Supercritical variational problems with unique critical points

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Abstract. We study how a simple class of one-parameter transformation-groups interacts with functionals $\mathcal{L}[u] = \int_\Omega L(x, u, \nabla u)\, dx$. We show how one can understand Pohožaev's identity as a formula for the rate-of-change of $\mathcal{L}$ under the action of the transformation group. Uniqueness of the critical point of $\mathcal{L}$ will follow provided the transformation group reduces the values of $\mathcal{L}$. As an application we find a class of conformally contractible domains, which includes the class of star-shaped domains, and on which Pohožaev's uniqueness result remains valid.

1 Transformation groups

We begin our study with one-parameter transformation groups. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain with exterior unit normal $\nu(x)$ for $x \in \partial\Omega$. We consider a one-parameter family of maps $G = \{g_\epsilon : \mathrm{dom} g_\epsilon \subset \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}\}$ enjoying the group-property $g_{\epsilon_1} \circ g_{\epsilon_2} = g_{\epsilon_1+\epsilon_2}$, $g_0 = \mathrm{Id}$. A particularly nice example of such a group arises when g_ϵ is defined as the flow-map at time ϵ of the dynamical system

$$\dot{x} = \xi(x, u), \qquad \dot{u} = \phi(x, u) \tag{1}$$

with smooth functions ξ, ϕ. To be precise, let $(\chi_\epsilon(x_0, u_0), \psi_\epsilon(x_0, u_0))$ denote the solution of (1) at time ϵ with initial conditions (x_0, u_0). Then a one-parameter group G arises by setting $g_\epsilon(x_0, u_0) = (\chi_\epsilon(x_0, u_0), \psi_\epsilon(x_0, u_0))$ for every $(x_0, u_0) \in \Omega \times \mathbb{R}$. In a forthcoming publication [5] we investigate transformation groups arising from the dynamical system (1) together with their interactions with variational problems.

In this paper we study a simple type of transformation group arising from the following dynamical system:

$$\dot{x} = \xi(x), \qquad \dot{u} = a(x)u \tag{2}$$

The system (2) is a special case of (1) and can be integrated semi-explicitly. Since the two equations in (2) can be decoupled, we denote by $\chi_\epsilon(x)$ the solution of the first equation at time ϵ starting with initial condition x at time $\epsilon = 0$. Then $\psi_\epsilon(x,u) = \exp(\int_0^\epsilon a(\chi_\tau(x))\, d\tau)u$ solves the second equation with initial condition u at time $\epsilon = 0$. Let us now take an initial function $u(x)$. We can insert its graph $(x, u(x))$ into the flow and obtain after time ϵ the transformed graph $(\chi_\epsilon(x), \psi_\epsilon(x, u(x))$. For sufficiently small ϵ this will represent the graph of a new function $(\tilde{x}, \tilde{u}(\tilde{x}))$ with new independent variable $\tilde{x} = \chi_\epsilon(x)$. Expressing $x = \chi_{-\epsilon}(\tilde{x})$ we find the following explicit formula for the transformed function

$$
\begin{aligned}
\tilde{u}(\tilde{x}) &= \exp\Big(\int_0^\epsilon a(\chi_{\tau-\epsilon}(\tilde{x}))\, d\tau\Big) u(\chi_{-\epsilon}(\tilde{x})) \\
&= \exp\Big(\int_{-\epsilon}^0 a(\chi_\tau(\tilde{x}))\, d\tau\Big) u(\chi_{-\epsilon}(\tilde{x})),
\end{aligned} \tag{3}
$$

where $\tilde{x}$ belongs to $\Omega_\epsilon = \chi_\epsilon(\Omega)$. We will write $g_\epsilon u$ for the new function $\tilde{u}(\tilde{x})$. In the following we will use summation convention, i.e., double indices will automatically be summed from 1 to n.

