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On the total variation of the Jacobian

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Abstract

A characterization of the total variation $TV(u,\Omega)$ of the Jacobian determinant $\det Du$ is obtained for some classes of functions $u:\Omega\subset\mathbb{R}^2\to\mathbb{R}^2$ outside the traditional regularity space $W^{1,2}(\Omega;\mathbb{R}^2)$. In particular, explicit formulas are deduced for functions that are locally Lipschitz continuous away from a given one point singularity $x_0\in\Omega$, i.e., $u\in W^{1,p}(\Omega;\mathbb{R}^2)\cap W^{1,\infty}(\Omega\setminus\{x_0\};\mathbb{R}^2)$ for some $p>1$.

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1. Introduction

Well-established theories in the calculus of variations and in partial differential equations have been challenged in recent years by new phenomena in solid physics and in materials sciences which demand innovative approaches and new ideas. In this paper, we address the study of the *Jacobian determinant* $\det Du$ of fields $u:\Omega\to\mathbb{R}^n$ outside the traditional regularity space $W^{1,n}(\Omega;\mathbb{R}^n)$, where $\Omega\subset\mathbb{R}^n$ is an open, bounded set. The analysis will be mostly centered on the plane, i.e., $n=2$,

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where more general results may be obtained and the arguments of the proofs are geometrically more intuitive. A forthcoming paper [14] will address the n-dimensional setting.

The role of the distributional determinant in the study of harmonic mappings with singularities was first identified by Brezis et al. in a seminal paper (see [7]) that paved the way for a wealth of developments in the subject, with relevance in many areas of applications such as the study of liquid crystals and Ginzburg–Landau type theories. In [7,8], the authors bridge the notion of topological degree to the appearance of Dirac measures as singular parts of the underlying generalized, measure-valued determinant.

To fix the notations, we consider a map $u:\Omega\subset\mathbb{R}^2\to\mathbb{R}^2$, defined in an open set Ω of $\mathbb{R}^2$. If $u\in W^{1,2}(\Omega;\mathbb{R}^2)$, since $|\det Du(x)|\leqslant 1/2|Du(x)|^2$, then the Jacobian determinant $\det Du$ is a function of class $L^1(\Omega)$. When $u\notin W^{1,2}(\Omega;\mathbb{R}^2)$, it may still be possible to consider the *distributional Jacobian determinant*

$$\operatorname{Det} Du := \frac{\partial}{\partial x_1}\left(u^1\frac{\partial u^2}{\partial x_2}\right)-\frac{\partial}{\partial x_2}\left(u^1\frac{\partial u^2}{\partial x_1}\right). \tag{1}$$

An equivalent definition may be obtained by interchanging the roles of u^1 and u^2 with signs reversed accordingly. The definition of the distributional Jacobian determinant $\operatorname{Det} Du$ is based on integration by parts of the formal expression in (1), after multiplication by a test function. To render definition (1) mathematically precise it is necessary to make some assumptions on u. We may take u^1 to be bounded and the gradient Du to be of class L^1, i.e., $u\in L^\infty(\Omega;\mathbb{R}^2)\cap W^{1,1}(\Omega;\mathbb{R}^2)$. Another possibility is to require that $u\in W^{1,p}(\Omega;\mathbb{R}^2)$ for some $p>\frac{4}{3}$. In fact, in this case by the Sobolev Imbedding Theorem we have $u\in L^4(\Omega;\mathbb{R}^2)$ and the products in (1) are well defined in L^1. In this paper, we assume that $u\in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^2)\cap W^{1,p}(\Omega;\mathbb{R}^2)$ for some $p>1$, and we focus in particular on those maps u which are locally Lipschitz-continuous away from a given point $x_0\in\Omega$ (and thus the Jacobian determinant $\det Du$ may only be singular at x_0). These maps were treated also in the book by Bethuel et al. (see [5]) where a detailed study of one-point singularities (vortices) of stationary solutions for complex-valued Ginzburg–Landau equations may be found. Here, again, the notion of topological degree comes into play. Along the same lines, we refer also to the study of density results of smooth functions in $H^1(B(0,1);S^2)$, where $B(0,1)\subset\mathbb{R}^3$. Bethuel [4] showed that this density result holds for $u\in H^1(B(0,1);S^2)$ if $\det Du=0$.

Since the fundamental work of Morrey [24], who treated weak continuity properties of $\operatorname{Det} Du$ in (1) (see also [26]), $\operatorname{Det} Du$ has played a pivotal role in the *calculus of variations* (see, also [3,25]). In recent years, several attempts have been made to establish relations between $\operatorname{Det} Du$ and the *total variation* of the Jacobian determinant $\det Du(x)$. One possible definition for the latter is based on the following limit formula: given $u\in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^2)\cap W^{1,p}(\Omega;\mathbb{R}^2)$ for some $p>1$, the

total variation $\mathrm{TV}(u,\Omega)$ *of the Jacobian determinant* is defined by

$$\mathrm{TV}(u,\Omega)=\inf\left\{\liminf_{h\to+\infty}\int_\Omega |\det Du_h(x)|\,dx:\right.$$
$$\left. u_h\rightharpoonup u \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^2),\ u_h\in W^{1,2}(\Omega;\mathbb{R}^2)\right\}. \tag{2}$$

Note that, a priori, definition (2) may depend on p and, more precisely, we should use the notation $\mathrm{TV}_p(u,\Omega)$ instead of $\mathrm{TV}(u,\Omega)$. However, the representation formulas for $\mathrm{TV}(u,\Omega)$ given in this paper turn out to be independent of p, and, surprisingly, it can be shown that *weak* convergence in $W^{1,p}(\Omega;\mathbb{R}^2)$ may be equivalently replaced by *strong* convergence in $W^{1,p}(\Omega;\mathbb{R}^2)$ for certain classes of functions u. This approach has been considered by Marcellini [23], Giaquinta et al. [18,19], Fonseca and Marcellini [16], Bouchitté et al. [6], among others. In particular, Marcellini [23] and Fonseca and Marcellini [16] noticed that the total variation of the Jacobian determinant may have a nonzero singular part, while Bouchitté et al. [6] proved that this singular part is a measure. Giaquinta et al. [18,19] found that the lower limit in (2) can be different from the total variation of the measure Det Du.

It has been first noted by Malý [21] and by Giaquinta et al. [18] (see also [20]) that, for some maps $u\in L^\infty(\Omega;\mathbb{R}^2)\cap W^{1,p}(\Omega;\mathbb{R}^2)$ with $p\in(1,2)$, it may happen that the distribution Det Du is *identically equal* to zero while the *total variation* of the Jacobian determinant is *different* from zero. Also, when Det Du is a *measure*, it turns out that, in general, $\mathrm{TV}(u,\Omega)$ *is not* the total variation of the measure Det Du. Some examples illustrating this phenomenon may found in Section 3.

