Parabolic equations with $p, q$-growth

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Received 21 February 2012
Available online 29 January 2013

Abstract
We consider parabolic equations of the type

$$\partial_t u - \text{div} a(x,t,Du) = 0$$

on a parabolic space–time cylinder $\Omega_T$. The vector field $a$ is assumed to satisfy a non-standard $p, q$-growth assumption. When

$$2 \leq p \leq q < p + \frac{4}{n}$$

it is established that any weak solution $u \in L^p(0,T; W^{1,p}(\Omega)) \cap L^q_{\text{loc}}(0,T; W^{1,q}_{\text{loc}}(\Omega))$ admits a locally bounded spatial gradient $Du$. Moreover, it is shown that the stronger assumption

$$2 \leq p \leq q < p + \frac{4}{n + 2}$$

guarantees an existence result for the Cauchy–Dirichlet problem associated to the parabolic equation from above. The results cover for example equations of the type

$$\partial_t u - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left((\mu^2 + |D_i u|^2)^{\frac{p_i-2}{2}} D_i u \right) = 0$$

with $\mu \in [0, 1]$ and suitable growth exponents $p_i$. We emphasize that the results include the degenerate case $\mu = 0$.

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Résumé
On considère l’équation parabolique

$$\partial_t u - \text{div} a(x,t,Du) = 0$$

sur le cylindre parabolique espace–temps $\Omega_T$. Le champ vectoriel $a$ satisfait les conditions non standard de croissance $p, q$. Lorsque

$$2 \leq p \leq q < p + \frac{4}{n}$$

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http://dx.doi.org/10.1016/j.matpur.2013.01.012
on montre que n’importe quelle solution faible
\[ u \in L^p(0, T; W^{1,p}(\Omega)) \cap L^{q}_{\text{loc}}(0, T; W^{1,q}_{\text{loc}}(\Omega)) \]
a un gradient spatial \( Du \) localement borné. De plus on établit que l’hypothèse plus forte
\[ 2 \leq p \leq q < p + \frac{4}{n+2} \]
\[ \partial_t u - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \left( \mu^2 + |Du|^2 \right)^{\frac{p-2}{2}} D_i u \right) = 0 \]
avec \( \mu \in [0, 1] \) et des conditions appropriées sur les \( p_i \). Soulignons encore que le cas dégénéré \( \mu = 0 \) est contenu dans notre analyse.
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**MSC:** 35K20; 35K55; 35K65

**Keywords:** Parabolic equations; Nonstandard \( p,q \)-growth; Existence; Regularity

### 1. Introduction

#### 1.1. The setting

Nous nous intéressons à l’existence et régularité de solutions faibles de l’équation parabolique de type
\[ \partial_t u - \text{div} \, a(x,t,Du) = 0 \quad \text{in} \quad \Omega_T, \quad (1.1) \]

Où \( \Omega_T := \Omega \times (0,T) \) est un cylindre espace-temps ouvert à propre. Points dans \( \mathbb{R}^{n+1} \) sont notés par \( z = (x,t) \) avec \( x \in \mathbb{R}^n \) et \( t \in \mathbb{R} \) et \( \partial_t u \), respectivement \( u \), dénote la dérivée par rapport au temps \( t \),

while by \( Du \) we mean the spatial gradient with respect to \( x \). The vector field \( a : \Omega_T \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is assumed to be differentiable with respect to the spatial variable \( x \) and with respect to the gradient variable \( w \in \mathbb{R}^n \), and to satisfy the following non-standard growth and ellipticity conditions:

\[
\begin{align*}
|a(x,t,\xi)| + (\mu^2 + |\xi|^2)^{\frac{q}{2}} |\partial_\xi a(x,t,\xi)| &\leq L(\mu^2 + |\xi|^2)^{\frac{q-1}{2}}, \\
|\partial_\xi a(x,t,\xi)\xi,\xi| &\geq v(\mu^2 + |\xi|^2)^{\frac{q-2}{2}} |\xi|^2, \\
|\partial_\xi a_j(x,t,\xi)| &\leq L(\mu^2 + |\xi|^2)^{\frac{q+1}{4}},
\end{align*}
\]

(1.2)

for any \((x,t) \in \Omega_T, \xi,\tilde{\xi} \in \mathbb{R}^n, j,\ell = 1, \ldots, n\), \(2 \leq p < q \) et \( v, L > 0 \) et \( \mu \in [0, 1]\). De toute façon on supposera que les cartes partielles \((x,t,\xi) \mapsto a(x,t,\xi), (x,t,\xi) \mapsto \partial_\xi a(x,t,\xi) \) et \((x,t,\xi) \mapsto \partial_\xi a(x,t,\xi)\) sont des Carathéodory. Notez que toute régularité en dehors de la mesurabilité est assumée sur la carte partielle \( t \mapsto a(x,t,\xi) \).

Le problème elliptique correspondant a été étudié intensivement dans le passé, et l’histoire commence avec deux papers of the third author [16,17]. Tapez autonomes fonctionnels variatifs et équations elliptiques du type
\[ -\text{div} \, a(x,Du) = h \quad \text{in} \quad \Omega \quad (1.3) \]

avec non-standard \( p,q \)-growth are considered. Moreover, everywhere regularity for elliptic systems of Uhlenbeck-type was considered in [19,20]; in this case the vector field \( a \) in (1.3) has the form
\[ a(Du) := g(|Du|)Du, \quad (1.4) \]

where \( g : [0, +\infty) \rightarrow [0, +\infty) \) is a function with general growth at \( +\infty \). After developing the concept of a weak solution it is shown that those weak solutions exhibit a local \( L^\infty \)-bound for the gradient, provided \( p \) and \( q \) are not
too far from each other (see Remarks 1.4 and 1.7, respectively Refs. [16–20] for more precise details). This is done in form of an a priori estimate. Afterwards, the a priori estimate is used to establish the existence of a weak solution to the associated Dirichlet problem (under certain assumptions on the Dirichlet data). As said before, there is a large literature on elliptic problems with non-standard $p, q$-growth, cf. [3,5,6,11,14,18,22,23] and the references therein.

In the parabolic case much less is known. Here the standard case $p = q$ is well understood and we refer to the famous results in [7–9], while for partial regularity results of parabolic systems we refer to [1,2,10]. In contrast to the elliptic setting not much is known for the non-standard case $p < q$. To our knowledge, only the fundamental results of DiBenedetto and Friedman have been extended in [26] to treat everywhere regularity of systems with a vector field $a$ of Uhlenbeck-type as in (1.4), satisfying suitable Orlicz-type growth conditions.

This work is concerned with a local $L^\infty$-bound and the Hölder-continuity of the solution, and moreover a local $L^\infty$-bound for the spatial gradient and its Hölder-continuity. Any of these results requires the solution to exhibit the natural kind of regularity and integrability from the underlying energy space. Furthermore, in [21,13] gradient estimates for bounded solutions to certain anisotropic parabolic equations are derived. The question of existence of solutions to the Cauchy–Dirichlet problem associated to a parabolic equation with a non-standard growth condition has up to now not been addressed in the literate, except in the recent paper [4] in which the Cauchy–Dirichlet problem is treated for equations of non-standard $p, q$-growth with a variational structure of the form $\partial_t u - \text{div} a(x,t,Du) = 0$ and homogeneous Dirichlet data $u = 0$ on the lateral boundary $\partial \Omega \times (0,T)$ and an initial condition $u(\cdot,0) = u_0$.

The aim of the present paper is twofold: we want to develop on the one hand a general concept of weak solution which allows good a priori estimates, as for example a local $L^\infty$-bound for the spatial gradient, and on the other hand is general enough to establish (with these a priori estimates at hand) the existence of weak solutions to the associated Cauchy–Dirichlet problem. With this respect we follow the strategy developed in [17]. Furthermore, we want to treat parabolic equations of $p, q$-growth which might degenerate in the gradient variable, similarly to the degenerate $p$-Laplacian equation.

1.2. Results

One could think of several strategies to define a weak solution to the Cauchy–Dirichlet problem (1.1). Here, we shall use the following one which is inspired from the elliptic case [17].

**Definition 1.1.** We identify a function $u \in L^q_{\text{loc}}(0,T; W^{1,q}_{\text{loc}}(\Omega))$ as a weak solution of the parabolic equation (1.1) if and only if

$$\int_{\Omega_T} (u \cdot \varphi_t - \langle a(x,t,Du), D\varphi \rangle) \, dz = 0 \quad (1.5)$$

holds, whenever $\varphi \in C^\infty_0(\Omega_T)$.

Then, the first main result of the paper is the following local regularity result for weak solutions.

**Theorem 1.2 (A priori estimate).** Let $u \in L^q_{\text{loc}}(0,T; W^{1,q}_{\text{loc}}(\Omega))$ be a weak solution to the parabolic equation (1.1) where the structure conditions (1.2) are in force and assume that

$$2 \leq p \leq q < p + \frac{4}{n}. \quad (1.6)$$

Then, we have $Du \in L^\infty_{\text{loc}}(\Omega_T, \mathbb{R}^n)$ and for any parabolic cylinder $Q_\rho(z_0) \subseteq \Omega_T$ and $s \in (0,1)$ there holds

$$\sup_{Q_s\rho(z_0)} |Du| \leq c \left[ \frac{1}{[(1-s)\rho]^2 + 2} \int_{Q_\rho(z_0)} (1 + |Du|^2)^{\frac{q}{2}} \, dz \right]^{\frac{1}{q} - \frac{2q}{4 - \alpha(q-p)}}$$

for a constant $c = c(\bar{n}, p, q, v, L)$, where
Definition 1.5. We identify a function \( \partial \Omega \) parabolic deficit 2 (which is due to the different scaling in time) and on the other hand due to the growth in the diffusion term the to (1.8) as follows.

Remark 1.3. The exponent

\[ d_{p,q} := \frac{2q}{4 - n(q - p)} \]

is the natural scaling deficit which prevents the sup-estimate from Theorem 1.2 to be homogeneous. When the exponent \( q \) approaches the upper bound, i.e. \( q \uparrow p + \frac{4}{n} \) we have \( d_{p,q} \to \infty \). On the other hand in the special case \( q = p \) we have \( d_{p,p} = \frac{2}{n} \) which is in perfect accordance with the classical sup-estimate from [7, Chapter VIII, Theorem 5.1]. The exponent \( \frac{2}{n} \) reflects in a natural way the space–time inhomogeneity of a parabolic equation of type (1.1) with standard \( p \)-growth.

Remark 1.4. We compare the upper bounds on \( q \) occurring in the elliptic and parabolic setting. The elliptic bound in [17, Theorem 2.1] is

\[ 2 \leq p \leq q < \frac{np}{n-2} \equiv p + \frac{2p}{n-2}, \]

while the parabolic bound reads as

\[ 2 \leq p \leq q < p + \frac{4}{n} \equiv p + \frac{2}{n+2} - \frac{2}{p}. \]

With this respect the parabolic restriction is the natural one, because on the one hand \( n \) has to be replaced by \( n + 2 \) (which is due to the different scaling in time) and on the other hand due to the growth in the diffusion term the parabolic deficit \( \frac{2}{p} \) shows up. This is in complete accordance with similar phenomena occurring for example in the higher integrability for solutions to differentiable systems with \( p \)-growth. In the elliptic case there is a gain of integrability from \( p \) to \( \frac{np}{n-2} \) due to the Sobolev embedding. This is exactly the elliptic upper bound for \( q \). In the parabolic setting one receives instead the gain from \( p \) to \( p + \frac{4}{n} \), which is just the parabolic upper bound for \( q \), cf. [10, Lemma 5.4].

We are also interested in the existence of solutions to the Cauchy–Dirichlet problem associated to the parabolic equation (1.1), i.e.

\[
\begin{align*}
\partial_t u - \text{div} a(x,t,Du) &= 0 \quad \text{in } \Omega_T, \\
u &= g \quad \text{on } \partial_{\text{par}} \Omega_T,
\end{align*}
\]

where \( \partial_{\text{par}} \Omega_T := (\partial \Omega \times (0,T)) \cup (\Omega \times \{0\}) \) is the parabolic boundary of \( \Omega_T \). The initial condition \( u = g \) on \( \Omega \times \{0\} \) has to be understood in the usual \( L^2 \)-sense (see (1.10) below), while the condition \( u = g \) on the lateral boundary \( \partial \Omega \times (0,T) \) has to be understood in the sense of traces, that is \( (u - g)(\cdot,t) \in W^{0,1,r}(\Omega) \) for almost every \( t \in (0,T) \). For the boundary data \( g \) we assume that

\[ g \in C^0([0,T]; L^2(\Omega)) \cap L^r(0,T; W^{1,r}(\Omega)), \quad g_t \in L^{p'}(0,T; W^{-1,1,p'}(\Omega)), \]

where \( p' := \frac{p}{p-1} \) is the Hölder conjugate to \( p \) and \( r := p'(q - 1) \). Note that \( r \geq q \) and \( g_t \in L^{q'}(0,T; W^{-1,1,q'}(\Omega)) \) since \( q \geq p \). Moreover, the assumption \( g \in C^0([0,T]; L^2(\Omega)) \) is somewhat redundant due to the embedding \( L^r(0,T; W^{1,r}(\Omega)) \cap W^{1,p'}(0,T; W^{-1,1,p'}(\Omega)) \hookrightarrow C^0([0,T]; L^2(\Omega)) \). In this framework we define a weak solution to (1.8) as follows.

Definition 1.5. We identify a function

\[ u \in L^p(0,T; W^{1,p}(\Omega)) \cap L^q_{\text{loc}}(0,T; W^{1,1,q}_{\text{loc}}(\Omega)) \]

For the constant \( c \) there holds: \( c \uparrow \infty \) when \( q \uparrow p + \frac{4}{n} \).
as a weak solution of the Cauchy–Dirichlet problem (1.8) if and only if (1.5) holds and moreover, 
\( u \in g + L^p(0, T; W^{1, p}(\Omega)) \) and \( u(\cdot, 0) = g(\cdot, 0) \) in the \( L^2 \)-sense, i.e.

\[
\lim_{h \downarrow 0} \frac{1}{h} \int_0^h \int_\Omega |u(\cdot, t) - g(\cdot, 0)|^2 \, dx \, dt = 0.
\]  

(1.10)

The second main result of the paper ensures the existence of a weak solution.