Proposition 1 *Let $G = \{g_\epsilon\}_{\epsilon \geq 0}$ be a transformation group generated by (2). Then for every $i = 1, \dots, n$ and every $\tilde{x} \in \Omega_\epsilon$ the following holds*

$$
\begin{aligned}
\frac{\partial \tilde{u}(\tilde{x})}{\partial \tilde{x}_i} &= \exp\Big(\int_{-\epsilon}^0 a(\chi_\tau(\tilde{x}))\, d\tau\Big)\Big) \frac{\partial u}{\partial x_j}(\chi_{-\epsilon}(\tilde{x})) \frac{\partial \chi^j_{-\epsilon}}{\partial x_i}(\tilde{x}) \\
&+ \int_{-\epsilon}^0 \frac{\partial a}{\partial x_j}(\chi_\tau(\tilde{x})) \frac{\partial \chi^j_\tau}{\partial x_i}(\tilde{x})\, d\tau \exp\Big(\int_{-\epsilon}^0 a(\chi_\tau(\tilde{x}))\, d\tau\Big) u(\chi_{-\epsilon}(\tilde{x})).
\end{aligned} \tag{4}
$$

Proof. (4) follows directly from (3). ∎

In order to state the following Proposition let us introduce the space $C^1_0(\overline{D})$ of continuously differentiable functions on D with zero boundary-values and the space $C^{0,1}_0(\overline{D})$ of uniformly Lipschitz-continuous functions on D with zero boundary values.

Proposition 2 *Suppose $\xi(x) \cdot \nu(x) \leq 0$ for all $x \in \partial\Omega$. Then for $\epsilon \geq 0$ we have $\Omega_\epsilon \subset \Omega$. Extending u by zero outside Ω, the transformed function $g_\epsilon u$ is zero in $\Omega \setminus \Omega_\epsilon$. Moreover, if the initial function u belongs to the space $C^1_0(\overline{\Omega})$ or $C^{0,1}_0(\overline{\Omega})$ then for small $\epsilon \geq 0$ we have $g_\epsilon u \in C^1_0(\overline{\Omega}_\epsilon), C^{0,1}_0(\overline{\Omega}_\epsilon)$, respectively. In both cases we have $g_\epsilon u \in C^{0,1}_0(\overline{\Omega})$ for small $\epsilon \geq 0$.*

A transformation half-group $\{g_\epsilon\}_{\epsilon \geq 0}$ arising from a flow of type (2) and with the property $\xi(x) \cdot \nu(x) \leq 0$ on $\partial\Omega$ will be called a **contracting** transformation half-group.

2 Rate of change of a functional under a transformation group

We consider a functional $\mathcal{L}[u] = \int_\Omega L(x, u, \nabla u)\,dx$ where $L : \overline{\Omega} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuously differentiable Lagrangian. Let us suppose w.l.o.g that $L(x, 0, 0) = 0$ for all $x \in \Omega$. It is simple to see that $\mathcal{L}$ is Fréchet-differentiable on the space $C_0^{0,1}(\overline{\Omega})$.

Our goal is to find a formula for the rate of change of $\mathcal{L}$ under the action of a contracting transformation group. The theory of interaction of functionals with transformation groups is well developed, see Olver [2]. In order to keep our point of view accessible for non-experts we decided not just to quote the following theorem from the literature but to include a proof for simple transformation groups generated by (2).

Theorem 3 *Let $G = \{g_\epsilon\}_{\epsilon \geq 0}$ be a contracting transformation half-group generated by (2). Suppose the initial function u belongs to $C_0^1(\overline{\Omega})^\dagger$. Then the rate of change of the functional $\mathcal{L}$ under the transformation group can be computed as follows:*

$$(5)\quad \frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = \int_\Omega \frac{\partial L}{\partial x_i}\xi^i + \frac{\partial L}{\partial u}au + \frac{\partial L}{\partial u_i}\Big(\frac{\partial a}{\partial x_i}u + a\frac{\partial u}{\partial x_i} - \frac{\partial \xi^j}{\partial x_i}\frac{\partial u}{\partial x_j}\Big) + L\mathrm{div}\,\xi\,dx,$$

where the Lagrangian L and its derivatives are evaluated at $(x, u(x), \nabla u(x))$, and the functions a, ξ, u and their derivatives are evaluated at x.

Remark. It turns out that a formula similar to (5) holds for more general transformation groups generated by (1). The proof requires more in-depth analysis of transformation groups and can be found in Olver [2].