In this paper, we give an explicit characterization of the total variation $\mathrm{TV}(u,\Omega)$ of the Jacobian determinant for maps u as described above (see Theorem 1). We relate the total variation of the Jacobian determinant $|\mathrm{Det}\,Du|(\Omega)$ to $\mathrm{TV}(u,\Omega)$, and, in turn, $\mathrm{TV}(u,\Omega)$ is expressed in terms of the *topological degree* (see Remark 3). In particular, denoting by B_1 the unit ball of $\mathbb{R}^2$ and by $S^1:=\partial B_1$ its boundary, we prove that, if $v:S^1\to S^1$ is a map of class C^1 onto S^1, locally invertible with local inverse of class C^1 at any point of S^1, and if $u:B_1\backslash\{0\}\to S^1$ is defined by $u(x):=v\left(\frac{x}{|x|}\right)$, then the total variation $\mathrm{TV}(u,B_1)$ may be expressed in terms of the *topological degree* of the maps v and $\tilde v$, where $\tilde v:B_1\to B_1$ is any *Lipschitz-continuous extension* of v to the unit ball B_1. Precisely,

$$\mathrm{TV}(u,B_1)=\omega_2|\deg v|=\omega_2|\deg\tilde v|. \tag{3}$$

Note that formula (3) *does not hold*, in general, if the map $v:S^1\to\mathbb{R}^2$ takes values on a set $v(S^1)$ not diffeomorphic to S^1 (see Theorem 12 and the examples of Section 3).

2. Maps with values in a curve diffeomorphic to S^1

In the sequel Ω is an open, bounded subset of $\mathbb{R}^2$, we denote by $v:[0,2\pi]\to\Gamma\subset\mathbb{R}^2$ a Lipschitz-continuous map with values on a curve Γ, with $v(0)=v(2\pi)$, and with

components $v(\vartheta)=(v^1(\vartheta),v^2(\vartheta))$. We assume that Γ may be parametrized as

$$\Gamma := \{\xi + r(\vartheta)(\cos\vartheta,\sin\vartheta) : \vartheta\in[0,2\pi]\}, \tag{4}$$

where $r(\vartheta)$ is a periodic piecewise C^1-function such that $r(0)=r(2\pi)$, and $r(\vartheta)\geqslant r_0$ for every $\vartheta\in[0,2\pi]$ and for some $r_0>0$. Condition (4) reduces to saying that Γ is the boundary of a domain

$$D := \{\xi + \varrho(\cos\vartheta,\sin\vartheta) : \vartheta\in[0,2\pi],\ 0\leqslant\varrho\leqslant r(\vartheta)\}, \tag{5}$$

starshaped with respect to a point ξ in the *interior* of D. We have the following result.

Theorem 1. *Let u be a function of class $W^{1,p}(\Omega;\mathbb{R}^2)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega\backslash\{0\};\mathbb{R}^2)$ for some $p\in(1,2)$. Let $v:[0,2\pi]\to\Gamma$, $v(\vartheta)=(v^1(\vartheta),v^2(\vartheta))$, $\vartheta\in[0,2\pi]$, be a Lipschitz-continuous map, with $v(0)=v(2\pi)$ and Γ as in* (4)*, and such that*

$$\lim_{\varrho\to 0}\ ||u(\varrho,\cdot)-v(\cdot)||_{L^\infty((0,2\pi);\mathbb{R}^2)}=0. \tag{6}$$

If the tangential derivative $D_\tau u$ of u satisfies the bound

$$\sup_{\varrho>0}\frac{1}{\varrho^{2-p}}\int_{B_\varrho}|D_\tau u|^p\,dx=\sup_{\varrho>0}\frac{1}{\varrho^{2-p}}\int_0^\varrho r^{1-p}\,dr\int_0^{2\pi}|u_\vartheta(r,\vartheta)|^p\,d\vartheta\leqslant M_0 \tag{7}$$

for some positive constant M_0, then the total variation of u is given by

$$\mathrm{TV}(u,\Omega)=\int_\Omega|\det Du(x)|\,dx+\frac{1}{2}\left|\int_0^{2\pi}\{v^1(\vartheta)v^2_\vartheta(\vartheta)-v^2(\vartheta)v^1_\vartheta(\vartheta)\}\,d\vartheta\right|. \tag{8}$$

In order to illustrate a class of functions squarely fitting into the hypotheses of Theorem 1, consider the particular case in which the map $u=u(\varrho,\vartheta)$ does not depend on ϱ, that is $u=u(\vartheta)$. Then, as a function of ϑ, $u=u(\vartheta):[0,2\pi]\to\mathbb{R}^2$ is a Lipschitz-continuous map and $u(0)=u(2\pi)$. Considered as a function of two variables, i.e., $u:\Omega=B_1\to\mathbb{R}^2$ constant with respect to $\varrho\in(0,1]$, it turns out that $u\in L^\infty(\Omega;\mathbb{R}^2)\cap W^{1,p}(\Omega;\mathbb{R}^2)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega\backslash\{0\};\mathbb{R}^2)$ for every $p\in[1,2)$, but $u\notin W^{1,2}(\Omega;\mathbb{R}^2)$ unless $u(\vartheta)$ is constant. From the previous result, with $u=v$, we immediately obtain the following consequence.

Corollary 2. *Let Γ be as in* (4)*, and let $u=v:[0,2\pi]\to\Gamma$ be a Lipschitz-continuous map such that $v(0)=v(2\pi)$. Then* $\det Du(x)=0$ *for almost every $x\in\mathbb{R}^2$ and the total variation of the Jacobian determinant is given by*

$$\mathrm{TV}(u,\Omega)=\frac{1}{2}\left|\int_0^{2\pi}\{v^1(\vartheta)v^2_\vartheta(\vartheta)-v^2(\vartheta)v^1_\vartheta(\vartheta)\}\,d\vartheta\right|. \tag{9}$$

Remark 3. (i) We observe that formula (9) has a relevant geometrical meaning. In fact, the right-hand side is equal to π times the "winding number" of the curve $v = (v^1, v^2)$, i.e., $\pi|\deg v|$.

(ii) A careful scrutiny of the proof of Theorem 1 yields easily that an analog result stiff holds if we assume that u is in $W^{1,p}(\Omega;\mathbb{R}^2)$ for some $p\in(1,2)$ and is locally Lipschitz outside a finite number of points $a_i\in\Omega$, $i=1,\dots,N$, provided that u satisfies in a neighborhood of each a_i both assumptions (7) and (6) for suitable functions v_i. In this case, the total variation of the Jacobian of u is given by (see [15])

$$\mathrm{TV}(u,\Omega) = \int_\Omega |\det Du(x)|\,dx + \sum_{i=1}^{N} \pi|\deg v_i|.$$

For possible extensions of this formula to more general spaces we refer to [9,10].

(iii) A suitable adaptation of the proof enables us to replace the assumption $u\in W^{1,\infty}_{\mathrm{loc}}(\Omega\backslash\{0\};\mathbb{R}^2)$ by the weaker condition $u\in W^{1,2}_{\mathrm{loc}}(\Omega\backslash\{0\};\mathbb{R}^2)$. On the other hand, the requirement that $p>1$ is essential since our proof of the lower estimate for $\mathrm{TV}(u,\Omega)$ does not work in the case $p=1$.

