STATIONARY ISOTHERMIC SURFACES FOR UNBOUNDED DOMAINS

ROLANDO MAGNANINI AND SHIGERU SAKAGUCHI

Abstract. The initial temperature of a heat conductor is zero and its boundary temperature is kept equal to one at each time. The conductor contains a stationary isothermic surface, that is, an invariant spatial level surface of the temperature. In a previous paper, we proved that, if the conductor is bounded, then it must be a ball. Here, we prove that the boundary of the conductor is either a hyperplane or the union of two parallel hyperplanes when it is unbounded and satisfies certain global assumptions.

1. Introduction

Let $u = u(x,t)$ be the unique (bounded) solution of the following problem for the heat equation:

$$\partial_t u = \Delta u \quad \text{in} \quad \Omega \times (0,+\infty), \tag{1.1}$$

$$u = 1 \quad \text{on} \quad \partial\Omega \times (0,+\infty), \tag{1.2}$$

$$u = 0 \quad \text{on} \quad \Omega \times \{0\}, \tag{1.3}$$

where Ω is a domain in $\mathbb{R}^N$, $N \geq 2$.

¿From a physical point of view, $u(x,t)$ can be regarded as the normalized temperature of a conductor Ω at the point $x \in \Omega$ and time $t > 0$. Keeping this in mind, we will say that an $(N-1)$-dimensional manifold $\Gamma \subset \Omega$ is an *isothermic surface* if u is constant on Γ for some time $t_0 > 0$; also, Γ is said to be a *stationary isothermic surface* if Γ is an isothermic surface for every $t > 0$, that is

$$u(x,t) = a(t), \quad (x,t) \in \Gamma \times (0,+\infty), \tag{1.4}$$

for some function $a : (0,+\infty) \to (0,+\infty)$.

Stationary isothermic surfaces were considered in [2], [3], [11] and [9]. In [2], by using Serrin's symmetry result on overdetermined boundary value problems (see [13]), Alessandrini proved that, if Ω is bounded and every point in $\partial\Omega$ is regular for the Dirichlet problem for the Laplace equation, then the requirement that *all* isothermic surfaces of u be stationary implies that Ω must be a ball. In [11], a different proof of Alessandrini's result was given with the aid of the classification theorem for isoparametric hypersurfaces in Euclidean space due to Levi-Civita [8] and Segre [12]. Such a proof extends to the case where the Dirichlet condition (1.2) is replaced by the homogeneous Neumann condition.

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A substantial improvement to Alessandrini's result was given in [9], in which the authors proved that, if Ω is bounded, $\partial\Omega$ satisfies the exterior sphere condition and u has *at least one* stationary isothermic surface Γ which is the boundary of a domain D compactly contained in Ω satisfying the interior cone condition, then Ω must be a ball (see Theorem 1.1 in [9]).

This result is based on the fact that, if Ω contains a stationary isothermic surface, then there exists a positive constant $c > 0$ such that

$$\prod_{j=1}^{N-1} (1 - R\,\kappa_j) = c, \quad \text{on } \partial\Omega, \tag{1.5}$$

where $R > 0$ is the distance between the stationary isothermic surface and $\partial\Omega$, and where $\kappa_j,\ j = 1, \dots, N-1$ are the principal curvatures of $\partial\Omega$ with respect to the interior normal ν to $\partial\Omega$ (with this orientation of ν, if all κ_j's are non-negative, then Ω is convex).

The proof of (1.5) essentially relies on the presence of a boundary layer when $t \to 0^+$ and on two observations: if Γ is a stationary isothermic surface for u, then (i) Γ is an analytic surface parallel to $\partial\Omega$; (ii) if r and t are fixed, the *heat content* $\int_{B(x,r)} u(y,t)\ dy$ of any ball $B(x,r)$ contained in Ω is constant for $x \in \Gamma$, where $B(x,r)$ denotes an open ball with radius $r > 0$ and centered at $x \in \mathbb{R}^N$.

When $\partial\Omega$ is bounded, (1.5) implies that $\partial\Omega$ is a sphere, by a general version of Aleksandrov's Soap Bubble Theorem (see [1]).

In this paper we study unbounded domains which contain a stationary isothermic surface.

We immediately observe that, by the same arguments employed in [9], we can prove spherical symmetry for $\partial\Omega$ if Ω is an *exterior domain,* that is a domain whose complement is bounded (Theorem 3.1).

The situation substantially changes when also $\partial\Omega$ is unbounded. Even if, by adjusting the techniques used in [9], we can show that (1.5) holds for a large class of domains (see Lemma 2.4), it is not clear whether (1.5) is sufficient to infer some symmetry of Ω.

We incidentally observe that, in the context of unbounded domains, we see better the difference between Alessandrini's assumption and ours: the former carries a lot more information than the latter. In fact, as we prove in Theorem 3.5, if we assume that u admits $N-1$ distinct isothermic surfaces $\Gamma_1, \dots, \Gamma_{N-1}$, then (1.5) holds for R that takes the values $R_k = \text{dist}\,(\Gamma_k, \partial\Omega),\ k = 1, \dots, N-1$ and c that possibly takes $N-1$ distinct positive values $c_1, \dots, c_{N-1}$. Thus, we can conclude that each principal curvature of $\partial\Omega$ is constant, which implies that every connected component of $\partial\Omega$ is an isoparametric surface, that is it is either a sphere or a spherical cylinder or an hyperplane. It is evident that the same conclusions hold if all isothermic surfaces are stationary.