Proof. Since $L(x, 0, 0) = 0$ and since $\mathrm{supp}(g_\epsilon u) \subset \Omega_\epsilon$ we may write $\mathcal{L}[g_\epsilon u] = \int_{\Omega_\epsilon} L(\tilde{x}, \tilde{u}, \nabla_{\tilde{x}}\tilde{u})\,d\tilde{x}$. If we consider $\tilde{x} = \chi_\epsilon(x)$ as a change of variables we obtain $\mathcal{L}[g_\epsilon u] = \int_\Omega L(\chi_\epsilon(x), \tilde{u}(\chi_\epsilon(x)), (\nabla_{\tilde{x}}\tilde{u})(\chi_\epsilon(x)))\det(D\chi_\epsilon(x))\,dx$. Differentiation w.r.t. ϵ at $\epsilon = 0$ yields

$$(6)\quad \frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = \int_\Omega \frac{\partial L}{\partial x_i}\xi^i + \frac{\partial L}{\partial u}au + \frac{\partial L}{\partial u_i}\frac{d}{d\epsilon}\Big(\frac{\partial \tilde{u}}{\partial \tilde{x}_i}(\chi_\epsilon(x))\Big)\Big|_{\epsilon=0} + L\mathrm{div}\,\xi\,dx,$$

[†] It is important to have u continuously differentiable. For Lipschitz functions, the formula does not hold in general.

where we have used that $\frac{d}{d\epsilon}\det D\chi_\epsilon(x) = \operatorname{div}\xi(\chi_\epsilon(x))\det D\chi_\epsilon(x)$ and $\chi_0 = \mathrm{Id}$. It remains to express $\partial\tilde{u}/\partial\tilde{x}_i$ in terms of $\chi_\epsilon(x)$ by the help of (4), and to differentiate the result w.r.t. ϵ. Doing so we obtain first by (4)

$$\begin{aligned}\frac{\partial\tilde{u}}{\partial\tilde{x}_i}(\chi_\epsilon(x)) &= \exp\Big(\int_0^\epsilon a(\chi_\tau(x))\,d\tau\Big)\frac{\partial u}{\partial x_j}(x)\frac{\partial\chi^j_{-\epsilon}}{\partial x_i}(\chi_\epsilon(x))\\ &\quad + \int_{-\epsilon}^0 \frac{\partial a}{\partial x_j}(\chi_{\tau+\epsilon}(x))\frac{\partial\chi^j_\tau}{\partial x_i}(\chi_\epsilon(x))\,d\tau\,\exp\Big(\int_0^\epsilon \chi_\tau(x)\,d\tau\Big)u(x),\end{aligned}$$

and after differentiation w.r.t. ϵ at $\epsilon = 0$ and using $\chi_0 = \mathrm{Id}$, $\partial\chi_0^j/\partial x_i = \delta_{ij}$ we get

$$\begin{aligned}(7)\quad \frac{d}{d\epsilon}\Big(\frac{\partial\tilde{u}}{\partial\tilde{x}_i}(\chi_\epsilon(x))\Big)\Big|_{\epsilon=0} &= a(x)\frac{\partial u}{\partial x_i}(x)\\ &\quad + \frac{\partial u}{\partial x_j}(x)\Big(-\frac{\partial\xi^j}{\partial x_i}(x) + \frac{\partial^2\chi^j_\epsilon}{\partial x_i\partial x_l}(\chi_\epsilon(x))\frac{\partial\chi^l_\epsilon(x)}{\partial\epsilon}\Big|_{\epsilon=0}\Big) + \frac{\partial a}{\partial x_i}(x)u(x).\end{aligned}$$

The last relation simplifies since $\partial^2\chi_0^j/\partial x_i\partial x_l = 0$. The result follows by inserting (7) back into (6). ∎