(iv) It is also interesting to observe that, under the assumptions of Theorem 1, the distributional determinant $\mathrm{Det}\,Du$ is a measure and that its total variation $|\mathrm{Det}\,Du|(\Omega)$ coincides with $\mathrm{TV}(u,\Omega)$. This property is proved, in a more general context, in the forthcoming paper [14]. However, the equality between the total variation of $\mathrm{Det}\,Du$ and $\mathrm{TV}(u,\Omega)$ is due to the fact that here we are essentially dealing with maps valued into S^1. The examples given at the end of Section 3 show that this equality may no longer be true otherwise, and in particular this equality fails for maps valued on the "eight" curve.

The remainder of this section is devoted to the proof of Theorem 1 and Corollary 2.

For every $\xi = (\xi^1,\xi^2)\in\mathbb{R}^2$, $\xi\neq 0$, we denote by $\mathrm{Arg}\,\xi$ the unique angle in $[-\pi,\pi)$ such that

$$\cos\mathrm{Arg}\,\xi = \frac{\xi^1}{|\xi|}, \quad \sin\mathrm{Arg}\,\xi = \frac{\xi^2}{|\xi|}.$$

By B_r, we denote the circle in $\mathbb{R}^2$ with center in 0 and radius $r>0$, B_1 is the circle of radius $r=1$, and $\partial B_1 = S^1$ is its boundary. If $\alpha,\beta\in[0,2\pi]$, $\alpha<\beta$, then $S(\alpha,\beta)$ represents the *polar sector*

$$S(\alpha,\beta) := \{\xi = \varrho(\cos\vartheta,\sin\vartheta)\in\mathbb{R}^2 : \quad \varrho\leqslant 1, \vartheta\in[\alpha,\beta]\}.$$

Let $v:[0,2\pi]\to\mathbb{R}^2$ be a Lipschitz-continuous *closed* curve, i.e., $v(0)=v(2\pi)$. We denote by $v_\vartheta := (v^1_\vartheta, v^2_\vartheta)$ the gradient of v, which exists for almost every $\vartheta\in[0,2\pi]$, and

if $v(\vartheta)\neq 0$ for every $\vartheta\in[0,2\pi]$, then $A_v(\vartheta)$ stands for

$$A_v(\vartheta) := \operatorname{Arg} v(0) + \int_0^{\vartheta} \frac{v^1(t)v_\vartheta^2(t) - v^2(t)v_\vartheta^1(t)}{|v(t)|^2}\,dt.$$

There exists a simple relation between A_v and $\operatorname{Arg} v$, which may be inferred from the next lemma. Its simple proof is omitted.

Lemma 4. *If $v:[0,2\pi]\to\mathbb{R}^2$ is a Lipschitz-continuous curve such that $v(\vartheta)\neq 0$ for every $\vartheta\in[0,2\pi]$, then for every $\alpha,\beta\in[0,2\pi]$ with $\alpha<\beta$, there exists $k\in\mathbb{Z}$ such that*

$$A_v(\beta) - A_v(\alpha) = \operatorname{Arg} v(\beta) - \operatorname{Arg} v(\alpha) + 2k\pi. \tag{10}$$

Lemma 5. *Let Γ be as in* (4) *and let $v:[0,2\pi]\to\Gamma$ be a Lipschitz-continuous map. If $\operatorname{Arg}(v(0)-\xi)=0$ then the curve v may be represented as*

$$v(\vartheta) := \xi + r(A_{v-\xi}(\vartheta))(\cos A_{v-\xi}(\vartheta), \sin A_{v-\xi}(\vartheta)) \tag{11}$$

for all $\vartheta\in[0,2\pi]$.

Proof. Since $A_{v-\xi}(0) = \operatorname{Arg}(v(0)-\xi) = 0$, by Lemma 4 for every $\vartheta\in[0,2\pi]$ there exists $k\in\mathbb{Z}$ such that $A_{v-\xi}(\vartheta) = \operatorname{Arg}(v(\vartheta)-\xi) + 2k\pi$. Also, as $v(\vartheta)\in\Gamma$ for all ϑ, we have

$$r(A_{v-\xi}(\vartheta)) = r(\operatorname{Arg}(v(\vartheta)-\xi)) = |v(\vartheta)-\xi|,$$

and we obtain

$$\begin{aligned} &r(A_{v-\xi}(\vartheta))(\cos A_{v-\xi}(\vartheta), \sin A_{v-\xi}(\vartheta)) \\ &\quad = |v(\vartheta)-\xi|(\cos\operatorname{Arg}(v(\vartheta)-\xi), \sin\operatorname{Arg}(v(\vartheta)-\xi)) = v(\vartheta)-\xi. \end{aligned}$$

This concludes the proof. □

The next lemma plays a central role in this section.

Lemma 6 (The "umbrella" lemma). *Let Γ be as in* (4) *and let $v:[0,2\pi]\to\Gamma$ be a Lipschitz-continuous map. If $\alpha,\beta\in[0,2\pi]$, $\alpha<\beta$, are such that $A_{v-\xi}(\alpha) = A_{v-\xi}(\beta)$, then for every $\varepsilon>0$ there exists a Lipschitz-continuous map $w:S(\alpha,\beta)\to\mathbb{R}^2$ satisfying the boundary conditions*

$$\begin{cases} w(1,\vartheta) = v(\vartheta) & \forall\vartheta\in[\alpha,\beta], \\ w(\varrho,\alpha) = w(\varrho,\beta) = \xi + \varrho(v(\alpha)-\xi) & \forall\varrho\in[0,1], \end{cases} \tag{12}$$

and such that

$$\int_{S(\alpha,\beta)} |\det Dw(x)|\,dx < \varepsilon. \tag{13}$$

Proof. Without loss of generality we can assume that $\mathrm{Arg}(v(0)-\xi)=0$. Fix $h\in\mathbb{N}$ and set

$$w_h(\varrho,\vartheta) := \xi + \varrho r(\varphi_h(\varrho,\vartheta))(\cos\varphi_h(\varrho,\vartheta), \sin\varphi_h(\varrho,\vartheta)), \tag{14}$$

where, for every $\varrho\in[0,1]$ and for every $\vartheta\in[\alpha,\beta]$,

$$\varphi_h(\varrho,\vartheta) := \varrho^h A_{v-\xi}(\vartheta) + (1-\varrho^h)A_{v-\xi}(\alpha).$$

Since $\varphi_h(1,\vartheta)=A_{v-\xi}(\vartheta), \varphi_h(\varrho,\alpha)=\varphi_h(\varrho,\beta)=A_{v-\xi}(\alpha)$, by the representation formula (11) of Lemma 5 we obtain the validity of the boundary conditions (12).

