On Complex-valued 2D Eikonals. Part Four: Continuation Past a Caustic

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Abstract. Theories of monochromatic high-frequency electromagnetic fields have been designed by Felsen, Kravtsov, Ludwig and others with a view to portraying features that are ignored by geometrical optics. These theories have recourse to eikonals that encode information on both phase and amplitude — in other words, are complex-valued. The following mathematical principle is ultimately behind the scenes: any geometric optical eikonal, which conventional rays engender in some light region, can be consistently continued in the shadow region beyond the relevant caustic, provided an alternative eikonal, endowed with a non-zero imaginary part, comes on stage.

In the present paper we explore such a principle in dimension 2. We investigate a partial differential system that governs the real and the imaginary parts of complex-valued two-dimensional eikonals, and an initial value problem germane to it. In physical terms, the problem in hand amounts to detecting waves that rise beside, but on the dark side of, a given caustic. In mathematical terms, such a problem shows two main peculiarities: on the one hand, degeneracy near the initial curve; on the other hand, ill-posedness in the sense of Hadamard. We benefit from using a number of technical devices: hodograph transforms, artificial viscosity, and a suitable discretization. Approximate differentiation and a parody of the quasi-reversibility method are also involved. We offer an algorithm that restrains instability and produces effective approximate solutions.

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1. Introduction

1.1. Geometrical optics fits well a variety of issues, but especially survives as an asymptotic theory of monochromatic high-frequency electromagnetic fields — [14], [56], [76], [80], [98], [99], [100], [105] and [106], [116], [137], [145], [184] are selected apropos references. Generalizations have been worked out by Felsen, Kravtsov, Ludwig and their followers — see e.g. [39] and [40], [60], [71] and [72], [87], [112],[113], [114] and [115], [131], [135] and [136], or consult [26], [37], [117]. One is enough for successfully modeling basic optical processes, such as the propagation of light and the development of caustics. The others embrace geometrical optics and are additionally apt to account for certain optical phenomena — for instance, the rise of evanescent waves past a caustic — that are beyond the reach of geometrical optics. A leitmotif of these is allowing a keynote parameter to adjust itself to a standard equation, and simultaneously take complex values.

The partial differential equation

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = n^2(x, y) \tag{1.1}$$

underlies the mentioned theories in case the spacial dimension is 2. Here x and y denote *rectangular coordinates* in the Euclidean plane; n is a *real-valued, strictly positive* function of x and y; w is allowed to take both *real* and *complex* values. Function n represents the *refractive index* of an appropriate (isotropic, non-conducting) two-dimensional medium — its reciprocal stands for velocity of propagation. Function w is named *eikonal* according to usage, and relates to the asymptotic behavior of an electromagnetic field as the wave number grows large — the real part of w accounts for oscillations, the imaginary part of w accounts for damping. Throughout the present paper we assume the refractive index is conveniently smooth, and consider sufficiently smooth eikonals.

1.2. Geometrical optics deals exclusively with real-valued eikonals, by definition. The partial differential system

$$\begin{aligned} u_x^2 + u_y^2 - v_x^2 - v_y^2 &= n^2(x, y) \\ u_x v_x + u_y v_y &= 0 \end{aligned} \tag{1.2}$$

governs complex-valued eikonals, i.e. those solutions to (1.1) that obey

$$u = \operatorname{Re} w, \quad v = \operatorname{Im} w.$$

Observe the architecture of (1.2): gradients are involved through their orthogonal invariants — lengths and inner product — only. Observe also the following properties, which result from a standard test and easy algebraic manipulations. First, system (1.2) is *elliptic-parabolic* or *degenerate elliptic.* Second, a solution array $[u\ v]$ to (1.2) is *elliptic* if and only if its latter entry v is free from critical points.

The *Bäcklund transformation,* which relates u and v through

$$\begin{gathered}\nabla v = f \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla u, \\ f^2 = 1 - \frac{n^2}{|\nabla u|^2}, \quad \operatorname{sgn} f = \operatorname{sgn}(u_x v_y - u_y v_x),\end{gathered} \tag{1.3}$$

and implies both

$$|\nabla u| \geq n$$

and

$$\operatorname{div}\left\{\sqrt{1 - \frac{n^2}{|\nabla u|^2}}\ \nabla u\right\} = 0, \tag{1.4}$$

is another, decoupled form of (1.2).

System (1.2) discloses two scenarios — the former is tantamount to conventional geometrical optics, the latter opens up new vistas. Either the equations

$$u_x^2 + u_y^2 = n^2 \quad \text{and} \quad v_x = v_y = 0$$

hold, or the inequalities

$$|\nabla u| > n \quad \text{and} \quad |\nabla v| > 0,$$

and the following equations prevail.

$$\nabla u = \frac{1}{f}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \nabla v, \quad \frac{1}{f^2} = 1 + \frac{n^2}{|\nabla v|^2}, \tag{1.5}$$

$$\operatorname{div}\left\{\sqrt{1 + \frac{n^2}{|\nabla v|^2}}\ \nabla v\right\} = 0, \tag{1.6}$$

$$(|\nabla u|^4 - n^2 u_y^2) u_{xx} + 2n^2 u_x u_y u_{xy} + (|\nabla u|^4 - n^2 u_x^2) u_{yy} = n|\nabla u|^2 \langle \nabla n, \nabla u\rangle, \tag{1.7}$$

$$(|\nabla v|^4 + n^2 v_y^2) v_{xx} - 2n^2 v_x v_y v_{xy} + (|\nabla v|^4 + n^2 v_x^2) v_{yy} + n|\nabla v|^2 \langle \nabla n, \nabla v\rangle = 0. \tag{1.8}$$

Equations (1.5) represent another Bäcklund transformation, which inverts the previous one and imply (1.6). Quasi-linear partial differential equations of the second order in non-divergence form appear in (1.7) and (1.8). The former has a *mixed elliptic-hyperbolic* character: a solution u is *elliptic* or *hyperbolic* depending on whether the length of ∇u exceeds, or is smaller than n. The latter is *elliptic-parabolic* or *degenerate elliptic:* a solution v such that ∇v is free from zeros is *elliptic,* degeneracy occurs at the *critical points* of v.

Any sufficiently smooth solution to (1.4) satisfies (1.7). An elliptic solution to (1.7) coincides with the former entry of an elliptic solution to (1.2). A real-valued function u, smooth and without critical points, is a hyperbolic solution to (1.7) if and only if two real-valued smooth, essentially distinct eikonals φ and ψ exist such that u is the average of φ and ψ, i.e.

$$\varphi_x^2 + \varphi_y^2 = n^2, \quad \psi_x^2 + \psi_y^2 = n^2, \quad \varphi_x \psi_y - \varphi_y \psi_x \neq 0,$$

and

$$u = \frac{1}{2}(\varphi + \psi).$$

Any sufficiently smooth solution to (1.6) satisfies (1.8). Any *elliptic* solution to (1.8) satisfies (1.6). However, a solution to (1.8) *need not* satisfy (1.6): for instance, perfectly smooth solutions to (1.8) exist whose gradient vanishes exclusively in a set of measure 0, and which make the left-hand side of (1.6) a non-zero distribution.

A variational approach to equations (1.6) and (1.8) can be summarized thus. Let J be endowed with an appropriate domain and obey

$$J(v) = \iint j\left(\frac{|\nabla v|}{n}\right) n^2\, dxdy$$

for any v from that domain — here j is the arc length along a parabola, videlicet

$$j(\rho) = \frac{\rho}{2}\sqrt{1+\rho^2} + \frac{1}{2}\log(\rho + \sqrt{1+\rho^2})$$

for any real ρ. The following properties hold. (i) J is convex, coercive and sub-differentiable, but not Fréchet-differentiable. (ii) Any critical point of J, i.e. any function v such that

$$\partial J(v) \ni 0,$$

satisfies (1.6) in any open set $\mathcal{O}$ such that

$$\mathcal{O} \text{ is essentially contained in } \{(x,y) \in \text{domain of } v : \nabla v(x,y) \neq 0\}.$$

Consequently, a critical point of J solves a *free-boundary problem* for equation (1.6) — the relevant free boundary is

$$(\text{domain of } v) \cap \partial\{(x,y) \in \text{domain of } v : \nabla v(x,y) \neq 0\}$$

and plays the role of a *caustic.* (iii) Any function v such that

$$J(v) = \text{minimum}$$

satisfies (1.8) in an appropriate *viscosity sense.* In other words, a minimizer of J solves in a generalized sense a *boundary value problem* for equation (1.8).

An early treatment of (1.2) traces back to [61]. Further apropos information is offered in [138], [139], [140], and [141], where solutions in closed form, qualitative features, exterior boundary value problems, related free boundaries, variational and viscosity methods are discussed.

1.3. Geometrical optics ultimately amounts to manipulating: (i) the Riemannian metric known as *travel time,* videlicet

$$n(x,y)\sqrt{dx^2+dy^2};$$

(ii) appropriate one-parameter families of travel time geodesics — whose members are nicknamed *rays;* (iii) the envelopes of rays — called *caustics.*

Geometric optical eikonals are entirely controlled by rays. They shine in light regions (those spanned by relevant rays), burn out beside caustics (where the ray system breaks down), and shut down in shadow regions (that rays avoid). As a consequence, geometrical optics is unable to account for any optical process that takes place beyond a caustic, on the dark side of it. Essentials of two-dimensional geometrical optics (which are well-established but instrumental here) are outlined in an appendix for reader's convenience.

Complex-valued eikonals prove more flexible. The cited work of Felsen, Kravtsov and Ludwig comprehends the following manifesto among other things: complex-valued eikonals are apt to *consistently continue* geometric optical cognates *into shadow regions.*

Such a continuation is the main theme of the present paper.

2. Heuristics

Let us pave the way by heuristically considering the case where refractive index n is 1. A classical recipe informs how general geometric optical eikonals can be cooked up: start from a complete integral, derive a one-parameter family of solutions, take the relevant envelope, and shake well. Let f be an arbitrary, but sufficiently smooth, real function. The pair

$$w = x\cos t + y\sin t + f(t), \quad 0 = -x\sin t + y\cos t + f'(t) \tag{2.1}$$

causes w and t to enjoy the following properties:

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = 1,$$

the pertinent *eikonal equation* governing w;

$$\cos t \, \frac{\partial t}{\partial x} + \sin t \, \frac{\partial t}{\partial y} = 0,$$

a *Burgers-type equation* governing t;

$$w_x t_x + w_y t_y = 0,$$

showing that the gradients of w and t are *orthogonal;*

$$w_x = \cos t, \quad w_y = \sin t,$$

a *Bäcklund transformation* further relating w and t. Both the rays of w and the level lines of t are the straight-lines where

$$-x \sin t + y \cos t + f'(t) = 0$$

and t equals a constant. Such straight-lines span the *light region* and envelope the *caustic.* We have

$$-f' = \text{the support function of the caustic,}$$

and

$$x = f''(t) \cos t + f'(t) \sin t, \quad y = f''(t) \sin t - f'(t) \cos t, \quad w = f(t) + f''(t)$$

along the caustic. Therefore the second-order derivatives of w and the gradient of t simultaneously blow up there — in particular, the caustic of w is also the shock-line of t.

We claim that both w and t can be *continued beyond the caustic,* in a subset of the shadow region, if *complex values* are allowed. Suppose f can be continued by a holomorphic function of a complex variable. (Information on analytic continuation is in the next section.) Let $i = \sqrt{-1}$, the unit imaginary number. Let u, v, λ, μ be real; put the equations

$$w = u + iv, \quad t = \lambda + i\mu,$$

and equations (2.1) together, but force (2.1) to produce real x and y. The formulas

$$\begin{aligned}
x &= \frac{\sin\lambda}{\cosh\mu} \, \mathrm{Re} f'(t) + \frac{\cos\lambda}{\sinh\mu} \, \mathrm{Im} f'(t), \\
y &= -\frac{\cos\lambda}{\cosh\mu} \, \mathrm{Re} f'(t) + \frac{\sin\lambda}{\sinh\mu} \, \mathrm{Im} f'(t), \\
u &= \mathrm{Re} f(t) + (x\cos\lambda + y \sin\lambda) \cosh\mu, \\
v &= \mathrm{Im} f(t) + (-x \sin\lambda + y\cos\lambda) \sinh\mu,
\end{aligned} \tag{2.2}$$

ensue, then an inspection testifies that the claimed continuation ensues too.

Incidentally, we have also shown that solutions to the non-viscous Burgers equation can be continued past the shock-line by suitable complex-valued solutions to the same equation.

Note the following. In case

$$f(t) = t$$

for every real or complex t, the caustic is the unit circle, the rays are half-lines tangent to it, the shadow region is the unit disk. Formulas (2.1) and (2.2) become transparent when recast as shown below. Let rectangular and polar coordinates be related by

$$x = \rho \sin\varphi, \ y = -\rho\cos\varphi \quad (0 \le \rho < \infty, -\pi < \varphi \le \pi).$$

A geometrical optical eikonal happens to satisfy

$$w = \varphi \pm \left\{ \arctan\sqrt{\rho^2 - 1} - \sqrt{\rho^2 - 1} \right\}$$

in the region where

$$1 \le \rho < \infty;$$

the same eikonal is ipso facto continued by

$$w = u + iv, \ u = \varphi, \ v = \pm\left\{ \log\left(\frac{1}{\rho} + \sqrt{\frac{1}{\rho^2} - 1}\right) - \sqrt{1 - \rho^2} \right\}$$

in the region where

$$0 < \rho < 1$$

— u is a *helicoid,* v is a *pseudosphere.*

A more exhaustive analysis is carried out in the next section.

3. Analytic continuation

3.1. Here we sketch a special method of continuing a two-dimensional geometric optical eikonal past a caustic. Though rigorous, such a method is slightly reminiscent of the so-called theory of complex rays — cf. [37] or [115], for instance. It applies in the case where *the refractive index equals* 1, and can be used for simultaneously continuing solutions to non-viscous Burgers equation beyond shock lines. Let us mention that complex-valued solutions to viscous and non-viscous Burgers equation are dealt with in [102], [174] and [101].

