Mathematicum

Electrostatic characterization of spheres

Osvaldo Mendez and Wolfgang Reichel

Dedicated to the memory of Eugene B. Fabes
(Communicated by Giorgio Talenti)

Abstract. We characterize the sphere in $\mathbb{R}^n$, $n \geq 2$ as the unique member in the class of surfaces of convex bodies with the property that, in the absence of exterior fields, electric charges distribute homogeneously (i.e., with constant density) on the surface.

1991 Mathematics Subject Classification: 31B15, 35J25; 52A20.

1 Introduction and main result

Consider a bounded body $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$, whose surface $\partial\Omega$ consists of metal. We put a charge distribution $\rho : \partial\Omega \to [0, \infty)$ with total charge $\int_{\partial\Omega} \rho \, d\sigma$ onto the surface. The electric field of this charge distribution is described by the single-layer potential

$$S(\rho)(X) = \begin{cases} \dfrac{1}{(n-2)\omega_n} \displaystyle\int_{\partial\Omega} \dfrac{\rho(Q)}{|X-Q|^{n-2}} \, d\sigma(Q) & n \geq 3, \\[2ex] \dfrac{-1}{2\pi} \displaystyle\int_{\partial\Omega} \rho(Q) \log|X-Q| \, d\sigma(Q) & n = 2, \end{cases}$$

where $d\sigma$ is the $(n-1)$-dimensional Hausdorff-measure in $\mathbb{R}^n$ and ω_n is the surface-area of the unit sphere in $\mathbb{R}^n$. With N we denote the exterior unit normal on $\partial\Omega$, with $+$ the exterior and with $-$ the interior side of $\partial\Omega$. If Ω is a Lipschitz-domain and $\rho \in L^p(\partial\Omega)$, $1 < p < \infty$, then the potential energy is expressed as follows

$$E(\rho) = \begin{cases} \dfrac{1}{(n-2)\omega_n} \displaystyle\int_{\partial\Omega} \int_{\partial\Omega} \dfrac{\rho(X)\rho(Q)}{|X-Q|^{n-2}} \, d\sigma(X) \, d\sigma(Q) & n \geq 3, \\[2ex] \dfrac{-1}{2\pi} \displaystyle\int_{\partial\Omega} \int_{\partial\Omega} \rho(X)\rho(Q) \log|X-Q| \, d\sigma(X) \, d\sigma(Q) & n = 2. \end{cases}$$

For $n \geq 3$ we can use almost everywhere on the boundary the jump condition

$(\partial S/\partial N)^+ - (\partial S/\partial N)^- = -\rho$, cf. Verchota [16], to write the energy as $E(\rho) = \int_{\mathbb{R}^n} |\nabla S(\rho)|^2 \, dx$.

In the absence of any exterior field the charges distribute on $\partial\Omega$ according to a minimal-energy law

$$\min E(\rho) \text{ subject to } \int_{\partial\Omega} \rho \, d\sigma = C = \text{const.}, \quad \rho \geq 0.$$

The minimizer ρ^* is called the *equilibrium distribution.*

In a more general framework, where $\rho \, d\sigma$ is replaced by a probability measure $d\mu$ on $\partial\Omega$, the existence and uniqueness of such minimizing measures $d\mu^*$ was shown by J. Wermer [17].

In the framework of charge distributions $\rho \in L^p(\partial\Omega)$, $1 < p < \infty$, on Lipschitz domains one can follow the arguments in Wermer [17], Sections 6–10, to show that the existence of a minimizer is equivalent to finding a distribution ρ^* such that $S(\rho^*) = \text{const.}$ on $\partial\Omega$. Physically this means, that the equilibrium is characterized by the fact that there is no potential difference between any two points on $\partial\Omega$, or, in other words, any potential difference is leveled out by an electric current. Hence, the equilibrium distribution can be found by solving the eigenvalue problem

$$(1) \qquad \mathscr{K}\rho^* + \tfrac{1}{2}\rho^* = 0 \quad \text{on } \partial\Omega,$$

where $(\mathscr{K}\rho)(X) = \text{p.v.} \frac{1}{\omega_n} \int_{\partial\Omega} \frac{(Q-X)\cdot N(X)}{|Q-X|^n} \rho(Q) \, d\sigma(Q)$. This is true, because following [16] the jump-condition $(\partial S(\rho)/\partial N)^{\pm} = \mathscr{K}\rho \mp \frac{1}{2}\rho$ on $\partial\Omega$ shows that (1) is equivalent to $(\partial S(\rho^*)/\partial N)^- = 0$ on $\partial\Omega$, i.e., $S(\rho^*) \equiv \text{const.}$ in $\bar{\Omega}$.

If Ω is a ball, then it is easy to determine the equilibrium distribution as the constant distribution.[1] The following conjecture has been formulated by P. Gruber (Univ. Wien): *The equilibrium distribution is constant on the boundary of a domain if and only if the domain is a ball.*

Our main result is the following

Theorem 1. *For $n = 2$ the conjecture is true if Ω is in the class of bounded Lipschitz domains. For $n \geq 3$ the conjecture is true if Ω is in the class of bounded convex domains.*

The result is also known to be true in the class of $C^{2,\alpha}$-domains, as proved by Reichel [12], [13]. Previous results of Martensen [6] covered the class of piecewise smooth domains in $\mathbb{R}^2$, and the tools developed by Philippin [9] and Payne, Philippin [8] covered the case of star-shaped $C^{2,\alpha}$-domains in $\mathbb{R}^n$ for $n \geq 3$. Payne and Philippin utilize the isoperimetric inequality, a Pohožaev-identity and maximum principles.

[1] If ρ^* is an equilibrium distribution, then for any rotation matrix $R^T R = \text{Id}$ the distribution $\rho^*(Rx)$ is also an equilibrium distribution; and hence they are equal.

For the proof of our result we combine their method with convex analysis and a careful potential theoretic analysis of $S(\rho^*)$.

We point out, that for $n \geq 3$ the conjecture is now true for the union of the classes of $C^{2,\alpha}$-domains and convex domains, but remains open for bigger classes of domains.

Finally we note, that Henrot et al. [3] have obtained an electrostatic characterization of a ring-shaped conductor, where the inner surface is a sphere and charges distribute uniformly on the convex outer surface.

The paper is structured as follows: In Section 2 we prove an integral identity for the equilibrium potential. This already provides the proof of Theorem 1 in dimension $n = 2$. In Section 3 we introduce a function Φ as a combination of $S(\rho^*)$ and $\nabla S(\rho^*)$ which satisfies a maximum principle. Ultimately we reduce the proof that Ω is a ball to showing that Φ is constant. This requires a better understanding of the derivatives of $S(\rho^*)$. In Section 4 we establish that $\partial\Omega$ is C^1 (here we use heavily the convexity of Ω) and that $S(\rho^*)$ extends as a C^1-function onto $\partial\Omega$. In Section 5 we investigate the second-order derivatives of $S(\rho^*)$. This knowledge of the first and second order behavior of $S(\rho^*)$ near $\partial\Omega$ enables us to complete the proof of Theorem 1 in Section 6. In the Appendix we give the proofs of some technical lemmas: estimates for the singular kernels of layer-potentials and an extension of a formula of Minkowski.