Now we evaluate the left-hand side in (13). We observe that, if $u(x)=(u^1(\varrho,\vartheta),u^2(\varrho,\vartheta))$, and using the notation $\frac{\partial u^i}{\partial\varrho}=u^i_\varrho$, $\frac{\partial u^i}{\partial\vartheta}=u^i_\vartheta$ $(i=1,2)$, we have

$$\det Du(x) = \frac{1}{\varrho}\begin{vmatrix} u^1_\varrho(\varrho,\vartheta) & u^1_\vartheta(\varrho,\vartheta) \\ u^2_\varrho(\varrho,\vartheta) & u^2_\vartheta(\varrho,\vartheta)\end{vmatrix}. \tag{15}$$

For the function w_h we obtain

$$\int_{S(\alpha,\beta)} |\det Dw_h(x)|\,dx = \int_0^1 d\varrho \int_\alpha^\beta \left|\frac{\partial(w_h^1,w_h^2)}{\partial(\varrho,\vartheta)}\right| d\vartheta.$$

Now the Jacobian determinant of w_h is

$$\frac{\partial(w_h^1,w_h^2)}{\partial(\varrho,\vartheta)} = \varrho r^2(\varphi_h)\frac{\partial\varphi_h}{\partial\vartheta} = \varrho^{h+1} r^2(\varphi_h) A'_{v-\xi}(\vartheta)$$

and we conclude that

$$\int_{S(\alpha,\beta)} |\det Dw_h(x)|\,dx = \int_0^1 \varrho^{h+1}\,d\varrho \int_\alpha^\beta r^2(\varphi_h)\,|A'_{v-\xi}(\vartheta)|\,d\vartheta \leqslant \frac{c}{h+2},$$

where we denote by c a suitable constant. The conclusion follows by choosing $h\in\mathbb{N}$ sufficiently large. □

The following elementary lemma is stated without proof.

Lemma 7. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, piecewise strictly monotone in $[a,b]$ (with a finite number of monotonicity intervals) and such that $f(a)<f(b)$. Then there exists a partition $a=\alpha_0<\alpha_1<\cdots<\alpha_N=b$ of $[a,b]$ such that, for every $i=1,2,\ldots,N$, either f is strictly increasing in $[\alpha_{i-1},\alpha_i]$, or $f(\alpha_{i-1})=f(\alpha_i)$.*

Lemma 8. *Let $v:[0,2\pi]\to\Gamma$ be a Lipschitz-continuous map, with Γ as in* (4). *Let $\alpha,\beta\in[0,2\pi]$, $\alpha<\beta$, be such that $A_{v-\xi}(\alpha)=A_{v-\xi}(\beta)$. If $A_{v-\xi}(\vartheta)$ is piecewise strictly monotone in $[\alpha,\beta]$ (with a finite number of monotonicity intervals) then*

$$\int_\alpha^\beta \{(v^1(\vartheta)-\xi^1)v^2_\vartheta(\vartheta)-(v^2(\vartheta)-\xi^2)v^1_\vartheta(\vartheta)\}\,d\vartheta=0.$$

Proof. Without loss of generality we may assume that $\xi=(0,0)$. Since $A_v(\vartheta)$ is piecewise strictly monotone in $[\alpha,\beta]$ and $A_v(\alpha)=A_v(\beta)$, there exists a partition of the interval $[\alpha,\beta]$, $\alpha=\vartheta_0<\vartheta_1<\cdots<\vartheta_N=\beta$, $N\geqslant 2$, such that, for every $i=1,2,\ldots,N$, the real function $A_v(\vartheta)$ is strictly increasing in $[\vartheta_{i-1},\vartheta_i]$ and is strictly decreasing in $[\vartheta_i,\vartheta_{i+1}]$ (or vice versa).

The lemma can be proved via an induction argument based on the number N of these maximal intervals of monotonicity. However, in order to simplify the proof, we consider here only the case $N=2$. Hence, there exists $\vartheta_1\in(\alpha,\beta)$ such that $A_v(\vartheta)$ is strictly increasing in $[\alpha,\vartheta_1]$ and is strictly decreasing in $[\vartheta_1,\beta]$, or conversely. To fix the ideas, let us assume that $A_v(\vartheta)$ is strictly increasing in $[\alpha,\vartheta_1]$. For every $(\varrho,\vartheta)\in S(\alpha,\beta)$ let us define $\tilde v(\varrho,\vartheta):=\varrho v(\vartheta)$. If $A_v(\vartheta_1)-A_v(\alpha)\leqslant 2\pi$, then $\tilde v$ restricted to the interior of $S(\alpha,\vartheta_1)$ and $S(\vartheta_1,\beta)$ is one-to-one. Moreover, the images $\tilde v(S(\alpha,\vartheta_1))$ and $\tilde v(S(\vartheta_1,\beta))$ are equal, and by the area formula,

$$\int_{S(\alpha,\vartheta_1)}|\det D\tilde v(x)|\,dx=\operatorname{area}(\tilde v(S(\alpha,\vartheta_1)))=\operatorname{area}(\tilde v(S(\vartheta_1,\beta)))=\int_{S(\vartheta_1,\beta)}|\det D\tilde v(x)|\,dx.$$

Since $\det D\tilde v\geqslant 0$ in $S(\alpha,\vartheta_1)$ and $\det D\tilde v\leqslant 0$ in $S(\vartheta_1,\beta)$, we obtain

$$\begin{aligned}\int_{S(\alpha,\vartheta_1)}\det D\tilde v(x)\,dx&=\operatorname{area}(\tilde v(S(\alpha,\vartheta_1)))=\operatorname{area}(\tilde v(S(\vartheta_1,\beta)))\\&=-\int_{S(\vartheta_1,\beta)}\det D\tilde v(x)\,dx.\end{aligned}$$

Using again (15), we have

$$\det D\tilde v(\varrho,\vartheta)=v^1(\vartheta)v^2_\vartheta(\vartheta)-v^2(\vartheta)v^1_\vartheta(\vartheta)=A_v(\vartheta)|v(\vartheta)|^2. \tag{16}$$

Therefore, as claimed,

$$\begin{aligned}0&=\int_{S(\alpha,\beta)}\det D\tilde v(x)\,dx=\int_0^1\varrho\,d\varrho\int_\alpha^\beta\{v^1(\vartheta)v^2_\vartheta(\vartheta)-v^2(\vartheta)v^1_\vartheta(\vartheta)\}\,d\vartheta\\&=\frac12\int_\alpha^\beta\{v^1(\vartheta)v^2_\vartheta(\vartheta)-v^2(\vartheta)v^1_\vartheta(\vartheta)\}\,d\vartheta.\end{aligned}$$

If $2k\pi<A_v(\vartheta_1)-A_v(\alpha)\leqslant 2\pi(k+1)$ for some $k\geqslant 1$, then we denote by $\vartheta'\in(\alpha,\vartheta_1)$, $\vartheta''\in(\vartheta_1,\beta)$ the points such that $A_v(\vartheta')=A_v(\vartheta'')=2k\pi$. Again, using

the area formula, we have

$$\int_{S(\alpha,\vartheta_1)} |\det D\tilde{v}(x)|\,dx$$

$$= \int_{S(\alpha,\vartheta')} |\det D\tilde{v}(x)|\,dx + \int_{S(\vartheta',\vartheta_1)} |\det D\tilde{v}(x)|\,dx$$

$$= k \text{ area } D + \text{area } E,$$

where D is the domain in (5) enclosed by Γ and E is the domain represented in polar coordinates by

$$E = \{\varrho(\cos A_v(\vartheta), \sin A_v(\vartheta)) : \vartheta \in [\vartheta', \vartheta_1],\ 0 \leqslant \varrho \leqslant r(\vartheta)\}$$

$$= \{\varrho(\cos A_v(\vartheta), \sin A_v(\vartheta)) : \vartheta \in [\vartheta_1, \vartheta''],\ 0 \leqslant \varrho \leqslant r(\vartheta)\}.$$

