}(w)}}\right)}.$$

A priori, Φ is well defined on $\partial \Omega$. Given a straight line $L \subset \mathbb{C}$, and a point $z_0 \notin L$, we denote by $\Lambda(L; z_0)$ the open half-plane lying on one side of L and containing z_0 . We also let

$$E(L; z_0) = \Lambda(L; z_0) \cap \Omega$$
.

The main result in this paper is given by the following

Theorem 1. Suppose that there exist $z_0 \in \Omega$ and a straight line $L \subset \mathbb{C}$ such that:

(i) Φ can be extended to a holomorphic function in $E(L; z_0) \setminus \{z_0\}$ having a pole in z_0 ;

(ii) Φ is not a Möbius transformation.

Then, Ω has the Pompeiu property.

Figure 1 below illustrates the situation.

We now state two remarkable consequences of Theorem 1.

Corollary 2. Suppose that h is univalent in D and meromorphic in \mathbb{C} , with at least one pole in $\overline{\mathbb{C}}\setminus\overline{D}$. If, moreover, h is not a Möbius transformation, then $\Omega = h(D)$ has the Pompeiu property.

If we specialize Theorem 1 to the class of convex domains we obtain the following partial solution of the Pompeiu problem.

Corollary 3. Suppose that $\Omega = h(D)$ be a convex set. Assume that h has a pole on the boundary of the circle of convergence relative to its Taylor expansion at z = 0. If h is not a Möbius transformation, then Ω has the Pompeiu property.

Remark. Corollary 2 contains the result in our paper [GS2] (see also [GS3]) concerned with the case

$$h(z) = \sum_{k=0}^{N} a_k z^k \, .$$



FIGURE 1

Furthermore, it contains a result in a recent paper by Ebenfelt [E]. The latter has proved that if h is a univalent function in D such that $h(z) = \frac{p(z)}{q(z)}$, with p and q polynomials, then $\Omega = h(D)$ has the Pompeiu property, unless h is a Möbius transformation.

Our strategy to prove Theorem 1 is to study, by Riemann's method of the steepest descent, the asymptotic behavior of the (complexified) Fourier transform of the characteristic function of Ω , $\hat{\chi}_{\Omega}$, along the algebraic variety of \mathbb{C}^2 , $M_{\alpha} = \{\zeta_1^2 + \zeta_2^2 = \alpha\}$, $\alpha > 0$. This is due to an important characterization of the Pompeiu property established in 1973 by Brown, Schreiber, and Taylor [BST], see Theorem A in the next section. We mention that Berenstein [B] was the first one to use asymptotic expansions of $\hat{\chi}_{\Omega}$ in connection with the Pompeiu problem.

2. PRELIMINARY REDUCTIONS

We begin this section by recalling the above-mentioned characterization of the Pompeiu property due to Brown, Schreiber, and Taylor [BST].

Theorem A. A bounded domain $\Omega \subset \mathbb{R}^2$ has the Pompeiu property if and only if there exists no $\alpha \in \mathbb{C} \setminus \{0\}$ such that the complexified Fourier transform of the characteristic function of Ω , $\hat{\chi}_{\Omega}$, vanishes identically on

$$M_{\alpha} = \left\{ (\zeta_1, \zeta_2) \in \mathbb{C}^2 | \zeta_1^2 + \zeta_2^2 = \alpha \right\}.$$

It was observed by Berenstein [B] that, when Ω is simply connected, $\alpha \in \mathbb{C}\setminus\{0\}$ in the statement of Theorem A can be replaced by $\alpha > 0$. Furthermore, when $\partial \Omega$ is a rectifiable Jordan curve, then the divergence theorem allows to replace $\hat{\chi}_{\Omega}$ with $\hat{\chi}_{\partial\Omega}$. Note that for $\zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2$

(2.1)
$$\hat{\chi}_{\partial\Omega} = \int_{\partial\Omega} e^{i\langle\zeta,x\rangle} (dx_1 + idx_2)$$

where we have let $\langle \zeta, x \rangle = \zeta_1 x_1 + \zeta_2 x_2$. Changing ζ in $-i\zeta$ in (2.1) we are thus led to study the following oscillatory integral

