ON ISOPHONIC SURFACES *

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We present some remarks about the conjecture *Drums in the night.*

1. Introduction

In this note we will present some remarks on a conjecture that was posed by L. Zalcman under the title *Drums in the night.*

> *Drums in the night* [8]. A thin elastic membrane M of uniform areal density σ is stretched to a uniform tension T and held fixed at its boundary Γ, a simple closed curve. The small transverse vibrations of M can be modeled as solutions $u(x,t)$ of the wave equation in D, the region bounded by Γ, which vanish on Γ :
>
> $$\Delta u = \frac{1}{c^2}\, u_{tt} \qquad x \in D,\ t > 0, \tag{1}$$
>
> $$u(x,t) = 0 \qquad x \in \Gamma,\ t > 0. \tag{2}$$
>
> Here $c = \sqrt{T/\sigma}$ is the wave velocity and Δ is the Laplacian with respect to $x = (x_1, x_2)$.
>
> Suppose some solution u of (1) and (2) has the property that ∇u vanishes identically on a simple closed curve $\gamma \subset D \cup \Gamma$. Must Γ be a circle?
>
> In case Γ *is* a circle (of radius R, say, about the origin), the function $u(x,t) = J_0(k\,|x|)\, e^{ickt}$ will satisfy (1) and (2) if kR is a zero of the Bessel function J_0. Since $J_0' = -J_1$, $\nabla u = -kJ_1(k\,|x|)\, e^{ickt}\frac{x}{|x|}$.

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Thus if $J_1(kr) = 0$, ∇u will vanish on the circle of radius r concentric with Γ. Choosing k sufficiently large yields solutions of (1) and (2) which vanish on a family of such circles.

2. *Drums in the night,* Schiffer's conjecture and Pompeiu's problem

A solution of (1)-(2) can be written as a series expansion,

$$u(x,t) = \sum_{n=1}^{+\infty} u_n(x)\ [a_n \cos(c\sqrt{\lambda_n}t) + b_n \sin(c\sqrt{\lambda_n}t)], \tag{3}$$

where $u_n,\ n = 1, 2, \dots$ are the Dirichlet eigenfunctions of the Laplacian, and $\lambda_n,\ n = 1, 2, \dots$ the corresponding eigenvalues, that is, u_n and λ_n satisfy the problem:

$$\Delta u + \lambda u = 0 \ \text{ on } \ D, \quad u = 0 \ \text{ on } \ \Gamma. \tag{4}$$

If we require that the gradient of the function u defined in (3) vanishes identically on γ,

$$\nabla u(x,t) = 0 \qquad x \in \gamma,\ t > 0, \tag{5}$$

then we obtain that

$$\nabla u_n(x) = 0 \qquad x \in \gamma,\ n \in \mathcal{N}, \tag{6}$$

where

$$\mathcal{N} = \{n \in \mathbb{N} : (a_n, b_n) \neq 0\}. \tag{7}$$

The set $\mathcal{N}$ can be finite or infinite. If Ω denotes the interior of γ, then each $u_n, n \in \mathcal{N}$, satisfies *Schiffer's overdetermined boundary value problem:*

$$\begin{array}{ll} \Delta u + \lambda u = 0 & \text{in } \ \Omega, \\ u = \text{constant} & \text{on } \ \gamma, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \ \gamma, \end{array} \tag{8}$$

where ν is the exterior normal unit vector to γ.

It is well-known [5] [1] that if a non-trivial solution of problem (8) exists, then the set Ω does not enjoys the *Pompeiu property* [4] [7], that is there exists a function $f : \mathbb{R}^2 \to \mathbb{R}$, f not identically zero, such that

$$\int_{\sigma(\Omega)} f(x)\ dx = 0 \ \text{ for all rigid motions } \sigma. \tag{9}$$

Viceversa, if $f \neq 0$ exists such that (9) holds, then a non-trivial solution of (8) exists.

An old conjecture [4] states that the only domain not enjoying the Pompeiu property is the disk. Although this conjecture has not been proved or disproved up to now, a great variety of results are known on domains not satisfying the Pompeiu property. Having established a connection between the overdetermined problems (1)-(2)-(5) and (8), we can claim, for instance, that, if u is a solution of (1), (2) satisfying (5), then γ is a real analytic curve, by invoking Williams's result [6].

Moreover, a symmetry result can be drawn.