3.2. The present method involves *analytic continuation* from the real-number axis into the complex plane — an ill-posed process in the sense of Hadamard. Let h be a real or complex-valued function of a real variable, or even a list of samples. An analytic continuation of h is a holomorphic function of a complex variable, whose domain surrounds the real axis and whose restriction to the real axis fits h well — in other words, a solution H of the following initial value problem for Cauchy-Riemann equation

$$\frac{\partial H}{\partial y} = i\,\frac{\partial H}{\partial x}, \quad H(\cdot, 0) \simeq h.$$

If h is an *analytic function,* and is *not* polluted by noise, an analytic continuation H of h results from obvious formulas. For example,

$$H(x,y) = \sum_{k=0}^{\infty} \frac{d^k h(x)}{dx^k} \frac{(iy)^k}{k!};$$

or

$$H(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\xi(x+iy)]\, \widehat{h}(\xi)\, d\xi,$$

where hat denotes Fourier transformation.

If h collects *gross data,* an effective analytic continuation H of h can be obtained by analytically continuing an appropriate, smoothed and denoised version of h. Consider for instance the case where

$$\begin{aligned} -\infty < a < b < \infty, \\ N = \text{ an integer larger than 1,} \end{aligned}$$

a *mesh size* is given by

$$\Delta x = (b-a)/(N-1),$$

nodes are given by

$$x_j = a + (j-1)\Delta x \quad (j = 1, \ldots, N);$$

and h stands for

$$h_1, \ldots, h_N,$$

a sequence of *noisy samples.*

An ad hoc analytic continuation H solves the following least square problem

$$\Delta x \sum_{j=1}^{N} [H(x_j, 0) - h_j]^2 + \frac{\lambda}{2L} \int\limits_{a-\Delta x/2}^{b+\Delta x/2} dx \int\limits_{-L}^{L} |H(x,y)|^2 dy = \text{minimum}$$

in a convenient class of holomorphic functions — e.g. the class of trigonometric polynomials of a suitable degree. Here

$$\frac{\lambda}{2L} \int\limits_{a-\Delta x/2}^{b+\Delta x/2} dx \int\limits_{-L}^{L} |H(x,y)|^2 dy$$

plays the role of a *penalty;* λ and L are *regulating parameters* — λ is related to *noise,* L is related to *a priori information.*

An explicit expression of H can be derived via discrete Fourier transforms. Suppose for simplicity that N is odd, say

$$N = 2n + 1 \quad (n = 1, 2, \dots);$$

let

$$T = N\,\Delta x,$$

and let DFT be the discrete Fourier transform that obeys the equations

$$DFT_k(h) = \sum_{j=1}^{N} h_j \exp\left(-2\pi i k \frac{x_j}{T}\right) \quad (k = -n, \dots, n),$$

$$h_j = \frac{1}{N} \sum_{k=-n}^{n} DFT_k(h) \exp\left(2\pi i k \frac{x_j}{T}\right) \quad (j = 1, \dots, N),$$

$$\frac{1}{N} \sum_{k=-n}^{n} |DFT_k(h)|^2 = \sum_{j=1}^{N} |h_j|^2$$

— cf. [31], for instance. If

$$C_0 = 1,\ C_k = \frac{\sinh(4\pi k L/T)}{4\pi k L/T},\ C_{-k} = C_k \quad (k = 1, \dots, n),$$

then

$$H(x,y) = \sum_{k=-n}^{n} (1 + \lambda C_k)^{-1} DFT_k(h) \exp\left(2\pi i k \frac{x + iy}{T}\right).$$

This is a T-periodic trigonometric polynomial of degree n that enjoys the properties

$$\sum_{j=1}^{N} |H(x_j,0) - h_j|^2 \le \left(\frac{\lambda}{\lambda + 1/C_n}\right)^2 \sum_{j=1}^{N} |h_j|^2,$$

$$\frac{1}{2L} \int\limits_{a-\Delta x/2}^{b+\Delta x/2} dx \int\limits_{-L}^{L} |H(x,y)|^2 dy \le \frac{\Delta x}{4\lambda},$$

and

$$H(x,y) = \sum_{j=1}^{N} H(x_j,0)\, D_N\left(2\pi \frac{x - x_j + iy}{T}\right)$$

— here D_N denotes the Dirichlet, or periodic sinc function obeying

$$D_N(x) = \frac{\sin(Nx/2)}{N\sin(x/2)}$$

if $x/(2\pi)$ is not an integer.

More information on analytic continuation can be found in [1], [4], [23], [25], [35], [41]-[42], [50], [54]-[55], [69], [70], [74], [75], [81], [88], [124], [125], [130], [126], [149]-[150], [152], [187], [192], [194], [211], [215], [221], [223], [226].

3.3. Consider a plane curve C that either is inherently smooth or results from a suitable smoothing process of raw data. Assume C is *analytic* and *its curvature vanishes nowhere.* For simplicity, assume C is the graph of the equation

$$y = f(x), \tag{3.1}$$

and f is convex.

Alternative parametric representations of C, which are instrumental throughout, include

$$x = t, \quad y = f(t), \tag{3.2}$$

where parameter t coincides with the abscissa; and

$$x = g'(t), \quad y = tg'(t) - g(t), \tag{3.3}$$

where parameter t is the slope of the tangent straight-line. Here g denotes the *Legendre conjugate* of f — recall from e.g. [190] that f and g are related by

$$t = f'(x),\ x = g'(t),\ t\,x = f(x) + g(t),\ 1 = f''(x)\,g''(t).$$

Curve C changes into a *caustic* under the following modus operandi. Let

$$x = \alpha(t), \quad y = \beta(t) \tag{3.4}$$

be any parametric representation of C, where α and β are analytic. Let γ and κ stand for arc length and curvature, respectively — in other words,

$$\gamma(t) = \int \sqrt{\alpha'(t)^2 + \beta'(t)^2}\, dt,$$

$$\kappa = \frac{\alpha'\beta'' - \alpha''\beta'}{[(\alpha')^2 + (\beta')^2]^{3/2}}.$$

The pair

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha(s) \\ \beta(s) \end{bmatrix} + \frac{w - \gamma(s)}{\sqrt{\alpha'(s)^2 + \beta'(s)^2}} \begin{bmatrix} \alpha'(s) \\ \beta'(s) \end{bmatrix} \tag{3.5}$$

makes s and w curvilinear coordinates. As is easy to see, the level lines of s are *tangent straight-lines* to C, the level lines of w are *involutes* of C — orthogonal to one another.

Equations (3.5) imply

$$y \leq f(x),$$

and

$$s(x, y) = t, \quad w(x, y) = \gamma(t),$$

if x, y and t obey (3.4) — s and w live below C and satisfy precise conditions along C.

We compute

$$\frac{\partial(x, y)}{\partial(s, w)} = \begin{bmatrix} -\beta'(s) & \alpha'(s) \\ \alpha'(s) & \beta'(s) \end{bmatrix} \begin{bmatrix} \kappa(s)[w - \gamma(s)] & 0 \\ 0 & [(\alpha')^2 + (\beta')^2]^{-1/2} \end{bmatrix}$$

to draw the following set:

$$\frac{\partial}{\partial x}\alpha(s) + \frac{\partial}{\partial y}\beta(s) = 0, \tag{3.6}$$

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = 1, \tag{3.7}$$

$$\frac{\partial s}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial s}{\partial y}\frac{\partial w}{\partial y} = 0, \tag{3.8}$$

$$\nabla w = [\alpha'(s)^2 + \beta'(s)^2]^{-1/2} \begin{bmatrix} \alpha'(s) \\ \beta'(s) \end{bmatrix}, \tag{3.9}$$

$$|\nabla s|^{-2} = [w - \gamma(s)]^2 \kappa(s)^2 [\alpha'(s)^2 + \beta'(s)^2], \tag{3.10}$$

$$\begin{bmatrix} w_{xx} & w_{xy} \\ w_{xy} & w_{y} \end{bmatrix} = \frac{1}{\gamma(s) - w} \begin{bmatrix} -w_y \\ w_x \end{bmatrix} [-w_y \ w_x]. \tag{3.11}$$

Equation (3.6) is a *conservation law;* it reads

$$\frac{\partial s}{\partial x} + s\,\frac{\partial s}{\partial y} = 0,$$

the standard *Burgers equation,* if (3.3) is in force. Equation (3.7) is the equation of geometrical optics in hand. Equation (3.8) shows that the gradients of s and w are orthogonal. Equation (3.9) shows that both C and the tangent straight-lines to C are lines of steepest descent of w; it also shows that the straight-lines in question are isoclines of w. Equation (3.9) can be viewed as a *Bäcklund transformation,* which converts any solution to (3.6) into a solution to (3.7). It reads

$$\nabla w = [1 + f'(s)^2]^{-1/2} \begin{bmatrix} 1 \\ f'(s) \end{bmatrix}, \quad s = g'\left(\frac{w_y}{w_x}\right),$$

or simply

$$\nabla w = [1 + s^2]^{-1/2} \begin{bmatrix} 1 \\ s \end{bmatrix}, \quad s = \frac{w_y}{w_x},$$

depending on whether (3.2) or (3.3) is in effect. Equations (3.10) and (3.11) show that both the gradient of s and the second-order derivatives of w blow up near C.

We infer that s is governed by a Burgers-type equation, and develops shocks along C. The following objects — w, C, the tangent straight-lines to C, and the region below C — are a *geometric optical eikonal,* the relevant *caustic,* the *rays,* and the *light region,* respectively.

We now claim: (i) s and w can be *continuously extended* into the region where

$$y > f(x), \tag{3.12}$$

the *dark side* of C, if suitable imaginary parts are provided; (ii) the relevant extensions obey equations (3.6) to (3.11).

The points above C are reached by no tangent straight-line to C, of course. We insist in drawing tangent straight-lines from these points, but allow complex slopes. In other words, we recast (3.5) this way

$$-\beta'(s)[x - \alpha(s)] + \alpha'(s)[y - \beta(s)] = 0,$$
$$[w - \gamma(s)][\alpha'(s)^2 + \beta'(s)^2]^{1/2} = \alpha'(s)[x - \alpha(s)] + \beta'(s)[y - \beta(s)],$$

and force such equations to hold in the situation where

$$\mathrm{Re}(s) = \lambda, \quad \mathrm{Im}(s) = \mu, \quad \mathrm{Im}(x) = \mathrm{Im}(y) = 0.$$

The following formulas result:

$$
\begin{aligned}
&x = \frac{\mathrm{Im}[\overline{\alpha'(s)}(\alpha(s)\beta'(s) - \alpha'(s)\beta(s))]}{\mathrm{Im}[\overline{\alpha'(s)}\beta'(s)]},\\
&y = \frac{\mathrm{Im}[\overline{\beta'(s)}(\beta(s)\alpha'(s) - \beta'(s)\alpha(s))]}{\mathrm{Im}[\overline{\beta'(s)}\alpha'(s)]},\\
&s = \lambda + i\mu, \qquad (3.13)\\
&w = \gamma(s) - \frac{1}{\sqrt{\alpha'(s)^2 + \beta'(s)^2}}\left\{\overline{\alpha'(s)}\frac{\mathrm{Im}\,\beta(s)}{\mathrm{Im}[\overline{\alpha'(s)}\beta'(s)]} + \overline{\beta'(s)}\frac{\mathrm{Im}\,\alpha(s)}{\mathrm{Im}[\overline{\beta'(s)}\alpha'(s)]}\right\},
\end{aligned}
$$

where α, β and γ stand for the analytic continuations of the original objects.

Formulas (3.13) answer the claim. Among other things, they give

$$
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha(\lambda) \\ \beta(\lambda) \end{bmatrix} + \frac{dA(\lambda)}{d\lambda}\begin{bmatrix} \alpha'(\lambda) \\ \beta'(\lambda) \end{bmatrix}\mu^2 + B(\lambda)\begin{bmatrix} -\beta'(\lambda) \\ \alpha'(\lambda) \end{bmatrix}\mu^2 + O(\mu^4),
$$

$$
A = -\frac{1}{3}\log|\kappa| - \frac{1}{2}\log\sqrt{(\alpha')^2 + (\beta')^2}, \quad B = \frac{\kappa}{2}\sqrt{(\alpha')^2 + (\beta')^2},
$$

as μ approaches zero, and

$$
\det\frac{\partial(x,y)}{\partial(\lambda,\mu)} = \frac{|\alpha'(s)\mathrm{Im}\,\beta(s) - \beta'(s)\mathrm{Im}\,\alpha(s)|^2|\alpha'(s)\beta''(s) - \alpha''(s)\beta'(s)|^2}{\left[\mathrm{Im}\,\overline{\alpha'(s)}\beta'(s)\right]^3};
$$

consequently

$$
y - f(x) = \frac{1}{2}f''(\alpha(\lambda))\,[\mathrm{Im}\,\alpha(\lambda + i\mu)]^2 + O(\mu^4)
$$

and

$$
\det\frac{\partial(x,y)}{\partial(\lambda,\mu)} = \mu\,[\alpha'(\lambda)\beta''(\lambda) - \alpha''(\lambda)\beta'(\lambda) + O(\mu^2)]
$$

as μ approaches zero. Thus (3.13) imply (3.12), as well as

$$
\det\frac{\partial(x,y)}{\partial(\lambda,\mu)} \neq 0
$$

if μ is different from, and sufficiently close to 0.

3.4. Here is an example. A *catenary* is the graph of either the equation

$$
y = \cosh x \qquad (3.14)
$$

or the equations

$$
x = \log\bigl(t + \sqrt{1 + t^2}\bigr), \quad y = \sqrt{1 + t^2}, \qquad (3.15)
$$

where parameter t coincides with both an arc length and the slope of the tangent straight-line.

Consider solutions to the equations

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = 1, \qquad \frac{\partial s}{\partial x} + s\,\frac{\partial s}{\partial y} = 0,$$

which obey the conditions

$$w(x,y) = s(x,y) = t$$

as x and y obey (3.15). Function w and the catenary in question are an *eikonal* and the relevant *caustic,* respectively; s obeys *Burgers equation,* takes a constant value on each tangent straight-line to the catenary and develops shocks along the catenary.