Corollary 4 *Let $G = \{g_\epsilon\}_{\epsilon\geq 0}$ be a contracting transformation half-group generated by* (2). *Suppose that the initial function u belongs to $C^2(\Omega)\cap C_0^1(\overline{\Omega})$. Then the rate of change of the functional under the transformation group satisfies*

$$(8)\quad \frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = \int_\Omega (au - \xi\cdot\nabla u)\mathcal{E}[u]\,dx + \int_{\partial\Omega}(\xi\cdot\nu)(L - \frac{\partial u}{\partial x_i}\frac{\partial L}{\partial u_i})\,d\sigma,$$

where $\mathcal{E}[u] = -\frac{d}{dx_i}(\frac{\partial L}{\partial u_i}) + \frac{\partial L}{\partial u}$ is the Euler-operator associated to the functional $\mathcal{L}$. As before the Lagrangian L and its derivatives are evaluated at $(x, u(x), \nabla u(x))$, the functions a, u, ξ and their derivatives at x.

Proof. The proof hinges on a formula of E. Noether, which she used in the derivation of her famous theorem on symmetry and conservation laws, cf. Olver [2], Section 4.4. Noether's formula expresses the integrand in (5) as a multiple of the Euler-equation plus a divergence-term, i.e.,

$$\begin{aligned}(9)\quad &\frac{\partial L}{\partial x_i}\xi^i + \frac{\partial L}{\partial u}au + \frac{\partial L}{\partial u_i}(\frac{\partial a}{\partial x_i}u + a\frac{\partial u}{\partial x_i} - \frac{\partial\xi^j}{\partial x_i}\frac{\partial u}{\partial x_j}) + L\operatorname{div}\xi\\ &\qquad = (au - \xi\cdot\nabla u)\mathcal{E}[u] + \frac{d}{dx_i}(\xi^i L + (au - \xi\cdot\nabla u)\frac{\partial L}{\partial u_i}).\end{aligned}$$

Formula (9) is verified by explicit calculation. If we recall that $u = 0$ on $\partial\Omega$ and hence $\nabla u = \nu \partial u / \partial \nu$, an integration by parts results in (8) and completes the proof. ∎

Remark. For solutions of the Euler-equation $\mathcal{E}[u] = 0$ the identity between the volume integral (5) in Theorem 3 and the surface integral (8) in Corollary 4 is known as Pohožaev's identity, cf. Pohožaev [3] and Pucci and Serrin [4]. It has long been noted, cf. Pucci, Serrin [4] and Van der Vorst [7], that some kind of connection between one-parameter transformation groups, Noether's formula and Pohožaev's identity exists. For yet another variational approach for deriving Pohožaev's identity, which is especially useful for free-boundary problems, see Wagner [8].

3 Variational sub-symmetries and uniqueness of critical points

We can now begin to formulate our main result. Loosely speaking it states under mild assumptions on the Lagrangian that a contracting transformation half-group, which reduces the values of $\mathcal{L}$, implies the uniqueness of the critical point of $\mathcal{L}$.

Definition 5 *Let the functional $\mathcal{L}[u] = \int_\Omega L(x, u, \nabla u)\, dx$ be defined on $C_0^{0,1}(\overline{\Omega})$. A contracting transformation half-group $G = \{g_\epsilon\}_{\epsilon \geq 0}$ generated by* (2) *is called a* **variational sub-symmetry** *for $\mathcal{L}$ if for all $u \in C_0^{0,1}(\overline{\Omega})$ there exist $\epsilon_0(u) > 0$ such that $\mathcal{L}[g_\epsilon u] \leq \mathcal{L}[u]$ for all $\epsilon \in (0, \epsilon_0(u))$.*

Proposition 6 *Let $G = \{g_\epsilon\}_{\epsilon \geq 0}$ be a contracting transformation half-group generated by* (2)*. Then*

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} \leq 0 \text{ for all } u \in C_0^1(\overline{\Omega}) \tag{10}$$

is necessary and sufficient for G to be a variational sub-symmetry. If (10) *holds with strict inequality for all $u \in C_0^1(\overline{\Omega})$ unless $u \equiv 0$ then G is called a strict variational sub-symmetry w.r.t* 0*.*

The importance of the proposition lies in the fact that one can use the explicit formula for $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0}$ given in (5) to verify that G is a variational sub-symmetry. We are now ready to state our main uniqueness result, which follows directly from Corollary 4.

