Arch. Rational Mech. Anal. 137 (1997) 381–394. © Springer-Verlag 1997

Radial Symmetry for Elliptic Boundary-Value Problems on Exterior Domains

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Dedicated to Wolfgang Walter on the occasion of his 70th birthday

Communicated by J. Serrin

Abstract

By the Alexandroff-Serrin method [2, 14] of moving hyperplanes we obtain radial symmetry for the domain and the solutions of $\Delta u + f(u, |\nabla u|) = 0$ on an exterior domain $\Omega = \mathbb{R}^n \setminus \overline{\Omega}_1$, subject to the overdetermined boundary conditions $\partial u/\partial \nu = \text{const.}$, $u = \text{const.} > 0$ on $\partial\Omega_1$, $u, |\nabla u| \to 0$ at ∞ and $0 \leqq u < u|_{\partial\Omega_1}$ in Ω. In particular, the following conjecture from potential theory due to P. Gruber (cf. [11, 8]) is proved: Let $\Omega_1 \subset \mathbb{R}^2$ or $\Omega_1 \subset \mathbb{R}^3$ be a bounded smooth domain with a constant source distribution on $\partial\Omega_1$ and let Ψ be the induced single-layer potential. If Ψ is constant in $\overline{\Omega}_1$, then Ω_1 is a ball.

1. Introduction and main results

In a seminal paper [14], Serrin proved that the following overdetermined boundary-value problem determines the geometry of the underlying set, i.e., if Ω is a bounded C^2 domain and $u \in C^2(\overline{\Omega})$ is a solution of

$$\Delta u + f(u, |\nabla u|) = 0 \text{ in } \overline{\Omega}, \quad u > 0 \text{ in } \Omega,$$

$$u = 0, \ \frac{\partial u}{\partial \nu} = \text{const.} \leqq 0 \text{ on } \partial\Omega,$$

where f is in C^1, then u is radially symmetric and Ω is a ball.

In a recent work [12], I considered a corresponding problem for domains with one cavity: Let Ω_0, Ω_1 be bounded C^2 domains and let $\Omega = \Omega_0 \setminus \overline{\Omega}_1$ be connected. Then for the boundary-value problem (with $f \in C^1$)

$$\Delta u + f(u, |\nabla u|) = 0 \text{ in } \overline{\Omega}, \quad 0 < u < a \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial\Omega_0, \quad u = a \text{ on } \partial\Omega_1,$$

$$\frac{\partial u}{\partial \nu} = \text{const.} = c_i \text{ on } \partial\Omega_i \ (i = 0, 1)$$

it is proved that Ω is an annulus and every solution u is radially symmetric and decreasing in r.

In this paper, we address a corresponding problem on an exterior domain: Let Ω_1 be a bounded domain with a C^2 boundary. If $\Omega = \mathbb{R}^n \setminus \overline{\Omega}_1$ is connected, then we call it an exterior domain and study solutions $u \in C^2(\overline{\Omega})$ of the problem

$$\Delta u + f(u, |\nabla u|) = 0 \text{ in } \overline{\Omega}, \quad 0 \leqq u < a \text{ in } \Omega, \tag{1}$$

$$u = a, \ \frac{\partial u}{\partial \nu} = \text{const.} \leqq 0 \text{ on } \partial\Omega_1, \tag{2}$$

$$u, \nabla u = 0 \text{ at } \infty, \tag{3}$$

where ν is the exterior normal with respect to Ω_1.

In the following considerations monotonicity is understood in the weak sense, e.g., a function h is increasing if $s \leqq t$ implies $h(s) \leqq h(t)$. For the nonlinearity f we consider the hypotheses:

(H_1) $f(p, q) = f_1(p, q) + f_2(p)$ where f_1 is Lipschitz continuous in p and q, and f_2 is increasing in p. Furthermore, f is decreasing in p for small positive values of p, q.

(H_2) $f(p, q)$ is Lipschitz continuous in p and q, and is decreasing in p.

Note that (H_2) is the special case $f_2 \equiv 0$ of (H_1).

Theorem 1. *Let u be a solution of* (1)–(3) *and let f satisfy* (H_1). *Then Ω_1 is a ball, and u is radially symmetric and decreasing in r.*

Remark. If f is independent of $|\nabla u|$, then the condition $\nabla u = 0$ at ∞ is not needed. Also, the proof simplifies considerably when (H_2) holds.

Corollary 1. *Under the hypotheses of Theorem 1 together with $f(a, 0) \leqq 0$, condition* (1) *can be relaxed to $0 \leqq u \leqq a$ in Ω.*

In [13] I treated the quasilinear analogue of (1)–(3), which includes Monge-Ampère operators, the capillary surface operator and degenerate quasilinear operators like the p-Laplacian.

As an application of Theorem 1 we consider the following problem from potential theory: Let $\Omega_1 \subset \mathbb{R}^n (n \geqq 2)$ be a bounded $C^{2,\alpha}$ domain. On $\partial\Omega_1$ we consider a constant source distribution $A > 0$, which induces a single-layer potential

$$\Psi(x) = A \int_{\partial\Omega_1} \gamma(|x - y|) \, d\sigma_y,$$

where $\gamma(r) = -(1/2\pi) \log r$ for $n = 2$ and $\gamma(r) = r^{2-n}/(n-2)\omega_n$ for $n \geqq 3$, where ω_n is the surface area of the n-dimensional unit sphere. Now suppose that Ψ is constant in $\overline{\Omega}_1$. For $n = 2$ it is well known (see Martensen [11]) that Ω_1 must be a disk. To my knowledge, the corresponding conjecture for $n \geqq 3$ (attributed to P. Gruber by Martensen [11] and Heil & Martini [8, p. 353]) has not yet been solved.