The *light region* is the set where

$$-\infty < x < \infty, \quad y \leq \cosh x;$$

the *shadow region* is the set above the catenary, where

$$-\infty < x < \infty, \quad \cosh x < y.$$

The pair

$$\begin{aligned} s\,x - y &= s\,\log\bigl(s + \sqrt{1+s^2}\bigr) - \sqrt{1+s^2}, \\ w &= \frac{1}{\sqrt{1+s^2}}\left[x + s\,y - \log\bigl(s + \sqrt{1+s^2}\bigr)\right], \end{aligned}$$

makes s and w implicit functions of x and y in the light region. Eikonal w and its partner s can be *continued* in a subset of the shadow region via the equations

$$\begin{aligned} x &= \lambda + \tanh\lambda\,(\mu\cot\mu - 1), \\ y &= \frac{1}{\cosh\lambda}\left(\sinh^2\lambda\,\frac{\mu}{\sin\mu} + \mu\sin\mu + \cos\mu\right), \\ s &= \sinh(\lambda + i\mu), \\ w &= \sinh\lambda\,\frac{\mu}{\sin\mu} + \frac{i}{\cosh\lambda}\,(\sin\mu - \mu\,\cos\mu) \end{aligned}$$

— here λ and μ are parameters such that

$$-\infty < \lambda < \infty, \quad 0 \leq \mu \leq \pi/2.$$

The foregoing equations imply that

$$y - \cosh x = (\cosh\lambda)\,\mu^2\left[\frac{1}{2} + O(\mu^2)\right],$$

as μ approaches 0. Furthermore,

$$\det \frac{\partial(x,y)}{\partial(\lambda,\mu)} = \frac{[\mu^2 + \tanh^2\lambda\,(1-\mu\cot\mu)^2](\sinh^2\lambda + \cos^2\mu)}{\cosh\lambda\sin\mu},$$

$$\frac{\partial w}{\partial x} = \frac{1}{\cosh(\lambda+i\mu)}, \quad \frac{\partial w}{\partial y} = \tanh(\lambda+i\mu)$$

— in particular, a singularity occurs at the point whose coordinates are

$$x = 0, \quad y = \pi/2.$$

Figures 1 and 2 show plots of the imaginary parts of w and s.

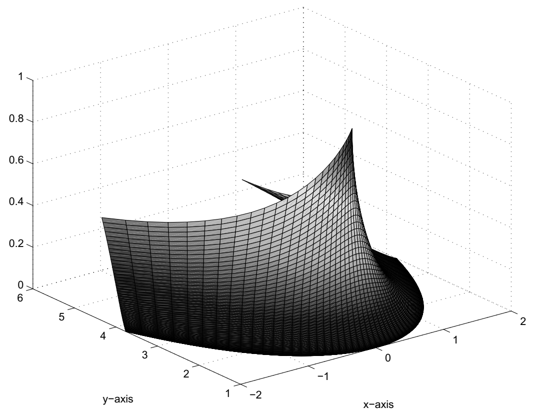

FIGURE 1. Eikonal equation: the imaginary part of w beyond a caustic.

4. Main statements

The most exhaustive method of continuing a two-dimensional geometric optical eikonal beyond a caustic consists perhaps in tackling a certain initial value problem — singular and ill-posed — for system (1.2). Such a problem is described in items (i) and (ii) below, and Figure 3.

(i) An initial curve IC is given.

The following alternative applies: either IC is specified *exactly* — no error infects the definition of IC; or else IC is a *phantom* — some coarse and polluted sampling of IC is gotten. In the former case assume IC is smooth enough. In the latter case *recover* IC, i.e. feed the available data into an

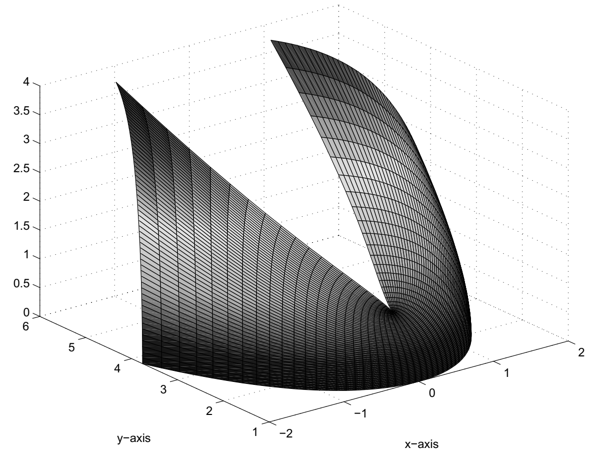

FIGURE 2. Burgers equation: the imaginary part of s beyond a shock-line.

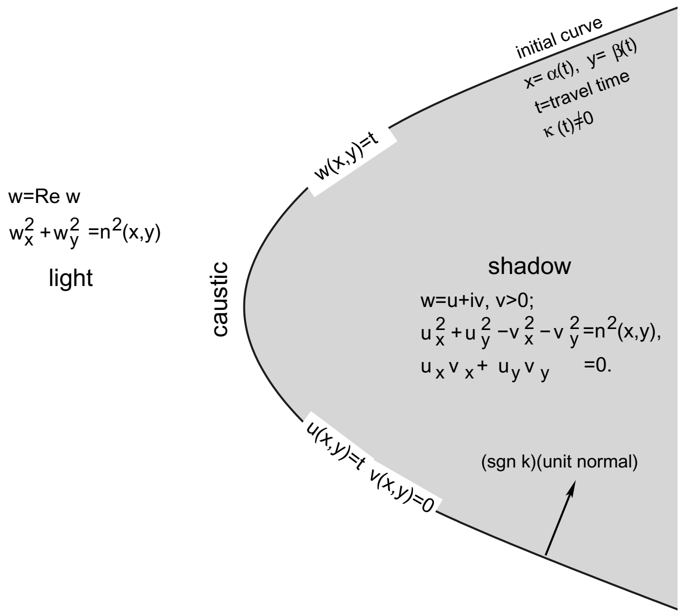

FIGURE 3. A geometric optical eikonal and its continuation past a caustic.

appropriate denoising process, and then elect the consequent output as an operative substitute of IC. (Ad hoc tools can be found in Section 7.)

Represent IC (either the authentic one, or else its surrogate) by the equations

$$x = \alpha(t), \quad y = \beta(t), \tag{4.1}$$

and adjust parameter t so as

$$t = \textit{a travel time} \tag{4.2}$$

without any loss of generality.

Assume travel time is an extra metric in action and the relevant *geodesic curvature* of IC is *free from zeroes.* In other words, postulate that (4.1) and either of the equations

$$\begin{gathered} \kappa\,(\text{velocity})^3 = \text{Geodesic curvature}, \\ \kappa\,(\text{velocity})^2 = \text{Euclidean curvature} - \langle \text{unit normal}, \nabla \log n(x,y) \rangle \end{gathered} \tag{4.3}$$

result in

$$\kappa \textit{ vanishes nowhere.} \tag{4.4}$$

(ii) A pair $[u\,v]$ is sought that obeys system (1.2) and fulfills the following conditions. First,

$$u(x,y) = t, \quad v(x,y) = 0, \tag{4.5}$$

if x, y and t are subject to (4.1). Second, u and v are defined in the side of IC that

$$(\text{sgn } \kappa) \times (\text{unit normal to IC})$$

points to.

Arguments from the appendix allow us to comment as follows. IC and the mentioned side of it can be viewed as a *caustic* and a *shadow region,* respectively. Any geometric optical eikonal, which makes IC a caustic, lives in the illuminated side of IC; the complex-valued eikonal, whose real part is u and whose imaginary part is v, lives in the opposite, dark side of IC instead. Both the former and the latter equal a travel time along IC. An *extension* of the geometric optical eikonal in hand ensues. Such an extension does obey the eikonal equation, lives in both the light region and a subset of the shadow region, and takes complex values where shadow prevails. In physical terms, problem (i) and (ii) accepts a caustic in input, then models *evanescent waves* that rise in the dark side of it.

In the present paper we focus our attention on solutions $[u\,v]$ to the problem (i) and (ii) that meet the following extra requirements:

(iii) they are *elliptic;*
(iv) their Jacobian determinant obeys

$$\operatorname{sgn}(u_x v_y - u_y v_x) = \operatorname{sgn}\kappa.$$

Condition (iii) ensures that the latter entry v is *not constant;* as will emerge from subsequent developments, initial conditions plus conditions (iii) and (iv) ensure that the same entry is *nonnegative.*

The solutions to problem (i)–(iv) develop *singularities* near IC, as any geometric optical eikonal does in the vicinity of the relevant caustic. We will show in Section 6 that they obey the expansions

$$u(x,y) = s + o(r), \;\; v(x,y) = \frac{2\sqrt{2}}{3}|\kappa(s)|^{1/2}|r|^{3/2} + o(r^{3/2}) \tag{4.6}$$

as (x,y) belongs to the appropriate side of, and is close enough to IC. Here r and s stand for the curvilinear coordinates described in the appendix — informally, r is a signed distance from IC, s is a lifting of a travel time inherent to IC. Note the physical meaning of (4.6): the damping effects, which are encoded in the imaginary part of a complex-valued eikonal, are tuned by the geodesic curvature of the relevant caustic.

Problem (i)–(iv) is recast in the next section into a more tractable form.

5. Framework

5.1. A convenient coordinate system must be called for. We choose to recast (1.2) by reversing the roles of dependent and independent variables — i.e. we think of u and v as curvilinear coordinates, and think of x and y as functions of u and v. In other words, we subject (1.2) to the change of variables that is called *hodograph transformation* at times — see [64] and [227], for instance.

Let $[u\,v]$ be any smooth elliptic solution to (1.2), and observe the following.
(i) The level lines of u and those of v are free from singular points, and cross at a right angle. Moreover, the Jacobian determinant

$$u_x v_y - u_y v_x \textit{ vanishes nowhere.} \tag{5.1}$$

Let

$[u\,v] \mapsto [x(u,v)\,y(u,v)]$ be a local *inverse* of $[x\,y] \mapsto [u(x,y)\,v(x,y)]$, then observe the following equation

$$(u_x v_y - u_y v_x)(x_u y_v - x_v y_u) = 1$$

and the propositions (ii)–(iv) below.
(ii) The following partial differential system holds:

$$1/E - 1/G = 1, \quad F = 0. \tag{5.2}$$

Here

$$E = n^2(x,y)(x_u^2 + y_u^2),\ F = n^2(x,y)(x_u x_v + y_u y_v),\ G = n^2(x,y)(x_v^2 + y_v^2)$$

— in other words,

$$n^2(x,y)((dx)^2 + (dy)^2) = E(du)^2 + 2F du\, dv + G(dv)^2.$$

(iii) The following systems and equations hold:

$$\frac{\partial}{\partial u}\begin{bmatrix} x \\ y \end{bmatrix} = f \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial v}\begin{bmatrix} x \\ y \end{bmatrix}, \tag{5.3}$$

$$\frac{\partial}{\partial v}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{f} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial u}\begin{bmatrix} x \\ y \end{bmatrix}, \tag{5.4}$$

$$\begin{gathered} \operatorname{sgn} f = \operatorname{sgn}(x_u y_v - x_v y_u), \\ \frac{1}{f^2} = 1 + n^2(x,y)(x_v^2 + y_v^2), \quad f^2 = 1 - n^2(x,y)(x_u^2 + y_u^2). \end{gathered} \tag{5.5}$$

(iv) The following equations hold:

$$f^2 = \frac{1 - n^2(x,y)x_u^2}{1 + n^2(x,y)x_v^2},\ f^2 = \frac{1 - n^2(x,y)y_u^2}{1 + n^2(x,y)y_v^2}, \tag{5.6}$$

$$\frac{\partial^2}{\partial u^2}\begin{bmatrix} x \\ y \end{bmatrix} + f^4 \frac{\partial^2}{\partial v^2}\begin{bmatrix} x \\ y \end{bmatrix} = (1 - f^2)\begin{bmatrix} x_u^2 - y_u^2 & -2x_u y_u \\ 2x_u y_u & x_u^2 - y_u^2 \end{bmatrix} \nabla \log n(x,y) \tag{5.7}$$

— in the event that n is identically 1, these equations read

$$x_{uu} + \left(\frac{1 - x_u^2}{1 + x_v^2}\right)^2 x_{vv} = 0, \quad y_{uu} + \left(\frac{1 - y_u^2}{1 + y_v^2}\right)^2 y_{vv} = 0. \tag{5.8}$$

Proof of (i). System (1.2) tells us that

$$|\nabla u| > 0,$$

and that the gradients of u and v are orthogonal. Ellipticity gives

$$|\nabla v| > 0.$$

The equation

$$(u_x v_y - u_y v_x)^2 = |\nabla u|^2 |\nabla v|^2 - \langle \nabla u, \nabla v\rangle^2$$

concludes the proof. $\square$

Proof of (ii). Since

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\frac{\partial(u,v)}{\partial(x,y)} \times \left[\frac{\partial(u,v)}{\partial(x,y)}\right]^T = \begin{bmatrix} |\nabla u|^2 & \langle \nabla u, \nabla v\rangle \\ \langle \nabla u, \nabla v\rangle & |\nabla v|^2 \end{bmatrix},$$

$$\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^T \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} x_u^2 + y_u^2 & x_u x_v + y_u y_v \\ x_u x_v + y_u y_v & x_v^2 + y_v^2 \end{bmatrix},$$

we have

$$\frac{1}{x_u^2 + y_u^2} = |\nabla u|^2 - \frac{\langle \nabla u, \nabla v\rangle^2}{|\nabla v|^2}, \quad \frac{1}{x_v^2 + y_v^2} = |\nabla v|^2 - \frac{\langle \nabla u, \nabla v\rangle^2}{|\nabla u|^2},$$
$$x_u x_v + y_u y_v = -\langle \nabla u, \nabla v\rangle (x_u y_v - x_v y_u)^2.$$

The conclusion ensues. $\square$

Proof of (iii) *and* (iv). The latter equation from (5.2) yields

$$x_u : y_v = y_u : (-x_v),$$

hence the following systems result:

$$\begin{bmatrix} x_u \\ y_u \end{bmatrix} = f \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_v \\ y_v \end{bmatrix}, \quad \begin{bmatrix} x_v \\ y_v \end{bmatrix} = \frac{1}{f} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_u \\ y_u \end{bmatrix}.$$

Factor f is easily identified. Both the above systems give

$$\operatorname{sgn} f = \operatorname{sgn}(x_u y_v - x_v y_u);$$

the former and system (5.2) imply

$$\frac{1}{f^2} = 1 + n^2(x,y)(x_v^2 + y_v^2);$$

the latter and system (5.2) imply

$$f^2 = 1 - n^2(x,y)(x_u^2 + y_u^2).$$

Another arrangement reads

$$\begin{bmatrix} x_u \\ x_v \end{bmatrix} = \begin{bmatrix} 0 & f \\ -1/f & 0 \end{bmatrix} \begin{bmatrix} y_u \\ y_v \end{bmatrix}, \quad \begin{bmatrix} y_u \\ y_v \end{bmatrix} = \begin{bmatrix} 0 & -f \\ 1/f & 0 \end{bmatrix} \begin{bmatrix} x_u \\ x_v \end{bmatrix}$$

— two Bäcklund transformations, inverse of one another. The former and system (5.2) imply

$$f^2 = \frac{1 - n^2(x,y)y_u^2}{1 + n^2(x,y)y_v^2},$$

the latter and system (5.2) imply

$$f^2 = \frac{1 - n^2(x,y)x_u^2}{1 + n^2(x,y)x_v^2}.$$

The integrability conditions, which pertain to the Bäcklund transformations in hand, read

$$\begin{bmatrix} \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} \end{bmatrix} \begin{bmatrix} 1/f & 0 \\ 0 & f \end{bmatrix} \begin{bmatrix} x_u & y_u \\ x_v & y_v \end{bmatrix} = 0$$

and result in equation (5.7) after algebraic manipulations.