Therefore, we also have

$$\int_{S(\vartheta_1,\beta)} |\det D\tilde{v}(x)|\,dx$$

$$= \int_{S(\vartheta_1,\vartheta'')} |\det D\tilde{v}(x)|\,dx + \int_{S(\vartheta'',\beta)} |\det D\tilde{v}(x)|\,dx$$

$$= \text{area } E + k \text{ area } D.$$

Arguing as before we get the thesis (with $N = 2$)

$$\frac{1}{2}\int_\alpha^\beta \left\{v^1(\vartheta)v^2_\vartheta(\vartheta) - v^2(\vartheta)v^1_\vartheta(\vartheta)\right\}\,d\vartheta = \int_{S(\alpha,\beta)} \det D\tilde{v}(x)\,\,dx$$

$$= \int_{S(\alpha,\vartheta_1)} |\det D\tilde{v}(x)|\,\,dx - \int_{S(\vartheta_1,\beta)} |\det D\tilde{v}(x)|\,\,dx = 0. \quad \square$$

Lemma 9. *Let $v : [0, 2\pi] \to \Gamma$ be a Lipschitz-continuous map, with Γ as in* (4). *Let $A_{v-\xi}(\vartheta)$ be piecewise strictly monotone in $[a, b]$ (with a finite number of monotonicity intervals). For every $\varepsilon > 0$, there exists a Lipschitz-continuous map $w : B_1 \to \mathbb{R}^2$ such that $w(1, \vartheta) = v(\vartheta)$ for every $\vartheta \in [0, 2\pi]$, and*

$$\int_{B_1} |\det Dw(x)|\,dx < \varepsilon + \frac{1}{2}\left|\int_0^{2\pi} \{v^1(\vartheta)v^2_\vartheta(\vartheta) - v^2(\vartheta)v^1_\vartheta(\vartheta)\}\,d\vartheta\right|.$$

Proof. If $A_{v-\xi}(0) = A_{v-\xi}(2\pi)$ then the result follows from Lemma 6. Otherwise, without loss of generality we may assume that $A_{v-\xi}(0) < A_{v-\xi}(2\pi)$. By Lemma 7, we can consider a partition of $[0, 2\pi]$ by means of points $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_N = 2\pi$ such that, for every $i = 1, 2, \ldots, N$, either $A_{v-\xi}$ is strictly increasing in $[\alpha_{i-1}, \alpha_i]$, or $A_{v-\xi}(\alpha_{i-1}) = A_{v-\xi}(\alpha_i)$. Denote by I the set of indices

$$I := \{i \in \{1, 2, \ldots, N\} : A_{v-\xi}(\alpha_{i-1}) = A_{v-\xi}(\alpha_i)\}.$$

Given $\varepsilon > 0$, if $i \in I$ then we denote by $w_i : S(\alpha_{i-1}, \alpha_i) \to \mathbb{R}^2$ the Lipschitz-continuous map provided by Lemma 6, satisfying the boundary conditions

$$\begin{cases} w_i(1, \vartheta) = v(\vartheta) & \forall \vartheta \in [\alpha_{i-1}, \alpha_i], \\ w_i(\varrho, \alpha_{i-1}) = w_i(\varrho, \alpha_i) = \xi + \varrho(v(\alpha_{i-1}) - \xi) & \forall \varrho \in [0, 1], \end{cases}$$

and the bound

$$\int_{S(\alpha_{i-1}, \alpha_i)} |\det Dw_i(x)|\, dx < \varepsilon. \tag{17}$$

For every $\varrho \in [0, 1]$ we define the Lipschitz-continuous map $w : B_1 \to \mathbb{R}^2$,

$$w(\varrho, \vartheta) := \begin{cases} \xi + \varrho(v(\vartheta) - \xi) & \forall \vartheta \in [\alpha_{i-1}, \alpha_i] \text{ if } i \notin I, \\ w_i(\varrho, \vartheta) & \forall \vartheta \in [\alpha_{i-1}, \alpha_i] \text{ if } i \in I. \end{cases}$$

Note that, in particular, w satisfies the boundary condition $w(1, \vartheta) = v(\vartheta)$. Moreover, if $\vartheta \in [\alpha_{i-1}, \alpha_i]$ for some $i \notin I$, then, in view of (16), we have

$$|\det Dw(x)| = |\det D[\xi + \varrho(v(\vartheta) - \xi)]| = (v^1(\vartheta) - \xi^1)v^2_\vartheta(\vartheta) - (v^2(\vartheta) - \xi^2)v^1_\vartheta(\vartheta), \tag{18}$$

where we have used the fact that $A_{v-\xi}(\vartheta)$ is strictly increasing for $\vartheta \in [\alpha_{i-1}, \alpha_i]$. By (17) and (18) we obtain

$$\begin{aligned} &\int_{B_1} |\det Dw(x)|\, dx \\ &\quad = \sum_{i \in I} \int_{S(\alpha_{i-1}, \alpha_i)} |\det Dw_i(x)|\, dx + \sum_{i \notin I} \int_{S(\alpha_{i-1}, \alpha_i)} |\det Dw(x)|\, dx \\ &\quad \leqslant \varepsilon \cdot \#(I) + \sum_{i \notin I} \int_0^1 \varrho\, d\varrho \int_{\alpha_{i-1}}^{\alpha_i} \{(v^1(\vartheta) - \xi^1)v^2_\vartheta(\vartheta) - (v^2(\vartheta) - \xi^2)v^1_\vartheta(\vartheta)\}\, d\vartheta \\ &\quad = \varepsilon \cdot \#(I) + \sum_{i \notin I} \frac{1}{2} \int_{\alpha_{i-1}}^{\alpha_i} \{(v^1(\vartheta) - \xi^1)v^2_\vartheta(\vartheta) - (v^2(\vartheta) - \xi^2)v^1_\vartheta(\vartheta)\}\, d\vartheta. \end{aligned}$$

By Lemma 8, for every $i \in I$ we have

$$\int_{\alpha_{i-1}}^{\alpha_i} \{(v^1(\vartheta) - \xi^1)v^2_\vartheta(\vartheta) - (v^2(\vartheta) - \xi^2)v^1_\vartheta(\vartheta)\}\, d\vartheta = 0;$$

hence,

$$\int_{B_1} |\det Dw(x)|\,dx \leqslant \varepsilon\cdot N + \frac{1}{2}\int_0^{2\pi} \{(v^1(\vartheta)-\xi^1)v^2_\vartheta(\vartheta) - (v^2(\vartheta)-\xi^2)v^1_\vartheta(\vartheta)\}\,d\vartheta$$
$$= \varepsilon\cdot N + \frac{1}{2}\int_0^{2\pi} \{v^1(\vartheta)v^2_\vartheta(\vartheta) - v^2(\vartheta)v^1_\vartheta(\vartheta)\}\,d\vartheta. \qquad \square$$

Next we consider maps $u = u(\varrho,\vartheta)$ depending explicitly on ϱ as well. The following result, valid for smooth maps, can also be easily obtained in the Lipschitz case by means of an approximation argument.