**Theorem 1.6 (Existence of weak solutions).** Suppose that assumptions (1.2) and (1.9) are in force and that

\[
2 \leq p < q < p + \frac{4}{n + 2}
\]

(1.11)

holds. Then there exists a weak solution \( u \) to the Cauchy–Dirichlet problem (1.8). Moreover its \( L^p(0, T; W^{1, p}(\Omega)) \)-norm is bounded by a constant depending only on \( n, p, q, v, L \) and \( \|Dg\|^L^{L'}(\Omega_T) \) and \( \|\partial_t\|^L^{L'}(0, T; W^{-1, p'}(\Omega)) \). Further, the solution \( u \) satisfies \( Du \in L^{\infty, \text{loc}}(\Omega_T, \mathbb{R}^p) \) and \( u_t \in L^{q, T}(0, T; W^{-1, p'}(\Omega)) \). Finally, we let

\[
\hat{n} := \begin{cases} n, & \text{if } n \geq 3, \\ \text{any number } \in (2, \frac{4}{q-p}-2), & \text{if } n = 2. 
\end{cases}
\]  

(1.12)

Then, there exists a constant \( c = c(\hat{n}, p, q, v, L) \) such that for any cylinder \( Q_{\hat{n}}(z_0) \subseteq \Omega_T \) and \( s \in (0, 1) \) there holds

\[
\sup_{Q_{\hat{n}}(z_0)} |Du| \leq c \left[ \frac{1}{((1-s)q)^{\hat{n}/2}} \int_{Q_{\hat{n}}(z_0)} (1 + |Du|^2)^{\frac{p}{2}} \, dz \right]^{\frac{1}{2}} 
\]

and if \( \mu > 0 \) or \( p = 2 \) we additionally have \( u \in L^{2, \text{loc}}(0, T; W^{2, 2}(\Omega)) \) and

\[
\int_{Q_{\hat{n}}(z_0)} |D^2u|^2 \, dz \leq \frac{c\hat{n}^\mu}{\mu^{p-2}} \left[ \frac{1}{\hat{n}^{\hat{n}/2}} \int_{Q_{\hat{n}}(z_0)} (1 + |Du|^2)^{\frac{p}{2}} \, dz \right]^{1 + \frac{2(p-1)}{4(p+2)(q-p)}}.
\]

**Remark 1.7.** Here, we compare the upper bound on \( q \) occurring in Theorem 1.6 with the one from the elliptic setting. The elliptic bound in [17, Theorem 4.1] is

\[
2 \leq p < q < \frac{(n+2)p}{n} \equiv p + \frac{2p}{n},
\]

while the parabolic one is given by

\[
2 \leq p < q < p + \frac{2}{n+2} \equiv p + \frac{2p}{n+2} \cdot \frac{2}{p}.
\]

Therefore, exactly the same transformations as described in Remark 1.4 lead from the elliptic to the parabolic bound.

### 1.3 Model examples

The previous results can for instance be applied to parabolic equations of the type

\[
\partial_t u = \text{div}(|Du|^{p-2} Du) + \text{div}(c(x, t)|Du|^{q-2} Du), \quad 0 \leq c(x, t) \leq L,
\]

with a suitable choice of \( p \) and \( q \). More precisely, in the particular case \( c(x, t) \equiv c(t) \) the results of Theorems 1.2 and 1.6 apply with the bounds (1.6), respectively (1.11) for \( p \) and \( q \). In the general case one has to sharpen these bounds in order to have the assumption (1.2)\(_3\) at hand. Then, the local regularity Theorem 1.2 applies for \( 2 \leq p \leq q < p + \frac{2}{n+2} \) while the existence result from Theorem 1.6 applies for \( 2 \leq p \leq q < p + \frac{2}{n+2} \). A second prominent model example which is included in our considerations is the parabolic \( p_1 \)-Laplacian equation

\[
\partial_t u - \sum_{i=1}^n \frac{\partial}{\partial x_i}(|D_i u|^{p_1-2} D_i u) = 0.
\]
Here, Theorems 1.2 and 1.6 apply under the conditions (1.6), respectively (1.11), where $p := \min_i \{ p_i \}$ and $q := \max_i \{ p_i \}$.

2. Preliminaries and notations

Here, we first introduce some notations used throughout the paper. By $\{ e_i \}_{i \leq n}$ we denote the standard basis of $\mathbb{R}^n$. Moreover, by

$$B_\varrho (x_0) := \{ x \in \mathbb{R}^n : |x - x_0| < \varrho \}$$

we denote the open ball in $\mathbb{R}^n$ with radius $\varrho > 0$ and center $x_0$. When dealing with parabolic regularity, instead of balls one uses parabolic cylinders, which are the balls with respect to the parabolic metric. In the following we shall work with the standard parabolic cylinders

$$Q_\varrho (z_0) := B_\varrho (x_0) \times (t_0 - \varrho^2, t_0), \quad \text{where } z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}.$$

Next, we state some well known results which will be useful in the sequel. In order to “reabsorb” certain terms, we will use the following iteration lemma, which is a standard tool and can for instance be found in [12].

Lemma 2.1. Let $\phi(\varrho)$ be a bounded, non-negative function on $0 \leq R_0 \leq \varrho \leq R_1$ and assume that for $R_0 \leq \varrho < r \leq R_1$ there holds

$$\phi(\varrho) \leq \vartheta \phi(r) + A \frac{r - \varrho}{\varrho^\alpha},$$

for some $\alpha > 0$, $\vartheta \in [0, 1)$ and a non-negative constant $A$. Then, there exists a constant $c = c(\alpha, \vartheta)$ such that for all $R_0 \leq \varrho < r \leq R_1$ we have

$$\phi(\varrho) \leq c A \frac{r - \varrho}{\varrho^\alpha}.$$

The following two lemmas can be found in [17], Lemmas 2.4, 2.5 and 4.4. Note that in [17] only the case $\mu = 1$ is treated; the general case $\mu \in [0, 1]$ can be retrieved by completely the same reasoning.

Lemma 2.2. Under the assumptions (1.2)1, (1.2)2 and (1.2)4 there exists a constant $c_1 = c_1(\nu, L)$ such that for any $(x, t) \in \Omega_T$ and $\xi, \lambda, \eta \in \mathbb{R}^n$ there holds

$$\left| \sum_{j, \ell = 1}^n \partial_{\xi_j} a_j (x, t, \xi) \lambda_j \ell \eta_j \right| \leq c_1 \left[ \sum_{j, \ell = 1}^n \partial_{\xi_j} a_j (x, t, \xi) \lambda_j \ell \lambda_j \ell \right]^{\frac{1}{2}} \left( \mu^2 + |\xi|^2 \right)^{\frac{q - 2}{4}} |\eta|. \quad (2.1)$$

Under the assumptions (1.2)2 and (1.2)3 we have for any $k \in \{1, \ldots, n\}$:

$$\left| \sum_{j = 1}^n \partial_{x_k} a_j (x, t, \xi) \lambda_j \lambda_j \right| \leq c_2 \left[ \sum_{j, \ell = 1}^n \partial_{\xi_j} a_j (x, t, \xi) \lambda_j \ell \lambda_j \ell \right]^{\frac{1}{2}} \left( \mu^2 + |\xi|^2 \right)^{\frac{q}{4}}, \quad (2.2)$$

for any $(x, t) \in \Omega_T$ and $\xi, \lambda \in \mathbb{R}^n$ and for a constant $c_2 = c_2(n, \nu, L)$.

Lemma 2.3. Under the assumptions (1.2)1 and (1.2)2 there exists a constant $c_3 = c_3(n, p, q, \nu, L)$ such that for any $(x, t) \in \Omega_T$ and $\xi, \tilde{\xi} \in \mathbb{R}^n$ there holds

$$|\xi|^p \leq c_3 \left[ (1 + |\xi|)^{p'(q - 1)} + \sum_{j = 1}^n a_j (x, t, \xi) (\xi_j - \tilde{\xi}_j) \right]. \quad (2.3)$$
3. Weak differentiability of $Du$

Here we derive some higher differentiability results for weak solutions to differentiable parabolic equations with non-standard $p, q$-growth satisfying (1.2). These results, especially the local Lipschitz bound are obtained by a Moser-iteration scheme following to a certain extend the arguments of [7, Chapter VIII]. We point out that the quantitative $L^\infty$-bound for the spatial gradient $Du$ is obtained in one step by a Moser iteration, similar to the reasoning in [24]. This can be achieved since the involved constants in the scheme are explicitly computed. We shall consider a weak solution

$$u \in L^q_{loc}(0, T; W^{1,q}_{loc}(\Omega))$$

of the nonlinear parabolic equation (1.1), i.e. of

$$\partial_t u - \text{div} \, a(x, t, Du) = 0 \quad \text{in } \Omega_T,$$

where the vector field $a$ is assumed to satisfy the non-standard $p, q$-growth and ellipticity conditions from (1.2).

We use the standard difference quotients technique in the spatial direction; to this aim, for $f \in L^1_{loc}(\Omega_T, \mathbb{R}^k)$ and $i \in \{1, \ldots, n\}$, $h \neq 0$ we define (when $x + he_i \in \Omega$)

$$\Delta_h^{(i)} f(x, t) := \frac{f(x + he_i, t) - f(x, t)}{h}.$$

Moreover, we set

$$\Delta_h f(x, t) := (\Delta_h^{(1)} f(x, t), \ldots, \Delta_h^{(n)} f(x, t))\right).$$

Now, in the weak formulation (1.5) of the parabolic system (1.1) we replace the testing function $\phi$ by $\Delta_h^{(i)} \phi$ where $0 < |h| < \text{dist}(\text{spt } \phi, \partial \Omega_T)$ and $i \in \{1, \ldots, n\}$ in order to get

$$\int_{\Omega_T} \left( \frac{\partial_t \Delta_h^{(i)} u \cdot \partial_t \phi - [A^{(i)}(\cdot, Du)] \cdot D\phi}{h} \right) dz = 0.$$

Replacing $\phi$ by $\phi_\varrho \equiv \phi \ast \varrho$ in the previous equation (here $\{\phi_\varrho\}, \varrho > 0$, denotes the family of standard, positive, radially symmetric mollifiers in $\mathbb{R}^{n+1}$) then yields after integration by parts with respect to $t$, for $0 < \varrho \ll 1$

$$\int_{\Omega_T} \left( \frac{\partial_t (\Delta_h^{(i)} u) \varrho \cdot \varphi + [A^{(i)}(\cdot, Du)] \varrho \cdot D\phi}{h} \right) dz = 0.$$

Now, in the last equation we choose the testing-function

$$\varphi \equiv \phi \left( |(\Delta_h^{(i)} u) \varrho|^2 \right),$$

where $\phi \in W^{1,\infty}_0(\Omega_T)$ is a cut-off function which will be specified later and $g \in W^{1,\infty}(\mathbb{R})$ is a bounded, non-decreasing, non-negative Lipschitz continuous function. For the term involving the time derivative we obtain

$$\int_{\Omega_T} \partial_t (\Delta_h^{(i)} u) \varrho \cdot \varphi \, dz = \frac{1}{2} \int_{\Omega_T} \partial_t \left[ \frac{1}{\varrho} |(\Delta_h^{(i)} u) \varrho|^2 \right] g \left( |(\Delta_h^{(i)} u) \varrho|^2 \right) \phi \, dz$$

$$= \frac{1}{2} \int_{\Omega_T} \partial_t \left[ \frac{1}{\varrho} \int_0^\varrho g(s) \, ds \right] \phi \, dz$$

$$= -\frac{1}{2} \int_{\Omega_T} \partial_t \left[ \int_0^\varrho g(s) \, ds \right] \phi \, dz.$$
The term involving the spatial derivatives is given by

\[
\sum_{j=1}^{n} \int_{\Omega_T} \left( \Delta_h^{(i)} \left[ a_j(\cdot, Du) \right] \right)_0 D_j \left[ \phi \left( \Delta_h^{(i)} u \right)_0 g \left( \| \Delta_h^{(i)} u \|_2^2 \right) \right] dz.
\]

Letting \( \varrho \downarrow 0 \) we get, using (1.2)_{1} to justify the passage to the limit,

\[
- \frac{1}{2} \int_{\Omega_T} \int_{0}^{1} g(s) ds \partial_t \phi \, dz + \sum_{j=1}^{n} \int_{\Omega_T} \Delta_h^{(i)} \left[ a_j(\cdot, Du) \right] D_j \left[ \phi \Delta_h^{(i)} u g \left( \| \Delta_h^{(i)} u \|_2^2 \right) \right] dz = 0.
\]

We rewrite the preceding equality in the form

\[
- \frac{1}{2} \int_{\Omega_T} \int_{0}^{1} g(s) ds \partial_t \phi \, dz + \sum_{j=1}^{n} \int_{\Omega_T} \Delta_h^{(i)} \left[ a_j(\cdot, Du) \right] D_j \left[ \phi \Delta_h^{(i)} u g \left( \| \Delta_h^{(i)} u \|_2^2 \right) \right] dz = - \sum_{j=1}^{n} \int_{\Omega_T} \Delta_h^{(i)} \left[ a_j(\cdot, Du) \right] D_j \left[ \phi \Delta_h^{(i)} u g \left( \| \Delta_h^{(i)} u \|_2^2 \right) \right] dz.
\]

At this stage we choose \( \phi(x,t) \equiv \varphi^2(x) \tilde{\chi}(t) \chi(t) \) where \( \chi \in W^{1,\infty}(0,T) \), \( \chi(0) = 0 \) and \( \partial_t \chi \geq 0 \), \( \varphi \in C_0^\infty(B_\varrho(x_0)) \) for some ball \( B_\varrho(x_0) \subseteq \Omega \), \( 0 \leq \varrho \leq 1 \), and \( \tilde{\chi} \) is a Lipschitz continuous function defined as follows, with \( 0 < \tau < \tau + \delta < T \) and \( \delta > 0 \)

\[
\tilde{\chi}(t) := \begin{cases} 
1 & \text{if } t \leq \tau, \\
1 - \frac{t - \tau}{\delta} & \text{if } \tau < t < \tau + \delta, \\
0 & \text{if } t \geq \tau + \delta.
\end{cases}
\]

With this choice of \( \phi \) and the abbreviation \( Q_T \equiv B_\varrho(x_0) \times (0, \tau) \) we obtain in the limit \( \delta \downarrow 0 \) for almost every \( \tau \in (0, T) \) that

\[
\frac{1}{2} \int_{B_\varrho(x_0)} \varphi^2(x) \chi(\tau) \int_{0}^{1} g(s) ds \, dx + \sum_{j=1}^{n} \int_{Q_T} \varphi^2 \chi \Delta_h^{(i)} \left[ a_j(\cdot, Du) \right] D_j \left[ \Delta_h^{(i)} u g \left( \| \Delta_h^{(i)} u \|_2^2 \right) \right] dz = -2 \sum_{j=1}^{n} \int_{Q_T} \varphi \chi \Delta_h^{(i)} \left[ a_j(\cdot, Du) \right] D_j \varphi \Delta_h^{(i)} u g \left( \| \Delta_h^{(i)} u \|_2^2 \right) dz + \frac{1}{2} \int_{Q_T} \varphi^2 \partial_t \chi \left[ \int_{0}^{1} g(s) ds \right] dz.
\]

(3.1)