Equations (5.5) and (5.6) are consistent with one another and with the early equations (1.3) and (1.5), as Proposition (ii) and its proof show.

The proof is complete. □

5.2. In view of the discussion above, problem (i)–(iv) stated in Section 4 amounts to looking for solutions $[x\,y]$ to the partial differential system

$$\begin{gathered} \frac{\partial}{\partial v}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{f}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial u}\begin{bmatrix} x \\ y \end{bmatrix}, \\ \operatorname{sgn} f = \operatorname{sgn} \kappa, \quad f^2 = 1 - n^2(x,y)(x_u^2 + y_u^2), \end{gathered} \tag{5.9}$$

that is defined either in the half-strip

$$a < u < b, \ 0 < v < \infty$$

or in an appropriate bounded subset if it, and satisfies the initial conditions

$$x(u,0) = \alpha(u), \quad y(u,0) = \beta(u) \ \text{ for } \ a \le u \le b. \tag{5.10}$$

In the remaining part of the paper we will concentrate on such a problem. A behavior of solutions $[x\,y]$ to (5.9) and (5.10) as v is close to 0 is fixed up in the next section. An algorithm for computing the same solutions is offered in Section 4. Section 5 is devoted to an example.

6. Behavior near the caustic

The state of affairs causes any solution of (5.9) and (5.10) to suffer from *singularities* near the initial line — indeed,

$$x_v^2(u,v) + y_v^2(u,v) \to \infty$$

as $a \le u \le b$ and v approaches 0. The proposition below offers more details on the subject, as well as a proof of expansions (4.6).

Proposition 6.1. *Let x and y obey system* (5.9) *and initial conditions* (5.10). *Assume $x(u,v)$ and $y(u,v)$ depend smoothly upon u for every nonnegative, sufficiently small v. Then the following asymptotic expansion holds:*

$$\begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix} = \begin{bmatrix} \alpha(u) \\ \beta(u) \end{bmatrix} + \frac{(3v)^{2/3} \mathrm{sgn}\ \kappa(u)}{2|\kappa(u)|^{1/3}\sqrt{\alpha'(u)^2 + \beta'(u)^2}} \begin{bmatrix} -\beta'(u) \\ \alpha'(u) \end{bmatrix} + o(v^{2/3})$$

as $a \le u \le b$ and v is positive and approaches 0.

Proof. Equation (3.2), equations (5.9) and initial conditions (5.10) tell us that

$$f(u,v) \to 0$$

as $a \le u \le b$ and v is positive and approaches 0. Therefore,

$$\lim_{v \downarrow 0} \frac{v^{1/3}}{f(u,v)} = \lim_{v \downarrow 0} \mathrm{sgn}\, f(u,v) \left\{ \mathrm{sgn}\, f(u,v) \Big/ \frac{\partial}{\partial u} f^3(u,v) \right\}^{1/3},$$

by L'Hospital's rule. Equations (5.9) give successively

$$\frac{\partial}{\partial v} n^2(x,y) = \frac{2\, n^2(x,y)}{f(u,v)} \left\langle \nabla \log n(x,y), \begin{bmatrix} -y_u \\ x_u \end{bmatrix} \right\rangle,$$

$$\frac{\partial}{\partial v}(x_u^2 + y_u^2) = -\frac{2}{f(u,v)}(x_u y_{uu} - x_{uu} y_u),$$

and

$$\frac{\partial}{\partial v} f^3(u,v) = 3\,(x_u^2 + y_u^2)^{3/2} n^2(x,y) \left\{ \frac{x_u y_{uu} - x_{uu} y_u}{(x_u^2 + y_u^2)^{3/2}} - \left\langle \nabla \log n(x,y), (x_u^2 + y_u^2)^{-1/2} \begin{bmatrix} -y_u \\ x_u \end{bmatrix} \right\rangle \right\}.$$

We infer

$$\lim_{v \downarrow 0} (3v)^{1/3} \begin{bmatrix} x_v \\ y_v \end{bmatrix} = \frac{\mathrm{sgn}\ \kappa(u)}{|\kappa(u)|^{1/3}\sqrt{\alpha'(u)^2 + \beta'(u)^2}} \begin{bmatrix} -\beta'(u) \\ \alpha'(u) \end{bmatrix}$$

because of (4.3) and (4.4). The conclusion follows. □

7. Differentiating discrete and polluted data

7.1. The present section is devoted to an auxiliary technique, which is a key to our main results.

Differentiating a real-valued function of one real variable is among the most elementary processes of mathematical and numerical analysis, but is also a significant prototype of those problems that are nowadays qualified ill-posed in the sense of Hadamard. On the other hand, differentiations with respect to those directions, which are tangent to the initial surface, are the main source of the ill-posed character of initial value problems for elliptic systems of partial differential equations. In the next section we tune a numerical approach to one of such problems, where a suitable approximation of tangential derivatives plays an important role.

Methods of approximating derivatives of smooth functions under non-exact data have been widely experimented over the years. Here we take the opportunity of sketching one more of such methods. We consider the case where data consist of *discrete* and *noisy* samples, nodes are *equally spaced,* and information is available on both the relevant noise and the underlying smoothness. Our method is inspired by ideas that the theory of statistical learning has recently revived — see e.g. [45], [65], [198]–[201], [217]–[220] — and of course mimics several of its ascendants — see e.g. [2], [3], [7]–[8], [9], [13], [19], [38], [44], [47], [48], [51], [52], [59], [73], [79], [97], [103], [110], [111], [133], [134], [146], [155]–[156], [157], [158], [162], [180]–[181], [182], [188], [193], [203], [204], [216], [225].

7.2. Items in input include:

(i) the *end points* of a bounded interval — a and b;

(ii) the *number* of both nodes and samples — an integer N, larger than 2;

(iii) *equally spaced nodes* from a to b — specifically,

$$x_k = a + (k-1)\,\frac{b-a}{N-1} \quad (k = 1, \dots, N);$$

(iv) *noisy samples* — a sequence

$$g_1, g_2, \dots, g_N$$

of real numbers whatever.

Goals include recovering some *noiseless* function f and the derivative f' of f based upon the following information only:

(v) f is *smooth;*

(vi) $f(x_k)$ is *close* to g_k, for $k = 1, 2, \dots, N$.

Our recipe segments into the following three steps. First, let λ and μ be *positive parameters,* and solve the variational problem

$$\sum_{k=1}^{N} [u(x_k) - g_k]^2 + \lambda \int_{-\infty}^{\infty} \left[\mu^7 (u'''')^2 + \mu^{-1} u^2\right] \, dx = \text{minimum}, \tag{7.1}$$

under the condition

$$u \;\text{ belongs to Sobolev space } W^{4,2}(-\infty, \infty). \tag{7.2}$$

Second, adjust λ and μ properly. Third, take u, u' as approximations of f, f'.

Observe that

$$\sum_{k=1}^{N} [u(x_k) - g_k]^2,$$

the beginning of line (7.1), is a *data-fidelity term;*

$$\int_{-\infty}^{\infty} \left[\mu^7 (u'''')^2 + \mu^{-1} u^2\right] \, dx,$$

a squared and conveniently scaled norm of u in $W^{4,2}(-\infty, \infty)$, plays the role of a *penalty.* Parameter λ balances the data-fidelity term and the penalty term; parameter μ balances

$$\int_{-\infty}^{\infty} (u'''')^2 dx \quad \text{and} \quad \int_{-\infty}^{\infty} u^2 dx,$$

hence tunes a Rayleigh quotient of u.

It could be shown that the above recipe has the potential of recovering not only f and f' as requested here, but f'' and f''' as well. Involving

$$\int_{-\infty}^{\infty} \left[\mu^3 (u'')^2 + \mu^{-1} u^2\right] \, dx$$

instead, and letting $W^{2,2}(-\infty, \infty)$ be the space of competing functions, would be enough for our present purposes; whereas involving

$$\int_{-\infty}^{\infty} \left[\mu (u')^2 + \mu^{-1} u^2\right] \, dx$$

and Sobolev space $W^{1,2}(-\infty,\infty)$ would not. Powers different from squares could be allowed, but would make the method impracticable.

7.3. Effective formulas read as follows. Rudiments of the calculus of variations demonstrate that Problem (7.1) & (7.2) possesses a unique solution; moreover that such a solution — named u throughout — obeys

$$\lambda\left(\mu^7\,\frac{d^8u}{dx^8}+\mu^{-1}u\right)+\sum_{k=1}^{N}[u(x_k)-g_k]\,\delta(x-x_k)=0 \tag{7.3}$$

for $-\infty<x<\infty$. The following features are decisive: equation (7.3) is *affine;*

$$\mu^7\,\frac{d^8}{dx^8}+\mu^{-1}$$

is a *positive* operator in $L^2(-\infty,\infty)$, whose inverse *mollifies*;

$$\sum_{k=1}^{N}[u(x_k)-g_k]\,\delta(\cdot-x_k)$$

is a *spike train* or a *shah-function.* (Shah stands for a letter of Cyrillic alphabet, Ш, which is suggestive of an assemblage of vertical needles.)

Condition (7.2) and equation (7.3) give

$$-\lambda\,u(x)=\sum_{k=1}^{N}[u(x_k)-g_k]\,K\left(\frac{x-x_k}{\mu}\right) \tag{7.4}$$

for $-\infty<x<\infty$. Equation (7.4) gives

$$A\begin{bmatrix}u(x_1)\\ \vdots\\ u(x_N)\end{bmatrix}+\lambda\begin{bmatrix}u(x_1)\\ \vdots\\ u(x_N)\end{bmatrix}=A\begin{bmatrix}g_1\\ \vdots\\ g_N\end{bmatrix}. \tag{7.5}$$

Here K denotes an appropriate *reproducing kernel* for the Sobolev space $W^{4,2}(-\infty,\infty)$ — in other words a *fundamental solution* to d^8/dx^8+1. K is the solution to

$$\frac{d^8K}{dx^8}+K=\delta(x)$$

that decays at infinity, is given by

$$\pi\,K(x)=\int_0^\infty\frac{\cos(x\xi)}{1+\xi^8}\,d\xi \tag{7.6}$$

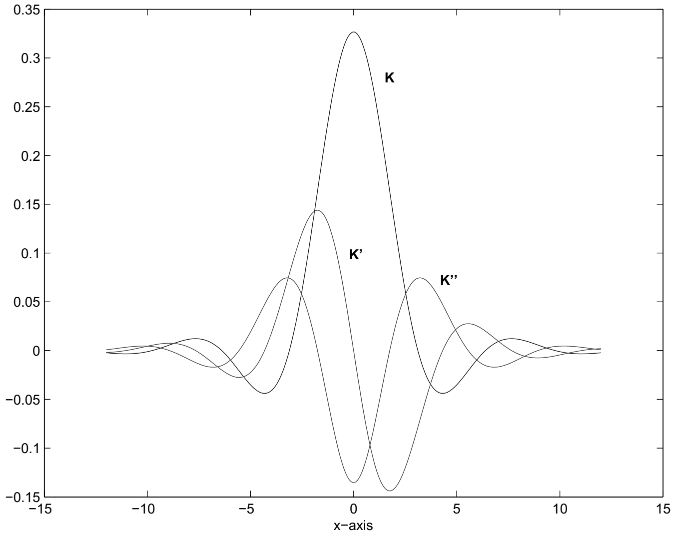

FIGURE 4. Plots of K, K' and K''.

for $-\infty < x < \infty$, is *even* and *positive definite.* Secondly,

$$A = \begin{bmatrix} K(0) & K\left(\dfrac{x_1 - x_2}{\mu}\right) & \dots & K\left(\dfrac{x_1 - x_N}{\mu}\right) \\ K\left(\dfrac{x_2 - x_1}{\mu}\right) & K(0) & \dots & K\left(\dfrac{x_2 - x_N}{\mu}\right) \\ \vdots & \vdots & \ddots & \vdots \\ K\left(\dfrac{x_N - x_1}{\mu}\right) & K\left(\dfrac{x_N - x_2}{\mu}\right) & \dots & K(0) \end{bmatrix} \tag{7.7}$$

— a *symmetric, positive definite* Toeplitz matrix.

Let $\nu = \pi/8$. Manipulating formula (7.6) gives

$$K(x) = \frac{1}{8\sin\nu} - \frac{1}{8\cos\nu}\frac{x^2}{2} + \frac{1}{8\cos\nu}\frac{x^4}{24} - \frac{1}{8\sin\nu}\frac{x^6}{720} + \frac{1}{2}\frac{|x|^7}{5040} + O(x^8)$$

as x approaches zero — K behaves near zero like a *spline* of order seven. Calculus of residues, or convenient formulas from [PBM, Section 2.5.10], give

$$4K(x) = e^{-|x|\cos\nu}\cos(|x|\sin\nu - \nu) + e^{-|x|\sin\nu}\sin(|x|\cos\nu + \nu)$$

as $-\infty < x < \infty$. Therefore,

$$\frac{d^n K}{dx^n}(x) = \frac{(-\operatorname{sgn} x)^n}{4} \Big\{ e^{-|x|\cos\nu} \cos(|x|\sin\nu - (n+1)\nu) \\ + e^{-|x|\sin\nu} \sin(|x|\cos\nu + (n+1)\nu - n\pi/2) \Big\}$$

as $x \neq 0$ and $n = 1, 2, 3, \ldots$. Figure 4 shows plots of K, K', K''.