The case in which only one isothermic surface of u is stationary needs more attention. We observe that, in order to derive spherical symmetry from (1.5), we needed that $\partial\Omega$ were bounded — a *global* assumption. When $\partial\Omega$ is unbounded, it is our opinion that some global information on $\partial\Omega$ can be derived from the behavior of u for large times. So far, we have not been able to exploit this idea efficiently.

However, in this paper, we prove some symmetry results for unbounded domains by adding extra global assumptions on their boundaries. We consider three cases that we summarize in Theorems 3.2, 3.3 and 3.4.

In Theorem 3.2, we assume that $\partial\Omega$ is the graph of locally Lipschitz continuous function $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ whose gradient does not grow too rapidly at infinity ($|\nabla\varphi(x')| = o(|x'|^{1/2})$ as $|x'| \to +\infty$), and prove that, if u has a stationary isothermic surface, then $\partial\Omega$ must be a hyperplane. In our proof, we apply to equation (1.5) a form of Bernstein's theorem due to Caffarelli, Nirenberg and Spruck [5].

In Theorem 3.3, we use (1.5) to show that $\partial\Omega$ must be either a hyperplane or the union of two parallel hyperplanes, when the stationary isothermic surface Γ is the boundary of a domain D satisfying the interior cone condition, if either Ω is convex or $\partial\Omega$ contains a relatively open subset whose principal curvatures are non-positive (with respect to the interior normal).

In Theorem 3.4 we need not use (1.5). Instead, if $\partial\Omega$ is the graph of a Lipschitz continuous function which behaves properly at infinity, we prove that $\partial\Omega$ must be a hyperplane. In the proof, we directly exploit property (ii) above with the help of an adaptation (due to Berestycki, Caffarelli and Nirenberg [4]) of the *sliding method*.

2. Preparatory lemmas

In order to prove the results contained in Section 3, we need to recall and, if necessary, adapt to the present situation some of the auxiliary Lemmas obtained in [9]. In fact, we need to extend the validity of (1.5) to domains with unbounded boundary.

We begin with some definitions. We recall that a domain $\Omega \subset \mathbb{R}^N$ satisfies the *exterior sphere condition* if for every $x \in \partial\Omega$ there exists a ball B such that $\overline{B} \cap \overline{\Omega} = \{x\}$; Ω satisfies the *uniform exterior sphere condition*, if there exists $r_0 > 0$ such that for every $x \in \partial\Omega$ there exists a ball $B(z, r_0)$ such that $\overline{B(z, r_0)} \cap \overline{\Omega} = \{x\}$. Also, a domain $D \subset \mathbb{R}^N$ satisfies the *interior cone condition* if for every $x \in \partial D$ there exists a finite right spherical cone K_x with vertex x such that $K_x \subset \overline{D}$ and $\overline{K_x} \cap \partial D = \{x\}$.

Also, we recall some facts from [7], § 14.6, on the distance function

$$d(x) = \text{dist}\,(x, \partial\Omega),\ \ x \in \Omega. \tag{2.1}$$

For $\delta > 0$ we set

$$\Omega_\delta = \{x \in \Omega : d(x) < \delta\}. \tag{2.2}$$

If $\partial\Omega$ is of class C^2, then there exists a positive number δ such that $d \in C^2(\overline{\Omega_\delta})$; moreover, $\nabla d(x) = \nu(x)$ for $x \in \partial\Omega$, where $\nu(x)$ is the interior unit normal to $\partial\Omega$ at x, and the eigenvalues of the matrix $-\nabla^2 d(x)$ are 0 and the *principal curvatures* $\kappa_1(x), \dots, \kappa_{N-1}(x)$ of $\partial\Omega$ at x. For $j \in \{1, \dots, N-1\}$ we will denote by K_j the j-th *symmetric invariant* of $\partial\Omega$ defined by

$$K_j(x) = \sum_{i_1 < \cdots < i_j} \kappa_{i_1(x)} \cdots \kappa_{i_j}(x),\ \ x \in \partial\Omega; \tag{2.3}$$

thus, $K_{N-1}(x)$ is the *Gauss curvature* and $K_1(x)/(N-1)$ is the *mean curvature* of $\partial\Omega$ at x.

We will also need later the following formula:

$$-\Delta d(x) = \sum_{j=1}^{N-1} \frac{\kappa_j(y)}{1 - \kappa_j(y) d(x)} \quad \text{for } x \in \Omega_\delta, \tag{2.4}$$

where y is the unique point in $\partial\Omega$ satisfying $d(x) = |x - y|$.

We now proceed to demonstrate (1.5) in the relevant cases in which $\partial\Omega$ is unbounded.