Theorem 2. *The only bounded $C^{2,\alpha}$ domain Ω_1 in $\mathbb{R}^n (n \geqq 3)$ that admits a non-trivial single-layer potential which is constant in $\overline{\Omega}_1$ and is induced by a constant source distribution on $\partial\Omega_1$ is a ball.*

Proof. Let A be the constant source distribution and Ψ be the induced potential. Then $\Psi \in C^{2,\alpha}(\overline{\Omega}_1)$; see Gilbarg & Trudinger [7, Theorem 6.14]. Clearly $\Delta\Psi = 0$ in $\mathbb{R}^n \setminus \Omega_1$, $\Psi = \text{const.}$ on $\partial\Omega_1$ and $\partial\Psi/\partial v_+ = 0$ on $\partial\Omega_1$, since Ψ is constant in $\overline{\Omega}_1$. By the jump condition for the normal derivative we find $\partial\Psi/\partial v_- = -A = \text{const.} \neq 0$ on $\partial\Omega_1$. Since Ω_1 is bounded, we also find that $\Psi \to 0$ at ∞. To show that $\Omega = \mathbb{R}^n \setminus \overline{\Omega}_1$ is connected suppose the contrary, i.e., $\overline{\Omega}_1^c$ has a bounded component Z. Since Ψ is harmonic in Z and constant on ∂Z, we deduce that $\Psi \equiv \text{const.}$ in Z, contradicting the Neumann conditions on ∂Z. Hence $\Omega = \mathbb{R}^n \setminus \overline{\Omega}_1$ is connected. To apply Theorem 1 it remains to show $0 \leqq \Psi < \Psi|_{\partial\Omega_1}$ in Ω. Since $\gamma(r) > 0$ in $\mathbb{R}^n \setminus \{0\}$ for $n \geqq 3$, we find $\Psi > 0$ by definition. Suppose Ψ attains larger values in Ω than $\Psi|_{\partial\Omega_1}$. Then Ψ must reach its maximum in Ω since $\Psi = 0$ at ∞. By the strong maximum principle and the connectedness of Ω, we find $\Psi \equiv \text{const.}$, which is a contradiction.

Remark 1. We discuss the case $n = 2$ after the proof of Theorem 1, in order to see which adjustments are needed to replace $u \to 0$ at ∞ by an appropriate condition which includes logarithmic potentials.

Remark 2. A similar uniqueness theorem for volume potentials (constant on $\partial\Omega_1$ and induced by a constant density) was given by Fraenkel as an application of results in [3] and [4].

2. The proof of Theorem 1

We use the following notation: $x = (x_1, \ldots, x_n) = (x_1, x')$ denotes a point in $\mathbb{R}^n$ with $x' = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$; $|x|$ is the Euclidean norm of x; and $B_r(x)$ is the open ball with radius r centered at x. For partial derivatives we use $\partial_{x_k} u = \partial_k u = u_k$ and for derivatives in direction $\eta \in \mathbb{R}^n \setminus \{0\}$ we use $\partial u/\partial\eta = \partial_\eta u$. Sometimes it is convenient to write $f = f(p, q_1, \ldots, q_n)$ instead of $f(p, q)$, where $q = |(q_1, \ldots, q_n)|$. With respect to Lipschitz continuity both notations are consistent, i.e., $f(p, q_1, \ldots, q_n)$ is Lipschitz continuous with respect to $q_1, \ldots, q_n$ if and only if $f(p, q)$ is Lipschitz continuous with respect to q. Finally, for a real-valued function c on a subset of $\mathbb{R}^n$ we use $c^-(x) = \min\{c(x), 0\}$ for the negative part of c.

We start with a brief outline of the proof: We show symmetry of Ω and u in the x_1-direction for any solution u of (1)–(3). Once this is done, we see that for any rotation M, the function $u(Mx)$ is also a solution of (1)–(3) on $M^\top\Omega$, i.e., $M^\top\Omega$ is symmetric in the x_1-direction. The radial symmetry of Ω and u follows. The x_1-symmetry will be proved by the method of reflection in hyperplanes, which was introduced by ALEXANDROFF [2] and later reintroduced and refined by SERRIN [14] in order to apply to partial differential equations on bounded domains, and which was finally used to great effect by GIDAS, NI & NIRENBERG [5]. For elliptic equations on $\mathbb{R}^n$, LI [9] and LI & NI [10] simplified the proofs and extended the original results of GIDAS, NI & NIRENBERG [6]. We use these simplifications together with a variant of SERRIN's method, which makes the method suitable for our exterior domain problem. Let us define

$T_\lambda = \{x \mid x_1 = \lambda\}$,	the hyperplane,
$H_\lambda = \{x \mid x_1 > \lambda\}$,	the right-hand half-space,
$x^\lambda = (2\lambda - x_1, x')$,	the reflection of x at T_λ,
$\Sigma_1(\lambda) = \{x \in \Omega_1 \mid x_1 > \lambda\}$,	the inner right-hand cap,
$\Gamma_1(\lambda) = \{x \in \partial\Omega_1 \mid x_1 > \lambda\}$,	the inner right-hand boundary,
$m_1 = \sup\{x_1 \mid x \in \Omega_1\}$,	the x_1–extent of Ω_1,
$\Sigma(\lambda) = H_\lambda \setminus \overline{\Omega}_1^\lambda$,	the reduced half-space.

For the geometry of inner right-hand caps, it is well known (see AMICK & FRAENKEL [3, Lemma A.1]) that for values of λ a little less than m_1, the reflection of $\Sigma_1(\lambda)$ lies in Ω_1, and the positive x_1-direction points out of $\Sigma_1(\lambda)$ at every point of $\Gamma_1(\lambda)$. Also it is well known that this remains true for decreasing values of λ until $\Sigma_1(\lambda)^\lambda$ becomes internally tangent to $\partial\Omega_1$, or the normal on $\Gamma_1(\lambda)$ becomes perpendicular to the x_1-direction (see Figure 2). We denote this critical value of λ by m. The admissible range for λ is then (m, ∞).