We now decompose \( \Delta_h^{(i)} [a_j(\cdot, \cdot, Du)] \) as follows:

\[
\Delta_h^{(i)} [a_j(\cdot, \cdot, Du(\cdot, \cdot))] (x, t) = \frac{1}{h} \int_{0}^{1} \frac{d}{ds} a_j \left( x + she_i, t, Du(x, t) + s \tau_h^{(i)} [Du](x, t) \right) ds
\]

\[= \int_{0}^{1} \left[ \partial_x a_j \left( x + she_i, t, Du(x, t) + s \tau_h^{(i)} [Du](x, t) \right) + \sum_{i=1}^{n} \partial_{x_i} a_j \left( x + she_i, t, Du(x, t) + s \tau_h^{(i)} [Du](x, t) \right) \Delta_h^{(i)} [D_{\ell} u](x, t) \right] ds.
\]

\[= \int_{0}^{1} \left[ \partial_x a_j + \sum_{i=1}^{n} \partial_{x_i} a_j \Delta_h^{(i)} [D_{\ell} u](x, t) \right] ds.
\]
With this notation (3.1) turns into

\[
\begin{align*}
\frac{1}{2} \int_{B_{\rho}(x_0)} \psi^2(x) \chi(\tau) & \int_0^1 g(s) ds \, dx \\
+ \sum_{j=1}^n \int_{Q_T} \varphi^2 \chi \left[ \partial_x a_j + \sum_{\ell=1}^n \partial_{\xi_\ell} a_j \Delta_h^{(i)} [D_t u] \right] ds \, D_j \left[ \Delta_h^{(i)} u \right] g(\Delta_h^{(i)} u^2) \, dz \\
+ \sum_{j=1}^n \int_{Q_T} \varphi^2 \chi \left[ \partial_x a_j + \sum_{\ell=1}^n \partial_{\xi_\ell} a_j \Delta_h^{(i)} [D_t u] \right] ds \, \Delta_h^{(i)} u \Delta_h^{(i)} [D_j u] g(\Delta_h^{(i)} u^2) \, dz \\
= -2 \sum_{j=1}^n \int_{Q_T} \varphi \chi \left[ \partial_x a_j + \sum_{\ell=1}^n \partial_{\xi_\ell} a_j \Delta_h^{(i)} [D_t u] \right] ds \, \Delta_h^{(i)} u \Delta_h^{(i)} [D_j u] g(\Delta_h^{(i)} u^2) \, dz \\
+ \frac{1}{2} \int_{Q_T} \varphi^2 \frac{\partial}{\partial \tau} [\Delta_h^{(i)} u^2] ds \, g(s) \, dz.
\end{align*}
\] 

(3.2)

We start by estimating the appearing integrals separately. In the integrals involving \( \partial_w a \) appearing on the left-hand side we rearrange the terms and denote the first one by \( I_1 \), that is

\[
I_1 := \int_{Q_T} \varphi^2 \chi \left[ \sum_{j=1}^n \partial_{\xi_j} a_j ds \Delta_h^{(i)} [D_t u] \Delta_h^{(i)} [D_j u] g(\Delta_h^{(i)} u^2) \right] dz
\]

and the second one by

\[
I_2 := \sum_{j=1}^n \int_{Q_T} \varphi^2 \chi \left[ \sum_{\ell=1}^n \partial_{\xi_\ell} a_j ds \Delta_h^{(i)} [D_t u] \Delta_h^{(i)} [D_j u] [g(\Delta_h^{(i)} u^2)] \right] dz
\]

\[
= 2 \int_{Q_T} \varphi^2 g'(\Delta_h^{(i)} u^2) \sum_{j,\ell=1}^n \int_0^1 \partial_{\xi_j} a_j ds \Delta_h^{(i)} [D_t u] \Delta_h^{(i)} [D_j u] [\Delta_h^{(i)} u^2] \, dz \geq 0,
\]

where we used the elementary computation

\[
D_j \left[ g(\Delta_h^{(i)} u^2) \right] = 2 g'(\Delta_h^{(i)} u^2) \Delta_h^{(i)} u \Delta_h^{(i)} [D_j u]
\] 

(3.3)

and the ellipticity condition \((1.2)_2\) as well as the fact that \( g \) is chosen non-decreasing. We shall keep this integral nevertheless on the left-hand side of our final estimate.

To bound the first integral appearing on the right-hand side of (3.2), i.e.

\[
II_1 := -2 \int_{Q_T} \varphi \chi \left[ \sum_{j=1}^n \partial_{\xi_j} a_j ds \Delta_h^{(i)} [D_t u] D_j \varphi \Delta_h^{(i)} u g(\Delta_h^{(i)} u^2) \right] dz
\]

we first rearrange the factors in the integrand and then estimate with the help of Lemma 2.2, that is (2.1), as follows:
\[
\sum_{j,\ell=1}^n \int_0^1 \partial_{x_l} a_j \, ds \, \Delta_h^{(i)} [D_{t\ell} u] D_j \varphi \Delta_h^{(i)} u \\
\leq c_1 \int_0^1 \left[ \sum_{j,\ell=1}^n \partial_{x_l} a_j \Delta_h^{(i)} [D_{t\ell} u] \Delta_h^{(i)} [D_j u] \right]^{1/2} \, ds \, \mathcal{D}^{(i)}(h)^{2} |D\varphi| |\Delta_h^{(i)} u| \\
\leq c_1 \left[ \int_0^1 \sum_{j,\ell=1}^n \partial_{x_l} a_j \Delta_h^{(i)} [D_{t\ell} u] \Delta_h^{(i)} [D_j u] \, ds \right]^{1/2} \mathcal{D}^{(i)}(h)^{2} |D\varphi| |\Delta_h^{(i)} u|,
\]

where \(c_1 = c_1(v, L)\) and we used the shorthand notation
\[
\mathcal{D}^{(i)}(h)(x, t) := \mu^2 + |Du(x, t)|^2 + |Du(x + he_t, t)|^2.
\]

We multiply both sides of (3.4) by \(\varphi \chi g(|\Delta_h^{(i)} u|^2)\), integrate with respect to \(z = (x, t)\) over \(Q_\tau\) and finally use Young's inequality for \(\varepsilon \in (0, 1)\), to obtain
\[
|I_1| \leq \varepsilon I_1 + \frac{c(q)c_1}{\varepsilon} \int_{Q_\tau} \chi \mathcal{D}^{(i)}(h)^{2} |D\varphi|^2 |\Delta_h^{(i)} u|^2 g(|\Delta_h^{(i)} u|^2) \, dz.
\]

We now turn our attention to the integrals involving the term \(\partial_x a\) in (3.2). The first integral is estimated with the help of inequality (2.2) from Lemma 2.2 as follows:
\[
|I_3| := \left| \sum_{j=1}^n \int_{\mathcal{Q}_\tau} \varphi^2 \chi \int_0^1 \partial_{x_l} a_j \, ds \, [\Delta_h^{(i)} u] D_j [\varphi (|\Delta_h^{(i)} u|^2)] \, dz \right| \\
\leq c_2 \sum_{i=1}^n \int_{\mathcal{Q}_\tau} \varphi^2 \chi \int_0^1 \left[ \sum_{j=1}^n \partial_{x_l} a_j \Delta_h^{(i)} [D_j u] \Delta_h^{(i)} [D_{t\ell} u] \right]^{1/2} \, ds \, \mathcal{D}^{(i)}(h)^{2} g(|\Delta_h^{(i)} u|^2) \, dz \\
\leq \varepsilon I_1 + \frac{c(q)c_2}{\varepsilon} \sum_{i=1}^n \int_{\mathcal{Q}_\tau} \varphi^2 \chi \mathcal{D}^{(i)}(h)^{2} g(|\Delta_h^{(i)} u|^2) \, dz,
\]

where \(c_2 = c_2(n, v, L)\). For the second term involving \(\partial_x a\) we use (3.3), Lemma 2.2, that is (2.2), and finally Young’s inequality to deduce
\[
I_4 := \sum_{j=1}^n \int_{\mathcal{Q}_\tau} \varphi^2 \chi \int_0^1 \partial_{x_l} a_j \, ds \, \Delta_h^{(i)} u D_j [g(|\Delta_h^{(i)} u|^2)] \, dz \\
= 2 \sum_{j=1}^n \int_{\mathcal{Q}_\tau} \varphi^2 \chi g'(|\Delta_h^{(i)} u|^2) \int_0^1 \partial_{x_l} a_j \, ds \, \Delta_h^{(i)} u \Delta_h^{(i)} u_D j [D_j u] \, dz \\
= 2 \int_{\mathcal{Q}_\tau} \varphi^2 \chi g'(|\Delta_h^{(i)} u|^2) \int_0^1 \sum_{j=1}^n \partial_{x_l} a_j \Delta_h^{(i)} [D_j u] \Delta_h^{(i)} u \, ds \, \Delta_h^{(i)} u \, dz \\
\leq 2c_2 \int_{\mathcal{Q}_\tau} \varphi^2 \chi g'(|\Delta_h^{(i)} u|^2) \int_0^1 \left[ \sum_{j,\ell=1}^n \partial_{x_l} a_j \Delta_h^{(i)} [D_j u] \Delta_h^{(i)} u \Delta_h^{(i)} [D_{t\ell} u] \Delta_h^{(i)} u \right]^{1/2} \, ds \, \mathcal{D}^{(i)}(h)^{2} |\Delta_h^{(i)} u| \, dz \\
\leq \varepsilon I_2 + \frac{c(q)c_2}{\varepsilon} \int_{\mathcal{Q}_\tau} \varphi^2 \chi \mathcal{D}^{(i)}(h)^{2} g'(|\Delta_h^{(i)} u|^2) \, dz.
\]
Finally, the term on the right-hand side involving $\partial_x a$ is much easier to estimate. Here we just have to use the bound for $\partial_x a$ from (1.2) to infer

$$|I_2| = 2 \sum_{j=1}^{n} \int_{\hat{Q}_\tau} \varphi \int_0^1 \partial_x a \, ds \, D_j \varphi \Delta^{(i)}_h u g(|\Delta^{(i)}_h u|^2) \, dz$$

$$\leq 2c(q)L \int_{\hat{Q}_\tau} \varphi |D\varphi| D^{(i)}(h) \frac{q+q^2}{2} |\Delta^{(i)}_h u| g(|\Delta^{(i)}_h u|^2) \, dz$$

$$= 2c(q)L \int_{\hat{Q}_\tau} \varphi |D\varphi| D^{(i)}(h) \frac{q^2}{2} + \frac{q}{2} |\Delta^{(i)}_h u| g(|\Delta^{(i)}_h u|^2) \, dz$$

$$\leq c(q)L \int_{\hat{Q}_\tau} \varphi [D\varphi]^2 D^{(i)}(h) \frac{q+q^2}{2} |\Delta^{(i)}_h u|^2 + \frac{q}{2} D^{(i)}(h) \frac{q^2}{2} g(|\Delta^{(i)}_h u|^2) \, dz.$$
As usual, for \( p \neq \tau \) force. Then

\[
\int_{B_{\varrho}(x_0)} |\Delta^{(i)} u(x,\tau)|^2 \, d\tau \int_{0}^{\varrho(s)} g(s) \, ds \, dx + \int_{Q_{\varrho}} \varphi^2 \chi D^{(i)}(h) \frac{p-2}{2} |\Delta^{(i)}[Du]|^2 g\left( |\Delta^{(i)} u|^2 \right) d\varrho \leq c \int_{Q_{\varrho}} \chi \left[ D^{(i)}(h)^{\frac{q-2}{2}} |\Delta^{(i)} u|^2 |D\varphi|^2 + (D^{(i)}(h)^{\frac{q}{2}} + D^{(i)}(h)^{\frac{q}{2}}) |\varphi|^2 \right] g\left( |\Delta^{(i)} u|^2 \right) d\varrho + c \int_{Q_{\varrho}} \varphi^2 \partial_\varrho \chi \left[ \int_{0}^{\varrho} g(x) \, ds \right] d\varrho \tag{3.5}
\]

for a constant \( c = c(n, p, q, v, L) \). We note that in (3.5) \( g \in W^{1,\infty}(\mathbb{R}) \) is an arbitrary bounded, non-decreasing, non-negative Lipschitz continuous function, \( \chi \in \mathcal{W}^{1,\infty}(0, T) \) satisfying \( \chi(0) = 0 \) and \( \partial_t \chi \geq 0 \) and \( \varphi \in C_0^\infty(B_{\varrho}(x_0)) \) with \( 0 \leq \varphi \leq 1 \) where \( B_{\varrho}(x_0) \subset \Omega \). Moreover, (3.5) holds whenever \( 0 < |h| \leq \text{dist}(B_{\varrho}(x_0), \partial\Omega) \). It is worth to note that we could have used in the derivation of (3.5) instead of \( B_{\varrho}(x_0) \) and \( \varphi \in C_0^\infty(B_{\varrho}(x_0)) \) any \( \varphi \in C_0^\infty(\Omega) \). In this case \( B_{\varrho}(x_0) \), \( Q_{\varrho} \) should be replaced by \( \text{spt} \varphi \), and \( \text{spt} \varphi \times (0, \tau) \); then (3.5) holds for any \( \varphi \in C_0^\infty(\Omega) \) with \( 0 \leq \varphi \leq 1 \) and any \( h \) satisfying \( 0 < |h| \leq \text{dist} \text{(spt}(\varphi, \partial\Omega) \).

In the following we will use the energy estimate (3.5) in several directions. The first application (with the choice \( g \equiv 1 \)) is in the following lemma which ensures in some special cases the existence of second spatial derivatives. As usual, for \( p \geq 2 \) and \( \mu \in [0, 1] \) fixed define the \( V \)-function by

\[
V^{(p)}_\mu(\xi) := (\mu^2 + |\xi|^2)^{\frac{p-2}{2}} \xi.
\]

**Lemma 3.1.** Let \( u \in L^p_{\text{loc}}(0, T; W^{1,p}_{\text{loc}}(\Omega)) \) be a weak solution to (1.1), where the structure conditions (1.2) are in force. Then

\[
\int_{Q_{\varrho}(z_0)} |D^{2} u|^2 \, dz \leq \frac{c}{\varrho^2 \mu^p - 2} \int_{Q_{\varrho}(z_0)} (1 + |Du|^2)^{\frac{q}{2}} \, dz.
\]

Moreover, there exists a constant \( c = c(n, p, q, v, L) \) such that for any cylinder \( Q_{\varrho}(z_0) \subset \Omega_T \) there holds

\[
\sup_{t_0-|q/2|^2 < t < t_0} \int_{B_{\varrho/2}(z_0)} |Du(\cdot, t)|^2 \, dx \leq \int_{Q_{\varrho/2}(z_0)} (1 + |Du|^2)^{\frac{q}{2}} \, dz.
\]

Moreover, if \( \mu > 0 \) or \( p = 2 \) we additionally have \( D^2 u \in L^2_{\text{loc}}(\Omega_T, \mathbb{R}^n) \), and there holds

\[
\int_{Q_{\varrho/2}(z_0)} |D^2 u|^2 \, dz \leq \frac{c}{\varrho^2 \mu^p - 2} \int_{Q_{\varrho}(z_0)} (1 + |Du|^2)^{\frac{q}{2}} \, dz.
\]

**Proof.** In (3.5) we choose \( g \equiv 1 \). Then (3.5) simplifies to

\[
\int_{B_{\varrho}(x_0)} \varphi^2 \chi(\tau) |\Delta^{(i)} u(\cdot, \tau)|^2 \, dx + \int_{Q_{\varrho}} \varphi^2 \chi D^{(i)}(h) \frac{p-2}{2} |\Delta^{(i)}[Du]|^2 d\varrho \leq c \int_{Q_{\varrho}} \chi \left[ D^{(i)}(h)^{\frac{q-2}{2}} |\Delta^{(i)} u|^2 |D\varphi|^2 + (D^{(i)}(h)^{\frac{q}{2}} + D^{(i)}(h)^{\frac{q}{2}}) |\varphi|^2 \right] d\varrho + c \int_{Q_{\varrho}} \varphi^2 \partial_\varrho \chi |\Delta^{(i)} u|^2 d\varrho \tag{3.6}
\]