As in [9], for $s > 0$, we define a function $W = W(x, s)$ by

$$W(x,s) = s \int_0^{+\infty} u(x,t)\, e^{-s\,t} dt; \tag{2.5}$$

W solves the following elliptic boundary value problem:

$$\Delta W - s\, W = 0 \quad \text{in } \Omega, \tag{2.6}$$

$$W = 1 \quad \text{on } \partial\Omega. \tag{2.7}$$

If Γ is a stationary isothermic surface for the solution u of (1.1)-(1.3), then from (1.4) we have that

$$W(x,s) = A(s),\ x \in \Gamma, \tag{2.8}$$

where $A(s) = s \int_0^{+\infty} a(t)\, e^{-s\,t} dt$.

Lemma 2.1 below is an easy consequence of the results in [14].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N, N \geq 2$, be a domain satisfying the uniform exterior sphere condition.*

Then

$$\lim_{s\to+\infty} -\frac{1}{\sqrt{s}} \log W(x,s) = d(x),\ x \in \overline{\Omega}, \tag{2.9}$$

where d is given by (2.1) and the convergence is uniform in the closure of every subset $\Omega_\delta, \delta > 0$, defined in (2.2).

Proof. Theorems 3.6 and 3.10, and Lemma 3.11 in [14], combined with the uniform exterior sphere condition for Ω, give the desired conclusion. □

Lemma 2.1 is needed to extend the validity of Lemma 3.1 in [9] to the cases considered in this paper. We consider two situations that are summarized in Lemma 2.2 below.

Lemma 2.2. *Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$, satisfying the uniform exterior sphere condition.*

Assume that the solution $u = u(x,t)$ to problem (1.1)-(1.3) satisfies condition (1.4) and define a positive constant R by

$$R = -\lim_{s\to+\infty} \frac{1}{\sqrt{s}} \log\left(\int_0^{+\infty} a(t)\, e^{-st}\, dt\right). \tag{2.10}$$

Then the assertions (i)-(vi) below hold if either (a) Γ is the boundary of a domain D satisfying the interior cone condition and such that $\overline{D} \subset \Omega$, or (b) Ω takes the form

$$\Omega = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > \varphi(x')\}, \tag{2.11}$$

where φ is a locally Lipschitz continuous function.

(i) *For every $x \in \Gamma$, $d(x) = R$, where d is defined by (2.1);*
(ii) *Γ is analytic;*
(iii) *$\partial\Omega$ is analytic and $\partial\Omega = \{x \in \mathbb{R}^N : \mathrm{dist}(x, \Gamma) = R\}$;*

(iv) *the mapping:* $\Gamma \ni x \mapsto y(x) \equiv x - R\nu^*(x) \in \partial\Omega$ *is a diffeomorphism; here* $\nu^*(x)$ *denotes the interior unit normal vector to* Γ *at* $x \in \Gamma$:

(v) *for every* $x \in \Gamma$, $\nabla d(y(x)) = \nu^*(x)$ *and* $\overline{B_R(x)} \cap \partial\Omega = \{y(x)\}$;

(vi) *let* $\kappa_j(y)$, $j = 1, \dots, N-1$ *denote the* j*-th principal curvature at* $y \in \partial\Omega$ *of the analytic surface* $\partial\Omega$; *then* $\kappa_j(y) < \frac{1}{R}$, $j = 1, \dots, N-1$, *for every* $y \in \partial\Omega$.

Proof. If (a) holds, the proof is a straightforward extension of that of Lemma 3.1 in [9].

If (b) is in force, we just need to prove (ii), for the proof of the other assertions runs exactly as in Lemma 3.1 in [9].

For every $h > 0$ and $x = (x', x_N)$, consider the function $v(x,t) = u(x', x_N + h, t) - u(x,t)$ and observe that v is a bounded solution of (1.1), $v(x,0) \equiv 0$ for every $x \in \Omega$ and $v < 0$ on $\partial\Omega \times (0, +\infty)$. The maximum principle implies that $v < 0$ on $\Omega \times (0, +\infty)$. Hence, $\frac{\partial u}{\partial x_N} \le 0$ in $\Omega \times (0, +\infty)$ and applying the maximum principle to $\frac{\partial u}{\partial x_N}$ yields that $\frac{\partial u}{\partial x_N} < 0$ in $\Omega \times (0, +\infty)$.

Then, by the implicit function theorem, the analyticity of Γ follows from the analyticity of u in the spatial coordinate. ☐

Lemma 2.2 can now be used to show that, for large values of s, the two functions

$$W_\varepsilon^\pm(x,s) = \exp\{-\sqrt{s(1 \mp \varepsilon)}\, d(x)\}, \ 0 < \varepsilon < 1, \tag{2.12}$$

provide respectively an upper and a lower barrier for W in the set Ω_{2R} defined in (2.2).