Lemma 10. *Let $u\in W^{1,\infty}(B_1;\mathbb{R}^2)$. For every $r\in(0,1]$ we have*

$$\int_{B_r} \det Du(x)\,dx = \frac{1}{2}\int_0^{2\pi}\left\{u^1(r,\vartheta)\frac{\partial u^2}{\partial\vartheta}(r,\vartheta) - u^2(r,\vartheta)\frac{\partial u^1}{\partial\vartheta}(r,\vartheta)\right\}d\vartheta. \qquad (19)$$

We start by focusing on maps u of class $W^{1,p}(\Omega;\mathbb{R}^2)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega\backslash\{0\};\mathbb{R}^2)$ for some $p\in(1,2)$.

Lemma 11. *Let u be a map satisfying the assumptions of Theorem* 1. *There exists a sequence $\varrho_j\to 0$ such that*

$$\lim_{j\to+\infty}\int_0^{2\pi}\left\{u^1(\varrho_j,\vartheta)\frac{\partial u^2}{\partial\vartheta}(\varrho_j,\vartheta) - u^2(\varrho_j,\vartheta)\frac{\partial u^1}{\partial\vartheta}(\varrho_j,\vartheta)\right\}d\vartheta$$
$$= \int_0^{2\pi}\left\{v^1(\vartheta)\frac{dv^2}{d\vartheta}(\vartheta) - v^2(\vartheta)\frac{dv^1}{d\vartheta}(\vartheta)\right\}d\vartheta.$$

Proof. First we use assumption (7), which implies that for every $j\geqslant 2$ we have

$$\int_{1/(2j)}^{1/j} d\varrho\int_{\partial B_\varrho}|D_\tau u|^p\,dH^1 \leqslant \int_{B_{1/j}}|D_\tau u|^p\,dx \leqslant \frac{M_0}{j^{2-p}}.$$

From this inequality, we immediately get that there exist $\varrho_j\in(\frac{1}{2j},\frac{1}{j})$ such that

$$(\varrho_j)^{p-1}\int_{\partial B_{\varrho_j}}|D_\tau u|^p\,dH^1 \leqslant cM_0.$$

Since

$$(\varrho_j)^{p-1}\int_{\partial B_{\varrho_j}}|D_\tau u|^p\,dH^1 = \int_0^{2\pi}\left|\frac{\partial u}{\partial\vartheta}(\varrho_j,\vartheta)\right|^p d\vartheta,$$

we deduce that $\{\frac{\partial u}{\partial \vartheta}(\varrho_j,\cdot)\}_{j\in\mathbb{N}}$ is a bounded sequence in $L^p(0,2\pi)$. By assumption (6), $\{u(\varrho_j,\cdot)\}_{j\in\mathbb{N}}$ converges to $v(\cdot)$ in $L^\infty((0,2\pi);\mathbb{R}^2)$. Since $\{\frac{\partial u}{\partial \vartheta}(\varrho_j,\cdot)\}_{j\in\mathbb{N}}$ remains bounded in $L^p((0,2\pi);\mathbb{R}^2)$, then it converges to $\frac{\partial v}{\partial \vartheta}$ weakly in $L^p((0,2\pi);\mathbb{R}^2)$ as $j\to+\infty$. We reach the conclusion

$$\lim_{j\to+\infty}\int_0^{2\pi}\left\{u^1(\varrho_j,\vartheta)\frac{\partial u^2}{\partial\vartheta}(\varrho_j,\vartheta)-u^2(\varrho_j,\vartheta)\frac{\partial u^1}{\partial\vartheta}(\varrho_j,\vartheta)\right\}d\vartheta$$
$$=\int_0^{2\pi}\left\{v^1(\vartheta)\frac{dv^2}{d\vartheta}(\vartheta)-v^2(\vartheta)\frac{dv^1}{d\vartheta}(\vartheta)\right\}d\vartheta. \qquad \square$$

We are now ready to give the proof of Theorem 1. Actually, we shall prove a stronger statement, i.e., not only that the representation formula (8) holds, but also that $\mathrm{TV}(u,\Omega)=\mathrm{TV}^{\mathrm{s}}(u,\Omega)$, where

$$\mathrm{TV}^{\mathrm{s}}(u,\Omega)=\inf\left\{\liminf_{h\to+\infty}\int_\Omega|\det Du_h(x)|\,dx:\right.$$
$$\left.u_h\to u \text{ strongly in } W^{1,p}(\Omega;\mathbb{R}^n),\ u_h\in W^{1,n}(\Omega;\mathbb{R}^n)\right\} \qquad (20)$$

is the *total variation of the Jacobian in the strong topology*. We divide the proof into four steps, and we will refer to the preliminary lemmas above and to Lemma 23 of Section 4.

Proof of Theorem 1. *Step* 1 (*lower bound*). Let u be a function in $W^{1,p}(\Omega;\mathbb{R}^2)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega\backslash\{0\};\mathbb{R}^2)$ for some $p\in(1,2)$. Observe that, by assumption (6), there exists $r>0$ such that $B_r\subset\Omega$ and $u\in L^\infty(B_r;\mathbb{R}^2)$. Let $\varrho_j\to0$ be the sequence provided by Lemma 11 and let $j\in\mathbb{N}$ be sufficiently large so that $B_{\varrho_j}\subset B_r$. For such values of $j\in\mathbb{N}$, we use the estimate (54) of Lemma 23 to obtain

$$\mathrm{TV}(u,\Omega)\geqslant\int_{\Omega\backslash B_{\varrho_j}}|\det Du(x)|\,dx+\left|\int_{B_{\varrho_j}}\det D\tilde u(x)\,dx\right|,$$

where $\tilde u: B_{\varrho_j}\to\mathbb{R}^2$ is any Lipschitz-continuous map such that $\tilde u(x)=u(x)$ on ∂B_{ϱ_j}. By formula (19) of Lemma 10 (valid on each ball B_{ϱ_j}), since $\tilde u=u,\ \partial\tilde u/\partial\vartheta=\partial u/\partial\vartheta$ on ∂B_{ϱ_j}, we have

$$\mathrm{TV}(u,\Omega)\geqslant\int_{\Omega\backslash B_{\varrho_j}}|\det Du(x)|\,dx$$
$$+\left|\frac12\int_0^{2\pi}\left\{u^1(\varrho_j,\vartheta)\frac{\partial u^2}{\partial\vartheta}(\varrho_j,\vartheta)-u^2(\varrho_j,\vartheta)\frac{\partial u^1}{\partial\vartheta}(\varrho_j,\vartheta)\right\}d\vartheta\right|.$$

Letting $j\to+\infty$, by Lemma 11 we obtain the lower bound

$$\mathrm{TV}(u,\Omega)\geqslant\int_\Omega|\det Du(x)|\,dx+\frac{1}{2}\left|\int_0^{2\pi}\left\{v^1(\vartheta)\frac{dv^2}{d\vartheta}(\vartheta)-v^2(\vartheta)\frac{dv^1}{d\vartheta}(\vartheta)\right\}d\vartheta\right|. \tag{21}$$

Step 2 (*upper bound—first part*). To assert the opposite inequality in (21), let us first assume that u is radially symmetric, i.e., $u=v=v(\vartheta)=u(\vartheta)$, and that it satisfies the assumptions of Lemma 9. By the conclusion of Lemma 9, given $\varepsilon>0$ there exists a Lipschitz-continuous map $w:B_1\to\mathbb{R}^2$ such that $w(1,\vartheta)=v(\vartheta)$ for every $\vartheta\in[0,2\pi]$ and

$$\int_{B_1}|\det Dw(x)|\,dx<\varepsilon+\frac{1}{2}\left|\int_0^{2\pi}\{v^1(\vartheta)v^2_\vartheta(\vartheta)-v^2(\vartheta)v^1_\vartheta(\vartheta)\}\,d\vartheta\right|.$$