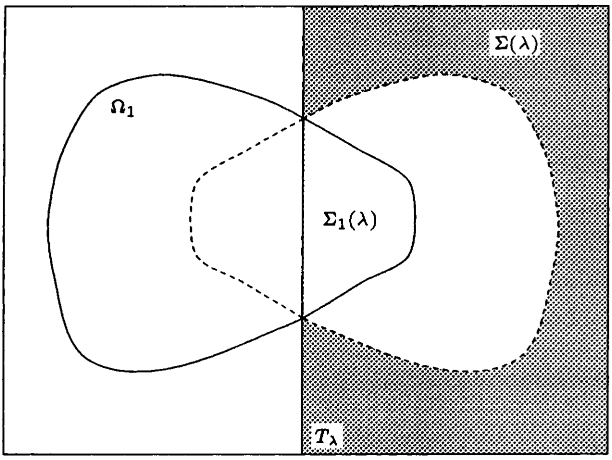

Figure 1. Illustration of the reduced half-space.

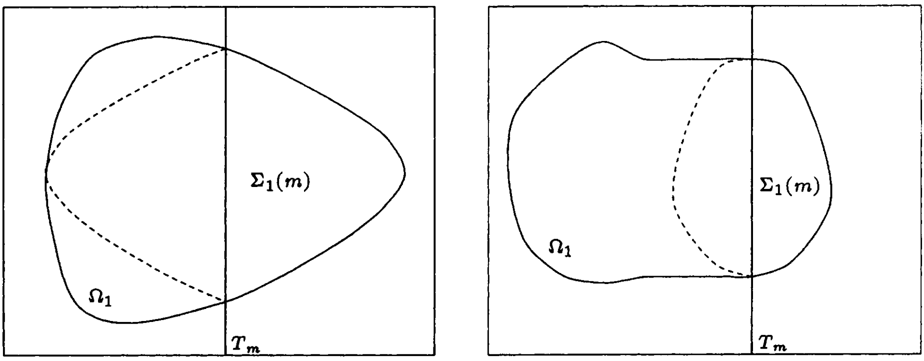

Figure 2. The critical positions of T_m.

For such λ the comparison function

$$w(x,\lambda) = v(x,\lambda) - u(x) = u(x^\lambda) - u(x)$$

is well defined on $\Sigma(\lambda)$. Notice that $\partial\Sigma(\lambda) \setminus T_\lambda \subset \partial\Omega_1^\lambda$ since the inner right-hand cap $\Sigma_1(\lambda)$ reflects into Ω_1. On $\partial\Sigma(\lambda) \cap T_\lambda$ we have $w(x,\lambda) = 0$, and on $\partial\Sigma(\lambda) \cap \partial\Omega_1^\lambda$ we have $w(x,\lambda) = a - u(x) \geqq 0$; i.e., $w \geqq 0$ on $\partial\Sigma(\lambda)$ for all $\lambda \in (m,\infty)$. The idea of reduced half-spaces $\Sigma(\lambda)$ is also used in the form of reduced right-hand caps for bounded domains in Alessandrini [1], Willms, Gladwell & Siegel [15] and Reichel [12].

We shall establish the following properties of $w(x,\lambda)$ for all $\lambda \in (m,\infty)$:

(i) $w(x,\lambda) > 0$ in $\Sigma(\lambda)$,
(ii) $\partial_1 w(x,\lambda) > 0$ on $\Omega \cap T_\lambda$.

Properties (i) and (ii) will be proved by an *initial step* for $\lambda \in (R,\infty)$ (R sufficiently large) and by a *continuation step* for all λ in a maximal interval (μ,∞) with $\mu = m$. We then have $w(x,m) \geqq 0$ in $\Sigma(m)$. From this we shall conclude that $w(x,m) \equiv 0$ on a component Z of $\Sigma(m)$, and finally we shall show that $\overline{Z \cup Z^m} = \overline{\Omega}$, which is the desired symmetry of Ω. In case of Hypothesis (H_2) the proof simplifies considerably. This will be discussed at the end of the proof.

The following boundary lemma will be proved in the Appendix.

Proposition 1. *Let u be a solution of* (1)–(3). *For $z \in \partial\Omega_1$ let $\eta \in \mathbb{R}^n$ be a fixed unit vector with $\nu(z) \cdot \eta > 0$, where $\nu(z)$ is the exterior normal to $\partial\Omega_1$ at z. Then there exists a radius $\rho = \rho(z)$ and a ball $B_\rho(z)$ such that $\partial_\eta u < 0$ in $B_\rho(z) \cap \Omega$.*

Remark. We always assume that the ball $B_\rho(z)$ is so small that the vector η points out of Ω_1 at all points of $\partial\Omega_1 \cap B_\rho(z)$.

Proof of Theorem 1.
Step (I). First we show that if $w \equiv 0$ on a component Z of $\Sigma(\lambda)$ for some $\lambda \geqq m$, then $\overline{Z \cup Z^\lambda} = \overline{\Omega}$, which is the desired symmetry of Ω. ∂Z decomposes

into two parts: the part on the hyperplane $\partial Z \cap T_\lambda$ and the remaining part $\partial Z \setminus T_\lambda$. From the assumption $w \equiv 0$ we find $u = a$ on $\partial Z \setminus T_\lambda$, i.e., $\partial Z \setminus T_\lambda$ is a subset of $\partial\Omega_1$ by (1) and of $\partial\Omega_1^\lambda$ by definition. This is the intrinsic reason for the symmetry of Ω. The following topological argument shows that $Z \cup Z^\lambda$ can be extended across T_λ to an open subset of Ω. We define

$$X = Z \cup Z^\lambda \cup (\partial Z \cap \Omega) \cup \left(\partial Z^\lambda \cap \Omega\right)$$

and show that X is open. Once this is proved, we have

$$\partial X \subset \left(\partial Z \cup \partial Z^\lambda\right) \setminus (\partial Z \cap \Omega) \setminus \left(\partial Z^\lambda \cap \Omega\right) \subset \partial\Omega,$$

which implies that $\Omega = X$ since X is also nonempty and Ω is connected. To show that X is open, observe first that $Z \cup Z^\lambda \subset \operatorname{int} X$.