Theorem 7 *Suppose G is a variational sub-symmetry generated by (2) for the functional $\mathcal{L}[u] = \int_\Omega L(x, u, \nabla u)\, dx$ on the space $C_0^{0,1}(\overline{\Omega})$. Suppose also that $L(x, 0, \mathbf{p})$ is convex in $\mathbf{p} \in \mathbb{R}^n$ for all $x \in \partial\Omega$. Then either of the following two conditions implies that $u = 0$ is the only critical point of $\mathcal{L}$:*

(i) G is a strict variational sub-symmetry w.r.t. 0,

(ii) if $u \in C^2(\Omega) \cap C_0^1(\overline{\Omega})$ is a solution of $\mathcal{E}[u] = 0$ and $(\xi \cdot \nu)(L(x, 0, \nabla u) - \frac{\partial u}{\partial x_i} \frac{\partial L}{\partial u_i}(x, 0, \nabla u)) = 0$ on $\partial\Omega$ then necessarily $u \equiv 0$.

Remark. (1) For a strict variational sub-symmetry the theorem can be understood heuristically: since a strict variational sub-symmetry strictly reduces the energy $\mathcal{L}[u]$ everywhere except at $u = 0$ it is only compatible with a (unique) global minimizer at $u = 0$. For example, a strict variational sub-symmetry applied to a mountain-pass reduces the energy everywhere along a minimizing path – which contradicts the definition of a mountain-pass.
(2) A verification of condition *(ii)* is usually achieved via a unique continuation principle, cf. Section 4.1.

4 Applications

4.1 A generalization of Pohožaev's result

In 1965 S.I. Pohožaev proved a remarkable identity [3], which allowed him to conclude that the problem

$$\Delta u + |u|^{p-1} u = 0 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega \tag{11}$$

has only the trivial solution if $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is a bounded, star-shaped domain and $p \geq \frac{n+2}{n-2}$. Theorem 7 recovers this result if we consider the transformation group generated by

$$\dot{x} = -x, \qquad \dot{u} = \frac{n-2}{2} u,$$

i.e., $g_\epsilon u(\tilde{x}) = e^{\frac{n-2}{2}\epsilon} u(e^\epsilon \tilde{x})$ with $\tilde{x} \in e^{-\epsilon}\Omega$ and if we assume w.l.o.g that Ω is star-shaped w.r.t. 0. It turns out that $G = \{g_\epsilon\}_{\epsilon \geq 0}$ is a variational sub-symmetry for $\mathcal{L}[u] = \int_\Omega \frac{1}{2}|\nabla u| - \frac{1}{p+1}|u|^{p+1}\, dx$. The functional verifies the conditions of Theorem 7. In particular property *(ii)* is fulfilled, since there exists a subset of $\partial\Omega$ of positive measure with $x \cdot \nu(x) > 0$ and therefore the vanishing of $x \cdot \nu(x)|\nabla u|^2$ on $\partial\Omega$ as assumed in *(ii)* implies that $|\nabla u|^2 = 0$ on a subset of $\partial\Omega$ of positive measure. By the unique continuations theorem $u \equiv 0$ follows. In case $p > \frac{n+2}{n-2}$ we could argue without the unique continuation theorem, since in that case G is a strict variational sub-symmetry.

Now we want to investigate more systematically what kind of vector-fields $\xi(x)$ and what kind of functions $a(x)$ can generate a variational sub-symmetry for the functional $\mathcal{L}[u]$ via

$$\dot{x} = \xi(x), \qquad \dot{u} = a(x)u.$$

To verify the infinitesimal criterion of Proposition 6 we use (5) and calculate

$$\begin{aligned} \frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} &= \int_\Omega -a|u|^{p+1} + u\nabla a\cdot\nabla u + a|\nabla u|^2 - \nabla u D\xi \nabla u\, dx \\ &\quad + \int_\Omega \operatorname{div}\xi(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1})\, dx \\ &= \int_\Omega (-a - \frac{\operatorname{div}\xi}{p+1})|u|^{p+1} + \operatorname{div}(\nabla a\frac{u^2}{2}) - \Delta a\frac{u^2}{2}\, dx \\ &\quad + \int_\Omega |\nabla u|^2(a + \frac{\operatorname{div}\xi}{2}) - \nabla u D\xi\nabla u\, dx, \end{aligned} \tag{12}$$