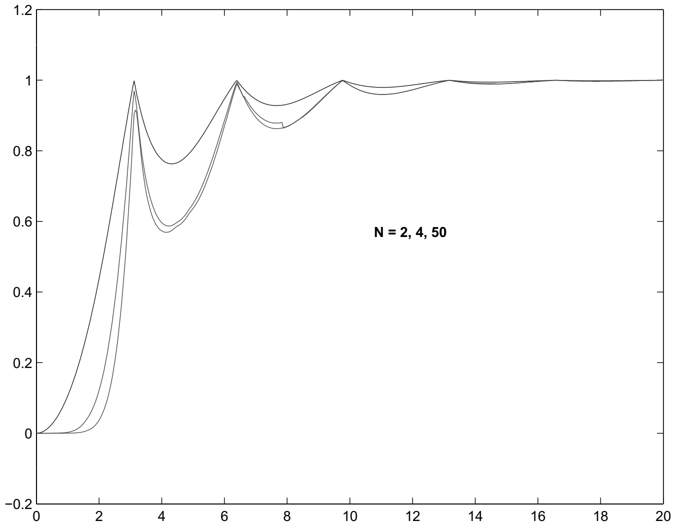

FIGURE 5. Reciprocal condition estimator of A, plotted versus the ratio $(b-a)/((N-1)\mu)$.

Analysis shows the following. The spectrum of A lies in the open interval $]0, N/(8\sin\nu)[$. All eigenvalues of A are close to $1/(8\sin\nu)$, if $\mu(N-1)/(b-a)$ is small; otherwise, the largest eigenvalue of A is close to $N/(8\sin\nu)$ and most remaining eigenvalues of A are close to 0. In particular, A is *invertible* anyway; A is either *well-conditioned* or *ill-conditioned* depending on whether $\mu(N-1)/(b-a)$ is small or large. Figure 5 shows plots of a reciprocal condition estimator of A versus $(b-a)/((N-1)\mu)$.

Equation (7.4) implies that u belongs to the *linear span* of

$$K\left(\frac{\cdot - x_1}{\mu}\right), \ldots, K\left(\frac{\cdot - x_N}{\mu}\right)$$

— *translations* and *dilations* of K. Such items own either a spike-shaped or a well-rounded profile depending on whether μ is small or large, inasmuch as

$$\frac{\int_{-\infty}^{\infty} |(d/dx)\mathrm{Item}|^2\,dx}{\int_{-\infty}^{\infty} |\mathrm{Item}|^2\,dx} = \frac{5}{7}\,\mu^{-2}\tan\nu, \quad \frac{\int_{-\infty}^{\infty} |(d^2/dx^2)\mathrm{Item}|^2\,dx}{\int_{-\infty}^{\infty} |\mathrm{Item}|^2\,dx} = \frac{3}{7}\,\mu^{-4}\tan\nu.$$

The same items are definitely *linearly independent,* although appropriate formulas and analysis show that their Gram matrix is well-conditioned only if $\mu(N-1)/(b-a)$ is small enough.

Equation (7.5) determines $u(x_1), \dots, u(x_N)$ in terms of data. Note that they solve

$$\sum_{k=1}^{N}[u(x_k)-g_k]^2 + \lambda\,[u(x_1)\ \cdots\ u(x_N)]\,A^{-1}\begin{bmatrix} u(x_1)\\ \vdots \\ u(x_N)\end{bmatrix} = \text{minimum}$$

— a standard *finite-dimensional least-square* problem, where λ and the inverse of A imitate a *regulating parameter* à la Tikhonov and a *penalty,* respectively.

Now we are in a position to draw conclusions. Let Id be the $N \times N$ *unit matrix,* and

$$R = (\lambda\,\mathrm{Id} + A)^{-1}$$

— a *resolvent.* Let B the vector-valued function such that

$$B(x) = R\begin{bmatrix} K\left(\dfrac{x-x_1}{\mu}\right) \\ \vdots \\ K\left(\dfrac{x-x_N}{\mu}\right)\end{bmatrix}$$

for $-\infty < x < \infty$ — an *alternative basis* in the linear span mentioned above. Define C and D by $C = A\,R$ and

$$D = \mu^{-1}\begin{bmatrix} 0 & K'\left(\dfrac{x_1-x_2}{\mu}\right) & \dots & K'\left(\dfrac{x_1-x_N}{\mu}\right) \\ K'\left(\dfrac{x_2-x_1}{\mu}\right) & 0 & \dots & K'\left(\dfrac{x_2-x_N}{\mu}\right) \\ \vdots & \vdots & \ddots & \vdots \\ K'\left(\dfrac{x_N-x_1}{\mu}\right) & K'\left(\dfrac{x_N-x_2}{\mu}\right) & \dots & 0 \end{bmatrix} R.$$

The following equations hold.

$$u(x) = [g_1 \, \cdots \, g_N] \, B(x)$$

for $-\infty < x < \infty$,

$$\begin{bmatrix} u(x_1) \\ \vdots \\ u(x_N) \end{bmatrix} = C \begin{bmatrix} g_1 \\ \vdots \\ g_N \end{bmatrix},$$

$$\begin{bmatrix} u'(x_1) \\ \vdots \\ u'(x_N) \end{bmatrix} = D \begin{bmatrix} g_1 \\ \vdots \\ g_N \end{bmatrix}.$$

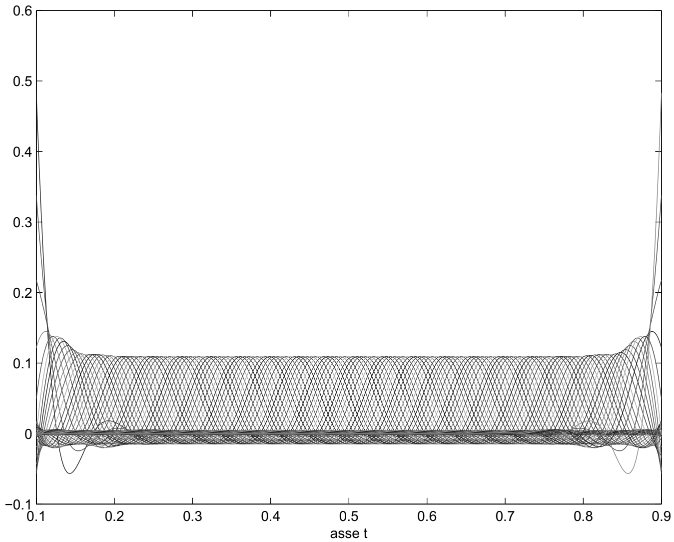

FIGURE 6. The alternative basis B : plots of $B_1(x), \ldots,$ $B_N(x)$ versus x.

As figures 6, 7 and 8 show, B and C mimic a typical *Green's function* from two different perspectives, D mimics a *derivative* of a Green's function. Observe incidentally that

$$\int_{-\infty}^{\infty} \left[\mu^7 (B'''')^T B'''' + \mu^{-1} B^T B\right] dx = \operatorname{tr} \left[A \, (\lambda \, \mathrm{Id} + A)^{-2}\right],$$

$$C = \mathrm{Id} - \lambda \, (\lambda \, \mathrm{Id} + A)^{-1},$$

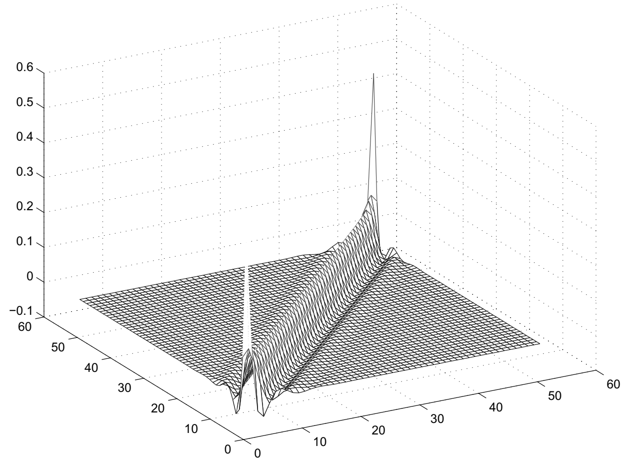

FIGURE 7. Matrix C playing the role of a regularizing filter.

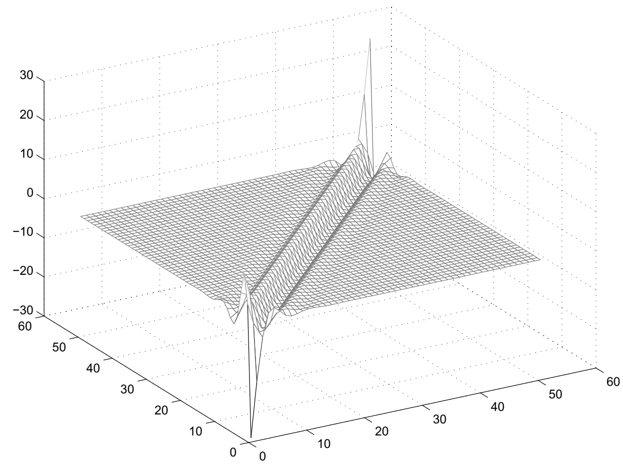

FIGURE 8. Matrix D, simulating differentiation.

and that B and C solve the variational problem

$$\lambda \int_{-\infty}^{\infty} \left[\mu^7 (B'''')^T B'''' + \mu^{-1} B^T B\right] dx + \operatorname{tr} \left[(C^T - \mathrm{Id})(C - \mathrm{Id})\right] = \text{minimum},$$

subject to the conditions

$$B \in [W^{4,2}(-\infty,\infty)]^N, \quad C = \begin{bmatrix} B_1(x_1) & B_2(x_1) & \dots & B_N(x_1) \\ B_1(x_2) & B_2(x_2) & \dots & B_N(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ B_1(x_N) & B_2(x_N) & \dots & B_N(x_N) \end{bmatrix}.$$

7.4. Here we offer directions for adjusting λ and μ properly. Parameter λ — *dimensionless* — discriminates whether solution u to Problem (7.1) & (7.2) either fits data well (but is simultaneously sensible of noise), or else is little affected by noise (but departs somewhat from data). In loose terms, the following statements hold. First, u virtually *interpolates* $g_1, \dots, g_N$ if λ is close to zero; however, u is liable to own an *irregular* profile at the same time. Second, u *quenches smoothly* if λ grows larger and larger. Indeed,

$$\int_{-\infty}^{\infty} \left[\mu^7 (B'''')^T B'''' + \mu^{-1} B^T\, B\right]\, dx = \operatorname{tr}\, A^{-1} + O(\lambda),$$

$$C = \mathrm{Id} + O(\lambda)$$

as λ approaches zero;

$$\|d^n B(x)/dx^n\| \le \lambda^{-1}\, \frac{\sqrt{N}}{8\sin[(n+1)\nu]}$$

as $-\infty < x < \infty$ and $n = 0, 1, \dots, 6$.

Parameter μ — making $\mu/(b-a)$ *dimensionless* — determines how much u is close to, or departs from a spike train. Indeed,

$$B(x) = \mu\, \frac{8\sin\nu}{1+\lambda\sin\nu} \begin{bmatrix} \delta(x-x_1) \\ \vdots \\ \delta(x-x_N) \end{bmatrix} + O(\mu^3)$$

as $-\infty < x < \infty$ and μ approaches zero;

$$B(x) = \frac{1}{N+8\lambda\sin\nu} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + O(\mu^{-2})$$

as $-\infty < x < \infty$ and μ grows larger and larger.

In the case where suitable information is available *a priori*, a theorem from Subsection 7 below suggests which values of λ and μ work properly. Otherwise, parameter λ may be determined based upon the *discrepancy*

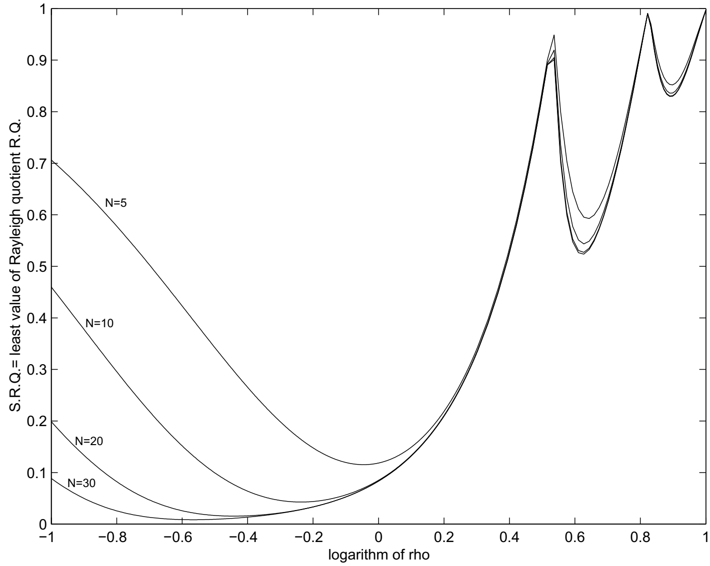

FIGURE 9. Plots of S.R.Q. versus $\log \rho$. Here $\rho = (b - a)/((N-1)\mu)$.

principle, the *cross-validation,* the *L-curve criterion,* or other customary devices — see e.g. [86, Chapter 7] for details.

Parameter μ is expediently identified by the following recipe. Let

$$\text{R.Q. of } v = \mu^{14} \, \frac{\int_{-\infty}^{\infty} \left[(d/dx)^7 v(x)\right]^2 dx}{\int_{-\infty}^{\infty} \left[v(x)\right]^2 dx}$$

be a dimensionless *Rayleigh quotient;* define a relevant minimum by

$$\text{S.R.Q.} = \min\left\{\text{R.Q. of } v : 0 \neq v \in \text{span of } K\left(\frac{\cdot - x_1}{\mu}\right), \ldots, K\left(\frac{\cdot - x_N}{\mu}\right)\right\};$$

then determine μ so that

$$\text{S.R.Q.} = \text{minimum}. \tag{7.8}$$

Equation (7.8) causes solution u of Problem (7.1) & (7.2) to retain its most favorable Rayleigh quotient. Equation (7.8) can be approached via equation (7.9) below and tools from linear algebra, although care must be taken of the ill-condition of the involved matrices.