Note that this inequality holds for any \( i \in \{1, \ldots, n\} \) and a.e. \( \tau \in (0, T) \). Next, we choose \( \chi \in W^{1,\infty}(\mathbb{R}) \) and \( \varphi \in C_0^\infty(B_{\varrho/4}(x_0)) \) such that \( \chi \equiv 0 \) on \((-\infty, t_0 - (q/4)^2]\), \( \chi \equiv 1 \) on \([t_0 - (q/2)^2, \infty) \), \( 0 \leq \partial_t \chi \leq (2/\varrho)^2 \) and \( \varphi \equiv 1 \) on \( B_{\varrho/2}(x_0) \), \( 0 \leq \varphi \leq 1 \) and \( |D\varphi| \leq 8/\varrho \). Using (3.6) with these choices of \( \chi \) and \( \varphi \) twice, i.e. taking first the supremum over \( \tau \in (t_0 - (q/2)^2, t_0) \), and then choosing \( t = t_0 \) and taking into account that

\[
D^{(i)}(h)^{\frac{p-2}{2}} |\Delta^{(i)}[Du]|^2 \geq \frac{1}{c(p)} |\Delta^{(i)}[V^{(p)}_\mu(Du)]|^2,
\]

(3.7)
which follows from a standard estimate for the function $V_{\mu}^{(p)}$, we arrive at

$$
\sup_{t_0 - (\varrho/2)^2 < t < t_0} \int_{B_{\varrho/2}(x_0)} |\Delta_h^{(i)} u(\cdot, t)|^2 \, dx + \int_{Q_{\varrho/2}(z_0)} |\Delta_h^{(i)} [V_{\mu}^{(p)}(Du)]|^2 \, dz \leq \frac{c}{\varrho^2} \int_{Q_{\varrho/4}(z_0)} |(D^{(i)}(h) + D^{(i)}(h))^2| + |\Delta_h u|^2 \, dz
$$

for any $0 < |h| < \varrho/4$. We note that the right-hand side is bounded by a constant independent of $h$ and therefore the standard estimates for difference quotients yield $D_i[V_{\mu}^{(p)}(Du)] \in L^2(Q_{\varrho/2}(z_0))$ as well as $Du \in L^\infty(t_0 - (\varrho/2)^2, t_0; L^2(B_{\varrho/2}(x_0)))$ with the quantitative estimate

$$
\sup_{t_0 - (\varrho/2)^2 < t < t_0} \int_{B_{\varrho/2}(x_0)} |D_i u(\cdot, t)|^2 \, dx + \int_{Q_{\varrho/2}(z_0)} |D_i [V_{\mu}^{(p)}(Du)]|^2 \, dz \leq \frac{c}{\varrho^2} \int_{Q_{\varrho}(z_0)} (1 + |Du|^2)^2 \, dz. \tag{3.8}
$$

Next, we consider the case where $\mu > 0$ or $p = 2$. Here, we once again start from (3.6). In the second integral on the left-hand side we have $D^{(i)}(h)^{\frac{p-2}{2}} \geq \mu^{p-2} > 0$. By the same reasoning as above this leads to the assertion that $D_i Du \in L^2(Q_{\varrho/2}(z_0))$ with the quantitative estimate

$$
\mu^{p-2} \int_{Q_{\varrho/2}(z_0)} |D_i Du|^2 \, dz \leq \frac{c}{\varrho^2} \int_{Q_{\varrho}(z_0)} (1 + |Du|^2) \, dz. \tag{3.9}
$$

Summing up over $i = 1, \ldots, n$ in the inequalities (3.8) and (3.9) yields the assertion of the lemma for a constant $c$ depending on $n, p, q, \nu$ and $L$. □

4. Boundedness of $Du$

The first step towards the local $L^\infty$-bound of $Du$ is an improvement of the Caccioppoli type inequality from Lemma 3.1 assuming a certain higher integrability of $Du$ in form of $Du \in L^{q+2\sigma}(\Omega_T, \mathbb{R}^n)$ for some $\sigma > 0$. This is usually one step in a Moser-iteration scheme leading eventually to the quantitative $L^\infty$-bound for $Du$.

**Lemma 4.1.** Let $u \in L^q_{loc}(0, T; \mathcal{L}^{1,q}_{loc}(\Omega))$ be a weak solution to (1.1), where the structure conditions (1.2) are in force. Further, assume that for some $\sigma > 0$ there holds $Du \in L^{q+2\sigma}(\Omega_T, \mathbb{R}^n)$. Then, for any cylinder $Q_{\varrho}(z_0) \subset \Omega_T$ and any cut-off functions $\varphi \in C_0^\infty(B_{\varrho}(x_0))$ satisfying $0 \leq \varphi \leq 1$ and $\|D\varphi\|_\infty \geq 1$ and $\chi \in W^{1,\infty}([t_0 - \varrho^2, t_0])$ satisfying $0 \leq \chi \leq 1, \chi(t_0 - \varrho^2) = 0, \partial_t \chi \geq 0$ and $\|\partial_t \chi\|_\infty \geq 1$ and any $i \in \{1, \ldots, n\}$ there holds

$$
\sup_{t_0 - \varrho^2 < t < t_0} \int_{B_{\varrho/2}(x_0)} \varphi^2 \chi(t) (\mu^2 + |D_i u(\cdot, t)|^2)^{1+\sigma} \, dx + \int_{Q_{\varrho}(z_0)} \varphi^2 \chi |D_i [\mu^2 + |Du|^2]^{\frac{p+\sigma}{p+2\sigma}}| \, dz \leq c(1 + \sigma)^3 \left[\|D\varphi\|_{L^\infty}^2 \int_{Q_{\varrho}(z_0)} (\mu^2 + |Du|^2)^{\frac{q+2\sigma}{p+2\sigma}} \, dz + \|\partial_t \chi\|_\infty \int_{Q_{\varrho}(z_0)} (\mu^2 + |Du|^2)^{1+\sigma} \, dz\right],
$$

for a constant $c = c(n, p, q, \nu, L)$. 

Proof. For fixed $k \in \mathbb{N}$ we define $T_k(s) := \min\{s, k\}$ for $s \geq 0$ and
\[ g_\sigma(s) := (\mu^2 + s)^\sigma \quad \text{for } s \geq 0. \]
In (3.5) we choose $g \equiv T_k \circ g_\sigma$. Note that this is possible, since $T_k \circ g_\sigma$ is a bounded, non-negative, non-decreasing Lipschitz continuous function. With $T_k \circ g_\sigma$ instead of $g$ (3.5) turns into
\[
\int_{B_\rho(x_0)} \phi^2(x) \chi(\tau) \int_0^T (T_k \circ g_\sigma)(s) \, ds \, dx + \int_{Q_\tau} \phi^2 \chi D^{(i)}(h) \frac{\sigma^2}{2} \Delta_h^{(i)}[\Delta^i_h u]^2 (T_k \circ g_\sigma) \, dz \\
\leq c \int_{Q_\tau} \chi \left[ D^{(i)}(h) \frac{\sigma^2}{2} \Delta_h^{(i)} u^2 |D\phi|^2 + (D^{(i)}(h) \frac{\sigma^2}{2} + D^{(i)}(h) \frac{\sigma^2}{2}) \phi^2 \right] g_\sigma \left( \Delta_h^{(i)} u^2 \right) \, dz \\
+ c \int_{Q_\tau} \phi^2 \chi D^{(i)}(h) \frac{\sigma^2}{2} \Delta_h^{(i)} u^2 (T_k \circ g_\sigma)' \left( \Delta_h^{(i)} u^2 \right) \, dz + c \int_{Q_\tau} \phi^2 \partial_t \chi \left[ \int_0^T (T_k \circ g_\sigma)(s) \, ds \right] \, dz
\]
(4.1)
for any $\tau \in (0, T)$, where $Q_\tau = B_\rho(x_0) \times (0, \tau)$. Now, since $T_k \circ g_\sigma \leq g_\sigma$ and $(T_k \circ g_\sigma)' \leq g_\sigma'$ we can bound the right-hand side of (4.1) from above by replacing $T_k \circ g_\sigma$ by $g_\sigma$; that is we have
\[
\int_{B_\rho(x_0)} \phi^2(x) \chi(\tau) \int_0^T (T_k \circ g_\sigma)(s) \, ds \, dx + \int_{Q_\tau} \phi^2 \chi D^{(i)}(h) \frac{\sigma^2}{2} \Delta_h^{(i)}[\Delta^i_h u]^2 (T_k \circ g_\sigma) \, dz \\
\leq c \int_{Q_\tau} \chi \left[ D^{(i)}(h) \frac{\sigma^2}{2} \Delta_h^{(i)} u^2 |D\phi|^2 + (D^{(i)}(h) \frac{\sigma^2}{2} + D^{(i)}(h) \frac{\sigma^2}{2}) \phi^2 \right] g_\sigma \left( \Delta_h^{(i)} u^2 \right) \, dz \\
+ c \int_{Q_\tau} \phi^2 \chi D^{(i)}(h) \frac{\sigma^2}{2} \Delta_h^{(i)} u^2 (T_k \circ g_\sigma)' \left( \Delta_h^{(i)} u^2 \right) \, dz + c \int_{Q_\tau} \phi^2 \partial_t \chi \left[ \int_0^T g_\sigma(s) \, ds \right] \, dz
\]
(4.2)
For the last term we note that $\partial_t \chi \geq 0$. At this stage we do not perform particular choices of $\chi$ and $\phi$. The only conditions are that $\phi \in C^0_0(B_\rho(x_0))$ satisfies $0 \leq \phi \leq 1$, $\|D\phi\|_\infty \geq 1$ and moreover that $\chi \in W^{1,\infty}(0, T)$ is non-decreasing, $\chi \equiv 0$ on $(0, t_0 - \rho^2]$, $0 \leq \chi \leq 1$ and $\|\partial_t \chi\|_\infty \geq 1$. Next, we use the preceding inequality twice: firstly we take the supremum over $\tau \in (t_0 - \rho^2, t_0)$, and secondly we let $\tau \uparrow t_0$. Proceeding this way leads to
\[
\sup_{t_0 - \rho^2 < \tau < t_0} \int_{B_\rho(x_0)} \phi^2(x) \chi(t) \int_0^T (T_k \circ g_\sigma)(s) \, ds \, dx \\
+ \int_{Q_{t_0}} \phi^2 \chi D^{(i)}(h) \frac{\sigma^2}{2} \Delta_h^{(i)}[\Delta^i_h u]^2 (T_k \circ g_\sigma) \, dz \\
\leq c \int_{Q_{t_0}} \chi \left[ D^{(i)}(h) \frac{\sigma^2}{2} \Delta_h^{(i)} u^2 |D\phi|^2 + (D^{(i)}(h) \frac{\sigma^2}{2} + D^{(i)}(h) \frac{\sigma^2}{2}) \phi^2 \right] \left( \mu^2 + \Delta_h^{(i)} u^2 \right)^\sigma \, dz \\
+ c \sigma \int_{Q_{t_0}} \phi^2 \chi D^{(i)}(h) \frac{\sigma^2}{2} \Delta_h^{(i)} u^2 \left( \mu^2 + \Delta_h^{(i)} u^2 \right)^{\sigma - 1} \, dz + \frac{c}{1 + \sigma} \int_{Q_{t_0}} \phi^2 \partial_t \chi \left( \mu^2 + \Delta_h^{(i)} u^2 \right)^{\sigma + 1} \, dz
\]
\[
\leq c \int_{Q_{t_0}} \left[ D^{(i)}(h) \frac{\sigma^2}{2} \Delta_h^{(i)} u^2 |D\phi|^2 + (1 + \sigma)D^{(i)}(h) \frac{\sigma^2}{2} + D^{(i)}(h) \frac{\sigma^2}{2} \right] \left( \mu^2 + \Delta_h^{(i)} u^2 \right)^\sigma \, dz
\]
\[
+ \frac{c}{1+\sigma} \int_{Q_{r}(z_0)} |\partial_t \chi| (\mu^2 + |\Delta_h^{(i)} u|^2)^{\sigma+1} \, dz
\]
\[
\leq c(1+\sigma) \|D\varphi\|_{\infty}^2 \int_{Q_{r}(z_0)} \mathcal{D}^{(i)}(h) \frac{\mu^2 + |\Delta_h^{(i)} u|^2}{1+\sigma} \frac{\mu^2 + |\Delta_h^{(i)} u|^2}{1+\sigma} \, dz
\]
\[
+ c \int_{Q_{r}(z_0)} \mathcal{D}^{(i)}(h) \frac{\mu^2 + |\Delta_h^{(i)} u|^2}{1+\sigma} \, dz + c \|\partial_t \chi\|_{\infty} \int_{Q_{r}(z_0)} (\mu^2 + |\Delta_h^{(i)} u|^2)^{\sigma+1} \, dz
\]
\[
\leq c(1+\sigma) \|D\varphi\|_{\infty}^2 \int_{Q_{r}+|h|\cdot r^2(z_0)} (\mu^2 + |Du|^2)^{\frac{q+2\sigma}{2}} \, dz
\]
\[
+ c \int_{Q_{r}+|h|\cdot r^2(z_0)} (\mu^2 + |Du|^2)^{\frac{q+2\sigma}{2}} \, dz + c \|\partial_t \chi\|_{\infty} \int_{Q_{r}+|h|\cdot r^2(z_0)} (\mu^2 + |Du|^2)^{1+\sigma} \, dz
\]
\[
\leq c(1+\sigma) \|D\varphi\|_{\infty}^2 \int_{Q_{r}+|h|\cdot r^2(z_0)} (\mu^2 + |Du|^2)^{\frac{q+2\sigma}{2}} \, dz + c \|\partial_t \chi\|_{\infty} \int_{Q_{r}+|h|\cdot r^2(z_0)} (\mu^2 + |Du|^2)^{1+\sigma} \, dz.
\]

for a constant \( c = c(n, p, q, \nu, L) \). Here, we abbreviated \( Q_{r+|h|\cdot r^2(z_0)} := B_{r+|h|}(z_0) \times (t_0 - r^2, t_0) \) and we used in turn Young’s inequality and a standard estimate for difference quotients. Using (3.7) as in the proof of Lemma 3.1 we can bound the second term of the left-hand side of the preceding inequality from below and infer

\[
\sup_{t_0 - r^2 < t < t_0} \int_{B_{r}(x_0)} \varphi^2(x) \chi(t) \left( T_k \circ g_\sigma(s) \right) ds \, dx + \int_{Q_{r}(z_0)} \varphi^2 \chi \left| \Delta_h^{(i)} \left[ V_{\nu}^{(p)}(Du) \right] \right|^2 (T_k \circ g_\sigma)(|\Delta_h^{(i)} u|^2) \, dz
\]
\[
\leq c(1+\sigma) \|D\varphi\|_{\infty}^2 \int_{Q_{r}+|h|\cdot r^2(z_0)} (\mu^2 + |Du|^2)^{\frac{q+2\sigma}{2}} \, dz + c \|\partial_t \chi\|_{\infty} \int_{Q_{r}+|h|\cdot r^2(z_0)} (\mu^2 + |Du|^2)^{1+\sigma} \, dz.
\]