(2.2)
$$\int_{\partial\Omega} e^{\langle\zeta,x\rangle} (dx_1 + idx_2)$$

for $\zeta \in M_{-\alpha}$, with $\alpha > 0$. We write ζ in the form

$$\zeta = r(\cos\theta,\,\sin\theta) + it(-\sin\theta,\,\cos\theta)\,.$$

The condition $\zeta \in M_{-\alpha}$ becomes

$$(2.3) t^2 = \alpha + r^2.$$

We have

(2.4)
$$\langle \zeta, x \rangle = x_1 (r \cos \theta - it \sin \theta) + x_2 (r \sin \theta + it \cos \theta)$$
$$= r x_1 e^{-i\theta} + ir x_2 e^{-i\theta} - i(t-r) x_1 \sin \theta + i(t-r) x_2 \cos \theta$$
$$= r e^{-i\theta} (x_1 + ix_2) - i(t-r) (x_1 \sin \theta - x_2 \cos \theta) .$$

Since from our assumptions in the introduction $\partial \Omega = h(\partial D)$, where h is univalent in $D = \{w \in \mathbb{C} | |w| < 1\}$, we have for $s \in [0, 2\pi]$

(2.5)
$$x_1(s) = \frac{1}{2}h(e^{is}) + \frac{1}{2}k(e^{is}).$$





FIGURE 2

Here, we have let

(2.6)
$$k(w) = h\left(\frac{1}{\overline{w}}\right).$$

Analogously, we have

(2.7)
$$x_2(s) = \frac{1}{2i}h(e^{is}) - \frac{1}{2i}k(e^{is}).$$

Inserting (2.5), (2.7) in (2.4) we obtain

$$\langle \zeta, x \rangle = r e^{-i\theta} h - i \frac{(t-r)}{2} [(h+k) \sin \theta + i(h-k) \cos \theta],$$

which, after some easy reductions, gives

(2.8)
$$\langle \zeta, x \rangle = \frac{t+r}{2} e^{-i\theta} h(e^{is}) - \frac{t-r}{2} e^{i\theta} k(e^{is}).$$

Taking (2.8) into account, we see that (up to a factor of i) the integral in (2.2) becomes

(2.9)
$$I(r) = \int_{\partial\Omega} \exp\left[\frac{t+r}{2}e^{-i\theta}w - \frac{t-r}{2}e^{i\theta}\Phi(w)\right] dw$$
$$= e^{i\theta}\int_{\partial\Sigma} \exp\left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w)\right] dw$$

where $\Sigma = e^{-i\theta}\Omega$, $\Psi(w) = e^{i\theta}\Phi(e^{i\theta}w)$. At this point we choose $\theta \in [0, 2\pi]$ in such a way that the straight line $e^{-i\theta}L$, where L is as in the statement of Theorem 1, becomes parallel to the imaginary axis. We let $w_0 = e^{-i\theta}z_0$, $M = e^{-i\theta}L$, where $z_0 \in \Omega$ is as in the assumption of Theorem 1, see Figure 2.

We now have from (2.9) (2.10)

$$e^{-i\theta}I(r) = \left(\int_{\partial E(M;w_0)} + \int_{\partial [\Sigma \setminus E(M;w_0)]}\right) \exp\left[\left(\frac{t+r}{2}\right)w - \frac{t-r}{2}\Psi(w)\right]dw.$$

3. Asymptotic expansion of $\hat{\chi}_{\partial\Omega}$ and the Pompeiu property

The aim of this section is to establish the asymptotic behavior as $r \to \infty$ of the integral in the right-hand side of (2.10). We begin by analyzing that

part of the integral on the set $\partial(\Sigma \setminus E(M; w_0))$. We let $A = \max |\Psi|$ on $\partial(\Sigma \setminus E(M; w_0))$. Then

$$\begin{aligned} \left| \int_{\partial(\Sigma \setminus E(M\,;\,w_0))} & \exp\left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w)\right] \, dw \right| \\ & \leq \int_{\partial(\Sigma \setminus E(M\,;\,w_0))} \exp\left[\frac{t+r}{2}\Re e \, w + \frac{t-r}{2}A\right] \, ds \, ds \end{aligned}$$