2 The Pohožaev-Rellich identity

From now on we assume that $\rho^* \equiv 1$ is the equilibrium distribution for the domain Ω, which is supposed to satisfy the hypotheses of Theorem 1. Then, the single-layer potential $S(1)$ satisfies the following overdetermined boundary-value problem

(2) $$\Delta u = 0 \text{ on } \mathbb{R}^n \setminus \partial\Omega,$$

(3) $$u = \text{const.} = c \text{ on } \partial\Omega \text{ (and hence in } \bar{\Omega}),$$

(4) $$\left(\frac{\partial u}{\partial N}\right)_e = -1, \quad \left(\frac{\partial u}{\partial T}\right)_e = 0 \quad \text{a.e. on } \partial\Omega,$$

where, for $p \in \partial\Omega$ with the normal $N(p)$, we denote by $(\frac{\partial S(1)}{\partial N}(p))_{e/i}$ the exterior/interior normal component of the gradient of $S(1)$ at p, i.e., $(\frac{\partial S(1)}{\partial N}(p))_{e/i} = \lim_{x \to p} \nabla u(x) \cdot N(p)$ and x varies in a nontangential exterior/interior cone at p. Likewise, for a direction T orthogonal to N, $(\frac{\partial S(1)}{\partial T}(p))_{e/i}$ is an exterior/interior tangential component of the gradient. In general, cf. Verchota [16]), we have

$$\left(\frac{\partial S(1)}{\partial N}\right)_e - \left(\frac{\partial S(1)}{\partial N}\right)_i = -1, \quad \left(\frac{\partial S(1)}{\partial T}\right)_e - \left(\frac{\partial S(1)}{\partial T}\right)_i = 0 \quad \text{a.e. on } \partial\Omega.$$

By vol Ω, $|\partial\Omega|$ we denote for $n \geq 3$ the volume and the surface area of Ω and for $n = 2$ the area and the perimeter of Ω. The asymptotic behavior of $S(1)$ at ∞ is given by

$$n=2: \quad S(1)(X) \approx \frac{-|\partial\Omega|}{2\pi}\log r, \qquad \nabla S(1)(X) \approx \frac{-|\partial\Omega|}{2\pi}\frac{X}{r^2},$$

$$n \geq 3: \quad S(1)(X) \approx \frac{|\partial\Omega|}{(n-2)\omega_n}\frac{1}{r^{n-2}}, \quad \nabla S(1)(X) \approx \frac{-|\partial\Omega|}{\omega_n}\frac{X}{r^n},$$

where "$\approx$" means that the (componentwise) quotient tends to 1 as $r = |X| \to \infty$.

The following Pohožaev-Rellich identity for harmonic functions h can be verified by straight-forward differentiation (cf. Rellich [14], Pohožaev [10], and also Pucci, Serrin [11])

$$\text{(5)} \qquad \operatorname{div}\left(X\frac{|\nabla h|^2}{2} - (X \cdot \nabla h)\nabla h\right) = \frac{n-2}{2}|\nabla h|^2.$$

For a smooth domain $\Omega' \supset\supset \Omega$ we will use this identity for $h = S(1)$ and integrate it in the domain $B_R(0)\backslash\Omega'$ and let R tend to ∞. We employ the divergence theorem and get from (5)

$$\int_{\partial\Omega'} + \int_{\partial B_R(0)} \left[X\frac{|\nabla S(1)|^2}{2} - (X \cdot \nabla S(1))\nabla S(1)\right] \cdot N\, d\sigma$$

$$= \frac{n-2}{2}\int_{B_R(0)\backslash\Omega'} |\nabla S(1)|^2\, dx$$

$$= \frac{n-2}{2}\left(\int_{\partial\Omega'} S(1)\nabla S(1) \cdot (-N)\, d\sigma + \int_{\partial B_R(0)} S(1)\nabla S(1) \cdot \frac{X}{R}\, d\sigma\right).$$

Using the asymptotics at ∞ we find that the boundary term coming from $\partial B_R(0)$ vanishes in the limit for $n \geq 3$, but gives a contribution for $n = 2$:

$$n=2: \quad \int_{\partial\Omega'}\left[X\frac{|\nabla S(1)|^2}{2} - (X \cdot \nabla S(1))\nabla S(1)\right] \cdot N\, d\sigma - \frac{|\partial\Omega|^2}{4\pi} = 0,$$

$$n \geq 3: \quad \int_{\partial\Omega'}\left[X\frac{|\nabla S(1)|^2}{2} - (X \cdot \nabla S(1))\nabla S(1)\right] \cdot N\, d\sigma$$

$$= \frac{2-n}{2}\int_{\partial\Omega'} S(1)\frac{\partial S(1)}{\partial N}\, d\sigma.$$

Choosing an approximating sequence of domains $\Omega_j' \searrow \Omega$ like in Verchota [16], Theorem 1.12, and using (4), we find for almost every $P \in \partial\Omega$ that $[X_j \frac{|\nabla S(1)|^2}{2} - (X_j \cdot \nabla S(1))\nabla S(1)] \cdot N_j(X_j)$ tends to $\frac{-1}{2} P \cdot N(P)$ in the nontangential exterior limit $X_j \to P$ as $j \to \infty$. Taking the limit $j \to \infty$ we get:

$$(6) \qquad n = 2: \quad \mathrm{vol}(\Omega) - \frac{|\partial\Omega|^2}{4\pi} = 0,$$

$$(7) \qquad n \geq 3: \quad n\,\mathrm{vol}(\Omega) = (2-n)\int_{\partial\Omega} S(1)\frac{\partial S(1)}{\partial N}\,d\sigma = (n-2)c|\partial\Omega|,$$

where c is the constant value of $S(1)$ on $\partial\Omega$. In case $n = 2$ the proof of Theorem 1 is finished, because (6) is the equality case of the isoperimetric inequality, and hence Ω must be a disk. In case $n \geq 3$, further work is needed.

3 An isoperimetric relation

From now on we assume $n \geq 3$. Payne and Philippin's approach [8], [9] was based on the fact that the following combination of $S(1)$ and $\nabla S(1)$ satisfies a maximum principle:

$$\Phi(X) = |\nabla S(1)(X)|^2 S(1)(X)^{(2-2n)/(n-2)}, \quad X \in \mathbb{R}^n \backslash \bar{\Omega}.$$

In [8], they proved that there exists a continuous vector-field b on $\mathbb{R}^n \backslash (\bar{\Omega} \cup E)$ such that

$$\Delta\Phi + b \cdot \nabla\Phi \geq 0 \quad \text{in } \mathbb{R}^n \backslash (\bar{\Omega} \cup E),$$

where $E = \{X \in \mathbb{R}^n \backslash \bar{\Omega} \mid S(1)(X) = 0 \text{ or } \nabla S(1)(X) = 0\}$. Since $S(1)$ is positive, the set E consists only of critical points of $S(1)$. The function Φ is not an arbitrary combination of $S(1)$ and $\nabla S(1)$. The exponent $(2-2n)/(n-2)$ is chosen such that if Ω is a ball, then $\Phi \equiv$ const. because $S(1)(X) = \text{const.}\,|X|^{2-n}$. Our ultimate goal is to prove $\Phi \equiv$ const. and then to deduce (in a yet non-trivial step) that Ω has to be a ball.

Using the asymptotics of $S(1)$, we first calculate the value of Φ at ∞:

$$\Phi(X) \to (n-2)^{(2n-2)/(n-2)}\left(\frac{\omega_n}{|\partial\Omega|}\right)^{2/(n-2)} \quad \text{as } |X| \to \infty.$$

In the next section, we shall prove that Φ extends continuously onto $\partial\Omega$ and that $\Phi|_{\partial\Omega} = c^{(2-2n)/(n-2)}$, where c is the constant value of $S(1)$ on $\partial\Omega$. If we assume this result, then we find by the Pohožaev-Rellich identity (7) that

$$\Phi|_{\partial\Omega} = \left(\frac{n-2}{n}\right)^{(2n-2)/(n-2)}\left(\frac{|\partial\Omega|}{\mathrm{vol}\,\Omega}\right)^{(2n-2)/(n-2)}.$$

Recalling the isoperimetric inequality $|\partial\Omega|^{n/(n-1)} \geq \omega_n^{1/(n-1)} n \operatorname{vol}\Omega$, we find that $\Phi|_{\partial\Omega} \geq \Phi_\infty$, where equality holds if and only if Ω is a ball. Therefore, the proof of Theorem 1 is reduced to showing that $\Phi \equiv \text{const}$.

Using the maximum principle, we find that Φ attains its maximum over $\mathbb{R}^n\backslash\Omega$ either on $\partial\Omega$, at ∞ or on E. Clearly, Φ cannot attain its maximum on E, since this would imply $\Phi \equiv 0$. Moreover we have just seen that $\Phi|_{\partial\Omega} \geq \Phi|_\infty$ and therefore

$$\max_{\mathbb{R}^n\backslash\Omega} \Phi = \Phi|_{\partial\Omega}.$$

If we assume for contradiction that $\Phi \not\equiv \text{const.}$, then the Hopf boundary-lemma applies to $P \in \partial\Omega$ since the convex domain Ω satisfies an exterior ball condition, and states

$$\liminf_{t\to 0+} \frac{\Phi(P + tN(P)) - \Phi(P)}{t} < 0 \quad \text{for all } P \in \partial\Omega. \tag{8}$$

This will lead to a contradiction in Section 6.