Since \( D[V_{\nu}^{(p)}(Du)] \in L^2_{\text{loc}}(\Omega_T) \) by Lemma 3.1 and \( Du \in L^{q+2\sigma}_{\text{loc}}(\Omega_T) \) by assumption we know that \( \Delta_h^{(i)} [V_{\nu}^{(p)}(Du)] \rightarrow D_I [V_{\nu}^{(p)}(Du)] \) converges strongly in \( L^2(Q_{r}(z_0)) \) and \( \Delta_h^{(i)} u \rightarrow D_I u \) strongly in \( L^{q+2\sigma}(Q_{r}(z_0)) \). Therefore, we can pass to a subsequence \( h_i \rightarrow 0 \) such that \( \Delta_h^{(i)} [V_{\nu}^{(p)}(Du)] \rightarrow D_I [V_{\nu}^{(p)}(Du)] \) and \( \Delta_h^{(i)} u \rightarrow D_I u \) a.e. on \( Q_{r}(z_0) \). Hence, by Fatou’s Lemma we conclude

\[
\sup_{t_0 - r^2 < t < t_0} \int_{B_{r}(x_0)} \varphi^2(x) \chi(t) \left( T_k \circ g_\sigma(s) \right) ds \, dx + \int_{Q_{r}(z_0)} \varphi^2 \chi \left| \Delta_h^{(i)} \left[ V_{\nu}^{(p)}(Du) \right] \right|^2 (T_k \circ g_\sigma)(|\Delta_h^{(i)} u|^2) \, dz
\]
\[
\leq \liminf_{\ell \rightarrow \infty} \left[ \sup_{t_0 - r^2 < t < t_0} \int_{B_{r}(x_0)} \varphi^2(x) \chi(t) \left( T_k \circ g_\sigma(s) \right) ds \, dx + \int_{Q_{r}(z_0)} \varphi^2 \chi \left| \Delta_h^{(i)} \left[ V_{\nu}^{(p)}(Du) \right] \right|^2 (T_k \circ g_\sigma)(|\Delta_h^{(i)} u|^2) \, dz \right]
\]
\[
\leq c(1+\sigma) \|D\varphi\|_{\infty}^2 \int_{Q_{r}(z_0)} (\mu^2 + |Du|^2)^{\frac{q+2\sigma}{2}} \, dz + c \|\partial_t \chi\|_{\infty} \int_{Q_{r}(z_0)} (\mu^2 + |Du|^2)^{1+\sigma} \, dz.
\]

Since \( T_k \circ g_\sigma \rightarrow g_\sigma \), again Fatou’s Lemma implies
Lemma 3.1 that
On the other hand, we have
Joining this inequality with (4.4) we find
\[ \sup_{t_0 - \theta < t < t_0} \left[ \int_{0}^{\infty} \varphi^2(t) \chi(t) \right]^{1/\alpha} \leq \liminf_{k \to \infty} \left[ \sup_{t_0 - \theta < t < t_0} \int_{0}^{\infty} \varphi^2(t) \chi(t) \right]^{1/\alpha} \]
Evaluating the integrals yields
\[ \left[ \int_{0}^{\infty} \varphi^2(t) \chi(t) \right]^{1/\alpha} = \left[ \int_{0}^{\infty} \varphi^2(t) \chi(t) \right]^{1/\alpha} \]
Next, we compute
\[ \left[ \int_{0}^{\infty} \varphi^2(t) \chi(t) \right]^{1/\alpha} = \left[ \int_{0}^{\infty} \varphi^2(t) \chi(t) \right]^{1/\alpha} \]
for a constant \( c = c(n, p, q, \nu, L) \). Next, we compute
\[ \left| D \left[ \left( \mu^2 + |D_i u|^2 \right)^{\frac{p+2\alpha}{p}} \right] \right|^2 = \left| D \left[ \left( \mu^2 + |D_i u|^2 \right)^{\frac{p}{p+2\alpha}} \right] \right|^2 \]
Our aim now is to bound \( |D[\mu^2 + |D_i u|^2]^\frac{p}{p+2\alpha}]^2 \) in terms of \( |D_i V_{\mu}^{(p)}(Du)|^2 \). In the case \( \mu > 0 \) we recall from Lemma 3.1 that \( D^2 u \in L^2_{\text{loc}}(\Omega_T, \mathbb{R}^{n-n}) \) and therefore the following computations are justified. In the case \( \mu = 0 \) the following computations are formal and can be made rigorous by an approximation argument. We first evaluate
\[ |D_i V_{\mu}^{(p)}(Du)|^2 = \left( \mu^2 + |Du|^2 \right)^{p+2\alpha} \left( \mu^2 + |Du|^2 \right)^{\frac{p}{p+2\alpha}} |D_i u|^2 \]
On the other hand, we have
\[ \left| D_i V_{\mu}^{(p)}(Du) \right|^2 = \left( \mu^2 + |Du|^2 \right)^{p+2\alpha} \left| \mu^2 + |Du|^2 \right|^2 \]
Together we have shown that
\[ \left| D \left[ \left( \mu^2 + |D_i u|^2 \right)^{\frac{p}{p+2\alpha}} \right] \right|^2 \leq \left( \frac{p}{2} \right)^2 \left| D_i V_{\mu}^{(p)}(Du) \right|^2 \]
Joining this inequality with (4.4) we find
\[ \left| D \left[ \left( \mu^2 + |D_i u|^2 \right)^{\frac{p+2\alpha}{p}} \right] \right|^2 \leq \left( \frac{p+2\alpha}{2} \right)^2 \left| D_i V_{\mu}^{(p)}(Du) \right|^2 \]
This allows us to bound the second term on the left-hand side of (4.3) from below. Taking also into account that
\[ \int_0^t |D_t u(x, t)|^2 g_\sigma(s) \, ds = \frac{1}{1 + \sigma} (m^2 + |D_t u(x, t)|^2)^{1+\sigma} \]
we arrive at
\[
\sup_{t_0 - \sigma^2 < t < t_0} \frac{1}{1 + \sigma} \int_{B_{\rho}(x_0)} \varphi^2 \chi (m^2 + |D_t u(x, t)|^2)^{1+\sigma} \, dx + \left( \frac{2}{p + 2\sigma} \right)^\sigma \int_{Q_\sigma(z_0)} \varphi^2 \chi |D[(m^2 + |D_t u|^2)^{\frac{p+2\sigma}{4}}]|^2 \, dz
\]
\[
\leq c(1 + \sigma) \|D\varphi\|^2_{\infty} \int_{Q_\sigma(z_0)} \left( m^2 + |D_t u|^2 \right)^{\frac{q+2\sigma}{4}} \, dz + c\|\varphi\|_{\infty} \int_{Q_\sigma(z_0)} \left( m^2 + |D_t u|^2 \right)^{1+\sigma} \, dz,
\]
for a constant \( c = c(n, p, q, v, L) \). This completes the proof of the lemma. \( \square \)

Now, we are ready to prove the sup-bound for \( Du \) from Theorem 1.2. Here we use the improved Caccioppoli type estimate from Lemma 4.1 in order to gain a quantitative improvement of the integrability exponent of \( Du \) by an iteration scheme starting from \( \sigma = 0 \); more precisely, we work with a sequence of integrability exponents \( \sigma_k = q + 2\sigma_k \)
with a choice of \( (\sigma_k)_{k \in \mathbb{N}} \) such that \( \sigma_0 = 0 \) and \( \sigma_k \to \infty \) so that \( d_k \to \infty \) when \( k \to \infty \). Here, one needs to analyze the dependence of the constants carefully in order to have control when \( k \to \infty \).

**Proof of Theorem 1.2.** We take the Caccioppoli inequality of Lemma 4.1 as a starting point, i.e. under the assumptions of Lemma 4.1 we have for any \( i \in \{1, \ldots, n\} \) that
\[
\sup_{t_0 - \sigma^2 < t < t_0} \int_{B_{\rho}(x_0)} \varphi^2 \chi(t) (m^2 + |D_t u(x, t)|^2)^{1+\sigma} \, dx + \int_{Q_\sigma(z_0)} \varphi^2 \chi |D[(m^2 + |D_t u|^2)^{\frac{p+2\sigma}{4}}]|^2 \, dz
\]
\[
\leq c(1 + \sigma)^3 \left( \|D\varphi\|^2_{\infty} + \|\varphi\|_{\infty} \right) \int_{Q_\sigma(z_0)} \left( m^2 + |D_t u|^2 \right)^{\frac{q+2\sigma}{4}} + 1 \, dz,
\]
where \( c = c(n, p, q, v, L) \). We now rewrite the integrand of the second term appearing on the left-hand side of (4.5) as follows: Inserting the estimate
\[
|D[(m^2 + |D_t u|^2)^{\frac{p+2\sigma}{4}}]|^2 \leq 2 |D[(m^2 + |D_t u|^2)^{\frac{p+2\sigma}{4}}]]^2 \varphi^2 + 2(m^2 + |D_t u|^2)^{\frac{p+2\sigma}{2}} |D\varphi|^2
\]
into (4.5) and rearranging terms we obtain
\[
\sup_{t_0 - \sigma^2 < t < t_0} \int_{B_{\rho}(x_0)} \varphi^2 \chi(t) (m^2 + |D_t u(x, t)|^2)^{1+\sigma} \, dx + \int_{Q_\sigma(z_0)} \varphi^2 \chi |D[(m^2 + |D_t u|^2)^{\frac{p+2\sigma}{4}}]|^2 \, dz
\]
\[
\leq c(1 + \sigma)^3 \left( \|D\varphi\|^2_{\infty} + \|\varphi\|_{\infty} \right) \int_{Q_\sigma(z_0)} \left( m^2 + |D_t u|^2 \right)^{\frac{q+2\sigma}{4}} + 1 \, dz
\]
for a constant \( c = c(n, p, q, v, L) \). We now define \( \hat{n} \) according to (1.7). On a fixed time slice by Hölder’s inequality (note that \( \hat{n} > 2 \)) and the Sobolev embedding we have for any \( \delta > 0 \):
\[
\int_{B_{\rho}(x_0)} \varphi^{2 + \frac{\delta}{\hat{n}}} (m^2 + |D_t u|^2)^{\frac{p+2\sigma}{4} + \delta} \, dx \leq \left[ \int_{B_{\rho}(x_0)} \left[ \varphi^2 (m^2 + |D_t u|^2)^{\frac{p+2\sigma}{4}} \right]^{\frac{\delta}{\hat{n}}} \, dx \right]^\frac{\hat{n}}{\delta} \left[ \int_{B_{\rho}(x_0)} \varphi^2 (m^2 + |D_t u|^2)^{\frac{p+2\sigma}{4}} \, dx \right]^\frac{\delta}{\hat{n}}
\]
\[
\leq c(\hat{n}) \int_{B_{\rho}(x_0)} |D[(m^2 + |D_t u|^2)^{\frac{p+2\sigma}{4}}]|^2 \, dx \left[ \int_{B_{\rho}(x_0)} \varphi^2 (m^2 + |D_t u|^2)^{\frac{p+2\sigma}{4}} \, dx \right]^\frac{\delta}{\hat{n}}.
\]
Here we choose \( \delta \) such that
\[
\delta = \frac{2(1 + \sigma)}{\hat{n}}.
\]
Next we multiply the last inequality by $\chi(t)^{\frac{q+2}{2}}$, integrate with respect to $t \in (t_0 - \varrho^2, t_0)$, take the supremum over $t \in (t_0 - \varrho^2, t_0)$ in the term $[\ldots]^2$ and finally estimate both terms by the right-hand side of the Caccioppoli type estimate (4.6). In this way we arrive at

$$
\int_{Q_{t_0}(z_0)} (\varphi^2 \chi)^{1 + \frac{2}{n}} \left( \mu^2 + |D_t u|^2 \right)^{\frac{q+2}{2}} \, dz
\leq c(\hat{n}) \int_{Q_{t_0}(z_0)} \chi |D(\varphi^2 + |D_t u|^2)^{\frac{p+2\sigma}{2}}|^2 \, dz \left[ \sup_{t \in (t_0 - \varrho^2, t_0)} \int_{B_{t_0}} \varphi^2 \chi(t) \left( \mu^2 + |D_t u(\cdot, t)|^2 \right)^{1 + \sigma} \, dx \right]^{\frac{2}{n}}
\leq c(\hat{n}) \left[ c(1 + \sigma)^3 \left( \|D\varphi\|_\infty^2 + \|\partial_t \chi\|_\infty \right) \int_{Q_{t_0}(z_0)} \left[ \left( \mu^2 + |D_t u|^2 \right)^{\frac{q+2\sigma}{2}} + 1 \right] \, dz \right]^{1 + \frac{2}{n}}.
$$

We compute the exponent appearing on the left, that is

$$
p + 2\sigma + \delta = \frac{p + 2\sigma}{2} + \frac{2(1 + \sigma)}{\hat{n}} = \frac{p + 2\sigma}{2} + \frac{2}{n} + \sigma \left( 1 + \frac{2}{n} \right)
= \frac{q}{2} + \sigma \left( 1 + \frac{2}{n} \right) - \frac{q - p}{2} + \frac{2}{n} > \frac{q}{2} + \sigma \left( 1 + \frac{2}{n} \right),
$$

provided we assume (1.6). With these abbreviations the preceding inequality turns into

$$
\int_{Q_{t_0}(z_0)} (\varphi^2 \chi)^{1 + \frac{2}{n}} \left( \mu^2 + |D_t u|^2 \right)^{\frac{q + 2\sigma(1 + \frac{2}{n}) - \frac{q - p}{2} + \frac{2}{n}}{2}} \, dz
\leq c(1 + \sigma)^3 \left( \|D\varphi\|_\infty^2 + \|\partial_t \chi\|_\infty \right) \int_{Q_{t_0}(z_0)} \left[ \left( \mu^2 + |D_t u|^2 \right)^{\frac{q + 2\sigma(1 + \frac{2}{n}) - \frac{q - p}{2} + \frac{2}{n}}{2}} + 1 \right] \, dz \right]^{1 + \frac{2}{n}}
$$

for a constant $c = c(\hat{n}, p, q, \nu, L)$. Next, we define by induction a sequence $(\sigma_k)_{k \in \mathbb{N}_0}$ by letting $\sigma_0 = 0$ and

$$
\sigma_k := \left( 1 + \frac{2}{n} \right) \sigma_{k-1} - \frac{q - p}{2}.
$$

By induction we have

$$
\sigma_k = \left[ \frac{2}{n} - \frac{q - p}{2} \right] \hat{n} \left[ \left( 1 + \frac{2}{n} \right) - 1 \right] = \left[ 1 - \frac{\hat{n}}{4}(q - p) \right] \left[ \left( 1 + \frac{2}{n} \right) - 1 \right],
$$

and obtain from (4.7) the following recursive reverse Hölder’s inequality:

$$
\int_{Q_{t_0}(z_0)} (\varphi^2 \chi)^{1 + \frac{2}{n}} \left( \mu^2 + |D_t u|^2 \right)^{\frac{q + 2\sigma_k}{2}} \, dz
\leq c(1 + \sigma_k)^3 \left( \|D\varphi\|_\infty^2 + \|\partial_t \chi\|_\infty \right) \int_{Q_{t_0}(z_0)} \left[ \left( \mu^2 + |D_t u|^2 \right)^{\frac{q + 2\sigma_k}{2}} + 1 \right] \, dz \right]^{1 + \frac{2}{n}}.
$$