We now choose $\beta > 0$ such that on the set $\partial(\Sigma \setminus E(M; w_0))$ we have (see Figure 2)

$$\Re e \ w \leq \Re e \ w_0 - \beta \, .$$

Noting that (2.3) gives

(3.1)
$$\frac{t+r}{2} = r\left(1+O\left(\frac{1}{r^2}\right)\right), \quad t-r = \frac{\alpha}{2r}\left(1+O\left(\frac{1}{r}\right)\right)$$

as $r \to \infty$, it follows that on the set $\partial(\Sigma \setminus E(M; w_0))$ we have uniformly as $r \to \infty$

$$\frac{t+r}{2}\Re e \ w + \frac{t-r}{2}A \le \frac{t+r}{2}(\Re e \ w_0 - \beta) + \frac{t-r}{2}A = r(\Re e \ w_0 - \beta)(1+o(1)).$$

From this we derive the estimate for $r \to \infty$

(3.2)
$$\left| \int_{\partial(\Sigma \setminus E(M;w_0))} \exp\left[\frac{t+r}{2} w - \frac{t-r}{2} (w) \right] dw \right| \leq C \exp\left[r \left(\Re e \ w_0 - \frac{\beta}{2} \right) \right].$$

We will now analyze the first integral in the right-hand side of (2.10). We have for $\delta > 0$ small by Cauchy's theorem (3.3)

$$\int_{\partial E(M;w_0)} \exp\left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w)\right] dw$$

$$= \exp\left(\frac{t+r}{2}w_0\right) \int_{|w-w_0|=\delta} \exp\left[\frac{t+r}{2}(w-w_0) - \frac{t-r}{2}\Psi(w)\right] dw$$

$$= \exp\left(\frac{t+r}{2}w_0\right) \int_{|w|=\delta} \exp\left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0+w)\right] dw.$$

By the assumptions on Φ in Theorem 1, there exists $m \in \mathbb{N}$ such that

(3.4)
$$\Psi(w_0+w) = \sum_{k=-m}^{\infty} a_k w^k$$

for $|w| \leq \delta$, with $a_{-m} \neq 0$. We now distinguish two cases.

First case. $m \geq 2$.

Using (2.3) we can write

$$\frac{t-r}{2} = \frac{\alpha}{2(t+r)}.$$

By (3.4), (3.5) we have on the circle $\{w \in \delta e^{i\tau} | -\pi \le \tau \le \pi\}$

(3.6)
$$\frac{\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0 + w)}{= \frac{t+r}{2}\delta e^{i\tau} - \frac{\alpha a_{-m}}{2(t+r)}\delta^{-m}e^{-im\tau} - \frac{\alpha}{2(t+r)}\sum_{k=-m+1}a_k\delta^k e^{ik\tau}.$$

We now choose

$$\delta = \left(\frac{t+r}{2}\right)^{-2/(m+1)}$$

Then, (3.6) becomes as $r \to \infty$

$$\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0+w) = \left(\frac{t+r}{2}\right)^{(m-1)/(m+1)} \left[e^{i\tau} - \frac{\alpha a_{-m}}{4}e^{-im\tau} + o(1)\right].$$

From the first equality in (3.1) we conclude that

$$(3.7) \quad \frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0+w) = r^{(m-1)/(m+1)}q(r)\left\{e^{i\tau} - \frac{\alpha a_{-m}e^{-im\tau}}{4} + o(1)\right\}$$

with $q(r) \to 1$ as $r \to \infty$, uniformly on the circle $\{w = \delta e^{i\tau} | -\pi \le \tau \le \pi\}$. Taking (3.7) into account, we obtain for (3.3) with some $p(r) \to 1$ as $r \to \infty$ (3.8)

$$\int_{\partial E(M;w_0)} \exp\left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w)\right] dw = ir^{-2/(m+1)}p(r)\exp\left(\frac{t+r}{2}w_0\right)$$
$$\cdot \int_{-\pi}^{\pi} \exp\left\{r^{(m-1)/(m+1)}q(r)\left[e^{i\tau} - \frac{\alpha a_{-m}}{4}e^{-im\tau} + o(1)\right]\right\}e^{i\tau} d\tau.$$

At this point we observe that the integral on the right-hand side of (3.8) is of the type studied in the paper [GS2]. By virtue of the work done in [GS2] we can conclude that the asymptotic behavior, as $r \to \infty$, of the above-mentioned integral is as follows