(a) Take $x \in \partial Z \cap \Omega$. Since $\partial Z \setminus T_\lambda \subset \partial\Omega_1 = \partial\Omega$, we see that $x \in T_\lambda \cap \Omega$. Since Ω is open there is a ball $B_\rho(x) \subset \Omega$, and we can define

$$\begin{aligned} B_> &= B_\rho(x) \cap \{x_1 > \lambda\}, \\ B_< &= B_\rho(x) \cap \{x_1 < \lambda\}, \\ E &= B_\rho(x) \cap \{x_1 = \lambda\}. \end{aligned}$$

Clearly $Z \cap B_> \neq \emptyset$. If also $Z^c \cap B_> \neq \emptyset$, then $\partial Z \cap B_> \neq \emptyset$, which is impossible since $\partial Z \cap B_> \subset (\partial\Omega \cup T_\lambda) \cap B_> = \emptyset$. Hence $B_> \subset Z$, $B_< \subset Z^\lambda$ and $E \subset \Omega \cap (\overline{Z} \cap T_\lambda) \subset \Omega \cap \partial Z$, which together imply that $B_\rho(x) \subset X$.

(b) Take $x \in \partial Z^\lambda \cap \Omega$. It follows from the definition of Z that $\partial Z \setminus T_\lambda \subset \partial\Omega_1^\lambda$ or equivalently $\partial Z^\lambda \setminus T_\lambda \subset \partial\Omega_1 = \partial\Omega$. As before, this implies that $x \in T_\lambda$. Hence there is a sequence $\{x_\nu\}_{\nu=1}^\infty \subset Z^\lambda \cap \Omega$ with $x_\nu \to x$. Also $x_\nu^\lambda \in Z$ converge to $x = x^\lambda$, which shows that $x \in \overline{Z} \cap T_\lambda \cap \Omega \subset \partial Z \cap \Omega$. By (a) we find $x \in \operatorname{int} X$.

This proof goes back to lecture notes of L. E. Fraenkel.

Finally note that it is impossible that $w \equiv 0$ on Z for $\lambda \in (m, \infty)$, since Ω can only be symmetric to T_λ for $\lambda = m$.

Step (II). Here we derive a differential equation for $w(x, \lambda)$. For $\lambda \in (m, \infty)$ the function $v(x, \lambda) = u(x^\lambda)$ satisfies

$$\Delta v + f(v, |\nabla v|) = 0 \text{ in } \Sigma(\lambda).$$

Hence we find that

$$\Delta w + f(v, |\nabla v|) - f(u, |\nabla u|) = 0; \tag{4}$$

if we define

$$c(x) = \frac{f(v, \nabla u) - f(u, \nabla u)}{v - u},$$

$$b_i(x) = \frac{f(v, u_1, \ldots, u_{i-1}, v_i, \ldots, v_n) - f(v, u_1, \ldots, u_i, v_{i+1}, \ldots, v_n)}{v_i - u_i},$$

where the quotients are zero if the denominators are zero, then (4) reads

$$\Delta w + b_i \partial_i w + c w = 0 \ \text{ in } \Sigma(\lambda) \tag{5}$$

with bounded functions b_i and a possibly unbounded function c. For later use, we introduce $\bar{w} = w/g$, where g is a C^2 function on $\Sigma(\lambda)$. After a little computation we obtain

$$(5') \quad \Delta\bar{w} + \left(b_i + 2\frac{\partial_i g}{g}\right)\partial_i\bar{w} + \left(c + \frac{b_i\partial_i g + \Delta g}{g}\right)\bar{w} = \Delta\bar{w} + \bar{b}_i\partial_i\bar{w} + \bar{c}\bar{w} = 0.$$

If $K > 0$ is an upper bound for the functions b_i, then for $\alpha > K$ a good candidate is $g(x) = g(x, \lambda) = \exp(-\lambda\alpha/2) - \exp(-\alpha x_1)$, which has the properties $g > 0$ and $b_i\partial_i g + \Delta g \leqq \alpha(K - \alpha)\exp(-\alpha x_1) < 0$ in $\overline{\Sigma(\lambda)}$.

Step (III). We show that for the proof of (i) and (ii) it suffices to show that $w(x, \lambda) \geqq 0$ in $\Sigma(\lambda)$ for all $\lambda \in (m, \infty)$. Suppose that $w(x, \lambda) \geqq 0$ in $\Sigma(\lambda)$. By (H_1) we derive from (4) that

$$(6) \qquad \Delta w + f_1(v, |\nabla v|) - f_1(u, |\nabla u|) \leqq 0.$$

If we define $c(x)$ and $b_i(x)$ as in (I), with f_1 instead of f, we find

$$(7) \qquad \Delta w + b_i\partial_i w + c^- w \leqq \Delta w + b_i\partial_i w + cw \leqq 0$$

with bounded functions b_i, c, c^-. By the strong maximum principle applied to $w(x, \lambda) \geqq 0$ on a component Z of $\Sigma(\lambda)$ we have either $w \equiv 0$ on Z or $w > 0$ in Z. In (I) we proved that $w \equiv 0$ in Z implies the x_1-symmetry of Ω, which is impossible for $\lambda > m$. Hence, for all $\lambda \in (m, \infty)$, it is enough to prove the weak inequality $w(x, \lambda) \geqq 0$ in $\Sigma(\lambda)$ since we can sharpen it to $w(x, \lambda) > 0$ in $\Sigma(\lambda)$ and $\partial_1 w(x, \lambda) > 0$ on $T_\lambda \cap \Omega$ by using (7) and the maximum principle. (Observe that for $\lambda > m$ the positive x_1-direction is non-tangent and points inside $\Sigma(\lambda)$ at $T_\lambda \cap \partial\Sigma(\lambda)$; see Figure 1.)

It is important to notice that the same reasoning applies to $\bar{w}(x, \lambda) = w(x, \lambda)/g(x, \lambda)$ if $\bar{w} \geqq 0$ (here g is chosen as in the remark following (5′)). In fact, the differential inequality corresponding to (7) now reads

$$(7') \qquad \Delta\bar{w} + \bar{b}_i\partial_i\bar{w} + \bar{c}^-\bar{w} \leqq 0,$$

with bounded functions $\bar{b}_i, \bar{c}^-$. For $\lambda > m$ we can use (7′) to sharpen $\bar{w} \geqq 0$ to $\bar{w} > 0$ in $\Sigma(\lambda)$, since $\bar{w} \equiv 0$ on a component Z implies $w \equiv 0$ on Z, which is impossible by (I).