4 Continuity of the gradient of the single-layer

The purpose of this section is to show that the function Φ extends continuously onto $\partial\Omega$. This is equivalent to showing that $\nabla S(1)$ extends continuously from $\mathbb{R}^n\backslash\bar{\Omega}$ to $\partial\Omega$. We shall achieve this in three steps:

4.1 Boundedness of $\nabla S(1)$,
4.2 C^1-regularity of $\partial\Omega$,
4.3 Continuity of $\nabla S(1)$.

The regularity gain in the second step is deeply connected with the assumption of convexity of Ω.

4.1 Boundedness of $\nabla S(1)$

A very important role in our estimates is played by the singular integral kernels

$$K(x, y, t; k) = [|x - y|^2 + (\varphi(x) - \varphi(y) + t)^2]^{-k/2}$$

defined for $x, y \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$ and $k \in \mathbb{N}$. Here φ plays the role of a Lipschitz function, whose graph locally describes $\partial\Omega$. The proof of the following two lemmas is carried out in the Appendix.

Lemma 2. *Let $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function with $\varphi(0) = 0$. For $\alpha, \beta \geq 0$ we consider the integrals*

$$I_-(x) = \int_{|y|\le 1} |t|^\alpha |x-y|^\beta (K(x,y,t;k) - K(x,y,-t;k))\,dy,$$

$$I_+(x) = \int_{|y|\le 1} |t|^\alpha |x-y|^\beta (K(x,y,t;k) + K(x,y,-t;k))\,dy.$$

Then we have the following bounds for $t \to 0$ (uniformly for $x \in \mathbb{R}^{n-1}$):

(a) $|I_-(x)| = \|\nabla\varphi\|_\infty O(|t|^{\alpha+\beta+n-k-1})$ *if* $n+\beta < k+2$,

(b) $|I_-(x)| = \|\nabla\varphi\|_\infty O(|t|^{\alpha+1}\log|t|)$ *if* $n+\beta = k+2$,

(c) $|I_-(x)| = \|\nabla\varphi\|_\infty O(|t|^{\alpha+1})$ *if* $n+\beta > k+2$,

(d) $|I_+(x)| = O(|t|^{\alpha+\beta+n-k-1})$ *if* $n+\beta < k+1$,

(e) $|I_+(x)| = O(|t|^\alpha \log|t|)$ *if* $n+\beta = k+1$,

(f) $|I_+(x)| = O(|t|^\alpha)$ *if* $n+\beta > k+1$,

(g) *If* $\varphi(s) = O(|s|^2)$ *for* $s \to 0$ *then* (a)–(c) *hold for* $I_-(0)$ *with* β *replaced by* $\beta+1$ *and* $\|\nabla\varphi\|_\infty$ *replaced by a constant.*

Lemma 3. *If* $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ *is a Lipschitz function with* $\varphi(s) = \frac{1}{2}s^T H s + o(|s|^2)$ *for* $s \to 0$ *then*

(a) $I = \int_{|y|\le 1} t|y|([|y|^2 + (\varphi(y)+t)^2]^{-(n+2)/2} - [|y|^2 + (\frac{1}{2}y^T H y + t)^2]^{-(n+2)/2})\,dy = o(1)$,

(b) $J = \int_{|y|\le 1} t|y|^2([|y|^2 + (\varphi(y)+t)^2]^{-(n+2)/2} - [|y|^2 + t^2]^{-(n+2)/2})\,dy = O(t)$,

(c) $L = \int_{|y|\le 1} t o(|y|^2)[|y|^2 + (\varphi(y)+t)^2]^{-(n+2)/2}\,dy = o(1)$,

where $o(1) \to 0$ *for* $t \to 0$.

From now on, one always has to keep in mind that $S(1) \equiv$ const. in Ω, which means that all derivatives of $S(1)$ vanish inside Ω. However, we state the following propositions and lemmas in this section and in Section 5 for situations, where $S(1)$ may be non-trivial inside Ω (in fact, the results can even be generalized to continuous densities).

Proposition 4. *If* $\nabla S(1) \in L^\infty(\Omega)$ *then* $\nabla S(1) \in L^\infty(\mathbb{R}^n \setminus \bar{\Omega})$.

The proof of this theorem is an immediate consequence of

Lemma 5. *Let* Ω *be a bounded Lipschitz domain,* $P \in \partial\Omega$ *and assume that for a fixed ball* $B(P)$ *of radius* r *around* P *we have*

$$A := \sup_{X \in B(P)\cap\Omega} |\nabla S(1)(X)| < \infty.$$

Then, there exists a positive constant C depending only on A, Ω, P and r such that:

(9) $$\sup_{X \in B(P)\cap(\mathbb{R}^n\setminus\bar{\Omega})} |\nabla S(1)(X)| \le C.$$

For the proof we need the notion of "local boundary reflection" across $\partial\Omega$.

Definition 6. Let $P \in \partial\Omega$ and $B(P)$ be a ball of radius $r > 0$ centered at P such that after a translation and rotation of coordinates,

$$\partial\Omega \cap B(P) = \{(x, \varphi(x)) \mid x \in U(0)\},$$

and

$$\Omega \cap B(P) \subset \{(x, \varphi(x) + t) \mid x \in U(0), t > 0\},$$

where $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function with $\varphi(0) = 0$, and $U(0) \subset \mathbb{R}^{n-1}$ is a neighborhood of 0. For any point $X \in B(P) \cap (\mathbb{R}^N\setminus\Omega)$ with local coordinates $(x, \varphi(x) - t)$, let $\tilde{X}$ be the point whose local coordinates are $(x, \varphi(x) + t)$. We call $\tilde{X}$ the *local boundary reflection* of X across $\partial\Omega$.

Remark. The "local boundary reflection" depends on the choice of the local representation φ of a boundary portion of $\partial\Omega$. In the case that φ is a C^1-function with $\varphi(0) = 0$, $\nabla\varphi(0) = 0$ we call the local coordinate-system a *tangential* coordinate-system. For C^1-domains, such tangential coordinate-systems always exist.

Proof of Lemma 5. We use the notation of Definition 6. The proof of the lemma will follow once we establish the inequality

(10) $$\sup_{X \in B(P)\cap(\mathbb{R}^n\setminus\bar{\Omega})} |\nabla S(1)(\tilde{X}) - \nabla S(1)(X)| \le C$$

for a constant $C > 0$ that depends on A, Ω and $B(P)$ only. Consider an open cover of $\partial\Omega$ of which $B(P)$ is a member and let $(\chi_i)_i$ be a finite partition of unity subordinated to that cover. It is easy to see that (10) will follow if we can bound the expression

$$\sum_j \left| \int_{\partial\Omega} \left(\frac{\tilde{X} - Q}{|\tilde{X} - Q|^n} - \frac{X - Q}{|X - Q|^n} \right) \chi_j(Q)\, d\sigma(Q) \right|$$

from above by a constant depending only on A, Ω and $B(P)$. After expressing this integral in local coordinates, it is clear that the quantity to be controlled is

$$\int_{|y|\le 1} \left(\frac{(x-y, \varphi(x)-\varphi(y)+t)}{[|x-y|^2+(\varphi(x)-\varphi(y)+t)^2]^{n/2}} - \frac{(x-y, \varphi(x)-\varphi(y)-t)}{[|x-y|^2+(\varphi(x)-\varphi(y)-t)^2]^{n/2}} \right) \cdot \sqrt{1+|\nabla\varphi(y)|^2}\, dy$$

This integral can be estimated by a constant $C(\varphi)$ times

$$\int_{|y|\le 1} |x-y|\,|K(x,y,t;n)-K(x,y,-t;n)|+t|K(x,y,t;n)+K(x,y,-t;n)|\,dy,$$

which is bounded by Lemma 2(a) and (d). □

4.2 C^1-regularity of $\partial\Omega$

In this section we show that the convexity of Ω and the boundedness of the gradient of $S(1)$ result in a regularity gain.