(3.9)
$$\int_{-\pi}^{\pi} \exp\left\{r^{(m-1)/(m+1)}q(r)\left[e^{i\tau}-\frac{\alpha a_{-m}}{4}e^{-im\tau}+o(1)\right]\right\}e^{i\tau}\,d\tau$$
$$=r^{-(m-1)/2(m+1)}A(r)\exp[r^{(m-1)/(m+1)}B(r)],$$

where, having let $\varphi(\tau) = e^{i\tau} - \frac{\alpha a_{-m}}{4} e^{-im\tau}$ for $\tau \in \mathbb{C}$, one has for $r \to \infty$

$$A(r)
ightarrow rac{e^{i au_0}}{\sqrt{2arphi''(au_0)}} = A_0
eq 0 , \quad B(r)
ightarrow arphi(au_0) \, .$$

Here, τ_0 is a suitable simple critical point of the function φ . Inserting (3.9) in (3.8) and recalling (3.1), we obtain

(3.10)
$$\int_{\partial E(M;w_0)} \exp\left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w)\right] dw$$
$$= r^{-(m+3)/2(m+1)}A_1(r)\exp[rw_0 + r^{(m-1)/(m+1)}B(r)],$$

where $A_1(r) \to iA_0$, as $r \to \infty$.

Using (3.2), (3.10) in (2.10) we finally conclude for $r \to \infty$

(3.11)
$$e^{-i\theta}I(r) = r^{-(m+3)/2(m+1)}A_1(r) \exp[rw_0 + r^{(m-1)/(m+1)}B(r)] \\ \cdot \left\{ 1 + O\left(r^{(m+3)/2(m+1)} \exp\left[-\frac{\beta}{2}r + Cr^{(m-1)/(m+1)}\right]\right) \right\},$$

for some number C > 0. Observing now that $0 < \frac{m-1}{m+1} < 1$, we infer that

$$O\left(r^{(m+3)/2(m+1)}\exp\left[-\frac{\beta}{2}r + Cr^{(m-1)/(m+1)}\right]\right) = o(1)$$

as $r \to \infty$. In conclusion, we obtain from (3.11)

(3.12)
$$e^{-i\theta}I(r) = r^{-(m+3)/2(m+1)}A_2(r)\exp[rw_0 + r^{(m-1)/(m+1)}B(r)],$$

where $A_2(r) \to iA_0$ as $r \to \infty$.

To conclude the study of the asymptotic behavior of the integral $e^{-i\theta}I(r)$ in (2.10) we need to analyze the case in which Ψ has as simple pole in w_0 , i.e., the case in which m = 1 in (3.4).

Second case. m = 1.

We consider again the integral on the circle $\{w \in \delta e^{i\tau} | -\pi \le \tau \le \pi\}$ in the right-hand side of (3.3). By the assumptions in Theorem 1, the function Ψ is not a Möbius transformation. If $w = \sigma e^{i\tau}$, with $|\sigma| = \delta$, then we have from (3.5)

(3.13)
$$\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0 + w) = \frac{1}{2}\left[(t+r)w - \frac{\alpha}{t+r}\Psi(w_0 + w)\right]$$
$$= \frac{1}{2}\left\{(t+r)\sigma e^{i\tau} - \frac{\alpha}{t+r}\left[\frac{a_{-1}}{\sigma}e^{-i\tau} + a_0 + \sum_{k=q}^{\infty}a_k\sigma^k e^{ik\tau}\right]\right\}$$

for some $q \in \mathbb{N}$, with $a_q \neq 0$. At this point we choose

(3.14)
$$\sigma = \frac{i\sqrt{\alpha a_{-1}}}{t+r}.$$

It follows from (3.13), (3.14) that for $w = \sigma e^{i\tau}$ one has

(3.15)
$$\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0+w) = -\frac{\alpha a_0}{t+r} + i\sqrt{\alpha a_{-1}}\cos\tau + \frac{C}{(t+r)^{q+1}}e^{iq\tau} + O\left(\frac{1}{(t+r)^{q+2}}\right),$$

for some $C \neq 0$. We conclude (3.16)

$$\int_{|w|=\delta} \exp\left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0+w)\right] dw$$
$$= i\sigma\varepsilon^{-\alpha a_0/(t+r)} \int_{-\pi}^{\pi} e^{i\sqrt{\alpha a_{-1}}\cos\tau} \left\{1 + \frac{C}{(t+r)^{q+1}}e^{iq\tau} + O\left(\frac{1}{(t+r)^{q+2}}\right)\right\} e^{i\tau} d\tau.$$

We now recall the integral representation of the Bessel function J_n (see [L])

$$J_n(z)=\frac{i^{-n}}{2\pi}\int_{-\pi}^{\pi}e^{iz\cos\tau}e^{in\tau}\,d\tau\,,\qquad n\in\mathbb{Z}\,.$$