Initial Step (IV). By (III) we have to show the existence of a large R such that $w \geqq 0$ in $\Sigma(\lambda)$ for $\lambda \in (R, \infty)$. Suppose for contradiction that there exist sequences $\lambda_k \to +\infty$ and $x^{(k)} \in \Sigma(\lambda_k)$ with $w(x^{(k)}, \lambda_k) < 0$. We take $g(x, \lambda_k)$ as in the remark following (5′) and choose $x^{(k)} \in \overline{\Sigma(\lambda_k)}$ such that $\bar{w}(x, \lambda_k) = w(x, \lambda_k)/g(x, \lambda_k)$ attains its negative minimum over $\overline{\Sigma(\lambda_k)}$ in $x^{(k)}$ ($\bar{w} \to 0$ at ∞, $\bar{w} \geqq 0$ on $\partial\Sigma(\lambda)$). The point $x^{(k)}$ does not lie on $\partial\Sigma(\lambda_k)$ since there $\bar{w} \geqq 0$. Hence $\nabla_x\bar{w}(x^{(k)}, \lambda_k) = 0$ and $\Delta\bar{w}(x^{(k)}, \lambda_k) \geqq 0$. Also $u(x^{(k)}) \to 0$, $\nabla u(x^{(k)}) \to 0$ since $x^{(k)} \to \infty$, and $u(x^{(k),\lambda_k}) \to 0$ since $0 \leqq u(x^{(k),\lambda_k}) < u(x^{(k)})$. It follows from (H_1) and the definition of $c(x)$ in (II) that for k large, $c(x^{(k)}) \leqq 0$ and $\bar{c}(x^{(k)}) < 0$ by the choice of the function g. Hence we have

$$\Delta\bar{w}\big(x^{(k)},\lambda_k\big)+\bar{b}_i\big(x^{(k)}\big)\partial_i\bar{w}\big(x^{(k)},\lambda_k\big)+\bar{c}(x_k)\bar{w}\big(x^{(k)},\lambda_k\big)$$
$$=\Delta\bar{w}\big(x^{(k)},\lambda_k\big)+\bar{c}(x_k)\bar{w}\big(x^{(k)},\lambda_k\big)>0,$$

which contradicts the differential equation (5′). This finishes the initial step.

Continuation Step (V). By the initial step, the following quantity is well defined

$$\mu=\inf\{\alpha>m : w(x,\lambda)\geqq 0 \text{ in } \Sigma(\lambda)\ \forall\ \lambda\in(\alpha,\infty)\},$$

and both (i) and (ii) hold for all $\lambda\in(\mu,\infty)$ by (7) and the maximum principle. We want to show that $\mu=m$, and therefore suppose for contradiction $\mu>m$. (Then the x_1-direction is non-tangent on $\partial\Sigma_1(\mu)$.) There exist sequences $\lambda_k\uparrow\mu$ and $x^{(k)}\in\Sigma(\lambda_k)$ such that $w\big(x^{(k)},\lambda_k\big)<0$. We suppose that these sequences are chosen such that $\bar{w}(x,\lambda_k)$ attains its negative minimum over $\overline{\Sigma(\lambda_k)}$ in $x^{(k)}$. The point $x^{(k)}$ is not in $\partial\Sigma(\lambda_k)$ since on $\partial\Sigma(\lambda_k)\cap T_{\lambda_k}$ we have $\bar{w}=0$, and on the remaining part $\partial\Sigma(\lambda_k)\setminus T_{\lambda_k}=\partial\Omega_1^{\lambda_k}\cap H_{\lambda_k}$ we find that $\bar{w}(x,\lambda_k)=(a-u(x))/g(x)>0$. Hence $x^{(k)}\in\Sigma(\lambda_k)$, $\nabla_x\bar{w}\big(x^{(k)},\lambda_k\big)=0$ and $\Delta_x\bar{w}\big(x^{(k)},\lambda_k\big)\geqq 0$. We claim that $x^{(k)}$ are bounded. Suppose not; then $u\big(x^{(k)}\big)\to 0$ and $v\big(x^{(k)},\lambda_k\big)\to 0$ since $0\leqq u(x^{(k),\lambda_k})<u\big(x^{(k)}\big)$. This implies that $c\big(x^{(k)}\big)\leqq 0$ (see (II)) for k large enough, and by the same reasoning as in the initial step we get a contradiction to the differential equation (5′). Now, the boundedness of $x^{(k)}$ allows us to take a convergent subsequence $x^{(k)}\to\bar{x}\in\overline{\Sigma(\mu)}$. At $\bar{x}$ we find that $\nabla_x\bar{w}(\bar{x},\mu)=0$, $\bar{w}(\bar{x},\mu)\leqq 0$ and also $\bar{w}(\bar{x},\mu)\geqq 0$ by the definition of μ. Hence $\bar{w}(\bar{x},\mu)=0$. By the maximum principle, its boundary version and (7′), $\bar{x}$ has to be on the non-smooth part of $\partial\Sigma(\mu)$, i.e., $\bar{x}\in\partial\Omega\cap T_\mu$ where, because $\mu>m$, the positive x_1-direction points out of Ω_1. Let us denote the reflection of $x^{(k)}$ at T_{λ_k} by $y^{(k)}$; clearly $x^{(k)},y^{(k)}\to\bar{x}$. By Proposition 1, there exists a ball $B_\rho(\bar{x})$ such that $\partial_1 u<0$ in $\Omega\cap B_\rho(\bar{x})$. For k suitably large we have $x^{(k)},y^{(k)}\in B_\rho(\bar{x})$ and

$$u(x^{(k)})-u(y^{(k)})=\int_{2\lambda_k-x_1^{(k)}}^{x_1^{(k)}}\partial_1 u(t,x^{(k)\prime})dt<0$$

in contradiction to $w\big(x^{(k)},\lambda_k\big)<0$. This shows that $\mu=m$ and finishes the continuation step.