Lemma 7. *The domain Ω of Theorem 1 is a C^1-domain.*

Proof. According to Leichtweiß [4], a point $P \in \partial\Omega$ is called a singular point of order $s \in \{1,2,\dots,n\}$, if the intersection of all support hyperplanes through P is an $(n-s)$-dimensional affine space. A singular point of order 1 is called a regular point. If all boundary points of the convex set Ω are regular, then a famous theorem of Alexandroff (cf. Leichtweiß [4]) states that Ω is a C^1-domain. Suppose for contradiction that $0 \in \partial\Omega$ is a singular point of order $s \ge 2$. Let us take two different $(n-1)$-dimensional support hyperplanes H_1, H_2 at 0. Then $B = H_1 \cap H_2$ is an $(n-2)$-dimensional space and we can write

$$H_1 = B \oplus h_1, \quad H_2 = B \oplus h_2$$

for some linearly independent unit vectors h_1, h_2 with $h_1 \cdot h_2 = \cos\gamma$ for some $\gamma \in (0,\pi)$. Moreover, we can assume that Ω is contained in the cone

$$\mathscr{C} = B \oplus \{\lambda_1 h_1 + \lambda_2 h_2, \lambda_1, \lambda_2 > 0\}$$

and that there exists an exterior direction ν to Ω at 0 with $-\nu \in \mathscr{C}$ and $\nu \perp B$. In $\mathscr{C}_{ex} = \mathbb{R}^n \setminus \bar{\mathscr{C}}$, the exterior of $\mathscr{C}$, we shall construct a harmonic function $w > 0$ with $w = 0$ on $\partial\mathscr{C}$. To this end, we choose an Euclidean coordinate system $\xi_1, \dots, \xi_n$ at 0 with $\xi_1, \dots, \xi_{n-2}$-direction in B and $\xi_{n-1} = h_1$ and span $\{\xi_{n-1}, \xi_n\} = \text{span}\,\{h_1, h_2\}$. With polar-coordinates (r,θ) in the ξ_{n-1}, ξ_n-plane the exterior cone is given as $\mathscr{C}_{ex} = \{(\xi_1, \dots, \xi_{n-2}, r, \theta) : \xi_1, \dots, \xi_{n-2} \in B,\ r > 0, \theta \in (0, 2\pi - \gamma)\}$. If we define the function

$$v(\xi_{n-1}, \xi_n) = r^\alpha \sin(\alpha\theta) \quad \text{for all } r > 0 \quad \text{and} \quad \theta \in (0, 2\pi - \gamma)$$

with $\alpha = \frac{\pi}{2\pi - \gamma} \in (0,1)$, then $w(\xi_1, \dots, \xi_n) = v(\xi_{n-1}, \xi_n)$ is the described function. If $\theta_0 \in (0, 2\pi - \gamma)$ is the polar angle of ν in the ξ_{n-1}, ξ_n-plane, then

$$\frac{\partial w}{\partial \nu}(\xi) = (\alpha - 1) r^{\alpha - 1} \sin(\alpha\theta_0) \to -\infty \quad \text{as } r = |\xi| \to 0.$$

Next we choose a large ball $B_R(0)$ such that $S(1) \le c/2$ on $\partial B_R(0)$. Then we choose $\lambda > 0$ so small that $\lambda R^\alpha \le c/2$. This implies $c - \lambda w \ge c/2 \ge S(1)$ on $\partial B_R(0) \cap \mathscr{C}_{ex}$. Moreover $c - \lambda w = c \ge S(1)$ on $\bar{B}_R(0) \cap \partial\mathscr{C}_{ex}$. By the maximum principle we get $c - \lambda w \ge S(1)$ in $B_R(0) \cap \mathscr{C}_{ex}$, and moreover $(c - \lambda w)(0) = c = S(1)(0)$. But this contradicts the fact that $\frac{\partial(c-\lambda w)}{\partial \nu}(0) = +\infty$ and that $\frac{\partial S(1)}{\partial \nu}(0)$ is finite. Hence there is no singular boundary point of order $s \ge 2$, the boundary of Ω is regular and $\partial\Omega \in C^1$.

□

4.3 Continuity of $\nabla S(1)$

Our next aim is to prove

Proposition 8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1-domain. Assume that $S(1)(X)$ is constant in Ω. Then, for every point $P \in \partial\Omega$,*

$$\lim_{X \to P,\, X \in \mathbb{R}^n \setminus \bar{\Omega}} \nabla S(1)(X) = -N(P). \tag{11}$$

This result is an immediate consequence of Lemma 9 and Lemma 10. It shows that $S(1)$ can be extended to $\partial\Omega$ as a C^1-function.

Lemma 9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1-domain and let $T(P)$ be a tangent vector at $P \in \partial\Omega$. Then,*

$$\lim_{X \to P,\, X \in \mathbb{R}^n \setminus \bar{\Omega}} (\nabla S(1)(X) - \nabla S(1)(\tilde{X})) \cdot T(P) = 0,$$

where $\tilde{X}$ is the "local boundary reflection" of X with respect to a tangential coordinate system.

Proof. Fix $\varepsilon > 0$. By the smoothness of $\partial\Omega$, a neighborhood $B(P)$ of P and a local coordinate-system with origin at P can be chosen (in the same notation as in Definition 6) in such a way that $\varphi(0) = \nabla\varphi(0) = 0$, and that $\sup_{y \in U(0)} |\nabla\varphi(y)| \le \varepsilon$. It follows that in the new coordinate-system, the unit vectors $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1, 0) \in \mathbb{R}^n$ are an orthonormal basis for the tangent space of $\partial\Omega$ at P. Hence we may assume that $T(P) = (0, \ldots, 0, 1, 0, \ldots, 0) = e_k$ for some k between 1 and $n - 1$. For X and $\tilde{X}$ as in Definition 6 we have

$$T(P) \cdot (\nabla S(1)(X) - \nabla S(1)(\tilde{X}))$$

$$= \frac{-1}{\omega_n} \left(\int_{\partial\Omega \setminus B(P)} + \int_{B(P) \cap \partial\Omega} \right) T(P) \left(\frac{X - Q}{|X - Q|^n} - \frac{\tilde{X} - Q}{|\tilde{X} - Q|^n} \right) d\sigma(Q).$$

The lemma will follow from the fact that the integrand in the first integral converges uniformly to 0 as $t \to 0$ and that the second integral is less than ε if X is taken close enough to P. To see the latter, we bound the second integral as in the proof of Lemma 5 by a constant that depends on the Lipschitz character of the domain times

$$(12)\qquad \int_{|y|\le 1} |x_k - y_k|(K(x,y,t;n) - K(x,y,-t;n))\,dy.$$

A straight-forward application of Lemma 2(a) shows that (12) is bounded in t by a multiple of $\|\nabla\varphi\|_\infty \le \varepsilon$. This finishes the proof of the lemma. □

Lemma 10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1-domain and let $N(P)$ be the exterior normal at $P \in \partial\Omega$. Then,*

$$\lim_{X\to P,\, X\in\mathbb{R}^n\setminus\bar\Omega} (\nabla S(1)(X) - \nabla S(1)(\tilde X))\cdot N(P) = -1,$$

where $\tilde X$ is the "local boundary reflection" of X with respect to a tangential coordinate system.

Remark. For C^2-domains this lemma is a consequence of the classical jump-conditions for single-layer potentials with continuous density. In the context of C^1-domains, the observation that the jump-conditions are still valid in a weaker sense was previously made by Martensen [7] in dimension $n = 3$. He showed that $\lim_{t\to 0_+}(\nabla S(1)(P + tN(P)) - \nabla S(1)(P - tN(P)))\cdot N(P) = -1$. This is a special case of Lemma 10, since $P - tN(P)$ is the "local boundary reflection" of $P + tN(P)$ with respect to a tangential coordinate-system.

Proof. We consider the expression

$$(13)\qquad \nabla S(1)(X)\cdot N(P) + \mathscr{D}(1)(X),$$

where

$$(14)\qquad \mathscr{D}(1)(X) = \frac{1}{\omega_n}\int_{\partial\Omega} \frac{N(Q)(X-Q)}{|X-Q|^n}\,d\sigma(Q)$$

denotes the double layer potential with density 1 on $\partial\Omega$. Pick a neighborhood $B(P)$ of P. We can write (13) for $X \in (\mathbb{R}^n\setminus\Omega)\cap B(P)$ and for $\tilde X \in \Omega$ defined as in the proof of Lemma 5, and take the difference. This results in

$$\frac{-1}{\omega_n}\int_{\partial\Omega}\left(\frac{X-Q}{|X-Q|^n} - \frac{\tilde X - Q}{|\tilde X - Q|^n}\right)(N(P) - N(Q))\,d\sigma(Q).$$

The lemma will follow if we can show that the previous expression tends to 0 as $X \to P$, $X \in \mathbb{R}^n\setminus\Omega$, since $\mathscr{D}(1)(\tilde X) = -1$ and $\mathscr{D}(1)(X) = 0$. On the set $\partial\Omega\setminus B(P)$ the integrand converges uniformly to 0. Therefore it is again sufficient to consider

$$\int_{\partial\Omega\cap B(P)}\left(\frac{X-Q}{|X-Q|^n}-\frac{\tilde{X}-Q}{|\tilde{X}-Q|^n}\right)(N(P)-N(Q))\,d\sigma(Q). \tag{15}$$