Proposition 2.1. *Let a solution u of (1) and (2) satisfy condition (5) and suppose that γ is a simple closed curve of class $C^{2,\varepsilon}, \varepsilon > 0$.*

If the set $\mathcal{N}$ defined in (7) is infinite, then D is a disk.

Proof. Proposition 2 [1] states that if the eigenvalue problem (8) has infinitely many solutions, then Ω must be a disk. Hence, each u_n is a Neumann eigenfunction for the disk Ω. By continuing analytically u_n to D, we infer that Γ is a circle. □

If Schiffer's conjecture for the domain Ω were true, we could also settle down the case where the set $\mathcal{N}$ is finite. It should be noticed though that, even in the least favourable case where set $\mathcal{N}$ is made of a single element n_0, the overdetermined problem (1)-(2)-(5) gives more information than Schiffer's eigenvalue problem (8). In fact, in problem (1)-(2)-(5), we assume the existence of a Dirichlet eigenfunction u_{n_0} in a domain D that *contains* Ω.

In the following result, we try to exploit this observation.

Proposition 2.2. *A solution u of (1) and (2) satisfies condition (5) if and only if, for every positive number r with $r <$ dist (γ, Γ), we have that*

$$\int_{|y-x|=r} u_n(y)\ (y-x)\ dS_y = 0, \quad \text{for every } x \in \gamma \text{ and } n \in \mathcal{N}. \tag{10}$$

Proof. Consider the function

$$h(x,t) = \sum_{n \in \mathcal{N}} c_n\ u_n(x)\ e^{-\lambda_n t}, \tag{11}$$

where the numbers $c_n, n \in \mathcal{N}$, are arbitrarily chosen; $h(x,t)$ is a solution of the heat equation

$$\Delta h = h_t \qquad x \in D,\ t > 0. \tag{12}$$

By Theorem 2 [2] or Corollary 2.2 [3], we have that $\nabla h(x,t) = 0$ for every $t > 0$ if and only if

$$\int\limits_{|y-x|=r} h(y,t)(y-x)\ dS_y = 0 \ \text{ for every } \ 0 < r < \text{dist } (x,\Gamma) \text{ and } t > 0. \quad (13)$$

Therefore, the assertion of Proposition 2.2 follows from (13) and the definition (11) of h by the arbitrary choice of the c_n's. ☐

3. *Drums in the night* and isophonic curves

We observe that, if the gradient of a solution u of (1) and (2) vanishes on γ, then, in particular, γ is a *stationary isophonic curve* for u, *i. e.*

$$u(x,t) = U(t), \ x \in \gamma, \ t > 0, \quad (14)$$

where U is some real-valued function. Of course, the requirement that γ be a stationary isophonic curve for u is less strict than asking that the gradient of u vanishes on γ; hence, it is less likely that the existence of a stationary isophonic curve for u imply that D is a disk. In order to get symmetry, we need some additional information on u, as the following result shows.

Proposition 3.1. *Let u be a solution u of (1) and (2) such that*

$$u(x,0) = 0 \ \text{ and } \ u_t(x,0) = 1, \qquad x \in D. \quad (15)$$

Assume that γ is a simple closed curve such that Ω satisfies the interior cone condition.

If u satisfies condition (14), then D must be a disk.

We recall that Ω satisfies the *interior cone condition* if for every $x \in \gamma$ there exists a finite right spherical cone K_x with vertex at x such that $K_x \subset \overline{\Omega}$ and $\overline{K_x} \cap \gamma = \{x\}$.

Proof. If we extend u by $-u(x,-t)$ for $t < 0$, then u satisfies (1) and (2) in $D \times (-\infty, +\infty)$. The function defined for $(x,t) \in D \times (0,+\infty)$ by

$$h(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-s^2/4t} u_t(x,s)\ ds$$

then satisfies the Cauchy-Dirichlet boundary value problem:

$$\begin{array}{lll} h_t = \Delta h & \text{in} & \Omega \times (0,+\infty), \\ h = 0 & \text{on} & \partial\Omega \times (0,+\infty), \\ h = 1 & \text{on} & \Omega \times \{0\}. \end{array}$$

Moreover, if γ is a stationary isophonic curve for w, then Γ is a *stationary isothermic curve* for h, since

$$h(x,t) = H(t) := \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-s^2/4t} U'(s) \; ds \quad x \in \gamma.$$

The conclusion then follows from Theorem 1.1 [3]. □

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