Let

$$G = \mu^{-1} \times \text{Gram matrix of } K\left(\frac{\cdot - x_1}{\mu}\right), \ldots, K\left(\frac{\cdot - x_N}{\mu}\right),$$

$$H = \mu^{-1} \times \text{Gram matrix of } K^{(7)}\left(\frac{\cdot - x_1}{\mu}\right), \ldots, K^{(7)}\left(\frac{\cdot - x_N}{\mu}\right);$$

let L denote the *autocorrelation function* of K, videlicet

$$\pi L(x) = \int_0^\infty (1+\xi^8)^{-2} \cos(x\xi)\, d\xi$$

for $-\infty < x < \infty$. We have

$$G = \left[L\left(\frac{x_j - x_k}{\mu}\right)\right]_{j,k=1,\ldots,N}, \quad H = \left[-L^{(14)}\left(\frac{x_j - x_k}{\mu}\right)\right]_{j,k=1,\ldots,N};$$

$$L(x) = \frac{7}{8}\,K(x) - \frac{x}{8}\,K'(x), \quad L^{(14)}(x) = \frac{7}{8}\,K^{(6)}(x) + \frac{x}{8}\,K^{(7)}(x)$$

for $-\infty < x < \infty$, and

$$\text{S.R.Q.} = \text{least eigenvalue of } H \text{ with respect to } G \tag{7.9}$$

— in other words,

$$\text{S.R.Q.} = \text{least eigenvalue of } G^{-1/2} H\, G^{-1/2}.$$

Figure 9 shows plots of S.R.Q. versus $(b-a)/((N-1)\mu)$. The following table lists sample solutions to equation (7.8).

N	minimum value	$(b-a)/((N-1)\mu)$	$\mu/(b-a)$
5	0.115320757000	0.900127730000	0.277738360532
10	0.042825969700	0.578865888000	0.191946206219
20	0.015529762000	0.361101444000	0.145752889726
30	0.008487160550	0.271108504000	0.127191726235
40	0.005502126710	0.220171280000	0.116459447577
50	0.003920874640	0.187117585000	0.109065982576
60	0.002967795760	0.163520070000	0.103651818045
70	0.002342495540	0.145755736000	0.099431789245
80	0.001906845850	0.131825897000	0.096022315313
100	0.001349762400	0.111466361000	0.090619358257
120	0.001016274330	0.097183852500	0.086468699567
140	0.000798662701	0.086282601400	0.083380015062
150	0.000716797334	0.081805225800	0.082041328416

The formula

$$\frac{\mu}{b-a} \approx 0.0415 + 0.5416 \times N^{-1/2} - 0.6426 \times N^{-1} + 1.3706 \times N^{-3/2} \quad (7.10)$$

gives an effectual estimate of such solutions.

7.5. The following theorem holds.

Theorem 7.1. *Suppose u solves problem (7.1) & (7.2); suppose f, ε and E obey*

$$|f(x_k) - g_k| \leq \varepsilon \quad (k = 1, \dots, N),$$

$$\mu^{-4} \int_{-\infty}^{\infty} f(x)^2 dx + \mu^4 \int_{-\infty}^{\infty} [f''''(x)]^2 dx \leq 2(b-a)^{-3} E^2;$$

let

$$\delta = \max \left\{ (1 - 1/N)^{-1/2} \frac{\varepsilon}{E}, (N-1)^{-2} \right\}.$$

If δ approaches zero, and

$$\frac{\lambda}{N} \left(\frac{\mu}{b-a} \right)^3 \left(\frac{\varepsilon}{E} \right)^{-2}$$

is bounded and bounded away from zero, then

$$E^{-1} \max\{|f(x) - u(x)| : a \leq x \leq b\} = O(\delta^{3/4}),$$

$$E^{-1}(b-a) \max\{|f'(x) - u'(x)| : a \leq x \leq b\} = O(\delta^{1/4}).$$

Proof. Let

$$v = f - u. \quad (7.11)$$

The Functional J, whose domain is $W^{4,2}(-\infty, \infty)$ and whose value at any trial function φ obeys

$$J(\varphi) = \sum_{k=1}^{N} [\varphi(x_k) - g_k]^2 + \lambda \int_{-\infty}^{\infty} \left[\mu^7 (\varphi'''')^2 + \mu^{-1} \varphi^2 \right] dx,$$

attains its minimum value at u. Consequently,

$$J(f) = J(u) + (\text{a remainder}),$$

$$\text{remainder } = \sum_{k=1}^{N} [v(x_k)]^2 + \lambda \int_{-\infty}^{\infty} \left[\mu^7 (v'''')^2 + \mu^{-1} v^2 \right] dx.$$

Since

$$J(f) \le N\varepsilon^2 + 2\lambda \left(\frac{\mu}{b-a}\right)^3 E^2,$$
$$J(u) \ge 0,$$
$$\int_{-\infty}^{\infty} \left[\mu^7 (v'''')^2 + \mu^{-1} v^2\right] \, dx \ge 2\,\mu^3 \int_{-\infty}^{\infty} (v'')^2 dx,$$

we infer

$$\sum_{k=1}^{N} [v(x_k)]^2 + 2\lambda\mu^3 \int_{-\infty}^{\infty} (v'')^2 dx \le E^2 \left\{ N \left(\frac{\varepsilon}{E}\right)^2 + 2\lambda \left(\frac{\mu}{b-a}\right)^3 \right\}. \quad (7.12)$$

Combining inequality (7.12) and Lemma 7.2 below results in

$$(b-a)^{-1} \int_a^b v^2 dx$$
$$\le E^2 \left\{ \frac{N}{N-1} + \frac{1}{\pi^4 (N-1)^4} \frac{N}{2\lambda} \left(\frac{\mu}{b-a}\right)^{-3} \right\} \left\{ \left(\frac{\varepsilon}{E}\right)^2 + \frac{2\lambda}{N} \left(\frac{\mu}{b-a}\right)^3 \right\};$$

inequality (7.12) also yields

$$(b-a)^3 \int_a^b (v'')^2 dx \le E^2 \left\{ 1 + \frac{N}{2\lambda} \left(\frac{\mu}{b-a}\right)^{-3} \left(\frac{\varepsilon}{E}\right)^2 \right\}.$$

The last two inequalities and a hypothesis imply

$$(b-a)^{-1} \int_a^b v^2 dx \le (\text{Const.})\, E^2\, \delta^2, \quad (7.13)$$

$$(b-a)^3 \int_a^b (v'')^2 dx \le (\text{Const.})\, E^2. \quad (7.14)$$

The conclusions follow from (7.11), (7.13) and (7.14), by virtue of Lemma 7.3 below. □

Lemma 7.2. *Let* $-\infty < a < b < \infty$, *and* $N = 2, 3, \ldots$; *let*

$$\Delta x = \frac{b-a}{N-1}, \quad x_k = a + (k-1)\,\Delta x \quad (k = 1, \ldots, N).$$

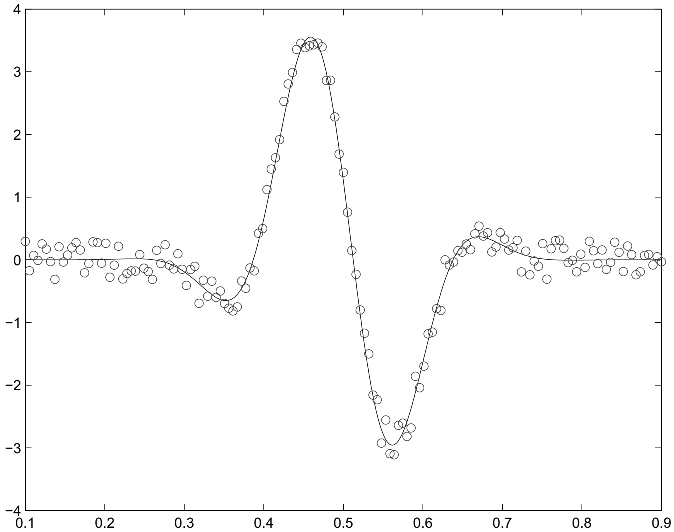

FIGURE 10. Original function (dotted), and samples polluted by noise (circled).

The inequality

$$\left\{\int_a^b v^2 dx\right\}^{1/2} \le (\Delta x)^{-1/2}\left\{\sum_{k=1}^{N} v(x_k)^2 dx\right\}^{1/2} + \left(\frac{\Delta x}{\pi}\right)^2 \left\{\int_a^b (v'')^2 dx\right\}^{1/2}$$

holds for every v from $W^{2,2}(a,b)$.

Lemma 7.3. *Suppose v is in $W^{2,2}(a,b)$, and*

$$(b-a)^{-1/2}\left\{\int_a^b v^2 dx\right\}^{1/2} = \|v\|, \quad (b-a)^3 \int_a^b (v'')^2 dx = 1.$$

The following inequalities hold:

$$\max|v| \le 2^{1/4}\,3^{-3/8}\|v\|^{3/4} + O(\|v\|),$$
$$(b-a)\,\max|v'| \le 2^{1/4}\,3^{-3/8}\|v\|^{1/4} + O(\|v\|).$$

The proofs of Lemma 7.2 and 7.3 are beyond the scope of the present paper, and are omitted.

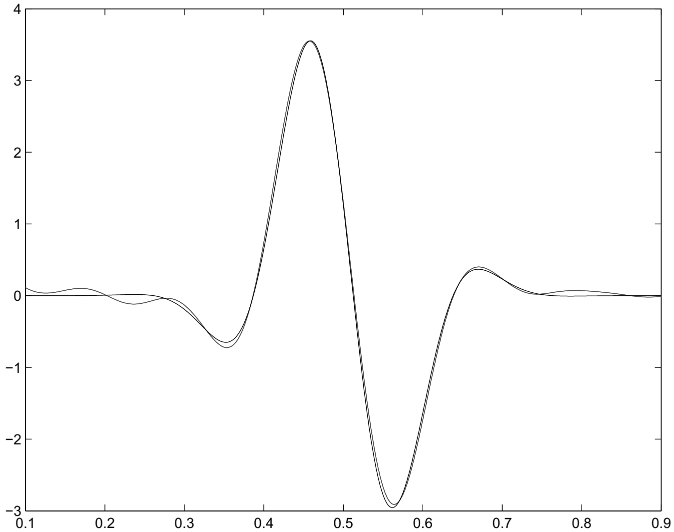

FIGURE 11. Original and recovered functions.

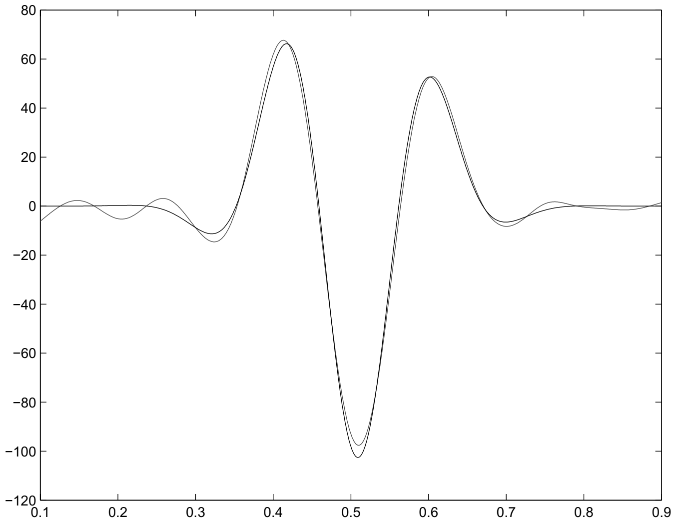

FIGURE 12. Original and recovered derivatives.

7.6. Here is an example, demonstrating how the present method works. Let

$$a = 0,\ b = 1,$$
$$f(x) = \exp\left[-72\,(x - \tfrac{1}{2})^2)\right]\,[\cos(25x) - 4\,\sin(25x)],$$

$$N = 150,$$
$$g_k = f(x_k) \pm (5\% \text{ random noise}) \quad (k = 1, \ldots, N).$$

Let parameter λ obey
$$\lambda = 10^{-6},$$
let parameter μ be given by formula (7.10), and let u solve problem (7.1) & (7.2). Figure 10 plots $g_1, g_2, \ldots, g_N$ versus $x_1, x_2, \ldots, x_N$; figure 11 plots f and u, and figure 12 plots f' and u'.

8. Discrete setting

8.1. Rendering problem (5.9) and (5.10) into an effective discrete form entails coping with *singularities* of solutions, *overflows,* and *ill-posedness* in the sense of Hadamard.

Singularities result from features of both the system and the initial conditions in hand, as already remarked in Section 6. Overflows take place whenever solutions stop to obey the constraint
$$n^2(x, y)(x_u^2 + y_u^2) < 1.$$

Note that the system
$$\frac{\partial}{\partial v}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial u}\begin{bmatrix} x \\ y \end{bmatrix}$$

— Cauchy-Riemann, a possible linearized version of (5.9) — possesses obvious solutions
$$\begin{bmatrix} x \\ y \end{bmatrix} = \exp(-\sqrt{t} + tv)\begin{bmatrix} \cos(tu) \\ \sin(tu) \end{bmatrix}$$
which are highly instable, i.e.
$$\sup\left\{\left[\frac{\partial^j x}{\partial u^j}(u, 0)\right]^2 + \left[\frac{\partial^j y}{\partial u^j}(u, 0)\right]^2 : -\infty < u < \infty\right\} \to 0$$
as $t \uparrow \infty$ and $j = 1, 2, \ldots,$ and
$$\inf\left\{x^2(u, v) + y^2(u, v) : -\infty < u < \infty\right\} \to \infty$$
as $t \uparrow \infty$ and v is positive.

Ill-posedness is a typical drawback of initial value problems for partial differential equations and systems of elliptic type. It was observed by Hadamard [82]–[83], and deeply investigated by John [94]–[96], Lavrentiev jr. [120]–[122], [127], Miller [147]–[148], Payne [163]–[166] and [171]–[172], Pucci [176]–[179], and Tikhonov [206]–[210]. Classical surveys on the subject have been authored by Lavrentiev jr., [123], [126], Payne [167], and Tikhonov [211]. Related information is in [19]–[20], [22], [15], [92], [93], [154], [159], [189] and [205]. A sample of more recent contributions includes [5], [6], [12], [17], [21], [16], [18], [29], [24], [30], [32], [36], [33], [34], [43], [46], [49], [62]–[63], [66]–[67], [84], [85], [89], [108], [109], [127], [132], [142]–[143], [153], [160]–[161], [168]–[170], [173], [183], [185], [186], [191], [202], [212], [213], [222], [224].

One might attempt to contend with ill-posedness via theorems involving *a priori* bounds on solutions and similar devices. We opt not to touch on this issue in the present paper, and focus our attention on constructive aspects instead.