For $k \in \mathbb{N}_0$ we set

$$
\varrho_k := s \varrho + \frac{1 - s}{2k} \varrho \quad \text{and} \quad Q_k := Q_{\varrho_k}(z_0)
$$

and choose cut-off functions $\varphi_k \in C_0^\infty(B_{\varrho_k}(x_0), [0, 1])$ such that $\varphi_k \equiv 1$ on $B_{\varrho_k+1}(x_0)$ and $|D\varphi_k| \leq \frac{\varrho}{(1-s)\varrho}$ and $\chi_k \in W^{1,\infty}((t_0 - \varrho^2, t_0), [0, 1])$ such that $\chi_k(t_0 - \varrho^2) = 0$, $\chi_k \equiv 1$ on $(t_0 - \varrho^2, t_0)$ and $0 \leq \partial_t \chi_k \leq \frac{\varrho}{(1-s)\varrho^2}$. Note
that this is possible since \( \varrho_k^2 - \varrho_{k-1}^2 \geq (\varrho_k - \varrho_{k-1})^2 = (\frac{(1-\varrho_k)}{2\varrho_{k+1}})^2 \). With these particular choices we infer from the previous inequality that for any \( k \in \mathbb{N} \) there holds:

\[
\int_{Q_k} \left( \mu^2 + |Du|^2 \right)^{\frac{q + 2\varrho_k}{2}} \, dz \leq \left[ \frac{4c^4 (1 + \sigma_k)^3}{(1-s)^2 2^{(2\varrho_{k-1})}} \right]^{1+\frac{2}{\varrho_{k-1}}} \left[ \left( \mu^2 + |Du|^2 \right)^{\frac{q + 2\varrho_k}{2}} \, dz + 1 \right]^{1+\frac{2}{\varrho_{k-1}}}. \tag{4.8}
\]

Taking mean values and summing up over \( i = 1, \ldots, n \), the last inequality turns into

\[
\int_{Q_k} \left( \mu^2 + |Du|^2 \right)^{\frac{q + 2\varrho_k}{2}} \, dz \leq \left[ \frac{c^4 (1 + \sigma_k)^3}{(1-s)^2 2^{(2\varrho_{k-1})}} \right]^{1+\frac{2}{\varrho_{k-1}}} \left[ \left( \mu^2 + |Du|^2 \right)^{\frac{q + 2\varrho_k}{2}} \, dz + 1 \right]^{1+\frac{2}{\varrho_{k-1}}}
\]

and holds for any \( k \geq 1 \). The constant \( c \) depends on \( \hat{n}, p, q, \nu, L \). For \( k \in \mathbb{N}_0 \) we define

\[
A_k := \int_{Q_k} \left( \mu^2 + |Du|^2 \right)^{\frac{q + 2\varrho_k}{2}} \, dz. \tag{4.9}
\]

With these notations inequality (4.8) can be rewritten in the form

\[
A_k \leq \left[ \frac{c^4 (1 + \sigma_k)^3}{(1-s)^2 2^{(2\varrho_{k-1})}} \right]^{1+\frac{2}{\varrho_{k-1}}} (A_{k-1} + 1)^{1+\frac{2}{\varrho_{k-1}}},
\]

for \( k \geq 1 \). We iterate the preceding inequality and obtain

\[
A_k \leq \left[ \frac{c^4 (1 + \sigma_k)^3}{(1-s)^2 2^{(2\varrho_{k-1})}} \right]^{1+\frac{2}{\varrho_{k-1}}} \left[ \left[ \frac{c^4 (1 + \sigma_{k-1})^3}{(1-s)^2 2^{(2\varrho_{k-2})}} \right]^{1+\frac{2}{\varrho_{k-2}}} (A_{k-2} + 1)^{1+\frac{2}{\varrho_{k-2}}} + 1 \right]^{1+\frac{2}{\varrho_{k-2}}}
\]

\[
\leq \cdots
\]

\[
\leq \prod_{j=1}^{k} \left[ \frac{c^4 (1 + \sigma_j)^3}{(1-s)^2 2^{(2\varrho_{j-1})}} \right]^{(1+\frac{2}{\varrho_{j-1}})^{k-j+1}} (A_0 + 1)^{(1+\frac{2}{\varrho_{k}})^k},
\]

for any \( k \in \mathbb{N} \). This inequality can be rewritten (after replacing \( 2c \) by \( c \)) as follows:

\[
A_k \leq \prod_{j=1}^{k} \left[ \frac{c^4 (1 + \sigma_j)^3}{(1-s)^2 2^{(2\varrho_{j-1})}} \right]^{(1+\frac{2}{\varrho_{j-1}})^{k-j+1}} (A_0 + 1)^{(1+\frac{2}{\varrho_{k}})^k}. \tag{4.10}
\]

Next, we observe that

\[
\lim_{k \to \infty} \frac{(1+\frac{2}{\varrho_{k}})^k}{q + 2\sigma_k} = \lim_{k \to \infty} \frac{(1+\frac{2}{\varrho_{k}})^k}{q + 2[1 - \frac{q}{2}(q - p)][(1+\frac{2}{\varrho_{k}})^k - 1]} = \frac{2}{4 - \hat{n}(q - p)},
\]

so that (recall that \( \sigma_0 = 0 \) and the definition of \( A_k \) from (4.9))

\[
\lim_{k \to \infty} (A_0 + 1)^{(1+\frac{2}{\varrho_{k}})^k} \leq \left[ \int_{Q_{\varrho(\varrho_0)}} \left( \mu^2 + |Du|^2 \right)^{\frac{q + 2\varrho_{k}}{2}} \, dz + 1 \right]^{\frac{2}{q - \varrho_{k}(q - p)}}.
\]

Next, with the abbreviations

\[
K := \frac{c}{(1-s)^2 2^{(2\varrho_{k-1})}} \geq 1 \quad \text{and} \quad \gamma := 2 - \frac{\hat{n}}{2}(q - p) \in (0, 1]
\]

we compute and estimate
\[
\prod_{j=1}^{k} \left[ \frac{c}{(1 - s)^{2Q^{\frac{2(k-j+1)}{n+2}}}} \right]^{(1 + \frac{2}{n})^{k-j+1}} = \exp \left[ \log K \cdot \sum_{j=1}^{k} \frac{(1 + \frac{2}{n})^{k-j+1}}{q + 2\sigma_k} \right]
\]
\[
= \exp \left[ \log K \cdot \sum_{j=1}^{k} \frac{(1 + \frac{2}{n})^{k-j+1}}{q + \gamma[(1 + \frac{2}{n})^{k} - 1]} \right]
\]
\[
\leq \exp \left[ \log K \cdot \sum_{j=1}^{k} \frac{(1 + \frac{2}{n})^{k-j+1}}{(1 + \frac{2}{n})^{k} - 1} \right]
\]
\[
= \exp \left[ \frac{(\hat{n} + 2) \log K}{2\gamma} \right] = \left[ \frac{c}{(1 - s)^{2Q^{\frac{2(k-j+1)}{n+2}}}} \right]^{\frac{4\hat{n}^2}{\gamma}}.
\]

Similarly, we compute
\[
\prod_{j=1}^{k} \frac{4}{\frac{q + 2\sigma_k}{(1 + \frac{2}{n})^{k-j+1}}} = \exp \left[ \log 4 \cdot \sum_{j=1}^{k} \frac{j(1 + \frac{2}{n})^{k-j+1}}{q + 2\sigma_k} \right]
\]
\[
= \exp \left[ \log 4 \cdot \sum_{j=1}^{k} \frac{j(1 + \frac{2}{n})^{k-j+1}}{q + \gamma[(1 + \frac{2}{n})^{k} - 1]} \right]
\]
\[
\leq \exp \left[ \log 4 \cdot \sum_{j=1}^{k} \frac{j(1 + \frac{2}{n})^{k-j+1}}{(1 + \frac{2}{n})^{k} - 1} \right]
\]
\[
= \exp \left[ \frac{(\hat{n} + 2)^2}{(\frac{1}{2})^3} \right] = 4^{\frac{1}{3}}(\frac{\hat{n} + 2}{2})^3.
\]

It remains to estimate the product involving the \(\sigma_j\). Here we obtain
\[
\prod_{j=1}^{k} (1 + \sigma_j)^{3 \frac{(1 + \frac{2}{n})^{k-j+1}}{q + 2\sigma_k}} = \exp \left[ 3 \sum_{j=1}^{k} \frac{(1 + \frac{2}{n})^{k-j+1}}{q + 2\sigma_k} \log(1 + \sigma_j) \right]
\]
\[
= \exp \left[ 3 \sum_{j=1}^{k} \frac{(1 + \frac{2}{n})^{k-j+1}}{q + \gamma[(1 + \frac{2}{n})^{k} - 1]} \log \left( 1 + \gamma \left[ (1 + \frac{2}{n})^{j} - 1 \right] \right) \right]
\]
\[
\leq \exp \left[ 3 \sum_{j=1}^{k} \frac{(1 + \frac{2}{n})^{k-j+1}}{(1 + \frac{2}{n})^{k} - 1} \log \left( 1 + \frac{2}{n} \right)^{j} \right]
\]
\[
= \exp \left[ \frac{3}{\gamma} \log \left( 1 + \frac{2}{n} \right) \sum_{j=1}^{k} \frac{j(1 + \frac{2}{n})^{k-j+1}}{(1 + \frac{2}{n})^{k} - 1} \right]
\]
Combining the last three inequalities we see that the product in (4.10) can be bounded as follows:

\[
\prod_{j=1}^{k} \left[ \frac{c4^{j}(1 + \sigma_j)^{4}}{(1 - s)^{2\beta + \rho}} \right]^{(1+\beta j+1)\frac{q}{\beta + 2q}} \leq c^{\frac{1}{(1 - s)^{2\beta + \rho} + 2}} \int_{Q_{\sigma_k}(z_0)} (\mu^2 + |Du|)^{\frac{q+2q_k}{q}} \, dz^{\frac{1}{q + 2q_k}}
\]

for a constant \( c = c(n, p, q, v, L) \). Since \( Q_k \downarrow sQ \) and \( \sigma_k \uparrow \infty \) as \( k \to \infty \) we arrive at

\[
\sup_{Q_{\sigma_k}(z_0)} (\mu^2 + |Du|)^{\frac{1}{q}} = \lim_{k \to \infty} \left[ \int_{Q_{\sigma_k}(z_0)} (\mu^2 + |Du|)^{\frac{q+2q_k}{q}} \, dz \right]^{\frac{1}{q + 2q_k}}
\]

for a constant \( c = c(n, p, q, v, L) \). We note that \( c \uparrow \infty \) when \( q \uparrow p + \frac{4}{n} \). This finishes the proof of the theorem. \( \square \)

5. Interpolation

In this section we use a sort of interpolation inequality in order to reduce the integrability exponent in Theorem 1.2 from \( q \) to \( p \). The idea is quite elementary and can roughly be summarized as follows: In the sup-bound for \( Du \) with cylinders \( Q_{\nu_0}, \sigma_0 \) we write the integrand in the from \( (1 + |Du|^2)^{\frac{q}{2}} (1 + |Du|^2)^{\frac{q}{2} - (q - p)} \), and than use Young’s inequality to have \( \frac{1}{2} \sup_{Q_0} (1 + |Du|^2)^{\frac{q}{2}} \) on the right-hand side. This is possible provided we impose a certain stronger bound on the size of \( q \). By an iteration procedure this leads to a sup-bound for \( Du \) in terms of the \( L^p \)-norm of \( Du \). An estimate of this type will play a crucial role in the proof of the existence result from Theorem 1.6, since it allows to establish uniform sup-bounds for the gradients of solutions to more regular problems, i.e. problems of standard \( q \)-growth, which eventually lead in the limit to a solution of the associated Cauchy–Dirichlet problem for a general parabolic equation with non-standard \( p, q \)-growth. The details are worked out in Section 6.

**Theorem 5.1.** Let \( u \in L^q_{loc}(0, T; W^{l-1,q}_{loc}(\Omega)) \) be a weak solution to (1.1) where the structure conditions (1.2) are in force. Further, assume that (1.11) holds. Then there exists a constant \( c = c(n, p, q, v, L) \) such that for any \( Q_k(z_0) \subseteq \Omega_T \) and \( s \in (0, 1) \) there holds

\[
\sup_{Q_{\sigma_k}(z_0)} |Du| \leq c \left[ \frac{1}{[(1 - s)q]^{\frac{n}{q} + 2}} \int_{Q_{\sigma_k}(z_0)} (1 + |Du|^2)^{\frac{q}{2}} \, dz \right]^{\frac{1}{2} \cdot \frac{n(q - p)}{q + 2q_k} + \frac{2q}{4 - n(q - p)}},
\]

where \( \hat{n} \) is defined according to (1.12). The constant \( c \) admits the following asymptotic behavior: \( c \uparrow \infty \) when \( q \uparrow p + \frac{4}{n+2} \).

**Proof.** We set

\[
\theta := \frac{2q}{4 - \hat{n}(q - p)}.
\]
where \( \hat{n} \) is defined according to (1.12). Starting with the sup-bound for \( Du \) from Theorem 1.2 in which we estimate for \( s \in (0, 1) \):

\[
\sup_{Q_{\epsilon}(z_0)} |Du| \leq c \left[ \frac{1}{[(1-s)\epsilon]^{\hat{n}+2}} \int_{Q_{\epsilon}(z_0)} (1 + |Du|^2)^{\frac{q}{2} + \frac{2}{\hat{n}+2}} \, dz \right]^\frac{\theta}{q}
\]

\[
\leq c \left[ \sup_{Q_{\epsilon}(z_0)} \left( 1 + |Du|^2 \right)^{\frac{1}{2}} \right]^\frac{q(q-p)}{q} \left[ \frac{1}{[(1-s)\epsilon]^{\hat{n}+2}} \int_{Q_{\epsilon}(z_0)} (1 + |Du|^2)^{\frac{q}{2}} \, dz \right]^\frac{\theta}{q}
\]

for a constant \( c = c(\hat{n}, p, q, v, L) \). Since we are assuming the stronger condition (1.11) on \( q \), that is \( p \leq q < p + \frac{4}{n+2} \), we have \( \frac{\theta(q-p)}{q} < 1 \).

Therefore we can apply Young’s inequality with exponents \( \frac{q}{q(q-p)} \) and \( \frac{q}{q(q-p)} \) to the right-hand side of the last inequality. This yields

\[
\sup_{Q_{\epsilon}(z_0)} |Du| \leq \frac{1}{2} \sup_{Q_{\epsilon}(z_0)} \left( 1 + |Du|^2 \right)^{\frac{1}{2}} + c \left[ \frac{1}{[(1-s)\epsilon]^{\hat{n}+2}} \int_{Q_{\epsilon}(z_0)} (1 + |Du|^2)^{\frac{q}{2}} \, dz \right]^\frac{\theta}{q(q-q-p)}
\]

\[
\leq \frac{1}{2} \sup_{Q_{\epsilon}(z_0)} |Du| + c \left[ \frac{1}{[(1-s)\epsilon]^{\hat{n}+2}} \int_{Q_{\epsilon}(z_0)} (1 + |Du|^2)^{\frac{p}{2}} \, dz \right]^\frac{\theta}{q(q-q-p)}
\]

At this stage we apply Lemma 2.1 with

\[\phi(\epsilon) := \sup_{Q_{\epsilon}(z_0)} |Du|, \quad \theta := \frac{1}{2}, \quad \alpha := \frac{\theta(\hat{n}+2)}{q-\theta(q-p)}, \quad A := c \left[ \frac{1}{\epsilon^{\hat{n}+2}} \int_{Q_{\epsilon}(z_0)} (1 + |Du|^2)^{\frac{p}{2}} \, dz \right]^\frac{\theta}{q(q-q-p)} \]

With these notations the application of Lemma 2.1 yields

\[
\sup_{Q_{\epsilon}(z_0)} |Du| \leq c \left[ \frac{1}{[(1-s)\epsilon]^{\hat{n}+2}} \int_{Q_{\epsilon}(z_0)} (1 + |Du|^2)^{\frac{p}{2}} \, dz \right]^\frac{\theta}{q(q-q-p)}
\]

for a constant \( c = c(\hat{n}, p, q, v, L) \). We note that \( c \uparrow \infty \) when \( q \uparrow p + \frac{4}{n+2} \). Recalling the definition of \( \theta \) the desired estimate follows.