Lemma 2.3. *Assume that* Ω *and* Γ *satisfy the hypotheses of Lemma 2.2.*

Then, for every $\varepsilon \in (0,1)$, *there exists a positive number* s_ε *such that*

$$W_\varepsilon^-(x,s) \le W(x,s) \le W_\varepsilon^+(x,s) \tag{2.13}$$

for every $x \in \overline{\Omega_{2R}}$ *and every* $s \ge s_\varepsilon$, *where* $W_\varepsilon^-(x,s)$ *and* $W_\varepsilon^+(x,s)$ *are defined in (2.12).*

Proof. It follows from (vi) of Lemma 2.2 and Lemma 14.16 of [7], p. 355 that the function $d = d(x)$ is of class C^∞ on the set $\overline{\Omega_{\frac{R}{2}}}$. A straightforward computation gives

$$\Delta W_\varepsilon^\pm - s\, W_\varepsilon^\pm = \mp\varepsilon\sqrt{s}\ \{\sqrt{s} \pm \frac{\sqrt{(1 \mp \varepsilon)}}{\varepsilon}\ \Delta d\}\ W_\varepsilon^\pm \quad \text{in} \ \ \Omega_{\frac{R}{2}}.$$

Lemma 2.2 (vi) and the uniform exterior sphere condition with radius $r_0 > 0$ for Ω imply that

$$-\frac{1}{r_0} \le \kappa_j(y) < \frac{1}{R} \quad \text{ for every } \ y \in \partial\Omega,$$

and hence we obtain that

$$-\frac{1}{r_0} \le \frac{\kappa_j(y)}{1 - \kappa_j(y) d(x)} < \frac{2}{R},$$

for every $x \in \Omega_{\frac{R}{2}}$ and every $j = 1, \dots, N-1$. By (2.4), we conclude that

$$|\Delta d(x)| \le (N-1)\max\{2/R,\ 1/r_0\} \ \text{ for every } \ x \in \overline{\Omega_{\frac{R}{2}}}.$$

Set $M = (N-1)\max\{\frac{2}{R},\ \frac{1}{r_0}\}$; if $s \ge \frac{1+\varepsilon}{\varepsilon^2}\ M^2$, then

$$\begin{array}{c} \Delta W_\varepsilon^+ - s\ W_\varepsilon^+ < 0 \\ \Delta W_\varepsilon^- - s\ W_\varepsilon^- > 0 \end{array} \quad \text{in} \quad \Omega_{\frac{R}{2}}. \tag{2.14}$$

Since the function $-\frac{1}{\sqrt{s}}\log W(x,s)$ converges uniformly on $\overline{\Omega_{2R}}$ to $d(x)$ as $s \to +\infty$, there exists a number $s^* > 0$ such that

$$-\frac{R}{2}(1-\sqrt{1-\varepsilon}) \le -\frac{1}{\sqrt{s}}\log W(x,s) - d(x) \le \frac{R}{2}(\sqrt{1+\varepsilon}-1),\ \ x \in \overline{\Omega_{2R}},$$

for every $s \ge s^*$. Hence, since $d(x) \ge \frac{R}{2}$ for every $x \in \overline{\Omega_{2R}} \setminus \Omega_{\frac{R}{2}}$, we obtain

$$W_\varepsilon^-(x,s) \le W(x,s) \le W_\varepsilon^+(x,s), \quad x \in \overline{\Omega_{2R}} \setminus \Omega_{\frac{R}{2}}, \tag{2.15}$$

for every $s \ge s^*$. Moreover,

$$W_\varepsilon^-(x,s) = W(x,s) = W_\varepsilon^+(x,s) = 1, \quad x \in \partial\Omega, \tag{2.16}$$

for every $s > 0$, clearly.

Choose $s_\varepsilon = \max(s^*, \frac{1+\varepsilon}{\varepsilon^2}\ M^2)$. Then with the help of the Phragmèn-Lindelöf principle (see [10], Corollary, p. 99), from (2.14), (2.15) and (2.16), we have

$$W_\varepsilon^-(x,s) \le W(x,s) \le W_\varepsilon^+(x,s), \quad x \in \Omega_{\frac{R}{2}}, \tag{2.17}$$

for every $s \ge s_\varepsilon$. Combining (2.17) with (2.15) yields (2.13). □

Lemma 2.4. *Let Ω be a domain in $\mathbb{R}^N$, $N \ge 2$, satisfying the uniform exterior sphere condition.*

Assume that the solution $u = u(x,t)$ to problem (1.1)-(1.3) satisfies condition (1.4). Then formula (1.5) holds if either (a) Γ is the boundary of a domain D satisfying the interior cone condition and such that $\overline{D} \subset \Omega$, or (b) Ω takes the form (2.11).

Proof. The proof runs exactly as the one of Theorem 3.2 in [9], where Lemmas 2.4 and 3.1 in [9] should be replaced by Lemmas 2.3 and 2.2 in the present paper, respectively. □

3. Symmetry results

The proof of the following theorem is a straightforward extension of Theorem 1.1 in [9].

Theorem 3.1. *Let Ω be an exterior domain in $\mathbb{R}^N$, $N \ge 2$, satisfying the exterior sphere condition and suppose that D is an exterior domain, with boundary Γ, satisfying the interior cone condition, and such that $\overline{D} \subset \Omega$.*

Assume that the solution u to problem (1.1)-(1.3) satisfies the condition (1.4) for some function $a : (0,+\infty) \to (0,+\infty)$.