So far we have the following conclusion: $w(x,m)\geqq 0$ in $\Sigma(m)$ and, in particular, either $w>0$ or $w\equiv 0$ for any component Z of $\Sigma(m)$. If $w\equiv 0$ on a component, then Ω is x_1-symmetric. To finish the proof it remains to show that there is such a component, i.e., that there is one point x in $\Sigma(m)$ with $w(x,m)=0$. In this final step the Neumann boundary condition comes into the play.

Step (VI). Suppose for contradiction that $w>0$ in $\Sigma(m)$. Recall here that the critical position of T_m originates either from $\Sigma_1(m)^m$ becoming internally tangent to $\partial\Omega_1$ or from the x_1-direction becoming tangent to the right-hand boundary $\Gamma_1(m)$.

(a) *Internal tangency*: Let $q = p^m$ be a point where $\partial\Sigma_1(m)^m$ meets $\partial\Omega_1$, $p \in \partial\Sigma_1(m) \setminus T_m$. Clearly $w(p,m) = 0$ so that by the differential inequality (7) and by Hopf's lemma we find $\partial_{\nu(p)} w(p,m) \neq 0$. By the internal tangency we find $\nu(q) = \nu(p)^m$, where $\nu(q)$ is the common normal to $\partial\Omega_1$ and $\partial\Sigma_1(m)^m$ at q, and it is easy to calculate that

$$\frac{\partial v}{\partial \nu(p)}(p,m) = \frac{\partial u}{\partial \nu(q)}(q) = \text{const.} = \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega_1}.$$

Hence $\partial_{\nu(p)} w(p,m) = 0$, in contradiction to our previous assertion.

(b) *There exists a point $p \in T_m \cap \partial\Omega_1$ with $\nu_1(p) = 0$*: Observe that p is a right-angled corner of $\Sigma(m)$ so that a direct application of Hopf's lemma as in (a) is not at hand. Instead we show that w has a zero of second order at p, which contradicts the corner version of Hopf's lemma due to Serrin [14] (see the Appendix). To calculate the derivatives of w we use a rectangular coordinate frame with origin at p, with the ξ_n-axis along the exterior normal $\nu(p)$ to $\partial\Omega_1$ and the ξ_1-axis collinear with the x_1-axis. In this frame $\partial\Omega_1$ is locally expressed by

$$\xi_n = h(\xi_1, \dots, \xi_{n-1}) = h(\xi'), \quad h \in C^2,$$

and the normal $\nu(\xi')$ at $(\xi', h(\xi'))$ is given by

$$\nu(\xi') = \frac{(-\nabla h(\xi'), 1)}{\left(|\nabla h(\xi')|^2 + 1\right)^{1/2}}.$$

That the exterior normal at p coincides with the ξ_n-axis means that $\nabla h(0) = 0$. The point $x = p$ corresponds to $\xi = 0$. Using the ξ-coordinates, we introduce new functions $\tilde{u}$, $\tilde{v}$, $\widetilde{w}$ by

$$\tilde{u}(\xi) = u(x),\ \tilde{v}(\xi) = v(x,m) = u(x^m),\ \widetilde{w}(\xi) = w(x,m)$$

and find the relations

$$\tilde{v}(\xi) = \tilde{u}(-\xi_1, \xi_2, \dots, \xi_n),$$

$$\widetilde{w}(\xi) = \tilde{u}(-\xi_1, \xi_2, \dots, \xi_n) - \tilde{u}(\xi_1, \dots, \xi_n).$$

The Dirichlet and Neumann boundary conditions on $\partial\Omega_1$ now read

$$\tilde{u}(\xi) = \text{const.},\ \frac{\partial \tilde{u}}{\partial \nu}(\xi) = \nabla_\xi \tilde{u}(\xi) \cdot \nu(\xi) = \text{const.},$$

and with the help of the parametrisation of $\partial\Omega_1$ we find

$$\tilde{u}(\xi', h(\xi')) = \text{const.},$$

$$-\sum_{l=1}^{n-1} \tilde{u}_l(\xi', h(\xi'))h_l(\xi') + \tilde{u}_n(\xi', h(\xi')) = \text{const.} \cdot (|\nabla h(\xi')|^2 + 1)^{1/2}.$$

Differentiation with respect to ξ_j for $j = 1, \ldots, n-1$ gives (we use the summation convention for $l = 1, \ldots, n-1$)

$$\tilde{u}_j + \tilde{u}_n h_j = 0,$$

$$-\tilde{u}_{lj} h_l - \tilde{u}_{ln} h_j h_l - \tilde{u}_l h_{lj} + \tilde{u}_{nj} + \tilde{u}_{nn} h_j = \text{const.} \cdot \frac{h_l h_{lj}}{(|\nabla h|^2 + 1)^{1/2}},$$

and evaluation at $\xi' = 0$ results in (recall that $\nabla h(0) = 0$)

$$\tilde{u}_j(0) = 0 \quad \text{for } j = 1, \ldots, n-1, \tag{8}$$

$$\tilde{u}_{nj}(0) = 0 \quad \text{for } j = 1, \ldots, n-1. \tag{9}$$

Let us now collect the results for the derivatives of $\widetilde{w}$ at $\xi = 0$. Notice that $\widetilde{w}_\alpha, \widetilde{w}_{\alpha\beta} = 0$ at $\xi = 0$ for $\alpha, \beta \in \{2, \ldots, n\}$, since at this point ξ and its reflected point coincide. For the same reason $\widetilde{w}_{11}(0) = 0$. Furthermore, $\widetilde{w}_1(0) = -2\tilde{u}_1(0) = 0$ by (8) and $\widetilde{w}_{1n}(0) = -2\tilde{u}_{1n}(0) = 0$ by (9). To obtain the second-order zero of $\widetilde{w}$ at 0 we need to show that the remaining derivatives $\partial_{1\alpha}\widetilde{w}(0)$ vanish for $\alpha = 2, \ldots, n-1$. We do this by using the following Taylor expansion of $\widetilde{w}$:

$$\widetilde{w}(\xi) = \widetilde{w}(0) + \sum_{j=1}^{n} \partial_j \widetilde{w}(0)\xi_j + \sum_{j,k=1}^{n} \partial_{jk}\widetilde{w}(0)\frac{\xi_j\xi_k}{2} + o\left(|\xi|^2\right)$$

$$= \sum_{\alpha=2}^{n-1} \partial_{1\alpha}\widetilde{w}(0)\xi_1\xi_\alpha + o\left(|\xi|^2\right). \tag{10}$$