For every $h\in\mathbb{N}$ we set

$$u_h(\varrho,\vartheta):=\begin{cases}w(\varrho h,\vartheta) & \text{if } 0\leqslant\varrho\leqslant 1/h,\\ v(\vartheta) & \text{if } \varrho\geqslant 1/h.\end{cases}$$

Then $\{u_h\}_{h\in\mathbb{N}}$ converges to u in $L^p(\Omega)$ and

$$\int_\Omega|Du_h(x)-Du(x)|^p\,dx=\int_{B_{1/h}}|hDw(\varrho h,\vartheta)|^p\,dx=h^{p-2}\int_{B_1}|Dw(x)|^p\,dx,$$

and so, since $1\leqslant p<2$, $\{Du_h\}_{h\in\mathbb{N}}$ converges to Du strongly in $L^p(B_1;\mathbb{R}^{2\times2})$ and, finally,

$$\begin{aligned}\int_\Omega|\det Du_h(x)|\,dx&=\int_{B_{1/h}}|h^2\det Dw(\varrho h,\vartheta)|\,dx=\int_{B_1}|\det Dw(x)|\,dx\\&<\varepsilon+\frac{1}{2}\left|\int_0^{2\pi}\{v^1(\vartheta)v^2_\vartheta(\vartheta)-v^2(\vartheta)v^1_\vartheta(\vartheta)\}\,d\vartheta\right|.\end{aligned}$$

Therefore, making use of definition (20) of the total variation of the Jacobian $\mathrm{TV}^{\mathrm{s}}(u,\Omega)$ in the *strong* topology, we can conclude that

$$\mathrm{TV}^{\mathrm{s}}(u,\Omega)\leqslant\frac{1}{2}\left|\int_0^{2\pi}\{v^1(\vartheta)v^2_\vartheta(\vartheta)-v^2(\vartheta)v^1_\vartheta(\vartheta)\}\,d\vartheta\right|.$$

This inequality, together with the obvious inequality

$$\mathrm{TV}(u,\Omega)\leqslant\mathrm{TV}^{\mathrm{s}}(u,\Omega),\quad\forall u\in L^\infty(\Omega;\mathbb{R}^n)\cap W^{1,p}(\Omega;\mathbb{R}^n),$$

and with Step 1, yields the conclusion

$$\mathrm{TV}(u,\Omega)=\mathrm{TV}^{\mathrm{s}}(u,\Omega)=\frac{1}{2}\left|\int_0^{2\pi}\{v^1(\vartheta)v_\vartheta^2(\vartheta)-v^2(\vartheta)v_\vartheta^1(\vartheta)\}\,d\vartheta\right| \tag{22}$$

whenever $u=v=v(\vartheta)=u(\vartheta)$ satisfies the assumptions of Lemma 9.

Step 3 (*upper bound—second part*). As in the previous Step 2, we still assume that $u=v=v(\vartheta)=u(\vartheta)$, but we no longer require that the conditions of Lemma 9 are satisfied. Without loss of generality, we can assume that $\mathrm{Arg}(v(0)-\xi)=0$ and thus, by Lemma 5, the map $v(\vartheta)$ may be represented as

$$v(\vartheta)=\xi+r(A_{v-\xi}(\vartheta))(\cos A_{v-\xi}(\vartheta),\sin A_{v-\xi}(\vartheta)).$$

Construct a sequence $\{A_k(\vartheta)\}_{k\in\mathbb{N}}$ of piecewise affine functions, Lipschitz-continuous with bounded Lipschitz constants, satisfying the conditions

$$\begin{cases} A_k(0)=0 & \forall k\in\mathbb{N},\\ A_k'(\vartheta)\neq 0 & \text{a.e. } \vartheta\in[0,2\pi],\quad \forall k\in\mathbb{N},\\ A_k\to A_{v-\xi} & \text{in } C^0([0,2\pi]),\\ A_k'(\vartheta)\to A_{v-\xi}'(\vartheta) & \text{a.e. } \vartheta\in[0,2\pi],\\ |A_k'(\vartheta)|\leqslant L_0 & \text{a.e. } \vartheta\in[0,2\pi],\quad \forall k\in\mathbb{N}, \end{cases}$$

and define

$$v_k(\vartheta):=\xi+r(A_k(\vartheta))(\cos A_k(\vartheta),\sin A_k(\vartheta)).$$

Then the map $v_k(\vartheta)$ satisfies all the assumptions of the previous Step 2 and

$$\begin{cases} v_k\to v & \text{in } C^0([0,2\pi]),\\ \left|\left|\frac{dv_k}{d\vartheta}\right|\right|_{L^\infty(0,2\pi)}\leqslant L_0 & \forall k\in\mathbb{N}. \end{cases}$$

We prove that $\{\frac{dv_k}{d\vartheta}\}_{k\in\mathbb{N}}$ converges to $\frac{dv}{d\vartheta}$ for almost every $\vartheta\in[0,2\pi]$, as $k\to+\infty$. To this aim, let us recall that

$$v,v_k:[0,2\pi]\to\Gamma=\{\xi+r(\vartheta)(\cos\vartheta,\sin\vartheta):\vartheta\in[0,2\pi]\},$$

where $r(\vartheta)$ is a piecewise C^1-function, i.e., there exist a finite number of points $0\leqslant a_0<a_1<\cdots<a_N\leqslant 2\pi$, such that $r(\vartheta)$ is a function of class $C^1([a_{j-1},a_j])$ for every $j=1,2,\ldots,N$. Define

$$E:=\{\vartheta\in[0,2\pi]:\exists j=1,2,\ldots,N:A_{v-\xi}(\vartheta)=a_j\}.$$

Then

$$A_{v-\xi}'(\vartheta)=0\ \text{ and }\ v'(\vartheta)=0,\quad \text{for a.e. } \vartheta\in E, \tag{23}$$

and for almost every $\vartheta\in[0,2\pi]$ we have also

$$\frac{dv_k}{d\vartheta}=A_k'(\vartheta)\{r'(A_k(\vartheta))(\cos A_k(\vartheta),\sin A_k(\vartheta)) + r(A_k(\vartheta))(-\sin A_k(\vartheta),\cos A_k(\vartheta))\} \tag{24}$$