where the integral over $\operatorname{div}(\nabla a u^2/2)$ vanishes. In Pohožaev's original result with $\xi = -x$ the term $\nabla u D\xi \nabla u$ turned out to be just $-|\nabla u|^2$ which equals $\frac{\operatorname{div}\xi}{n}|\nabla u|^2$. In order to obtain similar best possible results for more general ξ we are interested in those vector fields ξ such that

$$\mathbf{b}D\xi\mathbf{b} = \frac{\operatorname{div}\xi}{n}|\mathbf{b}|^2 \text{ for all } \mathbf{b}\in\mathbb{R}^n. \tag{13}$$

Lemma 8 *In $\mathbb{R}^n$, $n \geq 3$ all smooth vector-fields ξ which satisfy* (13) *are (up to a constant shift) given as linear combinations of the vector-fields*

$$\begin{aligned} X &= x_i e_i, \\ Y_{ij} &= x_j e_i - x_i e_j \text{ for } 1 \leq i < j \leq n, \\ Z_i &= (x_i^2 - \textstyle\sum_{j\neq i}^n x_j^2)e_i + 2\sum_{j\neq i}^n x_i x_j e_j \text{ for } i = 1\ldots n, \end{aligned}$$

where e_i denotes the unit-vector in the x_i coordinate-direction. The linear combinations of these vector-fields are called **conformal** *vector-fields.*

The proof is obtained by showing first that all component-functions of ξ are second-order polynomials. It is then easy to find the precise form of ξ. In dimension $n = 2$ one can show that the conformal vector-fields are exactly those vector-fields $\xi = (\xi^1, \xi^2)$ such that $\xi^1(x,y) + i\xi^2(x,y)$ is holomorphic in $x + iy$. Applications of this observation are discussed in Reichel [5].

We use such a conformal vector-field ξ and return to (12) to find

$$\begin{aligned}\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} &= \int_\Omega (-a-\frac{\operatorname{div}\xi}{p+1})|u|^{p+1} - \Delta a\frac{u^2}{2}\,dx \\ &\quad + \int_\Omega |\nabla u|^2(a+\frac{n-2}{2n}\operatorname{div}\xi)\,dx.\end{aligned}$$

In order to achieve $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} \leq 0$ it is helpful to choose $a = \frac{2-n}{2n}\operatorname{div}\xi$. If we note that all our conformal vector-fields satisfy $\Delta\operatorname{div}\xi = 0$ we finally obtain

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = \int_\Omega (\frac{n-2}{2n} - \frac{1}{p+1})|u|^{p+1}\operatorname{div}\xi\,dx.$$

Hence the sub-symmetry criterion of Proposition 6 will be fulfilled provided $\operatorname{div}\xi \leq 0$ and $p \geq \frac{n+2}{n-2}$ †. Uniqueness of the solution $u \equiv 0$ of (11) now follows from Theorem 7. We sum up our results:

Definition 9 *A bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is called* **conformally contractible** *if there exists a conformal vector-field ξ such that $\xi(x)\cdot\nu(x) \leq 0$ on $\partial\Omega$ with strict inequality on a subset of $\partial\Omega$ of positive measure. The vector-field ξ is called an* **associated** *vector-field to Ω.*

Theorem 10 *(i) Let $p \geq \frac{n+2}{n-2}$. If $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is a bounded conformally contractible domain with associated vector-field ξ and if additionally* $\operatorname{div}\xi \leq 0$ *in Ω then* (11) *has only the zero-solution.*
(ii) In the special case $p = \frac{n+2}{n-2}$ the additional assumption $\operatorname{div}\xi \leq 0$ *is not needed.*

Remark. Part *(i)* of the theorem holds if we replace the non-linearity $|u|^{p-1}u$ in (11) by $f(u)$ and assume that $F(tu)/t^{\frac{n+2}{n-2}}$ is non-decreasing in $t \in \mathbb{R}$ for all $u \in \mathbb{R}$, where $F(s) = \int_0^s f(\tau)\,d\tau$.