Then $\partial\Omega$ must be a sphere.

Proof. Since both $\partial\Omega$ and ∂D are compact, the barrier arguments by using Varadhan's result also works. Therefore, we get the same formula for the principal curvatures of $\partial\Omega$. Hence each connected component S of $\partial\Omega$ must be a sphere, and the component of ∂D parallel to S, say T, is a concentric sphere with S. Consider one such component S of $\partial\Omega$. Denote by $x^0 \in \mathbb{R}^N$ the center of S and denote by E

the annulus with boundary $\partial E = S \cup T$. Let A be an arbitrary orthogonal $N \times N$ matrix, and consider the function $w = w(x,t)$ defined by

$$w(x,t) = u(x^0 + A(x - x^0), t) - u(x,t) \quad \text{for } (x,t) \in E \times (0,+\infty).$$

We observe that $w \equiv 0$ in $E \times (0,+\infty)$ by uniqueness, since w is a bounded solution of the problem:

$$\begin{aligned} \partial_t w = \Delta w \quad & \text{in} \quad E \times (0,+\infty), \\ w = 0 \quad & \text{on} \quad \partial E \times (0,+\infty), \\ w = 0 \quad & \text{on} \quad E \times \{0\}. \end{aligned}$$

Therefore, for any pair of integers $i, j \in \{1, \cdots, N\}$ with $i \neq j$, the function $v = v(x,t)$ defined by

$$(3.1) \quad v(x,t) = -(x_j - x_j^0)\frac{\partial u(x,t)}{\partial x_i} + (x_i - x_i^0)\frac{\partial u(x,t)}{\partial x_j} \quad \text{for } (x,t) \in \Omega \times (0,+\infty).$$

is identically zero in $E \times (0,+\infty)$, since A is arbitrary. Furthermore, since v is analytic in x, we get that $v \equiv 0$ in $\Omega \times (0,+\infty)$. This implies that $u(x,t)$ must be radially symmetric with respect to the center x^0, and hence we conclude that $\partial\Omega$ must be a sphere. □

Theorem 3.2. *Let Ω be the unbounded domain in $\mathbb{R}^N$, $N \geq 2$, defined by (2.11) where φ is a locally Lipschitz continuous function on $\mathbb{R}^{N-1}$ such that*

$$(3.2) \qquad |\nabla\varphi(x')| = o(|x'|^{\frac{1}{2}}) \quad \textit{as} \quad |x'| \to +\infty,$$

and suppose that Ω satisfies the uniform exterior sphere condition.

Suppose that D is a domain with $\overline{D} \subset \Omega$ whose boundary Γ has non-positive mean curvature (with respect to the interior normal to Γ).

If the solution u to problem (1.1)-(1.3) satisfies the condition (1.4) for some function $a : (0,+\infty) \to (0,+\infty)$, then $\partial\Omega$ is a hyperplane.

Proof. By using the uniform exterior sphere condition for Ω, we get for every $j = 1, \ldots, N-1$,

$$(3.3) \qquad \kappa_j(x) \geq -\frac{1}{r_0} \quad \text{for every } x \in \partial\Omega.$$

Combining (1.5) and (3.3) yields that there exists a positive constant $\tau > 0$ such that for every $j = 1, \ldots, N-1$,

$$(3.4) \qquad -\frac{1}{r_0} \leq \kappa_j(x) \leq \frac{1}{R} - \tau \quad \text{for every } x \in \partial\Omega.$$

Now, we shall use a result of Caffarelli, Nirenberg and Spruck (see [5], Theorem 2"). To avoid misunderstandings, we will uniform our notations to those of [5], by setting $\ell_j = -\kappa_j,\ j = 1, \ldots, N-1$, $\ell = (\ell_1, \ldots, \ell_{N-1})$, and

$$f(\ell) = \sum_{j=1}^{N-1} \log(1 + R\,\ell_j).$$

Then

$$(3.5) \qquad f(\ell(x)) \equiv f(\ell_1(x), \ldots, \ell_{N-1}(x)) = \log c \quad \text{for every } x \in \partial\Omega.$$

Notice that f is concave and symmetric in ℓ_j, $j = 1, \dots, N-1$, and $\frac{\partial f}{\partial \ell_j} > 0$. Also, with the help of (3.4), we observe that

$$|\ell|^2 \sum_{j=1}^{N-1} \frac{\partial f}{\partial \ell_j}(\ell) \le \tau^{-1}(N-1)\ (1/R + 1/r_0) \sum_{j=1}^{N-1} \frac{\partial f}{\partial \ell_j}(\ell)\ \ell_j^2 \ \text{ on } \ \partial\Omega. \tag{3.6}$$

Since Γ is parallel to $\partial\Omega$ at distance R, its principal curvatures are

$$-\frac{\ell_j}{1 + R\ \ell_j}, \ j = 1, \dots, N-1;$$

thus,

$$\sum_{j=1}^{N-1} \frac{\ell_j}{1 + R\ \ell_j} \ge 0 \ \text{ on } \ \partial\Omega,$$

because Γ has non-positive mean curvature.