Since the normal on $\partial\Omega$ is a continuous function, and since the integral without the $N(P)-N(Q)$ contribution was already shown to be bounded in Lemma 5, we can make (15) arbitrarily small by choosing the neighborhood $B(P)$ sufficiently small. □

5 The second derivatives of the single-layer

The results in this section concern the second derivatives of $S(1)(X)$:

$$\frac{\partial^2 S(1)(X)}{\partial X_i \partial X_j}=\frac{-1}{\omega_n}\int_{\partial\Omega}\frac{\delta_{ij}\,d\sigma(Q)}{|X-Q|^n}+\frac{n}{\omega_n}\int_{\partial\Omega}\frac{(Q-X)_i(Q-X)_j}{|X-Q|^{n+2}}\,d\sigma(Q). \tag{16}$$

For $P\in\partial\Omega$ we also introduce the "second normal derivative"

$$\frac{\partial^2 S(1)}{\partial N^2(P)}(X)=N(P)^T\left(\frac{\partial^2 S(1)}{\partial X_i\partial X_j}(X)\right)_{ij}N(P).$$

Both expressions are defined for all $X\in\mathbb{R}^n\setminus\partial\Omega$. Our goal in this section is to prove

Proposition 11. *Let Ω be a bounded, convex C^1-domain in $\mathbb{R}^n$. Then for (surface measure) almost every $P\in\partial\Omega$, there exists a positive constant $C(\Omega,P)$ such that for $X^+=P+tN(P)\in\mathbb{R}^n\setminus\Omega$, $X^-=P-tN(P)\in\Omega$ and $0<t<t_0(P)$ we have*

$$\left|\frac{\partial^2 S(1)}{\partial X_i\partial X_j}(X^+)-\frac{\partial^2 S(1)}{\partial X_i\partial X_j}(X^-)\right|\le C. \tag{17}$$

Moreover, for almost all $P\in\partial\Omega$ the following holds

$$\lim_{t\to 0^+}\left(\frac{\partial^2 S(1)}{\partial N^2(P)}(X^+)-\frac{\partial^2 S(1)}{\partial N^2(P)}(X^-)\right)=(n-1)\kappa(P), \tag{18}$$

where $\kappa(P)\ge 0$ is the mean-curvature of $\partial\Omega$ at P.[2]

[2] For a convex domain Ω, the differential geometric mean-curvature κ exists almost everywhere and belongs to $L^1(\partial\Omega)$, cf. Appendix.

Unlike in Section 4 we cannot formulate Proposition 11 in terms of X and its local boundary reflection $\tilde{X}$. Instead we have to restrict the approach-direction of X to P to normal directions only.

Before we start to prove Proposition 11, we need the following Lemma which provides a suitable tangential coordinate-system for almost every point on $\partial\Omega$.

Lemma 12. *Let Ω be a bounded, convex C^1-domain in $\mathbb{R}^n$. For almost every $P \in \partial\Omega$ there exists a ball $B(P) \subset \mathbb{R}^n$, a neighborhood $U(0) \subset \mathbb{R}^{n-1}$ and a convex C^1-function $\varphi : U(0) \to [0, \infty)$ such that after a translation and rotation*

$$\partial\Omega \cap B(P) = \{(x, \varphi(x)) \mid x \in U(0)\},$$

$$\Omega \cap B(P) \subset \{(x, \varphi(x) + t) \mid x \in U(0), t > 0\}.$$

Moreover, $\varphi(0) = 0$, $\nabla\varphi(0) = 0$ and φ is twice differentiable at 0 in the sense that $\varphi(x) = \frac{1}{2}x^T Hx + o(x^2)$ as $x \to 0$ for a symmetric, positive semi-definite matrix $H = H(P)$.

The existence of a tangential coordinate frame holds for any point of a C^1-domain. For almost every $P \in \partial\Omega$, the second-order differentiability of φ at 0 follows from Alexandroff's Theorem, [1].

Proof of Proposition 11. Fix $\delta > 0$, and let $B(P)$ be a ball of radius δ around P. For any $P \in \partial\Omega$, the difference (17) is a sum of an integral on $B(P) \cap \partial\Omega$ (first type) and an integral on $\partial\Omega \backslash B(P)$ (second type). It is easy to see that the integrals of the second type will be bounded in absolute value by a constant C which depends on the Lipschitz character of Ω and P. It remains to analyze the integrals of the first type, which we do by localizing. For almost every $P \in \partial\Omega$, let $B(P)$ and φ be as in Lemma 12. Note that in these local coordinates, $|X^{\pm} - Q|^k = (|y|^2 + (\varphi(y) \pm t)^2)^{k/2} = K(0, y, \mp t; k)$. Therefore the integrals of the first type in the left hand side of (17) are of the form $I_{ij} + J_{ij}$ with

$$I_{ij} = \frac{-1}{\omega_n} \int_{|y| \leq 1} \delta_{ij}(K(0, y, -t; n) - K(0, y, t; n))\sqrt{1 + |\nabla\varphi(y)|^2}\, dy$$

and one of the following integrals:
If $1 \leq i, j \leq n - 1$,

$$J_{ij} = \frac{n}{\omega_n} \int_{|y| \leq 1} y_i y_j (K(0, y, -t; n + 2) - K(0, y, t; n + 2))\sqrt{1 + |\nabla\varphi(y)|^2}\, dy.$$

In the case $1 \leq i \leq n - 1$, $j = n$, the integral is:

$$J_{in} = \frac{n}{\omega_n}\int_{|y|\le 1} y_i((\varphi(y)+t)K(0,y,-t;n+2) - (\varphi(y)-t)K(0,y,t;n+2)) \cdot \sqrt{1+|\nabla\varphi(y)|^2}\,dy$$

$$= \frac{n}{\omega_n}\int_{|y|\le 1} y_i\varphi(y)(K(0,y,-t;n+2) - K(0,y,t;n+2))\sqrt{1+|\nabla\varphi(y)|^2}\,dy$$

$$+ \frac{n}{\omega_n}\int_{|y|\le 1} y_i t(K(0,y,-t;n+2) + K(0,y,t;n+2))\sqrt{1+|\nabla\varphi(y)|^2}\,dy.$$

Finally, when $i = j = n$:

$$J_{nn} = \frac{n}{\omega_n}\int_{|y|\le 1} ((\varphi(y)+t)^2K(0,y,-t;n+2) - (\varphi(y)-t)^2K(0,y,t;n+2)) \cdot \sqrt{1+|\nabla\varphi(y)|^2}\,dy$$

$$= \frac{n}{\omega_n}\int_{|y|\le 1} (\varphi(y)^2+t^2)(K(0,y,-t;n+2) - K(0,y,t;n+2))\sqrt{1+|\nabla\varphi(y)|^2}\,dy$$

$$+ \frac{n}{\omega_n}\int_{|y|\le 1} 2\varphi(y)t(K(0,y,-t;n+2) + K(0,y,t;n+2))\sqrt{1+|\nabla\varphi(y)|^2}\,dy.$$

By straight-forward application of Lemma 2(g) we find the boundedness of I_{ij} $(i,j = 1,\ldots,n)$, J_{ij} $(i,j = 1,\ldots,n-1)$ and J_{nn}. Also the first part for J_{in} is bounded by Lemma 2(g), and hence we only need to bound

$$\text{(19)}\qquad \int_{|y|\le 1} y_i t(K(0,y,-t;n+2) + K(0,y,t;n+2))\sqrt{1+|\nabla\varphi(y)|^2}\,dy$$

$$= O(|t|\log|t|) + \int_{|y|\le 1} y_i t(K(0,y,-t;n+2) + K(0,y,t;n+2))\,dy,$$

where we have used $\sqrt{1+|\nabla\varphi(y)|^2} - 1 = O(|y|^2)$ and Lemma 2(e). Applying Lemma 3(a) we find that

$$\text{(20)}\qquad \int_{|y|\le 1} t y_i\left(\frac{1}{[|y|^2+(\varphi(y)+t)^2]^{(n+2)/2}} - \frac{1}{[|y|^2+(\frac{1}{2}yHy^T+t)^2]^{(n+2)/2}}\right)dy = o(1)$$

for $t \to 0$, and the same holds if t is replaced by $-t$. Since the second integral in (20) vanishes by the oddness of the integrand, we conclude that the remaining integrals in (19) tend to zero for $t \to 0$. This completes the bound (17).