Fix $\alpha \in \{2, \ldots, n-1\}$, define $\xi = \sigma(\rho, 0, \ldots, 0, \overbrace{\pm\rho}^{\alpha}, 0, \ldots, 0, 1)$ for $\sigma, \rho > 0$, where the plus sign is taken if $\partial_{1\alpha}\widetilde{w}(0) \leqq 0$ and the minus sign if $\partial_{1\alpha}\widetilde{w}(0) > 0$. The point $\xi(\rho, \sigma)$ moves along a straight line through $\xi = 0$ (see Figure 3), which makes the angle

$$\cos\theta(\rho) = \frac{1}{\sqrt{2\rho^2 + 1}}$$

with $v(p)$. Clearly $\theta(\rho) \to 0$ as $\rho \to 0$. Since $\Sigma(m)$ has a rectangular corner at $x = p$ (that is, at $\xi = 0$) and since the ξ_1-component of $\xi(\rho, \sigma)$ is positive, we

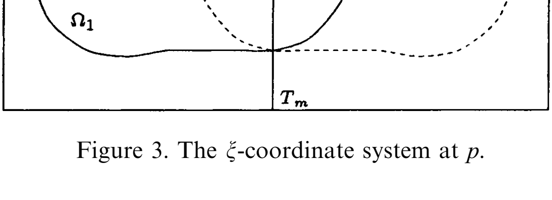

Figure 3. The ξ-coordinate system at p.

find that for ρ and σ small the point $\xi(\rho,\sigma)$ is in $\Sigma(\lambda)$. After this choice of ρ, σ it follows from (10) that

$$\widetilde{w}(\xi(\rho,\sigma)) = -\sigma^2\rho^2|\widetilde{w}_{1\alpha}(0)| + o(\sigma^2) \quad \text{for} \quad \sigma \to 0.$$

Since $\widetilde{w} > 0$ in the vicinity of the boundary point $\xi = 0$, we see that this forces $\widetilde{w}_{1\alpha} = 0$. Hence $\widetilde{w}$ and w have a second-order zero at p in contradiction to Serrin's corner lemma. This finishes the proof of Theorem 1.

Remarks on (*H2*). Under Hypothesis (H2) the coefficients b_i and c in (5) are bounded and c is non-positive. Since for all $\lambda \in (m,\infty)$, we know $w \geqq 0$ on $\partial\Sigma(\lambda)$ and $w = 0$ at ∞ and since $w \equiv 0$ on a component of $\Sigma(\lambda)$ is ruled out by (I), we find that $w(x,\lambda) > 0$ in $\Sigma(\lambda)$ and $\partial_1 w(x,\lambda) > 0$ on $\Omega \cap T_\lambda$ by a direct application of the strong maximum principle and its boundary-point version. Hence (III), the Initial Step (IV) and the Continuation Step (V) are not needed in the case of (H2).

Proof of Corollary 1. Suppose that $f(a,0) \leqq 0$ and $0 \leqq u(x) \leqq a$. Then

$$\begin{aligned} 0 &\leqq \Delta u + f(u,\nabla u) - f(a,0) \\ &\leqq \Delta u + f_1(u,\nabla u) - f_1(a,0) \text{ when } (H_1) \text{ holds,} \end{aligned} \tag{11}$$

and with the usual definitions of bounded functions

$$c(x) = \frac{f_1(u,0) - f_1(a,0)}{u-a},$$

$$b_i(x) = \frac{f_1(u,0,\ldots,0,u_i,\ldots,u_n) - f_1(u,0,\ldots,0,u_{i+1},\ldots,u_n)}{u_i},$$

we deduce from (11) the following linearized inequality with bounded coefficients

$$\begin{aligned} 0 &\leqq \Delta(u-a) + b_i\partial_i(u-a) + c(u-a) \\ &\leqq \Delta(u-a) + b_i\partial_i(u-a) + c^-(u-a). \end{aligned} \tag{12}$$

By the strong maximum principle we get $u < a$ in Ω, so that Theorem 1 applies.

3. The case $n = 2$

Here we prove the analogue of Theorem 2 in the case $n = 2$. Let $A > 0$ be the constant source density on the boundary of the domain Ω_1. Then the potential is given by

$$\Psi(x) = -\frac{A}{2\pi}\int_{\partial\Omega_1} \log|x-y|\, d\sigma_y,$$

and we want to prove that if $\Psi = \text{const.}$ in $\overline{\Omega}_1$, then Ω_1 is a ball in $\mathbb{R}^2$. As in the case $n \geqq 3$ it is straightforward to show that Ψ is harmonic in $\mathbb{R}^2 \setminus \Omega_1$, Ψ

and $\partial\Psi/\partial\nu$ are constant on $\partial\Omega_1$ and that (by the strong maximum principle) $\Psi < \Psi|_{\partial\Omega_1}$. But of course $\Psi(x) \to -\infty$ as $|x| \to \infty$. Hence Theorem 1 is not applicable directly. As before, we are in the situation of Hypothesis (H_2) since Ψ is harmonic. A close investigation of the relevant proof Steps (I), (II), (VI) and the preceding remark reveals that the condition that $u = 0$ at ∞ was only needed to show that $w(x, \lambda) = 0$ at ∞. Hence the proof of Theorem 1 would go through if we could establish that $w(x, \lambda) = 0$ at ∞ for our particular problem. Using the definition of Ψ, we have to investigate the behaviour of

$$\Psi(x^\lambda) - \Psi(x) = \text{const.} \int_{\partial\Omega_1} \log \frac{|x^\lambda - y|^2}{|x - y|^2} d\sigma_y \quad \text{for} \quad x \to \infty.$$

For y in bounded domains we find that

$$\begin{aligned} \log \frac{|x^\lambda - y|^2}{|x - y|^2} &= \log\left(1 + \frac{(2\lambda - x_1 - y_1)^2 - (x_1 - y_1)^2}{|x - y|^2}\right) \\ &= \log\left(1 + \frac{4\lambda^2 + 4x_1y_1 - 4\lambda x_1 - 4\lambda y_1}{|x - y|^2}\right) \\ &= O\left(\frac{1}{|x|}\right) \end{aligned}$$

as $|x| \to \infty$, uniformly in y (since $\partial\Omega_1$ is bounded). So indeed $\Psi(x^\lambda) - \Psi(x) \to 0$ at ∞, and we obtain Theorem 2 for $n = 2$ by the same method as before.