Therefore, we obtain that

$$\sum_{j=1}^{N-1} \frac{\partial f}{\partial \ell_j}(\ell)\ \ell_j \ge 0 \ \text{ on } \ \partial\Omega, \tag{3.7}$$

and, also,

$$\sum_{j=1}^{N-1} \ell_j \ge 0 \ \text{ on } \ \partial\Omega, \tag{3.8}$$

since $\ell_j - \frac{\ell_j}{1+R\ \ell_j} = \frac{R\ \ell_j^2}{1+R\ \ell_j} \ge 0, \ \ j = 1, \dots, N-1.$

The established properties (3.6), (3.7) and (3.8) allow us to use Theorem 2" in [5] and conclude that there exists a positive constant A depending only on N such that

$$\ell_j(x) \le \frac{A}{\sqrt{1 + |\nabla\varphi(x')|^2}}\ \frac{1}{\rho} \sup_{|y'-x'|<\rho} (1 + |\nabla\varphi(y')|^2),$$

for every $x = (x', \varphi(x')) \in \partial\Omega$, $\rho > 0$, and $j = 1, \dots, N-1$. This inequality and (3.2) then imply that

$$\ell_j(x) \le 0 \ \text{ for every } x = (x', \varphi(x')) \in \partial\Omega \ \text{ and } \ j = 1, \dots, N-1. \tag{3.9}$$

Combining (3.9) with (3.8) yields the desired result. □

Theorem 3.3. *Let Ω satisfy the uniform exterior sphere condition and suppose that for every $r > 0$, $\partial\Omega$ contains a graph over an $(N-1)$-dimensional ball with radius r.*

Let Γ be the boundary of a domain D satisfying the interior cone condition and such that $\overline{D} \subset \Omega$.

Suppose that the solution u to problem (1.1)-(1.3) satisfies the condition (1.4) for some function $a : (0, +\infty) \to (0, +\infty)$.

Assume that one of the following two conditions holds true:

(a) *Ω is convex;*
(b) *there exists a relatively open non-empty subset A in $\partial\Omega$ such that all the principal curvatures of $\partial\Omega$ are non-positive on A.*

Then $\partial\Omega$ is either a hyperplane or the union of two parallel hyperplanes.

Proof. By our assumptions, for every $r > 0$ there exist an $(N-1)$-dimensional ball B_r and a function $\psi_r : B_r \to \mathbb{R}$ such that $\partial\Omega$ contains the graph $\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N = \psi_r(x'),\ x' \in B_r\}$. Notice that

$$\sum_{j=1}^{N-1} \kappa_j = \operatorname{div}\left(\frac{\nabla\psi_r}{\sqrt{1+|\nabla\psi_r|^2}}\right). \tag{3.10}$$

(a) Since all principal curvatures of $\partial\Omega$ are non-negative, we infer that

$$\prod_{j=1}^{N-1}(1 - R\,\kappa_j) \geq 1 - R\sum_{j=1}^{N-1}\kappa_j \quad \text{on} \quad \partial\Omega. \tag{3.11}$$

Indeed, since each function $\kappa_i \mapsto \prod\limits_{j=1}^{N-1}(1 - R\,\kappa_j) - 1 + R\sum\limits_{j=1}^{N-1}\kappa_j$ is affine, then it will be non-negative on the interval $[0, 1/R]$ if and only if it is so at the endpoints 0 and $1/R$; (3.11) then easily follows by induction.

Since (1.5) holds, then $c \leq 1$ and, by (3.11),

$$\sum_{j=1}^{N-1}\kappa_j \geq c_R \quad \text{on} \quad \partial\Omega, \tag{3.12}$$

with $c_R = (1-c)/R$.

If $c = 1$, then by (1.5) all principal curvatures of $\partial\Omega$ are identically zero and every connected component of $\partial\Omega$ is a hyperplane.

If $c < 1$, by (3.10), we obtain the inequalities

$$c_R\,|B_r| \leq \int_{B_r}\operatorname{div}\left(\frac{\nabla\psi_r}{\sqrt{1+|\nabla\psi_r|^2}}\right)dx' = \int_{\partial B_r}\frac{\nabla\psi_r\cdot\nu}{\sqrt{1+|\nabla\psi_r|^2}}\,dS_{x'},$$

and hence

$$c_R\,|B_r| \leq |\partial B_r|, \tag{3.13}$$

which contradicts the fact that r can be chosen arbitrarily large.

(b) In this case, since all principal curvatures of $\partial\Omega$ are non-positive in A, we infer that $c \geq 1$. By the arithmetic-geometric mean inequality, we have that

$$c^{\frac{1}{N-1}} \leq \frac{1}{N-1}\sum_{j=1}^{N-1}(1 - R\,\kappa_j)$$

and hence

$$-\sum_{j=1}^{N-1}\kappa_j \geq c_R \quad \text{on} \quad \partial\Omega, \tag{3.14}$$

with $c_R = (N-1)(c^{\frac{1}{N-1}} - 1)/R$.

If $c > 1$, by the same arguments used in the proof of (a), we obtain (3.13) and hence a contradiction.