We rely upon: (i) *asymptotic expansions,* describing how relevant solutions behave near the initial line (see Section 6); (ii) a technique of *approximate differentiation,* especially designed for working in presence of errors (see Section 7); (iii) an appropriate injection of *artificial viscosity,* which softens a coefficient and protects against overflows; (iv) an imitation of the *quasi-reversibility method.*

8.2. Besides data exposed to view, our method takes seven parameters in input: $L, M, N, \lambda, \mu, \nu, \xi$. The first one stands for the expected span of the solution domain. Parameters number two and three are large integers that specify the number of samples in hand — e.g. $M = N = 100$. The remaining four parameters set the tone of smoothing processes: λ and μ relate to approximate differentiation; ν stands for viscosity; ξ relates to quasi-reversibility. Of course, a priori information (such as the smoothness of solutions, and the expected location of their singularities) are momentous for guessing effectual values of these parameters.

8.3. The equations

$$\begin{aligned} &ustep = (b-a)/(M-1), && vstep = L/(N-1), \\ &u_j = a + (j-1)\, ustep \quad (j = 1, \ldots, M), && v_k = (k-1)\, vstep \quad (k = 1, \ldots, N), \end{aligned}$$

will be in force throughout this section. We choose a mesh to consist of the points

$$(u_j, v_k) \quad (j = 1, \ldots, M; k = 1, \ldots, N),$$

and store sample values at mesh points in the matrices

$$[x(j,k)]_{j=1,\ldots,M;k=1,\ldots,N}, \quad [y(j,k)]_{j=1,\ldots,M;k=1,\ldots,N}.$$

The columns of these matrices, i.e.

$$x(\cdot,1), x(\cdot,2), \dots, x(\cdot,N), \quad y(\cdot,1), y(\cdot,2), \dots, y(\cdot,N),$$

are recursively generated in the following way.

First step.

$$x(j,1) = \alpha(u_j),\ y(j,1) = \beta(u_j) \ \ (j = 1, \dots, M),$$

according to initial conditions (5.10).

Second step.

$$x(j,2) = x(j,1) + \frac{\operatorname{sgn}\kappa(u_j)}{2|\kappa(u_j)|^{1/3}} \frac{-\beta'(u_j)}{\sqrt{\alpha'(u_j)^2 + \beta'(u_j)^2}} (3v_2)^{2/3},$$
$$y(j,2) = y(j,1) + \frac{\operatorname{sgn}\kappa(u_j)}{2|\kappa(u_j)|^{1/3}} \frac{\alpha'(u_j)}{\sqrt{\alpha'(u_j)^2 + \beta'(u_j)^2}} (3v_2)^{2/3}$$

$(j = 1, \dots, M)$, according to Proposition 6.1.

Further steps. For $k = 3, \dots, N$ do actions (i)–(iii) below.

(i) Mimic tangent differentiation $\partial/\partial u$. Avoid finite differences, use the following formulas instead:

$$X = x(\cdot, k-1),\ Y = y(\cdot, k-1),$$
$$A = DX,\ B = DY.$$

Here

$$\lambda = \text{a dimensionless positive parameter},$$
$$\frac{\mu}{b-a} \simeq 0.0415 + 0.5416 \times M^{-1/2} - 0.6426 \times M^{-1} + 1.3706 \times M^{-3/2},$$

according to Section 7;

$$\nu = \pi/8,$$
$$4K(u) = e^{-|u|\cos\nu} \cos(|u|\sin\nu - \nu) + e^{-|u|\sin\nu} \sin(|u|\cos\nu + \nu),$$

and

$$4K'(u) = \operatorname{sgn}(u) \Big\{ -e^{-|u|\cos\nu} \cos(|u|\sin\nu - 2\nu) + e^{-|u|\sin\nu} \cos(|u|\cos\nu + 2\nu) \Big\}$$

for every u; moreover

$$D = \frac{1}{\mu} \left[K'\left(\frac{u_i - u_j}{\mu}\right) \right]_{i,j=1,\dots,M} \left\{ \lambda \operatorname{Id} + \left[K\left(\frac{u_i - u_j}{\mu}\right) \right]_{i,j=1,\dots,M} \right\}^{-1}.$$

(ii) Enter viscosity. Modify an uncomfortable coefficient as follows:

$$f(j) = P(\nu, n(X(j), Y(j))\sqrt{A(j)^2 + B(j)^2}\,\mathrm{sgn}\,\kappa(u_j) \quad (j = 1, \dots, M),$$

and let

$$U = -\begin{bmatrix} f(1) & 0 & \dots & 0 \\ 0 & f(2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f(M) \end{bmatrix}^{-1} B,$$

$$V = \begin{bmatrix} f(1) & 0 & \dots & 0 \\ 0 & f(2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f(M) \end{bmatrix}^{-1} A.$$

Here

$$0 < \nu = \text{artificial viscosity} \le \pi/2,$$

and P is given by

$$P(\nu, \rho) = \begin{cases} \sqrt{1-\rho^2} & \text{if } \ 0 \le \rho \le (1 + \sin^2\nu)^{-1/2}, \\ \dfrac{\sin\nu}{\rho + \sqrt{\rho^2 - \cos^2\nu}} & \text{if } \ \rho > (1 + \sin^2\nu)^{-1/2} \end{cases}$$

— observe that

$0 \le \rho \mapsto P(\nu, \rho)$ is strictly positive and continuously differentiable,

$P(\nu, \rho)$ approaches $\sqrt{1-\rho^2}$ uniformly as $0 \le \rho \le 1$ and ν approaches 0.

(iii) Enter quasi-reversibility. Improve the conventional formulas

$$x(\cdot, k) = X + vstep\,U, \quad y(\cdot, k) = Y + vstep\,V,$$

as follows:

$$x(\cdot, k) = \varphi(v_k), \quad y(\cdot, k) = \psi(v_k).$$

Here

$$\xi = \text{a dimensionless positive parameter;}$$

$$R = (ustep)^{-2} \begin{bmatrix} 2 & -5 & 4 & -1 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & -1 & 4 & -5 & 2 \end{bmatrix},$$

a caricature of a second-order derivative; φ is the vector-valued mapping that obeys both the boundary conditions

$$\varphi(v_{k-1}) = X, \quad \varphi'(v_{k-1}) = U,$$

and a caricature of the variational constraint

$$\int_{v_{k-1}}^{v_k} \left\{ (vstep)^3 \|\varphi''(v)\|^2 + 3\xi \|R\varphi(v)\|^2 \right\} dv = \text{minimum},$$

namely

$$3\xi \, \|R\varphi(v_k)\|^2 + (vstep)^2 \int_{v_{k-1}}^{v_k} \|\varphi''(v)\|^2 dv = \text{minimum};$$

ψ obeys

$$\psi(v_{k-1}) = Y, \quad \psi'(v_{k-1}) = V,$$

$$3\xi \, \|R\psi(v_k)\|^2 + (vstep)^2 \int_{v_{k-1}}^{v_k} \|\psi''(v)\|^2 dv = \text{minimum}.$$

As is easy to see, the differential equations

$$\varphi''''(v) = 0 \;\text{ and }\; \psi''''(v) = 0 \;\text{ if }\; v_{k-1} < v < v_k,$$

and the extra boundary conditions

$$\begin{aligned} &\varphi''(v_k) = 0, \; \varphi'''(v_k) = 3\xi(vstep)^{-2}(R^T R)\,\varphi(v_k), \\ &\psi''(v_k) = 0, \; \psi'''(v_k) = 3\xi(vstep)^{-2}(R^T R)\,\psi(v_k), \end{aligned}$$

are in effect. The formulas

$$\begin{aligned} x(\cdot,k) &= \left(\mathrm{Id} + \xi\, vstep\,(R^T R)\right)^{-1} (X + vstep\, U), \\ y(\cdot,k) &= \left(\mathrm{Id} + \xi\, vstep\,(R^T R)\right)^{-1} (Y + vstep\, V) \end{aligned}$$

result — in other words $x(\cdot,k)$ and $y(\cdot,k)$ are *mollified versions* of $X + vstep\, U$ and $Y + vstep\, V$, respectively.

Last step. End.

8.4. As a matter of fact, the above process simulates the partial differential system

$$\frac{\partial}{\partial v}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{f}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial u}\begin{bmatrix} x \\ y \end{bmatrix} - \xi^4\frac{\partial^4}{\partial u^4}\begin{bmatrix} x \\ y \end{bmatrix},$$

$$f = P\left(\nu, n(x,y)\sqrt{x_u^2 + y_u^2}\right)\,\operatorname{sgn}\kappa(u),$$

where the modified, and extra, terms protect against overflows and instability. The methods based on either artificial viscosity or quasi-reversibility share basic features: they all suggest perturbing the underlying partial differential equation or system in a way or another, in order to palliate obstructions. The quasi-reversibility method was introduced in [119], and improved in [151], [77]; other references are [10], [11], [27]–[28], [53], [66], [78], [90], [91], [104], [118], [167], [196]–[197], [214].

9. Example

For simplicity, suppose refractive index n is 1. Consider the curve — known as *Tschirnhausen's cubic* or *trisectrix of Catalan* [128], [195] — whose parametric equations read

$$x = \frac{1}{2}(1 - 3t^2), \quad y = \frac{t}{2}(3 - t^2), \quad (t = \text{parameter}),$$

and imply

$$\text{arc length} = \frac{t}{2}(t^2 + 3), \quad t = 2\sinh\left(\frac{1}{3}\text{arcsinh(arc length)}\right).$$

A geometric optical eikonal w making Tschirnhausen's cubic a caustic is represented by the equations

$$x = \frac{1}{2}(1 - 3t^2) - \frac{2st}{1+t^2}, \quad y = \frac{t}{2}(3 - t^2) + \frac{s(1-t^2)}{1+t^2},$$

$$w = s + \frac{t}{2}(3 + t^2) \quad (s, t = \text{parameters})$$

in the light region. As arguments from Section 3 show, the same eikonal can be continued in the shadow region via the equations

$$x = \frac{1 - 2(s^2+t^2) + s^4 - 2s^2t^2 - 3t^4}{2(1+s^2+t^2)}, \quad y = \frac{t\,[3 - 2(s^2-t^2) - (s^2+t^2)^2]}{2(1+s^2+t^2)},$$

$$w = is + t - \frac{2(is+t)}{1+(is+t)^2}\,x + \frac{1-(is+t)^2}{1+(is+t)^2}\,y \quad (s, t = \text{real parameters}).$$

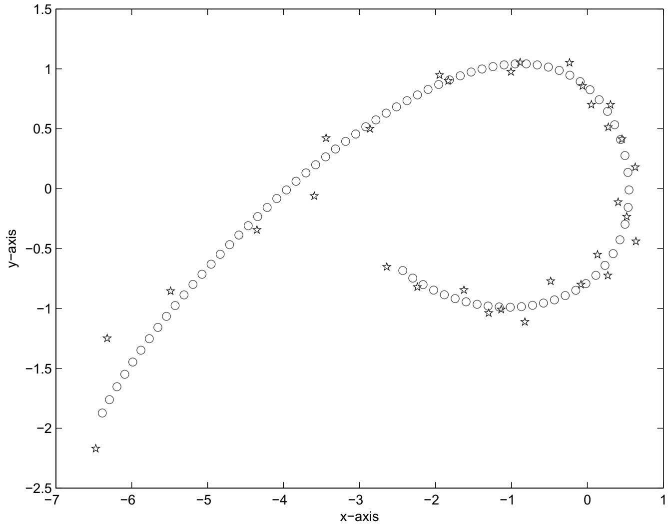

FIGURE 13. An arc of Tschirnhausen's cubic: a gross sampling (stars) and a denoised version (cicles).

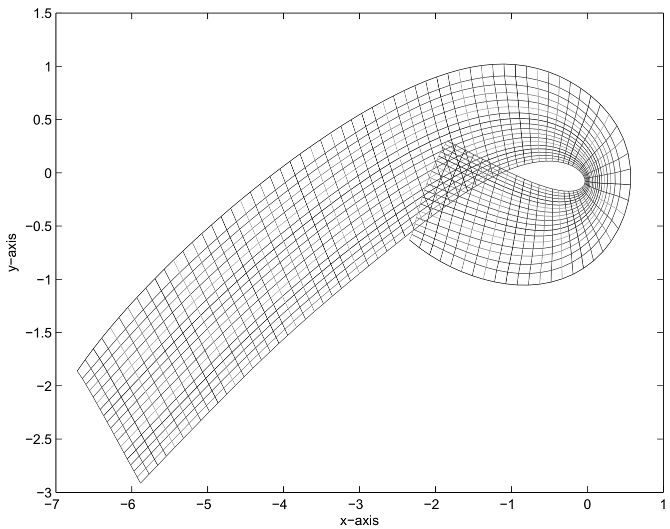

FIGURE 14. Level lines where either $u =$ constant or $v =$ constant.

The method from Sections 4 to 8 goes in the following way.

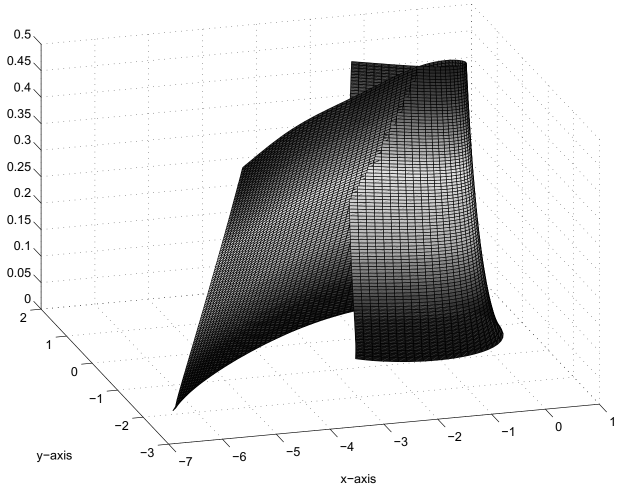

FIGURE 15. Plot of imaginary part v versus x and y.