For the proof of (18), we consider the expression

$$(21)\qquad \frac{\partial^2 S(1)}{\partial N^2(P)}(X^+) + \frac{\partial \mathscr{D}(1)}{\partial N(P)}(X^+) - \frac{\partial^2 S(1)}{\partial N^2(P)}(X^-) - \frac{\partial \mathscr{D}(1)}{\partial N(P)}(X^-),$$

where $\mathscr{D}(1)(X)$ stands for the double layer potential with density 1 over $\partial\Omega$, which is explicitly given in (14). Since $\mathscr{D}$ is constant both in Ω and $\mathbb{R}^n\backslash\bar{\Omega}$, the derivatives of $\mathscr{D}$ vanish in (21). However, it will be useful for our estimates to have added "zero" in (21). For the normal derivative of $\mathscr{D}(1)$ we find the expression

$$\frac{\partial \mathscr{D}(1)}{\partial N(P)}(X) = \frac{1}{\omega_n}\int_{\partial\Omega} \frac{N(P)\cdot N(Q)}{|X-Q|^n}\,d\sigma - \frac{n}{\omega_n}\int_{\partial\Omega} \frac{N(Q)\cdot(X-Q)N(P)\cdot(X-Q)}{|X-Q|^{n+2}}\,d\sigma.$$

Using for (surface measure) almost every P a coordinate system as described in Lemma 12 with $N(P)=(0,\dots,0,-1)$ and $N(Q)=(\nabla\varphi,-1)/\sqrt{1+|\nabla\varphi|^2}$, we can localize and express (21) as

$$\frac{1}{\omega_n}\int_{|y|\le 1}\left(\frac{-\sqrt{1+|\nabla\varphi(y)|^2}}{(|y|^2+(\varphi(y)+t)^2)^{n/2}} + \frac{1}{(|y|^2+(\varphi(y)+t)^2)^{n/2}}\right)dy$$

$$+\frac{n}{\omega_n}\int_{|y|\le 1}\frac{(\varphi(y)+t)^2\sqrt{1+|\nabla\varphi(y)|^2}+(\varphi(y)+t)(\nabla\varphi(y)y+(-1)(\varphi(y)+t))}{(|y|^2+(\varphi(y)+t)^2)^{(n+2)/2}}\,dy$$

$$-\ (\text{same integrals with } t \text{ replaced by } -t).$$

With the help of the singular kernels we can write these integrals as

$$\frac{1}{\omega_n}\int_{|y|\le 1}\left(1-\sqrt{1+|\nabla\varphi(y)|^2}\right)(K(0,y,-t;n)-K(0,y,t;n))\,dy$$

$$+\frac{n}{\omega_n}\int_{|y|\le 1}\left(\left(\sqrt{1+|\nabla\varphi(y)|^2}-1\right)(\varphi(y)^2+t^2)+y\nabla\varphi(y)\varphi(y)\right)\cdot(K(0,y,-t;n+2)-K(0,y,t;n+2))\,dy$$

$$+\frac{n}{\omega_n}\int_{|y|\le 1}\left(\sqrt{1+|\nabla\varphi(y)|^2}-1\right)2\varphi(y)t(K(0,y,-t;n+2)+K(0,y,t;n+2))\,dy$$

$$+\frac{n}{\omega_n}\int_{|y|\le 1} ty\nabla\varphi(y)(K(0,y,-t;n+2)+K(0,y,t;n+2))\,dy.$$

Since $1-\sqrt{1+|\nabla\varphi(y)|^2}=O(|y|^2)$ we can apply Lemma 2 to identify the first three integrals to be of order $O(t)$ as $t\to 0$. Therefore, the only term that remains to be analyzed is

$$(22)\qquad \frac{n}{\omega_n}\int_{|y|\le 1} ty\nabla\varphi(y)(K(0,y,-t;n+2)+K(0,y,t;n+2))\,dy.$$

Since $y\nabla\varphi(y)-y^THy=o(|y|^2)$ for $y\to 0$, (22) can be written by Lemma 3(c) as the sum of a $o(1)$-term and the integral

$$(23)\qquad \frac{n}{\omega_n}\int_{|y|\le 1} ty^THy(K(0,y,-t;n+2)+K(0,y,t;n+2))\,dy.$$

This integral can be transformed by a rotation of coordinates in such a way that in the new coordinate-system the Hessian is diagonal, its diagonal entries being the principal curvatures $\gamma_1,\dots,\gamma_{n-1}$ of $\partial\Omega$ at the point P. Therefore, it remains to compute

$$\frac{n}{\omega_n}\int_{|y|\le 1} t\gamma_i y_i^2(K(0,y,-t;n+2)+K(0,y,t;n+2))\,dy,$$

where repeated indices are summed from $i=1,\dots,n-1$. By Lemma 3(b), this equals

$$\frac{2n}{\omega_n}\int_{|y|\le 1} t\gamma_i y_i^2\frac{1}{(|y|^2+t^2)^{(n+2)/2}}\,dy+O(t).$$

Instead of integration over the ball $|y|\le 1$ we can change to integrating over the cube $[-1,1]\times\cdots\times[-1,1]$, so that it remains to compute

$$\lim_{t\to 0}\frac{2nt}{\omega_n}\int_{-1}^1 dy_1\cdots\int_{-1}^1 dy_{n-1}\frac{\gamma_i y_i^2}{(y_1^2+\cdots+y_{n-1}^2+t^2)^{(n+2)/2}}$$

After the change of variables $s_it=y_i$, the above limit becomes

$$\frac{n2^n}{\omega_n}\int_0^\infty ds_1\cdots\int_0^\infty ds_{n-1}\frac{\gamma_i s_i^2}{(s_1^2+\cdots+s_{n-1}^2+1)^{(n+2)/2}}.$$

Using properties of the Γ-function it is not hard to see that the above sum equals

na 2 to identify the first three
only term that remains to be

$n+2))\,dy.$

written by Lemma 3(c) as the

$+2))\,dy.$

rdinates in such a way that
ts diagonal entries being the
P. Therefore, it remains to

$2))\,dy,$

-1. By Lemma 3(b), this

nge to integrating over the
e

$)^{(n+2)/2}$

comes

at the above sum equals

$$\frac{n2^n\sum_{i=1}^{n-1}\gamma_i}{\omega_n(n-1)}\int_0^\infty dz_1\cdots\int_0^\infty dz_{n-1}\frac{|z|^2}{(1+|z|^2)^{(n+2)/2}}$$

$$=\frac{n2^n}{\omega_n 2^{n-1}}\kappa(P)\int_{\mathbb{R}^{n-1}}\frac{|z|^2}{(1+|z|^2)^{(n+2)/2}}dz=\frac{2n}{\omega_n}\kappa(P)\int_0^\infty \omega_{n-1}\frac{r^n}{(1+r^2)^{(n+2)/2}}dr$$

$$=\frac{2n\omega_{n-1}}{\omega_n}\kappa(P)\int_0^{\pi/2}\sin^n x\,dx=(n-1)\kappa(P).$$

This concludes the proof of Proposition 11. □

6 End of the proof of Theorem 1

In Section 3 we came to the following conclusion: if $\Phi\equiv$ const. in $\mathbb{R}^n\backslash\Omega$ then Ω is a ball. If we assume for contradiction that $\Phi\not\equiv$ const., then the Hopf boundary-lemma implies that

$$i(P)=\liminf_{t\to 0_+}\frac{\Phi(P_t)-\Phi(P)}{t}<0 \quad \text{for all } P\in\partial\Omega,$$

where $P_t=P+tN(P)$. For simplicity we write S for the single-layer potential $S(1)$, D^2S for its Hessian-matrix and α for the exponent $\frac{2-2n}{n-2}$. By the mean-value theorem there exists $\tau\in(0,t)$ such that for sufficiently small t

$$0>i(P)/2>\nabla\Phi(P_\tau)\cdot N(P)$$

$$=\{2\nabla S(P_\tau)D^2S(P_\tau)N(P)S(P_\tau)+\alpha|\nabla S(P_\tau)|^2\nabla S(P_\tau)\cdot N(P)\}S^{\alpha-1}(P_\tau).$$