Using this we can rewrite (3.16) as follows (3.17)

$$\int_{|w|=\delta} \exp\left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0+w)\right] dw$$

= $-\frac{\sqrt{\alpha a_{-1}}}{t+r}e^{-\alpha a_0/(t+r)}\left\{2\pi i J_1(\sqrt{\alpha a_{-1}}) + \frac{2\pi C i^q J_q(\sqrt{\alpha a_{-1}})}{(t+r)^{q+1}} + O\left(\frac{1}{(t+r)^{q+2}}\right)\right\}.$

Since, from (3.1), t + r(1 + o(1)) as $r \to \infty$, and by a theorem of Siegel, J_1 and J_q have no common zeros (see [W, p. 485]). (3.17) implies

(3.18)
$$\int_{|w|=\delta} \exp\left[\frac{t+r}{2}w - \frac{t-r}{2}\Psi(w_0+w)\right] dw$$
$$= \frac{E_1(r)}{r} \left[J_1(\sqrt{\alpha a_{-1}}) + \frac{E_2}{r^{q+1}}J_q(\sqrt{\alpha a_{-1}})\right],$$

where $E_1(r) \rightarrow E_0 \neq 0$ as $r \rightarrow \infty$, and $E_2 \neq 0$. From (2.10), (3.2) and (3.18) we finally obtain

(3.19)
$$\varepsilon^{-i\theta}I(r) = \varepsilon^{rw_0}\frac{E_3(r)}{r}\left[J_1(\sqrt{\alpha q_{-1}}) + \frac{E_2}{r^{q+1}}J_q(\sqrt{\alpha a_{-1}})\right]$$

with $E_3(r) \to E_0$ as $r \to \infty$ (of course, in this estimate we have used again (3.1)).

We are now ready to conclude the proof of Theorem 1. We recall that from the reductions in §2, the oscillatory integral $\hat{\chi}_{\partial\Omega}(\zeta)$, with ζ moving out to infinity along a special path of $M_{-\alpha}$, was shown to equal $e^{-i\theta}I(r)$ in (2.10) (up to a factor of i).

From (3.12), (3.19) we see that, under the assumptions in Theorem 1, the latter cannot vanish identically on $M_{-\alpha}$. From Theorem A we conclude that Ω has the Pompeiu property. \Box

4. Proofs of Corollaries 2 and 3

The proof of Corollary 2 follows immediately from Theorem 1 by observing that if Φ is a Möbius transformation, then so is h. Moreover, if h has at least one pole, then Φ has at least one pole and at most one essential singularity (w = 0).

As for the proof of Corollary 3 we observe that if $\Omega = h(D)$ is convex, then by Study's theorem [S, Theorem 2.4] so is $h(D_r)$ for $0 < r \le 1$, where $D_r = \{z \in \mathbb{C} | |z| \le r\}$. Set $S = \{x \in \mathbb{C} | |z| < R\}$ with R > 1, and denote $S^{-1} = \{\frac{1}{2} | z \in S\}$. Assume that h is holomorphic in S with a pole z_0 on ∂S . Then Φ is holomorphic on $h(S^{-1})$ and has a pole in $h(\frac{1}{z_0})$. Since $h(S^{-1})$ is convex we are in a position to apply Theorem 1, see Figure 3.

We close this paper with an example of a one-parameter family of domains which fall within the scope of Theorem 1, but are not included in any previous result on the Pompeiu problem.

Example. Consider for $0 < \lambda < 2$ the map $h_{\lambda} : D \to \mathbb{C}$ given by $h_{\lambda}(x) = \frac{e^{\lambda}}{\lambda_{z}-2}$. In Figure 4, we represent $\Omega_{\lambda} = h_{\lambda}(D)$ for some values of λ . There exists $\lambda_{0} \in (0, 2)$ such that for $0 < \lambda < \lambda_{0}$ the domain Ω_{λ} is convex. Furthermore, one verifies that for $\lambda \in (0, \lambda_{0})$

$$\min \operatorname{diam} \Omega_{\lambda} > \frac{1}{2} \max \operatorname{diam} \Omega_{\lambda}$$

so that the result in [BK] cannot be invoked.





FIGURE 3



FIGURE 4

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References

^{GS2} New Results on the Pompeiu Problem
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