We give two examples of domains in $\mathbb{R}^3$, which are not star-shaped but conformally contractible, and the associated vector-field ξ satisfies $\operatorname{div}\xi \leq 0$.

Example 1: We take $\xi = (-x+y, -y-x, -z)$ with $\operatorname{div}\xi = -3$. The vector-field ξ generates a composition of a dilation and a rotation in the x,y-plane. We construct a conformally contractible domain by extending a 2d-domain cylindrically in the z-direction. Both the 2d-cut and the 3d-domain are shown in Figure 1. In the 2d-domain the trajectories of the flow $(\dot{x},\dot{y}) = (-x+y, -y-x)$ starting from the boundary are shown. Clearly the 2d-domain is positively-invariant under the flow, i.e. our transformation

† The alternative $\operatorname{div}\xi \geq 0$ and $p \leq \frac{n+2}{n-2}$ must be discarded for bounded domains since we also require $\xi\cdot\nu \leq 0$ on $\partial\Omega$. It does however make sense if Ω is the complement of a bounded domain.

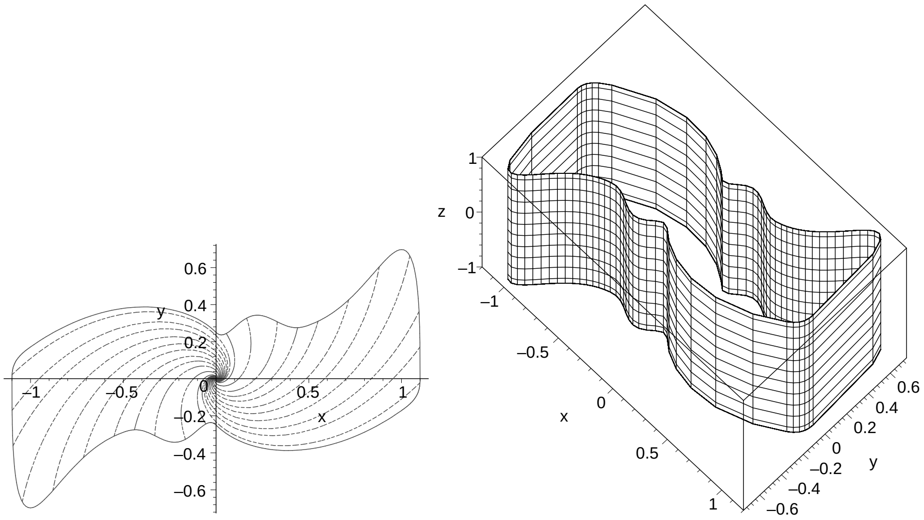

Figure 1: 2d-cut and 3d-domain

group is contracting. By the cylindrical extension this remains true for the 3d-domain.

Example 2: We take $\xi = (-2xz, -2yz, -z^2 + x^2 + y^2)$ with $\operatorname{div} \xi = -6z$. The vector-field ξ is the infinitesimal generator of a one-parameter group of conformal maps involving inversions. We construct a conformally contractible domain by rotating a planar domain around the z-axis. The flow $(\dot{x}, \dot{y}, \dot{z}) = \xi(x, y, z)$ is also rotationally symmetric around the z-axis. In Figure 2 the 2d-cut and the trajectories starting from the boundary are visualized. Again the 2d-domain is positively-invariant, and due to the rotation-symmetry this remains true for the 3d-domain. Hence our transformation

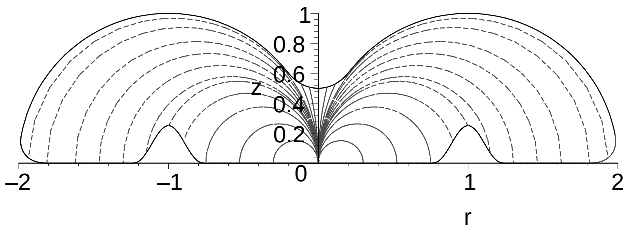

Figure 2: 2d-domain and trajectories

group is contracting, and since the domain lies in the region $z \geq 0$ we have $\operatorname{div} \xi \leq 0$. In Figure 3 the 3d-domain is visualized from above and from below.