Using Proposition 8 we find that the right-hand side equals

$$\{-2(N(P)+o(1))D^2S(P_\tau)N(P)(c+o(1))+\alpha(1+o(1))(-1+o(1))\}$$

$$\times(c^{\alpha-1}+o(1))$$

where $o(1)\to 0$ as $t\to 0$. Using the boundedness of D^2S as formulated in Proposition 11 we get for sufficiently small t

$$-2\frac{\partial^2 S}{\partial N^2(P)}(P_\tau)c-\alpha\le\frac{i(P)}{4c^{\alpha-1}}<0.$$

Since, for almost every $P\in\partial\Omega$, we have by Proposition 11 that $\frac{\partial^2 S}{\partial N^2(P)}(P_\tau)\to(n-1)\kappa(P)$ for $t\to 0$, the previous estimate implies

$$\kappa(P)>\frac{1}{(n-2)c}\quad\text{for almost all } P\in\partial\Omega. \tag{24}$$

Next we use the inequality

$$(25)\qquad \int_{\partial\Omega}(X\cdot N(X))\kappa(X)\,d\sigma(X)\le|\partial\Omega|$$

which is valid for general convex domains with $0\in\Omega$ as we prove in the Appendix (in the case of a general C^2-domain, equality holds in (25); in this case the relation is known as one of Minkowski's integral formulas). Combining (24) and (25) we get

$$\frac{n\operatorname{vol}\Omega}{(n-2)c}\le\int_{\partial\Omega}(X\cdot N)\kappa\,d\sigma\le|\partial\Omega|.$$

But from the Pohožaev-Rellich identity (7) in section 2 we already know that equality has to hold in the previous relation. This implies that for almost every boundary point $\kappa(X)=[(n-2)c]^{-1}$ in contradiction to (24). Therefore the assumption $\Phi\not\equiv$ const. cannot hold, and the proof of Theorem 1 is complete. □

Appendix

Proof of Lemma 2. We give the proof for positive t only. We set $M=\|\nabla\varphi\|_\infty$ and split the domain $D:|y|\le 1$ into the subsets $D_1:|x-y|\le t/(2M)$ and $D_2:|x-y|\ge t/(2M)$. In D_1 we have $|\varphi(x)-\varphi(y)|/t\le 1/2$ and hence we may estimate

$$\begin{aligned}&|K(x,y,t;k)-K(x,y,-t;k)|\\&\quad=t^{-k}\left[\left(\frac{x-y}{t}\right)^2+\left(\frac{\varphi(x)-\varphi(y)}{t}+1\right)^2\right]^{-k/2}\\&\qquad-t^{-k}\left[\left(\frac{x-y}{t}\right)^2+\left(\frac{\varphi(x)-\varphi(y)}{t}-1\right)^2\right]^{-k/2}\\&\quad\le\text{const.}\frac{|\varphi(x)-\varphi(y)|}{t^{k+1}},\end{aligned}$$

where the constant only depends on φ and k. In D_2 we find

$$\begin{aligned}&|K(x,y,t;k)-K(x,y,-t;k)|\\&\quad=|x-y|^{-k}\left[1+\left(\frac{\varphi(x)-\varphi(y)+t}{|x-y|}\right)^2\right]^{-k/2}\end{aligned}$$

prove in the Appendix (in
in this case the relation is
ing (24) and (25) we get

lready know that equality
r almost every boundary
fore the assumption $\Phi \not\equiv$
e. □

We set $M = \|\nabla\varphi\|_\infty$ and
$t/(2M)$ and $D_2 : |x - y|$
we may estimate

$$- |x - y|^{-k}\left[1 + \left(\frac{\varphi(x) - \varphi(y) - t}{|x - y|}\right)^2\right]^{-k/2}$$

$$\leq \text{const.}\,\frac{t|\varphi(x) - \varphi(y)|}{|x - y|^{k+2}}.$$

Similarly, for the sum of the singular kernels we obtain

$$|K(x, y, t; k) + K(x, y, -t; k)| \leq \begin{cases} \text{const. } t^{-k} & \text{in } D_1, \\ \text{const. } |x - y|^{-k} & \text{in } D_2. \end{cases}$$

Hence integration gives

$$|I_-(x)| \leq \text{const.}\|\nabla\varphi\|_\infty\left(\int_{D_1} t^{\alpha-k-1}|x - y|^{\beta+1}\,dy + \int_{D_2} t^{\alpha+1}|x - y|^{\beta-k-1}\,dy\right)$$

$$= \text{const.}\|\nabla\varphi\|_\infty\left(O(t^{\alpha+\beta+n-k-1}) + t^{\alpha+1}\int_t^1 r^{\beta+n-k-3}\,dr\right)$$

and if $\varphi(s) = O(|s|^2)$ then we get

$$|I_-(0)| = \text{const.}\left(O(t^{\alpha+\beta+n-k}) + t^{\alpha+1}\int_t^1 r^{\beta+n-k-2}\,dr\right).$$

Likewise

$$|I_+(x)| \leq \text{const.}\left(\int_{D_1} t^{\alpha-k}|x - y|^\beta\,dy + \int_{D_2} t^\alpha|x - y|^{\beta-k}\,dy\right)$$

$$= \text{const.}\left(O(t^{\alpha+\beta+n-k-1}) + t^\alpha\int_t^1 r^{\beta+n-k-2}\,dr\right).$$

This finishes the proof of the lemma. □

Proof of Lemma 3. As in the previous lemma we carry out the proof for $t > 0$, and we split the domain of integration into $D_1 : |y| \leq t/(2M)$ and $D_2 : |y| \geq t/(2M)$. We write $I = \int t|y|A$, $J = \int t|y|^2 B$ and $L = \int t o(|y|^2)C$. To estimate A in D_1 we use

$$A = \frac{1}{t^{n+2}}\left(\left[\left(\frac{|y|}{t}\right)^2 + \left(\frac{\varphi(y)}{t} + 1\right)^2\right]^{-(n+2)/2} - \left[\left(\frac{|y|}{t}\right)^2 + \left(\frac{y^T H y}{2t} + 1\right)^2\right]^{-(n+2)/2}\right)$$

$$= \frac{o(|y|^4) + t o(|y|^2)}{t^{n+4}},$$

and for B we use

$$B = \frac{1}{t^{n+2}}\left(\left[\left(\frac{|y|}{t}\right)^2 + \left(\frac{\varphi(y)}{t}+1\right)^2\right]^{-(n+2)/2} - \left[\left(\frac{|y|}{t}\right)^2 + 1\right]^{-(n+2)/2}\right)$$

$$= \frac{O(|y|^4) + tO(|y|^2)}{t^{n+4}}.$$

With the same method C can be estimated in D_1 by $t^{-n-2}O(1)$. In D_2 we estimate A by

$$A = \frac{1}{|y|^{n+2}}\left(\left[1 + \left(\frac{|\varphi(y)+t|}{|y|}\right)^2\right]^{-(n+2)/2} - \left[1 + \left(\frac{y^T H y + 2t}{2|y|}\right)^2\right]^{-(n+2)/2}\right)$$

$$= \frac{o(|y|^4) + to(|y|^2)}{|y|^{n+4}},$$

and B by

$$B = \frac{1}{|y|^{n+2}}\left(\left[1 + \left(\frac{|\varphi(y)+t|}{|y|}\right)^2\right]^{-(n+2)/2} - \left[1 + \left(\frac{t}{|y|}\right)^2\right]^{-(n+2)/2}\right)$$

$$= \frac{O(|y|^4) + tO(|y|^2)}{|y|^{n+4}}.$$

Likewise, we obtain $C = |y|^{-n-2}O(1)$ in D_2. Hence we find that the integrals are of the following type (we distinguish between $o(r)$ and $o(t)$ by a suffix)

$$I = \int_0^t tr\frac{o_r(r^{n+2}) + to_r(r^n)}{t^{n+4}}\,dr + \int_t^1 tr\frac{o_r(r^{n+2}) + to_r(r^n)}{t^{n+4}}\,dr,$$

$$J = O(1)\left(\int_0^t \frac{r^{n+4}}{t^{n+3}} + \frac{r^{n+2}}{t^{n+2}}\,dr + \int_t^1 t + \frac{t^2}{r^2}\,dr\right),$$

$$L = \int_0^t \frac{o_r(r^n)}{t^{n+1}}\,dr + \int_t^1 \frac{to_r(r^n)}{r^{n+2}}\,dr.$$

If one takes into account that $\int_0^t t^a o_r(r^b)\,dr = o_t(t^{a+b+1})$ if $b \geq 0$ and $a+b+1 \geq 0$, that $\int_t^1 \frac{t}{r} o_r(1)\,dr = o_t(1)$ and that $\int_t^1 \frac{t^{b-1}}{r^b} o_r(1)\,dr = o_t(1)$ if $b > 1$ for $t \to 0$, then we find that

$$I = o_t(1), \quad J = O_t(t) \quad \text{and} \quad L = o_t(1) \quad \text{for } t \to 0,$$

which was the claim of the corollary. □

An integral formula of Minkowski. If Ω is a bounded C^2-domain, then

$$\int_{\partial\Omega} (X \cdot N(X))\kappa(X)\, d\sigma(X) = |\partial\Omega|.$$

This integral formula of Minkowski can be found in Leichtweiß [5].