(i) Consider the arc of the Tschirnhausen's cubic where

$$-1.5 \le t \le 2.2,$$

for instance, and let

$$(\alpha_j, \beta_j) \quad (j = 1, \ldots, 31)$$

be a *gross* sampling of such an arc — in other words,

$$t_j = -1.5 + 3.7\,\frac{j-1}{30} \quad (j = 1, \ldots, 31),$$

$$\alpha_j = \frac{1}{2}(1 - 3t_j^2) + (5\% \text{ random noise}),$$

$$\beta_j = \frac{t_j}{2}(3 - t_j^2) + (5\% \text{ random noise}).$$

(ii) Plug gross data into the *denoising process* described in Section 7, i.e.

$$\lambda = 0.005, \quad \mu = 0.1260,$$

$$\sum_{j=1}^{31} \left[\alpha\left(\frac{j-1}{30}\right) - \alpha_j\right]^2 + \lambda \int_{-\infty}^{\infty} [\mu^7 (\alpha'''')^2 + \mu^{-1}\alpha^2]\, dt = \text{minimum},$$

$$\sum_{j=1}^{31} \left[\beta\left(\frac{j-1}{30}\right) - \beta_j\right]^2 + \lambda \int_{-\infty}^{\infty} [\mu^7 (\beta'''')^2 + \mu^{-1}\beta^2]\, dt = \text{minimum},$$

and let the path that is represented by the equations

$$x = \alpha(t), \quad y = \beta(t), \quad 0 \le t \le 1,$$

surrogate the original Tschirnhausen's cubic.
(iii) Adjust parametric equations as follows:

$$\frac{dt}{du} = [\alpha'(t)^2 + \beta'(t)^2]^{-1/2}, \; t(0) = 0,$$
$$x = \alpha(t(u)), \quad y = \beta(t(u)), \quad 0 \le u \le \text{Length}.$$

so as the working parameter become a travel time — a Runge-Kutta method fits well here.
(iv) Select requisite parameters by

$$M = 101, \quad N = 91, \quad vstep = 0.005, \quad \nu = 0.5, \quad \xi = 0.9;$$

and then set the algorithm from Section 8 to work.

Results are shown in Figures 13, 14 and 15, and comfortably agree with those drawn from closed formulas.

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Appendix

Basic mathematical lineaments of two-dimensional geometrical optics are outlined in the next paragraphs. Selected references on geometrical optics, and on some of its generalizations and applications, are [14], [56], [80], [98], [99], [100], [105]–[106], [107], [116], [137], [144], [145], [184].

A.1. Terminology

Let n be a *refractive index* — i.e. a tractable function of two real variables x and y, which takes positive values only and is bounded away from zero locally. Any real-valued, suitably smooth solution w to (1.1) is a *geometric optical eikonal (GOE).* The domain of w, plus parts of the relevant boundary, is a *light region;* the complement of it is a *shadow region.* The trajectories of ∇w, namely the orbits of the dynamical system

$$\begin{vmatrix} dx & dy \\ w_x(x,y) & w_y(x,y) \end{vmatrix} = 0$$

are called *lines of steepest descent* — irrespective of whether they are genuine lines or not. A line of steepest descent is a *ray* if w is twice continuously differentiable in some neighborhood of it; any line of steepest descent, which is not a ray, is a *caustic.* (Rays are smooth curves, which have one degree of freedom and travel all over areas without intersecting one another. Caustics are exceptions in a sense: loosely speaking, they can be thought of as envelopes of rays.) The Riemannian arc length, whose element takes the form

$$n(x, y)\sqrt{dx^2 + dy^2},$$

is known as *travel time* — travel time is an alias of the customary arc length in the case where $n \equiv 1$.

A.2. GOEs and travel time

(i) The restriction of any GOE w to either an appertaining ray or caustic automatically coincides with a properly rescaled travel time t.
(ii) Let *nodal line* be an alias of locus of zeros. The value of any GOE w at any point (x, y) equals either the travel time between (x, y) and a nodal line of w or the negative of such a travel time — provided (x, y) is not a long way off.

Proof of (i). By definition, both rays and caustics of w obey

$$dx : w_x(x, y) = dy : w_y(x, y);$$

the equation

$$t = \text{a rescaled travel time}$$

is an alias of

$$dt = \pm n(x, y)\sqrt{(dx)^2 + (dy)^2}.$$

Consequently,

$$(dw/dt)^2 = n^{-2}(x, y)(w_x^2 + w_y^2)$$

along any ray or caustic in question. Property (i) follows. □

Property (ii) is a consequence of (i) and Fermat's principle below.

A.3. Fermat's principle

The travel time geodesics, i.e. those curves which render

$$\int n(x,y)\sqrt{dx^2+dy^2}$$

either a minimum or stationary, are characterized by the following second-order ordinary differential equation:

$$\text{curvature} = \langle \text{unit normal}, \nabla \log n(x,y)\rangle$$

— they have geodesic curvature 0, and are perfect straight lines in the case where $n \equiv 1$. The rays of any GOE are geodesics with respect to travel time. The travel time geodesics that are trajectories of some proper vector field — i.e. have one degree of freedom and are free from mutual intersections — are the rays of some GOE.

The foregoing statements rest upon first principles of calculus of variations, differential geometry and ordinary differential equations. Recall that the following formulas apply to any smooth parametric curve:

$$\text{velocity} = \sqrt{(dx/dt)^2 + (dy/dt)^2},$$

$$\text{unit tangent} = (\text{velocity})^{-1}\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix},$$

$$\text{unit normal} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}(\text{unit tangent})$$

$$(\text{velocity})^{-1}\frac{d}{dt}(\text{unit tangent}) = (\text{curvature}) \times (\text{unit normal}),$$

$$(\text{velocity})^{-1}\frac{d}{dt}(\text{unit normal}) = -(\text{curvature}) \times (\text{unit tangent}),$$

$$\text{curvature} = (\text{velocity})^{-3}\begin{vmatrix} dx/dt & d^2x/dt^2 \\ dy/dt & d^2y/dt \end{vmatrix};$$

$$\text{curvature} = (\text{velocity})^{-3}h^{-3}\begin{vmatrix} x(t-h) & y(t-h) & 1 \\ x(t) & y(t) & 1 \\ x(t+h) & y(t+h) & 1 \end{vmatrix} + O(h^2),$$

$$\text{curvature} = (\text{velocity})^{-3} h^{-3} \left\{ \begin{vmatrix} x(t) & y(t) & 1 \\ x(t+h) & y(t+h) & 1 \\ x(t+2h) & y(t+2h) & 1 \end{vmatrix} \right.$$
$$\left. -\frac{1}{2} \begin{vmatrix} x(t+h) & y(t+h) & 1 \\ x(t+2h) & y(t+2h) & 1 \\ x(t+3h) & y(t+3h) & 1 \end{vmatrix} \right\} + O(h^2).$$

Recall that the formula

$$\begin{aligned} &\text{curvature of the lines of steepest descent} \\ &= \text{Jacobian determinant of } |\nabla w|^{-1} \ \&\ w \end{aligned}$$

applies whenever w is smooth and has no critical point. Recall also that

$$\begin{aligned} &n(x, y) \times (\text{Geodesic curvature}) \\ &= \text{Euclidean curvature} - \langle \text{unit normal}, \nabla \log n(x, y) \rangle \end{aligned}$$

if travel time is an alternative Riemannian metric in force.

A.4. Initial value problems and geometry of their solutions

The condition of taking given values along some given path is qualified *initial* according to usage. Seeking a GOE, which obeys some initial condition, is an *initial value problem.* Such a problem has either two different solutions or no solution at all, depending on whether the eikonal equation and the initial condition match or conflict. Generally speaking, a solution w can be detected in the former case by successively detecting the objects listed below, based upon the arguments provided.

- The values of ∇w along the initial curve.

Since the eikonal equation specifies the length of ∇w and the initial condition identifies a tangential component of ∇w, the normal component of ∇w along the initial curve comes out in two different modes.

- The rays of w.

An ODE reasoning demonstrates that the travel time geodesics, which live near the initial curve and leave it with the same direction as ∇w, are the trajectories of a smooth vector field. By Fermat's principle, these geodesics are the requested rays indeed.

- The values of w itself on each ray.

Property (i) from Subsection 2 fits the situation well.

A.5. Standard initial value problems

The present item concerns existence, regularity and the number of GOEs that satisfy orthodox initial conditions.

Assume henceforth all ingredients are smooth and let IC stand for initial curve in shorthand. Let

$$x = \alpha(t), \quad y = \beta(t), \quad a \le t \le b \tag{A.1}$$

be a parametric representation of IC such that

$$t = \textit{a travel time}. \tag{A.2}$$

Let the initial condition imply

$$w(x, y) = \gamma(t)$$

as x, y and t satisfy (A.1), and assume

$$\left| \frac{d\gamma}{dt}(t) \right| < 1 \tag{A.3}$$

for $a \le t \le b$. Then exactly two GOEs satisfy the initial condition displayed. Moreover, these eikonals are smooth in a *full* neighborhood of IC — the relevant light regions surround it completely.

The case where refractive index n is constant involves explicit formulas, as well as gives evidence to interactions among the eikonal equation, *Burgers-type equations* and *Bäcklund transformations.* Assume

$$n \equiv 1,$$

and let (A.1) to (A.3) be in force. Define $\varphi, \psi, \omega, p, q$ by

$$\begin{gathered} \cos\varphi = \alpha', \quad \sin\varphi = \beta'; \\ \cos\psi = \gamma', \quad \sin\psi = \pm\sqrt{1 - (\gamma')^2}; \\ \omega = \varphi + \psi; \\ p = \cos\omega, \quad q = \sin\omega. \end{gathered}$$

Since the appurtenant Jacobian matrix equals

$$\begin{bmatrix} q(s) & p(s) \\ -p(s) & q(s) \end{bmatrix} \begin{bmatrix} \sin\psi(s) - (w - \gamma(s))\, d\omega(s)/ds & 0 \\ 0 & 1 \end{bmatrix},$$

the pair

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha(s) \\ \beta(s) \end{bmatrix} + (w - \gamma(s)) \begin{bmatrix} p(s) \\ q(s) \end{bmatrix}$$

makes s and w implicit functions of x and y in a neighborhood of IC. The properties listed below ensue. Function s obeys the equations

$$-q(s)\,(x-\alpha(s)) + p(s)\,(y-\beta(s)) = 0,$$
$$p(s)\,\frac{\partial s}{\partial x} + q(s)\,\frac{\partial s}{\partial y} = 0,$$
$$\left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2 = \left[\sin\psi(s) - (w-\gamma(s))\,\frac{d\omega(s)}{ds}\right]^{-2}.$$

(The first assures that the level lines of s are straight, the second is a PDE of *Burgers type.*) Function w obeys the following *eikonal equation*:

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = 1.$$

Functions s and w are related by the equations

$$\nabla w = \begin{bmatrix} p(s) \\ q(s) \end{bmatrix},$$
$$\begin{bmatrix} w_{xx} & w_{xy} \\ w_{xy} & w_{yy} \end{bmatrix} = \frac{d\omega(s)}{ds} \begin{bmatrix} -w_y \\ w_x \end{bmatrix} [s_x \; s_y].$$

(The former is a *Bäcklund transformation,* which pairs solutions to the Burgers and the eikonal equations mentioned above. It assures that ∇s and ∇w are orthogonal, and the level lines of s are both lines of steepest descent and isoclines of w.) The Euclidean metric obeys

$$dx^2 + dy^2 = |\nabla s|^{-2} ds^2 + dw^2.$$

There holds

$$s(x,y) = t, \quad w(x,y) = \gamma(t)$$

as x, y and t satisfy (A.1). If ω is free from critical points, the line where

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha(s) \\ \beta(s) \end{bmatrix} + \frac{\sin\psi(s)}{d\omega(s)/ds} \begin{bmatrix} p(s) \\ q(s) \end{bmatrix}$$

is a caustic. (Such a line is an envelope of the level lines of s. Function s develops shocks along it, the restriction of w to it equals arc length, and the second-order derivatives of w blow up there.)

A.6. Borderline initial value problems and caustics

The present item is a recipe for producing caustics, which involves coupling the eikonal equation with borderline initial conditions. In the case where refractive index n is identically 1, any smooth convex curve can be viewed as a caustic provided a GOE is detected, whose restriction to the curve in hand equals a relevant arc length.

Let (A.1) specify IC and let (A.2) hold. Let the initial condition imply

$$w(x, y) = t$$

as x, y and t satisfy (A.1). Assume that the appropriate *geodesic curvature* of IC is free from zeros — in other words, let (A.1) and the equation

$$\kappa\,(\text{velocity})^2 = \text{Euclidean curvature} - \langle \text{unit normal}, \nabla \log n(x, y)\rangle$$

result in

$$\kappa \textit{ vanishes nowhere.}$$

Then exactly two GOEs satisfy the present initial condition. Both these eikonals fail to exist on both sides of, and be smooth near IC. They turn IC into a *caustic,* and make the side of it, which

$$(\operatorname{sgn} \kappa) \times (\text{unit normal})$$

points to, a *shadow* region. Either eikonal in hand obeys

$$w(x,y) = s \pm \frac{2\sqrt{2}}{3}|r|^{\frac{3}{2}}|\kappa(s)|^{\frac{1}{2}} + O(r^2), \quad \Delta w(x,y) = \pm\frac{1}{\sqrt{2}}|r|^{-\frac{1}{2}}|\kappa(s)|^{\frac{1}{2}} + O(1)$$

at every point (x, y) that belongs to the light region and is close enough to IC — in particular, the two-sheeted surface, made up of the two eikonals in hand, exhibits an *edge of regression* above IC.

Here r and s are the curvilinear coordinates that the pair

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha(s) \\ \beta(s) \end{bmatrix} + r\big(\alpha'(s)^2 + \beta'(s)^2\big)^{-\frac{1}{2}} \begin{bmatrix} -\beta'(s) \\ \alpha'(s) \end{bmatrix}$$

relates to rectilinear coordinates x and y. Coordinate r is a signed distance from IC, coordinate s makes $(\alpha(s(x, y)), \beta(s(x, y)))$ the orthogonal projection of (x, y) on IC. The former is constant on the parallel lines to IC and obeys the eikonal equation

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1;$$

the latter is constant on the normal straight-lines to IC and obeys the following Burgers-type equation:

$$\begin{vmatrix} \partial s/\partial x & \partial s/\partial y \\ \alpha'(s) & \beta'(s) \end{vmatrix} = 0.$$

Both are subject to the Bäcklund transformation

$$\nabla r = [\alpha'(s)^2 + \beta'(s)^2]^{-\frac{1}{2}} \begin{bmatrix} -\beta'(s) \\ \alpha'(s) \end{bmatrix}$$

and exhibit singularities along the evolute of IC.

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