If $c = 1$, then clearly $\kappa_j \equiv 0$ on A for $j = 1, \dots, N-1$. By analyticity, each κ_j must be zero on the connected component H of $\partial\Omega$ containing A, and hence H must be a hyperplane.

Consequently, in both cases (a) and (b), there exists a hyperplane H which is a connected component of $\partial\Omega$. Since Γ is parallel to $\partial\Omega$, there are two possibilities:

either Γ coincides with the hyperplane $\tilde{H} \subset \Omega$ at distance R from H, in which case $\partial\Omega$ would be a hyperplane, or Γ strictly contains $\tilde{H}$.

We shall show that, in the latter case, $\partial\Omega$ is the union of two parallel hyperplanes. For simplicity, assume that $H = \{x \in \mathbb{R}^N : x_N = 0\}$. Let Ω' be the strip between H and $\tilde{H}$; we observe that $u = u(x,t)$ satisfies (1.1) and (1.3) with Ω replaced by Ω', (1.2) with $\partial\Omega$ replaced by H and (1.4) replaced by $\tilde{H}$.

Let $a \in \mathbb{R}$ be an arbitrary number, and for every $j \in \{1, \dots, N-1\}$ consider the function $w_j = w_j(x,t)$ defined by

$$w_j(x,t) = u(x + a\ e_j, t) - u(x,t) \quad \text{for } (x,t) \in \Omega' \times (0,+\infty),$$

where $\{e_1, \cdots, e_N\}$ denotes the canonical basis of $\mathbb{R}^N$. By uniqueness, $w_j \equiv 0$ in $\Omega' \times (0,+\infty)$, since w_j is a bounded solution of the problem:

$$\begin{aligned} \partial_t w_j = \Delta w_j \quad & \text{in} \quad \Omega' \times (0,+\infty),\\ w_j = 0 \quad & \text{on} \quad \partial\Omega' \times (0,+\infty),\\ w_j = 0 \quad & \text{on} \quad \Omega' \times \{0\}. \end{aligned}$$

Thus, for every $j \in \{1, \dots, N-1\}$, the function $v_j = v_j(x,t)$ defined by

$$v_j(x,t) = \frac{\partial u(x,t)}{\partial x_j} \quad \text{for } (x,t) \in \Omega \times (0,+\infty). \tag{3.15}$$

is identically zero in $\Omega' \times (0,+\infty)$, since a is arbitrary. Furthermore, since v_j is analytic in x, we get that $v_j \equiv 0$ in $\Omega \times (0,+\infty)$.

Therefore, $u(x,t)$ depends only on t and x_N and hence $\partial\Omega$ is the union of two parallel hyperplanes. □

Theorem 3.4. *Let Ω be the unbounded domain in $\mathbb{R}^N$, $N \geq 2$, defined by (2.11), where φ is a globally Lipschitz continuous function on $\mathbb{R}^{N-1}$ such that*

$$\lim_{|x'|\to\infty} [\varphi(x'+\xi) - \varphi(x')] = 0 \quad \text{for each } \xi \in \mathbb{R}^{N-1}. \tag{3.16}$$

Let $u = u(x,t)$ be the solution of (1.1)-(1.3) and assume that Γ is an $(N-1)$-dimensional surface, contained in Ω, such that (1.4) holds for some function $a : (0,+\infty) \to (0,+\infty)$.

Then $\partial\Omega$ must be a hyperplane.

Proof. Consider the function $W = W(x,s)$ defined in (2.5); by Theorem 3.12 in [14], the number R in (2.10) is well defined. As observed in the proof of Theorem 3.2 in [9], since Γ is a stationary isothermic surface for u, for every $r \in (0,R)$ and every $s > 0$, we have that

$$\int_{B(x,r)} W(y,s)\ dy = c(r,s), \quad \text{for every } x \in \Gamma, \tag{3.17}$$

for some function $c = c(r,s)$.

Now, we follow the argument in [4], pp. 1108–1110. We fix a vector $\xi \in \mathbb{R}^{N-1}$, $\xi \neq 0$, and for $h \geq 0$ define the set

$$\Omega_{\xi,h} = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : (x'+\xi, x_N + h) \in \Omega\}.$$

Since φ is globally Lipschitz continuous, the set $\Omega_{\xi,h}$ must contain Ω if h is large enough. Furthermore, (3.16) yields that $\Omega_{\xi,h}$ does not equal Ω if $h > 0$. For such an h, we define the function

$$W_{\xi,h}(x,s) = W(x' + \xi, x_N + h, s) \quad \text{for} \quad x = (x', x_N) \in \Omega.$$

The function $Z_{\xi,h} = W_{\xi,h} - W$ is negative in Ω, by the maximum principle. In fact, $\Omega_{\xi,h}$ does not equal Ω, and $Z_{\xi,h}$ satisfies (2.6) in Ω and is non-positive on $\partial\Omega$.