Here we state and prove generalizations of this formula for convex domains with less smoothness. For convex sets (and more generally for sets of positive reach) the notion of curvature measures was introduced by Federer [2]. His construction is essentially the following, cf. Schneider [15]: Let K be compact convex subset of $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$, let $\pi(x) \in K$ be the unique nearest point to x. Let η be a Borel subset of $\mathbb{R}^n$. For $\varepsilon > 0$ the set

$$M_\varepsilon(K,\eta) = \{x \in \mathbb{R}^n : 0 < \operatorname{dist}(x,K) \le \varepsilon, \pi(x) \in \eta\}$$

is also a Borel set which is called the *local parallel subset* of K with respect to η at distance ε. A result of Steiner states, that the n-dimensional Lebesgue-measure $\mu_\varepsilon(K,\eta) = |M_\varepsilon(K,\eta)|$ is a polynomial in ε and depends only on $\varepsilon, \varepsilon^2, \dots, \varepsilon^n$. Hence we may write

$$\mu_\varepsilon(K,\eta) = \frac{1}{n}\sum_{j=0}^{n-1} \varepsilon^{n-j}\binom{n}{j} c_j(K,\eta),$$

where the Borel-measures $c_0(K,\cdot), \dots, c_{n-1}(K,\cdot)$ are called *curvature measures*. They are supported on ∂K. Moreover, $c_{n-1}(K,\cdot)$ is the $n-1$-dimensional Hausdorff-measure on ∂K and $c_{n-2}(K,\cdot)$ is called the *mean-curvature measure*. For convex C^2-domains one has $dc_{n-2} = \kappa\, d\sigma$, where σ is the $n-1$-dimensional Hausdorff-measure on ∂K, and κ is the differential-geometric mean curvature. The curvature measures have the property that $c_j(K_k,\cdot) \rightharpoonup c_j(K,\cdot)$ if K_k and K are compact, convex sets such that Hausdorff-distance $\rho(K_k,K) \to 0$ as $k \to \infty$; the Hausdorff-distance of two compact sets C, D being defined as $\min\{r \ge 0 : C \subset D + r\bar{B}_1(0), D \subset C + r\bar{B}_1(0)\}$ where $B_1(0)$ is the unit-ball. If Ω is a bounded, open, convex domain then we write $c_j(\Omega)$ for $c_j(\bar{\Omega})$.

Lemma. *Let Ω be a bounded convex domain in $\mathbb{R}^n$ which contains the origin. Then*

$$\int_{\partial\Omega} (X \cdot N)\kappa\, d\sigma \le |\partial\Omega|, \quad \text{where } \kappa = \frac{dc^{\mathrm{ac}}_{n-2}(\Omega)}{d\sigma} \in L^1(\partial\Omega)$$

is the Radon-Nikodym derivative of the absolutely-continuous part of $c_{n-2}(\Omega)$ with respect to the $n-1$ dimensional Hausdorff-measure. Moreover, for almost all boundary points $X \in \partial\Omega$ the value $\kappa(X)$ coincides with the differential-geometric mean-curvature of $\partial\Omega$ at X. If $\partial\Omega \in C^1$, then

$$\int (X \cdot N) dc_{n-2}(\Omega) = |\partial\Omega|.$$

Proof. Let Ω_k be a sequence of smooth convex domain such that $\rho(\Omega, \Omega_k) \to 0$ for $k \to \infty$. Since $\kappa_k\, d\sigma \rightharpoonup \kappa\, d\sigma + dc_{n-2}^{\text{sing}}$ as $k \to \infty$, we find that $\kappa_k \to \kappa$ σ-a.e. as $k \to \infty$. If $\tilde{N}_k$ is a continuous extension of N_k (the normal field on $\partial\Omega_k$), then $\tilde{N}_k \to N$ σ-a.e. on $\partial\Omega$. By the convexity of Ω_k and the fact that $0 \in \Omega_k$ we have that $(X \cdot N_k)\kappa_k \geq 0$. Applying Fatou's Lemma we get

$$\int_{\partial\Omega} (X \cdot N)\kappa\, d\sigma \leq \liminf_{k\to\infty} \int_{\partial\Omega_k} (X \cdot N_k)\kappa_k\, d\sigma = \liminf_{k\to\infty} |\partial\Omega_k| = |\partial\Omega|.$$

If $\partial\Omega \in C^1$ then $\tilde{N}_k \to N$ uniformly on $\partial\Omega$. Using again the weak convergence $\kappa_k\, d\sigma = dc_{n-2}(\Omega_k) \rightharpoonup c_{n-2}(\Omega)$ we find

$$|\partial\Omega_k| = \int (X \cdot \tilde{N}_k) dc_{n-2}(\Omega_k) \to \int (X \cdot N) dc_{n-2}(\Omega) \quad \text{as } k \to \infty,$$

which proves the result for convex C^1-domains. □

References

[1] Evans, L.C., Gariepy, R.F.: Measure Theory and Fine Properties of Functions. Studies in Advanced Mathematics. CRC Press, Boca Ranton 1992

[2] Federer, H.: Geometric Measure Theory. Die Grundlehren der mathematischen Wissenschaften 153. Springer-Verlag, New York 1969

[3] Henrot, A., Philippin, G.A., Prébet, H.: Overdetermined problems on ring shaped domains. Preprint (1998)

[4] Leichtweiß, K.: Konvexe Mengen. Hochschultext. Springer-Verlag, Berlin–New York 1980

[5] Leichtweiß, K.: Convexity and differential geometry. In: Handbook of convex geometry B (ed. by P. Gruber, J. Wills), pp. 1045–1080. North-Holland, Amsterdam 1993

[6] Martensen, E.: Eine Integralgleichung für die logarithmische Gleichgewichtsbelegung und die Krümmung der Randkurve eines ebenen Gebiets. Z. angew. Math. Mech. **72** (1992), T596–T599

[7] Martensen, E.: Eine Verallgemeinerung der Sprungbeziehungen der Potentialtheorie. Arch. Math. **26** (1975), 620–626

[8] Payne, L., Philippin, G.: On some maximum principles involving harmonic functions and their derivatives. SIAM J. Math. Anal. **10** (1979), 96–104

[9] Philippin, G.: On a free boundary problem in electrostatics. Math. Meth. Appl. Sci. **12** (1990), 387–392

[10] Pohožaev, S.I.: Eigenfunctions of the equation $\Delta u + f(u) = 0$. Soviet Math. Dokl. **6** (1965), 1408–1411. English translation, Dokl. Akad. Nauk. SSRR **165** (1965), 33–36

[11] Pucci, P., Serrin, J.: A general variational identity. Indiana Univ. Math. J. **35** (1986), 681–703

[12] Reichel, W.: Radial symmetry for elliptic boundary-value problems on exterior domains. Arch. Rational Mech. Anal. **137** (1997), 381–394

[13] Reichel, W.: Radial symmetry for an electrostatic, a capillarity and some fully nonlinear overdetermined problems on exterior domains. Z. Anal. Anwendungen **15** (1996), 619–635

[14] Rellich, F.: Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral. Math. Z. **46** (1940), 635–636

[15] Schneider, R.: Convex surfaces, curvature and surface area measures. In: Handbook of Convex Geometry A (ed. by P. Gruber, J. Wills), pp. 273–300. North-Holland, Amsterdam 1993

[16] Verchota, G.: Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz-domains. J. Func. Anal. **59** (1984), 572–611

[17] Wermer, J.: Potential Theory. Lecture Notes in Mathematics 408. Springer-Verlag, Berlin–Heidelberg–New York 1974

Received September 21, 1998

Osvaldo Mendez, School of Mathematics, University of Missouri at Columbia, U.S.A.
Wolfgang Reichel, Mathematisches Institut, Universität zu Köln, Germany; current address: Mathematisches Institut, Universität Basel, Switzerland