6. Existence

In this section we prove the existence result from Theorem 1.6. The proof of the theorem is divided in several steps, which we are going to present in the sequel.

6.1. Regularization

We will construct a solution \( u \) with the properties described in Theorem 1.6 as the limit of solutions to regularized parabolic equations satisfying standard \( q \)-growth and ellipticity conditions. For such kind of equations we have the
a priori estimates from Theorem 1.2 respectively Theorem 5.1 at hands and this allows us to pass to the limit in the approximation scheme. The regularized vector fields \( a_\varepsilon \) are defined by

\[
a_\varepsilon(x, t, \xi) = a(x, t, \xi) + \varepsilon \left( 1 + |\xi|^2 \right)^{\frac{p-2}{2}} \xi
\]

for \( \varepsilon \in (0, 1] \). (6.1)

If \( a \) satisfies (1.2) with structure constants \( \nu, L > 0 \) and \( 2 \leq p \leq q \), then the regularized vector fields \( a_\varepsilon \) fulfill the same growth, ellipticity, continuity and symmetry conditions for \( (\nu/2)^p, L + q - 1, \mu \) instead of \( (\nu, L, \mu) \). Therefore, we are allowed to apply the results of the preceding sections to parabolic equations with vector fields \( a_\varepsilon \). Associated to the vector fields \( a_\varepsilon \) we consider the following Cauchy–Dirichlet problems:

\[
\begin{align*}
\partial_t u_\varepsilon - \mathrm{div} a_\varepsilon(x, t, Du_\varepsilon) &= 0 & \text{in } \Omega_T, \\
u_L(\varepsilon) 
\end{align*}
\]

(6.2)

Note that

\[
\{\partial_t a_\varepsilon(x, t, \xi)\xi, \xi\} \geq \varepsilon \left( 1 + |\xi|^2 \right)^{\frac{p-2}{2}} |\xi|^2,
\]

which ensures that the regularized problem (6.2) satisfies standard \( q \)-growth conditions. The advantage of the Cauchy–Dirichlet problem (6.2) stems from the fact that the existence of a unique solution \( u_\varepsilon \in C^0([0, T]; L^2(\Omega)) \cap L^q(0, T; W^{1,q}(\Omega)) \) with \( \partial_t u_\varepsilon \in L^q(0, T; W^{-1,q}(\Omega)) \) can be ensured by the classic existence theory for parabolic equations, cf. [15]. The initial datum \( g(\cdot, 0) \) is taken in the usual \( L^2 \)-sense, the Dirichlet boundary data \( g \) on the lateral boundary \( \partial \Omega \times (0, T) \) are achieved in the sense of traces in \( W^{1,q}(\Omega) \). Note that this is possible, since we are assuming \( g \in L^r(0, T; W^{1,r}(\Omega)) \) for some \( r > q \).

6.2. Energy bound

In order to have an (uniform in \( \varepsilon \)) energy estimate for \( |Du_\varepsilon|^p \) we test the parabolic equation (6.2)1 with the testing function \( \varphi = \chi(t)(u_\varepsilon - g) \), where \( \chi \in W^{1,\infty}(0, T) \) is chosen to satisfy \( \chi \equiv 1 \) on \( (0, t_0), \chi \equiv 0 \) for \( t \in (t_0 + h, T) \) and \( \chi(t) := -\frac{1}{h}(t - t_0) + 1 \) for \( t \in [t_0, t_0 + h] \), where \( 0 < t_0 < t_0 + h < T \). We note that the following computations are somewhat formal concerning the use of the time derivative, but they can easily be made rigorous for example by the use of Steklov-averages. We skip this since this is a standard procedure. With the preceding choice of \( \varphi \) the weak form of (6.2)1 can be rewritten in the limit \( h \downarrow 0 \) as

\[
\frac{1}{2} \int_\Omega |u_\varepsilon - g|^2(\cdot, t_0) \, dx + \int_{t_0}^T \frac{1}{2} \int_\Omega |a_\varepsilon(\cdot, Du_\varepsilon)| \, dx + \frac{1}{2} \int_\Omega |Du_\varepsilon|^p \, dz = -\int_0^{t_0} \langle u_\varepsilon - g, g_t \rangle_{W^{1,p}_0(\Omega)} \, dt,
\]

where we abbreviated \( \Omega_0 := \Omega \times (0, t_0) \) and \( \langle \cdot, \cdot \rangle_{W^{1,p}_0(\Omega)} \) denotes the duality pairing between \( W^{1,p}_0(\Omega) \) and \( W^{-1,p'}(\Omega) \). Note that the preceding identity holds true for almost every \( t_0 \in (0, T) \). We now apply Lemma 2.3 and obtain

\[
\begin{align*}
\frac{c_3}{2} \int_\Omega |u_\varepsilon - g|^2(\cdot, t_0) \, dx + \int_{\Omega_0} |Du_\varepsilon|^p \, dz &\leq c_3 \left[ \frac{1}{2} \int_\Omega |u_\varepsilon - g|^2(\cdot, t_0) \, dx + \int_{\Omega_0} (1 + |Dg|)^{p/(q-1)} \, dz + \int_{\Omega_0} \langle a_\varepsilon(\cdot, Du_\varepsilon), Du_\varepsilon - Dg \rangle \, dz \right] \\
&= c_3 \left[ \int_{\Omega_0} (1 + |Dg|)^{p/(q-1)} \, dz + \int_0^{t_0} \langle u_\varepsilon - g, g_t \rangle_{W^{1,p}_0(\Omega)} \, dt \right] \\
&\leq c_3 \left[ \int_{\Omega_0} (1 + |Dg|)^{p/(q-1)} \, dz + \delta \int_{\Omega_0} |Du_\varepsilon - Dg|^p \, dz + c_3 \|g_t\|_{L^{p'}(0,t_0;W^{-1,p'}(\Omega))} \right] \\
&\leq c_3 \delta \int_{\Omega_0} |Du_\varepsilon|^p \, dz + c \int_{\Omega_0} (1 + |Dg|)^{p/(q-1)} \, dz + c_3 \|\varphi_{\partial_t} g\|_{L^{p'}(0,t_0;W^{-1,p'}(\Omega))}.
\end{align*}
\]
Reabsorbing as usual the first term of the right-hand side on the left and recalling the definition of $r = p'(q - 1)$, by a suitable choice of $\delta > 0$ we finally obtain for any $\varepsilon \in (0, 1]$ and $t_0 \in (0, T)$ the energy bound

$$\sup_{0 < t < T} \int_{\Omega} |u_{\varepsilon} - g|^p(\cdot, t) \, dx + \int_{\Omega_0} |Du_{\varepsilon}|^p \, dz \leq c \left( \int_{\Omega} (1 + |Dg|)^q \, dz + c\|\partial_t g\|_{L^p(0, T; W^{-1, p'}(\Omega))}^p \right),$$

for a constant $c = c(n, p, q, v, L)$ independent of $\varepsilon$.

### 6.3. Uniform bounds for $\|Du_{\varepsilon}\|_{L^\infty}$ and $\|D^2u_{\varepsilon}\|_{L^2}$

Having the uniform energy bound (6.3) at hand we can use the sup-estimate from Theorem 5.1 together with a covering argument to obtain for any $\tilde{\Omega} \Subset \Omega$ and $0 < t_1 < t_2 < T$ that there holds

$$\sup_{\tilde{\Omega} \times (t_1, t_2)} |Du_{\varepsilon}| \leq c \left[ \left( \int_{\tilde{\Omega}_T} (1 + |Du_{\varepsilon}|^2)^{\frac{p}{2}} \, dz \right)^{\frac{2}{p}} + \hat{c} \left( \int_{\tilde{\Omega}_T} (1 + |Du_{\varepsilon}|^2)^{\frac{2p}{q}} \, dz \right)^{\frac{q}{2p}} \right].$$

Moreover, we infer from Lemma 3.1 that in the case when $\mu > 0$ or $p = 2$ there holds for any cylinder $Q_{\rho}(z_0) \subseteq \Omega_T$ that

$$\int_{Q_{\rho/2}(z_0)} |D^2u_{\varepsilon}|^2 \, dz \leq \frac{c}{\rho^2 \mu^{p-2}} \int_{Q_{\rho/4}(z_0)} (1 + |Du_{\varepsilon}|^2)^{\frac{p}{2}} \, dz \leq \frac{c}{\rho^2 \mu^{p-2}} \sup_{Q_{\rho/4}(z_0)} \left( \int_{Q_{\rho/4}(z_0)} (1 + |Du_{\varepsilon}|^2)^{\frac{2p}{q}} \, dz \right)^{\frac{q}{2p}} \leq c \hat{\rho} \left[ \frac{1}{\rho^2 \mu^{p-2}} \int_{Q_{\rho}(z_0)} (1 + |Du_{\varepsilon}|^2)^{\frac{q}{2}} \, dz \right]^{\frac{2q-p}{q}}.$$

By a covering argument this implies for any $\tilde{\Omega} \Subset \Omega$ and $0 < t_1 < t_2 < T$ that

$$\sup_{t_1 < t < t_2} \int_{\tilde{\Omega}} \int_{\tilde{\Omega} \times (t_1, t_2)} |Du_{\varepsilon}|^2 \, dz \leq \frac{c}{\mu^{p-2}} \left[ \int_{\tilde{\Omega}_T} (1 + |Du_{\varepsilon}|^2)^{\frac{q}{2}} \, dz \right]^{\frac{2q-p}{q}} \leq c_2,$$

for a constant $c_2$ admitting the same dependencies as $c_1$.

### 6.4. Passing to the limit

Altogether we have shown that the sequence $(u_{\varepsilon})_{\varepsilon \in (0, 1]}$ is a bounded sequence in $L^p(0, T; W^{1, p}(\Omega))$, and also in $L^2_{\text{loc}}(0, T; W^{2, 2}_{\text{loc}}(\Omega))$ if $\mu > 0$ or $p = 2$. Moreover, $Du_{\varepsilon}$ is a bounded sequence in $L^\infty_{\text{loc}}(\Omega_T, \mathbb{R}^n)$. Thus we can conclude, passing to a subsequence, that there exists $u \in L^p(0, T; W^{1, p}(\Omega))$ with $Du \in L^\infty_{\text{loc}}(\Omega_T, \mathbb{R}^n)$ such that for any $\tilde{\Omega} \Subset \Omega$ and $0 < t_1 < t_2 < T$ there holds

$$\left\{ \begin{array}{ll}
    u_{\varepsilon} \rightharpoonup u & \text{weakly in } L^p(0, T; W^{1, p}(\Omega)), \\
    Du_{\varepsilon} \rightharpoonup^* Du & \text{weakly* in } L^\infty(\tilde{\Omega} \times (t_1, t_2), \mathbb{R}^n). 
\end{array} \right.$$

This implies in particular that $u = g$ on the lateral boundary of $\Omega_T$. By (6.4) we also have for any $\tilde{\Omega} \Subset \Omega$ and $0 < t_1 < t_2 < T$ that there holds

$$\sup_{\tilde{\Omega} \times (t_1, t_2)} |Du| \leq c_1,$$

where $c_1$ depends on $\tilde{\Omega}, t_1, t_2$ as described in (6.4). If $\mu > 0$ or $p = 2$ we furthermore have $u \in L^2_{\text{loc}}(0, T; W^{2, 2}_{\text{loc}}(\Omega))$. 
To proceed further we need to show that passing possibly to a further subsequence, which we still denote by \( \varepsilon_i \), that \( u_{\varepsilon_i} \to u \) strongly in \( L^p_{\text{loc}}(\Omega_T) \). We fix \( \Omega \subset \Omega \) and \( 0 < t_1 < t_2 < T \) and consider in the weak formulation of \((6.2)\)_1 testing functions \( \varphi \in C_0^\infty(\tilde{\Omega} \times (t_1, t_2)) \). This yields

\[
\left| \int_{\Omega_T} u_{\varepsilon} \partial_t \varphi \, dz \right| \leq \left| \int_{\Omega_T} (a_t(x, t, Du_{\varepsilon}), D\varphi) \, dz \right|
\leq (L + 1) \int_{\Omega_T} (1 + |Du_{\varepsilon}|^2)^{\frac{q-1}{2}} |D\varphi| \, dz
\leq (L + 1) |\text{spt} \varphi| \sup_{\tilde{\Omega} \times (t_1, t_2)} (1 + |Du_{\varepsilon}|^2)^{\frac{q-1}{2}} \|D\varphi\|_{L^\infty(\tilde{\Omega} \times (t_1, t_2))}
\leq c |\text{spt} \varphi| \|D\varphi\|_{L^\infty(\tilde{\Omega} \times (t_1, t_2))}
\]

for a constant \( c \) depending on \( n, p, q, \nu, \lambda, \text{dist}(\tilde{\Omega}, \partial \Omega), t_1, T - t_2, \|Dg\|_{L^r(\Omega_T)} \) and \( \|g_t\|_{L^p(0, T; \mathbb{W}^{-1, p'}(\Omega))} \). Now, for \( t_1 < s_1 < s_2 < t_2 \) and \( h > 0 \) small enough we choose

\[
\chi_h(t) := \begin{cases} 0, & \text{for } t_1 \leq t \leq s_1 - h, \\ \frac{1}{h}(t - s_1 + h), & \text{for } s_1 - h \leq t \leq s_1, \\ 1, & \text{for } s_1 \leq t \leq s_2, \\ -\frac{1}{h}(t - s_2 - h), & \text{for } s_2 \leq t \leq s_2 + h, \\ 0, & \text{for } s_2 + h \leq t \leq t_2, \end{cases}
\]

and let \( \varphi(x, t) := \chi_h(t)\varphi(x) \) with \( \varphi \in C_0^\infty(\tilde{\Omega}) \). Testing \((6.7)\) with \( \varphi \) (note that this is possible since \((6.7)\) can be seen to hold for any \( \varphi \in W^{1, \infty}_0(\tilde{\Omega} \times (t_1, t_2)) \) by an approximation argument) we obtain

\[
\left| \int_{\tilde{\Omega}} \frac{1}{h} \left( \int_{s_1 - h}^{s_1} u_{\varepsilon}(x, t) \, dt - \int_{s_2}^{s_2 + h} u_{\varepsilon}(x, t) \, dt \right) \psi(x) \, dx \right| \leq c(s_2 - s_1 + 2h) \|D\varphi\|_{L^\infty(\tilde{\Omega})}
\]

for any \( \varphi \in C_0^\infty(\tilde{\Omega}) \). Now, with \( \ell > \frac{n+2}{2} \) the Sobolev inequality yields

\[
\|D\varphi\|_{L^\infty(\tilde{\Omega})} \leq c(n, \ell, \tilde{\Omega}) \|\varphi\|_{W^{\ell, 2}(\tilde{\Omega})},
\]

provided \( \tilde{\Omega} \) is smooth enough. Using this above we arrive at

\[
\left| \int_{\tilde{\Omega}} \frac{1}{h} \left( \int_{s_1 - h}^{s_1} u_{\varepsilon}(x, t) \, dt - \int_{s_2}^{s_2 + h} u_{\varepsilon}(x, t) \, dt \right) \psi(x) \, dx \right| \leq c(s_2 - s_1 + 2h) \|\varphi\|_{W^{\ell, 2}(\tilde{\Omega})},
\]