Appendix

Proof of Proposition 1. Clearly $\partial_\eta u(z) \leqq 0$ since $u < a$ by (1). If $\partial_\eta u(z) < 0$, the statement follows by continuity; so let us suppose that $\partial_\eta u(z) = 0$. (This implies that $\nabla u(z) = 0$, since η is non-tangent and $u = \text{const.}$ on $\partial\Omega_1$.)

Case 1. $f(a, 0) \leqq 0$. As in the proof of Corollary 1, we find that

$$0 \leqq \Delta u + f(u, \nabla u) - f(a, 0),$$

which can be rewritten (by an appropriate choice of bounded functions b_i, c) as

$$\begin{aligned} 0 &\leqq \Delta(u - a) + b_i\partial_i(u - a) + c(u - a) \\ &\leqq \Delta(u - a) + b_i\partial_i(u - a) + c^-(u - a). \end{aligned}$$

Hopf's lemma at $z \in \partial\Omega_1$, where $u(z) = a$, yields $\partial_\eta u(z) < 0$ in contradiction to the assumption that $\partial_\eta u(z) = 0$. This shows that Case 1 cannot occur.

Case 2. $f(a, 0) > 0$. We first calculate $\Delta u(z)$ by summing the second directional derivatives with respect to the n vectors $\nu(z), \xi_2, \ldots, \xi_n$, where $\xi_2, \ldots, \xi_n$ are mutually orthogonal and tangent to $\partial\Omega_1$ at z. Since $u = \text{const.}$ on $\partial\Omega_1$, we find that $\Delta u(z) = \partial_\nu^2 u(z)$. Next we calculate $\partial_\eta^2 u(z)$. Using the summation convention for $i, j = 2, \ldots, n$, we have (with $\eta_i = \eta \cdot \xi_i$)

$$\partial_\eta = (\eta \cdot \nu)\partial_\nu + \eta_i \partial_{\xi_i},$$
$$\partial_\eta^2 = (\eta \cdot \nu)^2 \partial_\nu^2 + 2\eta_i \partial_\eta \partial_{\xi_i} - \eta_i \eta_j \partial_{\xi_i} \partial_{\xi_j}.$$

Taking into account that $\partial_{\xi_i}\partial_{\xi_j} u(z) = 0$ since $u = \text{const.}$ on $\partial\Omega_1$, and that $\partial_{\xi_i}\partial_\eta u(z) = 0$ since $\partial_\eta u$ has a local minimum on $\partial\Omega_1$ at z, we apply the operator ∂_η^2 to u at z and find

$$\begin{aligned}\partial_\eta^2 u(z) &= (\eta \cdot \nu(z))^2 \partial_\nu^2 u(z) = (\eta \cdot \nu(z))^2 \Delta u(z) \\ &= -\,(\eta \cdot \nu(z))^2 f(a, 0) < 0.\end{aligned}$$

By continuity, $\partial_\eta^2 u < 0$ in $B_\rho(z) \cap \Omega$. Next we take a smaller ball $B_{\rho_1}(z)$ such that for every $y \in B_{\rho_1}(z) \cap \Omega$ the straight line $y - t\eta$ $(t > 0)$ stays inside $B_\rho(z) \cap \Omega$ until it hits a point $y_0 \in B_\rho(z) \cap \partial\Omega_1$ (see Figure 4). Integrating $\partial_\eta^2 u(x) < 0$ along this straight line connecting y_0 to y, we get $\partial_\eta u(y) < \partial_\eta u(y_0) \leqq 0$ for all $y \in B_{\rho_1}(z) \cap \Omega$. This is the assertion of Proposition 1.

Finally, for completeness, we state (see [14])

Serrin's Corner Lemma. *Let D be a domain lying to the right of the hyperplane T_λ, and let $Q \in \partial D \cap T_\lambda$ be a point where ∂D intersects T_λ orthogonally. Suppose that $w \in C^2(\overline{D})$ satisfies*

$$\Delta w + b_i \partial_i w + c^- w \leqq 0 \quad \text{in} \quad D$$

(b_i, c^- bounded), while $w \geqq 0$ in D and $w(Q) = 0$. Let $m \in \mathbb{R}^n$ be a direction which enters D at Q non-tangentially. Then either

$$\frac{\partial}{\partial m} w(Q) > 0 \quad \text{or} \quad \frac{\partial^2}{\partial m^2} w(Q) > 0$$

unless $w \equiv 0$.

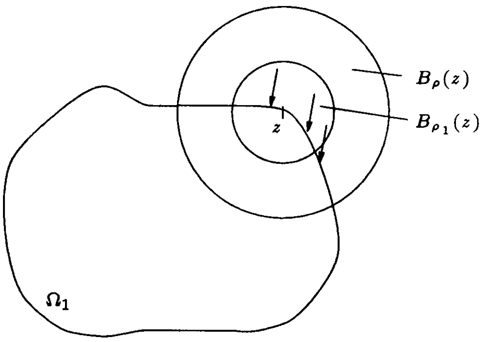

Figure 4. The choice of $B_{\rho_1}(z)$.

Acknowledgement. I thank JAMES SERRIN for many valuable comments and WOLFGANG STAHL (Karlsruhe) for pointing out Gruber's potential–theoretic conjecture and for a number of interesting discussions.

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(Accepted January 29, 1996)