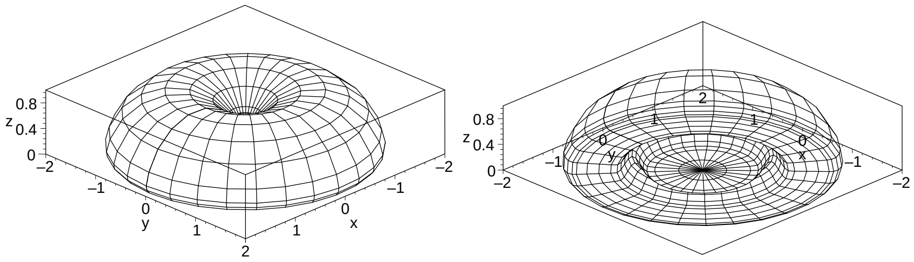

Figure 3: 3d-domain from above and below

4.2 A problem of Brezis and Nirenberg

In a famous paper [1] Brezis and Nirenberg showed that the problem

$$\Delta u + \lambda u + u^5 = 0 \text{ in } B_1(0) \subset \mathbb{R}^3, \qquad u = 0 \text{ on } \partial B_1(0)$$

has a positive solution precisely for $\lambda \in (\pi^2/4, \pi^2)$. The fact that no positive solution exists for $\lambda \leq 0$ follows from Theorem 10. The non-existence for $\lambda \in [0, \pi^2/4]$ was found by Brezis and Nirenberg via a Pohožaev-type identity. We indicate how this result follows by the method of transformation groups. Since all positive solutions are radially symmetric, a corresponding functional $\mathcal{L}[u] = \int_0^1 L(r, u, u')\, dr$ has Lagrangian $L(r, u, u') = (\frac{1}{2}|u'|^2 - \frac{\lambda}{2}u^2 - \frac{1}{6}u^6)r^2$. It turns out that for $\lambda \geq 0$ the dynamical system

$$\dot{r} = -\sin(2\sqrt{\lambda}r), \qquad \dot{u} = \Big(\frac{\sin(2\sqrt{\lambda}r)}{r} - \sqrt{\lambda}\cos(2\sqrt{\lambda}r)\Big)u$$

generates a transformation-half group which can be shown to be a variational sub-symmetry for the functional $\mathcal{L}$ by using Proposition 6 and (5). However, the half-group is contracting if and only if $\dot{r} \leq 0$, i.e. if and only if $\sin(2\sqrt{\lambda}r) \geq 0$, which restricts λ to the interval $[0, \pi^2/4]$. Uniqueness of the zero-solution for such values of λ is then obtained by Theorem 7.

4.3 Further applications

In our forthcoming work [5] we have included many applications of the interaction of one-parameter transformations groups and variational problems.

We end this paper with a list of interesting applications which can be treated by our method.

(1) For vector variational problems $\int_\Omega L(x, u^1, \dots, u^k, \nabla u^1, \nabla u^k)\, dx$ Theorem 7 will hold provided $L(x, 0, \dots, 0, \mathbf{p}^1, \dots, \mathbf{p}^k)$ is rank-one convex.

(2) For m-Laplace problems div $(|\nabla u|^{m-2}\nabla u) + |u|^{p-1}u = 0$ in Ω with $u = 0$ on $\partial\Omega$ and $p \geq \frac{nm}{n-m} - 1$ uniqueness of the zero-solution will hold for conformally contractible domains, where up to a constant shift the associated vector-field ξ is a linear combination of vector-fields X and Y_i from Lemma 8. In particular the domain of Example 1 is included.

(3) For the forth-order problem $\Delta^2 u = |u|^{p-1}u$ in Ω with $u = \nabla u = 0$ on $\partial\Omega$ the same class of conformally contractible domains as in Theorem 10 gives rise to uniqueness of the zero-solution provided $p > \frac{n+4}{n-4}$.

(4) Schaaf [6] proved uniqueness results in a different class of domains for exponents $p \geq p_c$, where p_c is larger then the critical Sobolev-exponent.

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