Here, we pass to the limit \( h \downarrow 0 \) and obtain for almost every \( t_1 < s_1 < s_2 < t_2 \) that

\[
\left| \int_{\tilde{\Omega}} (u_{\varepsilon}(x, s_1) - u_{\varepsilon}(x, s_2)) \psi(x) \, dx \right| \leq c(s_2 - s_1) \|\psi\|_{W^{\ell, 2}(\tilde{\Omega})}
\]

whenever \( \psi \in C_0^\infty(\tilde{\Omega}) \). By the density of \( C_0^\infty(\tilde{\Omega}) \) in \( W^{\ell, 2}(\tilde{\Omega}) \) the last inequality continues to hold also for any \( \psi \in W^{\ell, 2}_0(\tilde{\Omega}) \). At this stage we take the supremum over all \( \psi \in W^{\ell, 2}_0(\tilde{\Omega}) \) with \( \|\psi\|_{W^{\ell, 2}(\tilde{\Omega})} \leq 1 \). This gives

\[
\|u_{\varepsilon}(\cdot, s_1) - u_{\varepsilon}(\cdot, s_2)\|_{W^{-\ell, 2}(\tilde{\Omega})} \leq c|s_1 - s_2|,
\]

for almost every \( s_1, s_2 \in (t_1, t_2) \). At this stage we can apply \([25, \text{Theorem 5}]\) with \( X = W^{1,p}(\tilde{\Omega}), B = L^p(\tilde{\Omega}), Y = W^{-\ell, 2}(\tilde{\Omega}) \) to the sequence \((u_{\varepsilon_i})_{i \in \mathbb{N}}\) to conclude that the sequence \((u_{\varepsilon_i})_{i \in \mathbb{N}}\) is relatively compact in \( L^p(\tilde{\Omega} \times (t_1, t_2)) \). But this means that there exists a further subsequence, still denoted by \( \varepsilon_i \), such that \( u_{\varepsilon_i} \to u \) strongly in \( L^p(\tilde{\Omega} \times (t_1, t_2)) \) and since \( \tilde{\Omega} \subset \Omega \) and \( 0 < t_1 < t_2 < T \) this proves the claim \( u_{\varepsilon_i} \to u \) strongly in \( L^p_{\text{loc}}(\Omega_T) \).
Our next aim is to prove that $Du_{\epsilon} \to Du$ strongly in $L^p_{\text{loc}}(\Omega_T, \mathbb{R}^n)$. For this aim we first subtract the weak form of (6.2) and (1.5). This yields

$$
\int_{\Omega_T} [(u_e - u)\partial_t \varphi - \{a_e(\cdot, Du_e) - a(\cdot, Du), D\varphi\}] \, dz = 0 \quad \forall \varphi \in C^\infty_0(\Omega_T).
$$

(6.9)

In (6.9) we choose the test-function $\varphi = \psi(u_e - u)$, where $\psi \in C^\infty_0(\Omega_T)$ is a cut-off function with $\psi \geq 0$. Note that this choice is possible, after an approximation argument, since $Du_{\epsilon}, Du \in L^\infty_{\text{loc}}(\Omega_T, \mathbb{R}^n)$. Then, the first term in (6.9) turns into

$$
\int_{\Omega_T} (u_e - u)\partial_t \varphi \, dz = \int_{\Omega_T} \frac{1}{2} \partial_t |u_e - u|^2 \psi + |u_e - u|^2 \partial_t \psi \, dz
$$

$$
= \int_{\Omega_T} \frac{1}{2} \partial_t \left(|u_e - u|^2 \psi\right) + \frac{1}{2} |u_e - u|^2 \partial_t \psi \, dz
$$

$$
= \frac{1}{2} \int_{\Omega_T} |u_e - u|^2 \partial_t \psi \, dz.
$$

The second term in (6.9) we decompose as follows:

$$
\int_{\Omega_T} \langle a_e(\cdot, Du_e) - a(\cdot, Du), D\varphi \rangle \, dz
$$

$$
= \int_{\Omega_T} \langle a(\cdot, Du_e) - a(\cdot, Du), Du_e - Du \rangle \psi \, dz \quad (=: \text{I})
$$

$$
+ \int_{\Omega_T} \langle a(\cdot, Du_e) - a(\cdot, Du), D\psi \otimes (u_e - u) \rangle \, dz \quad (=: \text{II})
$$

$$
+ \epsilon \int_{\Omega_T} (1 + |Du_e|^2)^{\frac{p-2}{2}} Du_e \cdot D\varphi \, dz. \quad (=: \text{III})
$$

We now rewrite

$$
a(\cdot, Du_e) - a(\cdot, Du) = \int_0^1 \partial_w a(\cdot, Du + w(Du_e - Du)) \, dw(Du_e - Du).
$$

This together with the ellipticity bound (1.2) yields on the one hand the following bound for I from below:

$$
I \geq v \int_{\Omega_T} \left(\mu^2 + |Du + s(Du_e - Du)|^2\right)^{\frac{p-2}{2}} |Du_e - Du|^2 \psi \, dz
$$

$$
\geq \frac{v}{c(p)} \int_{\Omega_T} \left(\mu^2 + |Du|^2 + |Du_e - Du|^2\right)^{\frac{p-2}{2}} |Du_e - Du|^2 \psi \, dz
$$

$$
\geq \frac{v}{c(p)} \int_{\Omega_T} |Du_e - Du|^p \psi \, dz,
$$

and on the other hand, using the growth (1.2) instead of (1.2) we get
\[ |II| \leq 2L \int_{\Omega_T} \left( \mu^2 + |Du + s(Du_{\varepsilon} - Du)|^2 \right)^{\frac{q-2}{2}} ds |Du_{\varepsilon} - Du||u_{\varepsilon} - u||D\psi| \, dz \]
\[ \leq 2L \|D\psi\|_{L^\infty} \sup_{\text{spt } \psi} (1 + |Du| + |Du_{\varepsilon}|)^{q-1} \int_{\text{spt } \psi} |u_{\varepsilon} - u| \, dz. \]

Finally, the term III is estimated as follows:
\[ |III| \leq \varepsilon \int_{\Omega_T} \left( 1 + |Du_{\varepsilon}|^2 \right)^{\frac{q-2}{2}} Du_{\varepsilon} (|Du_{\varepsilon} - Du||\psi + |u_{\varepsilon} - u||D\psi|) \, dz \]
\[ \leq \varepsilon c (\|\psi\|_{L^\infty}, \|D\psi\|_{L^\infty}) \sup_{\text{spt } \psi} (1 + |Du| + |Du_{\varepsilon}|)^q \int_{\text{spt } \psi} (1 + |u_{\varepsilon} - u|) \, dz. \]

Combining the preceding estimates with (6.9) and recalling (6.4) as well as (6.6) we arrive at
\[ \int_{\Omega_T} \left| Du_{\varepsilon} - Du \right|^p \psi^2 \, dz \leq c_3 \int_{\text{spt } \psi} |u_{\varepsilon} - u| + |u_{\varepsilon} - u|^2 \, dz + \varepsilon c_3 \|\text{spt } \psi\|, \]

for a constant \( c_3 \) depending on \( n, \nu, L, p, q, \|Dg\|_{L^r(\Omega_T)}, \|g\|_{L^p(\Omega_T)}, \|D\psi\|_{L^\infty}, \|\partial_t \psi\|_{L^\infty}, \text{dist}(\text{spt } \psi, \partial \Omega_T), \) but independent of \( \varepsilon \). Since we have \( u_{\varepsilon} \rightarrow u \) strongly in \( L^p_{\text{loc}}(\Omega_T) \) and since \( \psi \in C_0^\infty(\Omega_T) \) with \( \psi \geq 0 \) is arbitrary, this implies the claim \( Du_{\varepsilon} \rightarrow Du \) strongly in \( L^p(\Omega_T, \mathbb{R}^n) \) and almost everywhere in \( Q_0 \) for any \( Q_0 \subseteq \Omega_T \). This allows us to pass to the limit in the weak form of (6.2), that is
\[ \int_{\Omega_T} \left( u_{\varepsilon, \partial_t \psi} - \left( a_{\varepsilon} (\cdot, Du_{\varepsilon}), D\psi \right) \right) \, dz = 0 \quad \forall \psi \in C_0^\infty(\Omega_T). \]

This can easily be inferred as follows: since \( |Du_{\varepsilon}(x, t)| \) is bounded independently of \( \varepsilon \) and \( (x, t) \in \text{spt } \psi \) by (6.4) we have that \( |a_{\varepsilon}(x, t, Du_{\varepsilon})| \) is also bounded independently of \( \varepsilon \) and \( (x, t) \). Moreover the almost everywhere convergence \( Du_{\varepsilon}(x, t) \rightarrow Du(x, t) \) in \( \text{spt } \psi \) and the convergence \( a_{\varepsilon}(x, t, w) \rightarrow a(x, t, w) \) imply that \( a_{\varepsilon}(x, t, Du_{\varepsilon}(x, t)) \rightarrow a(x, t, Du(x, t)) \) for almost every \( (x, t) \in \text{spt } \psi \). Therefore the dominated convergence theorem yields the convergence of the second integral. For the convergence of the first integral we recall that \( u_{\varepsilon} \rightarrow u \) strongly in \( L^2(\text{spt } \psi) \). Hence, we conclude that there holds
\[ \int_{\Omega_T} \left( u_{\partial_t \psi} - \left( a(\cdot, Du), D\psi \right) \right) \, dz = 0 \quad \forall \psi \in C_0^\infty(\Omega_T). \]

Moreover, from the almost everywhere convergence \( Du_{\varepsilon}(x, t) \rightarrow Du(x, t) \) in \( \text{spt } \psi \subseteq \Omega_T \) and Theorem 5.1 we conclude that for any cylinder \( Q_0(\varepsilon_0) \subseteq \Omega_T \) and any \( s \in (0, 1) \) there holds
\[ \sup_{Q_0(\varepsilon_0)} |Du| \leq \liminf_{i \rightarrow \infty} \sup_{Q_0(\varepsilon_0)} |Du_{\varepsilon_i}| \]
\[ \leq c \liminf_{i \rightarrow \infty} \left[ \frac{1}{(1 - s)^{n+2}} \int_{Q_0(\varepsilon_0)} (1 + |Du_{\varepsilon}|^2)^{\frac{q}{2}} \, dz \right]^{\frac{1}{p}} \left[ \frac{2p}{q(n+2)(q-p)} \right]^{\frac{1}{p}} \]
\[ = c \left[ \frac{1}{(1 - s)^{n+2}} \int_{Q_0(\varepsilon_0)} (1 + |Du|^2)^{\frac{q}{2}} \, dz \right]^{\frac{1}{p}} \left[ \frac{2p}{q(n+2)(q-p)} \right]^{\frac{1}{p}}. \]

Taking into account that in the case where \( \mu > 0 \) or \( p = 2 \) – possibly after passing to a further subsequence – we have \( D^2 u_{\varepsilon_i} \rightarrow D^2 u \) weakly in \( L^2(Q_0(\varepsilon_0), \mathbb{R}^{n \times n}) \) and using (6.5) instead of Theorem 5.1 we conclude by a similar reasoning that
\[ \int_{Q_{\frac{\rho}{2}}(z_0)} |D^2 u|^2 \, dz \leq \liminf_{i \to \infty} \int_{Q_{\frac{\rho}{2}}(z_0)} |D^2 u_{\epsilon_i}|^2 \, dz \]
\[ \leq c \frac{Q_{h_0}}{\frac{\rho}{2} - 2} \left( \frac{1}{Q_{\frac{\rho}{2}}(z_0)} \int (1 + |Du|)^{\frac{q}{2}} \, dz \right)^{1 + \frac{2(q-p)}{4-n}} \]
\[ = c \frac{Q_{h_0}}{\frac{\rho}{2} - 2} \left( \frac{1}{Q_{\frac{\rho}{2}}(z_0)} \int (1 + |Du|^2)^{\frac{q}{2}} \, dz \right)^{1 + \frac{2(q-p)}{4-n}} \cdot \]

again with a constant \( c = c(\hat{n}, p, q, \nu, L) \). We divide both sides by \( \rho_{\frac{\rho}{2} + 2} \) and obtain the desired estimate.

### 6.5. The initial boundary condition

Our next aim is to prove that \( u \) satisfies the initial boundary condition \( u(\cdot, 0) = g(\cdot, 0) \) in the sense of (1.10). For this we again take the energy estimate (6.3) as a starting point. Possibly passing to a further subsequence we may assume that \( u_{\epsilon_i} \rightharpoonup u \) weakly* in \( L^\infty(0, T; L^2(\Omega)) \). Therefore, using (6.3) we infer for any \( 0 < h < T \) that
\[ \sup_{0 < t < h} \int_{\Omega} |u - g|^2(\cdot, t) \, dx \leq \liminf_{i \to \infty} \left[ \sup_{0 < t < h} \int_{\Omega} |u_{\epsilon_i} - g|^2(\cdot, t) \, dx \right] \]
\[ \leq c \int_{\Omega_h} (1 + |Dg|)^{\frac{q}{2}} \, dz + c \| \partial_t g \|_{L^{\frac{q}{2}}(0, h; W^{-1, \frac{q}{2}}(\Omega))}, \]

which implies (1.10) since \( g \in C^0([0, T]; L^2(\Omega)) \).

### 6.6. Regularity of \( u_t \)

To prove some global regularity of \( u \) in time we let \( \varphi \in C_0^\infty(\Omega_T) \). Then, since \( q - 1 < p + \frac{4}{n+2} - 1 \leq p \) by Hölder’s inequality we get
\[ \left| \int_{\Omega_T} u \partial_t \varphi \, dz \right| = \left| \int_{\Omega_T} [a(\cdot, Du), D\varphi] \, dz \right| \]
\[ \leq L \int_{\Omega_T} (1 + |Du|)^{\frac{q}{2}} |D\varphi| \, dz \]
\[ \leq L \left( \int_{\Omega_T} (1 + |Du|^2)^{\frac{q}{2}} \, dz \right)^{\frac{q-1}{p}} \| D\varphi \|_{L^{\frac{p}{4-n}}(\Omega_T)}. \]

By the density of \( C_0^\infty(\Omega_T) \) in \( L^{\frac{p}{1+p-q}}(0, T; W^{1, \frac{p}{1+p-q}}(\Omega)) \) and since \( \left( \frac{p}{1+p-q} \right)' = \frac{p}{q-1} \) we conclude from the preceding inequality that
\[ u_t \in L^{\frac{p}{q-1}}(0, T; W^{-1, \frac{p}{q-1}}(\Omega)). \]

This concludes the proof of Theorem 1.6. \( \Box \)

### References