As in [4], we set

$$h^* = \inf\{h \geq 0 : \Omega \subset \Omega_{\xi,h}\}.$$

Suppose that $h^* > 0$; in view of (3.16), there exists a point p at a finite distance such that

$$p \in \partial\Omega \cap \partial\Omega_{\xi,h^*}.$$

Set

$$\Gamma_{\xi,h^*} = \{x = (x', x_N) : (x' + \xi, x_N + h^*) \in \Gamma\}$$

and let $p^* = p + R\nabla d(p)$: p^* belongs to $\Gamma \cap \Gamma_{\xi,h^*}$.

Choose an $r \in (0, R)$; we have that

$$\int_{B(p^*,r)} Z_{\xi,h^*}\, dx < 0,$$

since $Z_{\xi,h^*} < 0$ in Ω. On the other hand, (3.17) implies that

$$\int_{B(p^*,r)} Z_{\xi,h^*}\, dx = \int_{B(p^*,r)} W_{\xi,h^*}\, dx - \int_{B(p^*,r)} W\, dx = \tag{3.18}$$

$$= \int_{B(p^*+(\xi,h^*),r)} W\, dx - \int_{B(p^*,r)} W\, dx = \tag{3.19}$$

$$= 0, \tag{3.20}$$

since $p^* + (\xi, h^*) \in \Gamma$ because $p^* \in \Gamma_{\xi,h^*}$.

Therefore, we obtained a contradiction; it follows that $h^* = 0$, that is $\Omega \subset \Omega_{\xi,h}$ for all $h \geq 0$, which implies that $\Omega + (\xi, 0) = \Omega$ for all vectors $\xi \in \mathbb{R}^{N-1}$ (see [4]).

This last condition tells us that the function φ is periodic of period ξ; since this is true for all $\xi \in \mathbb{R}^{N-1}$, then φ must be constant. i.e. $\partial\Omega$ is a hyperplane. □

Finally, Theorem 3.5 extends Alessandrini's result to a quite general class of unbounded domains.

Theorem 3.5. *Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$, satisfying the uniform exterior sphere condition and suppose that $D_1, \dots, D_{N-1}$ are $N-1$ distinct domains, with boundaries $\Gamma_1, \dots, \Gamma_{N-1}$ satisfying the interior cone condition, and such that $\overline{D_k} \subset \Omega$, $k = 1, \dots, N-1$.*

Assume that the solution u to problem (1.1)-(1.3) satisfies the conditions

$$u(x,t) = a_k(t), \quad (x,t) \in \Gamma_k \times (0, +\infty),\ k = 1, \dots, N-1, \tag{3.21}$$

for some $N-1$ functions $a_k : (0, +\infty) \to (0, +\infty)$, $k = 1, \dots, N-1$.

Then $\partial\Omega$ is either a sphere, a spherical cylinder, a hyperplane, or the union of two parallel hyperplanes.

Proof. Applying Lemma 2.4 to each stationary surface Γ_k, we obtain that

$$\prod_{j=1}^{N-1} [1 - R_k\, \kappa_j(x)] = c_k,\ x \in \partial\Omega,\ k = 1, \dots, N-1, \tag{3.22}$$

where $R_k = \text{dist }(\Gamma_k, \partial\Omega),\ k = 1, \dots, N-1$, and $c_k,\ k = 1, \dots, N-1$, are positive constants; (3.22) is a linear, non-singular system where the unknown are the $N-1$ symmetric invariants of $\partial\Omega$. Since all the coefficients of such a system are constant on $\partial\Omega$, we obtain that all the symmetric invariants of $\partial\Omega$ are constant, and hence all the principal curvatures of $\partial\Omega$ are constant. By the classical results on isoparametric surfaces in $\mathbb{R}^N$ (see [8], [12]), we then infer that each connected component of $\partial\Omega$ must be either a sphere, a spherical cylinder or a hyperplane.

If one connected component of $\partial\Omega$ is a sphere or a hyperplane, we proceed as in Theorem 3.1 or 3.3, respectively, and conclude that $\partial\Omega$ must be a sphere, a hyperplane or the union of two hyperplanes. If one component of $\partial\Omega$ is a spherical cylinder, say $S_m = \{x \in \mathbb{R}^N : x_1^2 + \cdots + x_m^2 = r^2\}$ with $2 \le m \le N-1$, then a component T_m of the ∂D_k's will be a spherical cylinder parallel to S_m. Hence, in the cylindrical annulus E_m with boundary $\partial E_m = S_m \cup T_m$, we consider the auxiliary function (3.1) with $x^0 = 0,\ 1 \le i, j \le m$ and $i \ne j$ and the function (3.15) with $m + 1 \le j \le N$, by an argument similar to that of Theorem 3.1 we conclude that $\partial\Omega$ must be a spherical cylinder. □

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Dipartimento di Matematica "U. Dini", Università di Firenze, viale Morgagni 67/A, 50134 Firenze, Italy.
E-mail address: magnanin@math.unifi.it
URL: http://www.math.unifi.it/~magnanin

Department of Mathematics, Graduate School of Science and Engineering, Ehime University, 2-5 Bunkyo-cho, Matsuyama-shi, Ehime 790-8577 Japan.
E-mail address: sakaguch@dpc.ehime-u.ac.jp