As $k\to+\infty$, $\{r'(A_k(\vartheta))\}_{k\in\mathbb{N}}$ converges to r' $(A_{v-\xi})$ for every $\vartheta\notin E$. Thus, by (24) we have that $\{\frac{dv_k}{d\vartheta}\}_{k\in\mathbb{N}}$ converges to $\frac{dv}{d\vartheta}$ for almost every $\vartheta\in[0,2\pi]\backslash E$. On the other hand, by (23) for almost every $\vartheta\in E$, $\{A_k'(\vartheta)\}_{k\in\mathbb{N}}$ converges to $A'_{v-\xi}(\vartheta)=0$ and, since $\{r'(A_k(\vartheta))\}_{k\in\mathbb{N}}$ is uniformly bounded, by (24) we can conclude that $\{\frac{dv_k}{d\vartheta}\}_{k\in\mathbb{N}}$ converges to $0=\frac{dv}{d\vartheta}$ for almost every $\vartheta\in E$, as $k\to+\infty$. Therefore, $\{v_k(\vartheta)\}_{k\in\mathbb{N}}$ converges to $v(\vartheta)$ in the strong topology of $W^{1,q}(B_1;\mathbb{R}^2)$ for every $q\geqslant 1$, as $k\to+\infty$, and, in turn, this implies that the map $u_k(\varrho,\vartheta):=v_k(\vartheta)$ (independent of ϱ), which belongs to $W^{1,p}(B_1;\mathbb{R}^2)$ for every $p\in[1,2)$, converges to $u(\varrho,\vartheta)=v(\vartheta)$, as $k\to+\infty$, in the strong topology of $W^{1,p}(B_1;\mathbb{R}^2)$ for every $p\in[1,2)$. From Step 2, and in particular from (22), we deduce that

$$\begin{aligned}\mathrm{TV}(u,\Omega)\leqslant\mathrm{TV}^{\mathrm{s}}(u,\Omega)&\leqslant\liminf_{k\to+\infty}\mathrm{TV}^{\mathrm{s}}(u_k,\Omega)=\liminf_{k\to+\infty}\mathrm{TV}^{\mathrm{s}}(u_k,\Omega)\\&=\lim_{k\to+\infty}\frac{1}{2}\left|\int_0^{2\pi}\left\{v_k^1\frac{\partial v_k^2}{\partial\vartheta}-v_k^2\frac{\partial v_k^1}{\partial\vartheta}\right\}d\vartheta\right|\\&=\frac{1}{2}\left|\int_0^{2\pi}\left\{v^1\frac{dv^2}{d\vartheta}-v^2\frac{dv^1}{d\vartheta}\right\}d\vartheta\right|.\end{aligned}$$

By (21) of Step 1 we finally obtain

$$\mathrm{TV}(u,\Omega)=\mathrm{TV}^{\mathrm{s}}(u,\Omega)=\frac{1}{2}\left|\int_0^{2\pi}\{v^1v_\vartheta^2-v^2v_\vartheta^1\}\,d\vartheta\right|.$$

Step 4 (*upper bound—third part*). Here we study the general case where $u=u(\varrho,\vartheta)$ may depend explicitly on ϱ as well.

Using the argument of Lemma 22, for every $u\in L^\infty(\Omega;\mathbb{R}^2)\cap W^{1,p}(\Omega;\mathbb{R}^2)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega\backslash\{0\};\mathbb{R}^2)$ with $p>1$, it can be shown that admissible sequences for $\mathrm{TV}^{\mathrm{s}}(u,\Omega)$, defined in (20), may be required to assume prescribed boundary values, precisely

$$\mathrm{TV}^{\mathrm{s}}(u,\Omega)=\inf\left\{\liminf_{h\to+\infty}\int_\Omega|\det Du_h(x)|\,dx:\ u_h\to u\text{ strongly in }W^{1,p}(\Omega;\mathbb{R}^2),\ u_h\in u+W_0^{1,\infty}(\Omega;\mathbb{R}^2)\right\} \tag{25}$$

For every $\varepsilon>0$ there exists a map $w\in v+W_0^{1,\infty}(B_1;\mathbb{R}^2)$ such that

$$\int_{B_1}|\det Dw(x)|\,dx<\varepsilon+\mathrm{TV}^{\mathrm{s}}(v,B_1). \tag{26}$$

From the proof of Lemma 11, it follows that there exists a sequence $\{\varrho_h\}_{h\in\mathbb{N}}$ converging to zero, such that

$$(\varrho_j)^{p-1}\int_{\partial B_{\varrho_j}}|D_\tau u|^p\,dH^1=\int_0^{2\pi}|u_\vartheta(\varrho_h,\vartheta)|^p\,d\vartheta\leqslant M_0,\quad\forall h\in\mathbb{N}. \tag{27}$$

Introduce the sequence of functions

$$u_h(\varrho,\vartheta):=\begin{cases} w(\frac{\varrho}{\varrho_h(1-\sigma_h)},\vartheta) & \text{if } 0\leqslant\varrho\leqslant\varrho_h(1-\sigma_h),\\ \eta_h(\varrho)v(\vartheta)+[1-\eta_h(\varrho)]u(\varrho_h,\vartheta) & \text{if } \varrho_h(1-\sigma_h)<\varrho<\varrho_h,\\ u(\varrho,\vartheta) & \text{if } \varrho_h\leqslant\varrho,\end{cases}$$

where $\sigma_h:=(\varrho_h)^{\frac{2-p}{p-1}}$ and η_h is a *cut-off* function, i.e., $\eta_h(\varrho)=1$ if $0\leqslant\varrho\leqslant\varrho_h(1-\sigma_h)$, $\eta_h(\varrho)=0$ if $\varrho_h\leqslant\varrho\leqslant1$, $\eta_h(\varrho)$ is linear in the interval $[\varrho_h(1-\sigma_h),\varrho_h]$. Notice that $u_h\to u$ in $L^p(B_1;\mathbb{R}^2)$, as $h\to+\infty$, and that the sequence of gradients $(Du_h)_{h\in\mathbb{N}}$ converges in L^p to D_u. Indeed

$$\begin{aligned}\int_\Omega|Du_h-Du|^p\,dx\leqslant{}& c_1\int_{B_{\varrho_h(1-\sigma_h)}}\left|Dw\left(\frac{x}{\varrho_h(1-\sigma_h)}\right)\right|^p dx\\ &+c_1\int_{B_{\varrho_h}\setminus B_{\varrho_h(1-\sigma_h)}}\frac{|v_\vartheta|^p}{|x|^p}\,dx+c_1\int_{B_{\varrho_h}\setminus B_{\varrho_h(1-\sigma_h)}}\frac{|u_\vartheta(\varrho_h,\vartheta)|^p}{|x|^p}\,dx\\ &+\frac{c_1}{\varrho_h^p\sigma_h^p}\int_{B_{\varrho_h}\setminus B_{\varrho_h(1-\sigma_h)}}\left|u\left(\varrho_h\frac{x}{|x|}\right)-v\left(\frac{x}{|x|}\right)\right|^p dx\\ &+c_1\int_{B_{\varrho_h}}|Du|^p\,dx\\ \leqslant{}& c_2\varrho_h^{2-p}+c_2\varrho_h^{2-p}\sigma_h\int_0^{2\pi}|u_\vartheta(\varrho_h,\vartheta)|^p\,d\vartheta\\ &+c_2\frac{\varrho_h^{2-p}}{\sigma_h^{p-1}}\|u(\varrho_h,\vartheta)-v(\vartheta)\|^p_{L^\infty(0,2\pi)}+c_1\int_{B_{\varrho_h}}|Du|^p\,dx,\end{aligned}$$

and this quantity goes to zero since $\sigma_h = (\varrho_h)^{\frac{2-p}{p-1}}$. Therefore, by (26) we get

$$\begin{aligned}
\mathrm{TV}^{\mathrm{s}}(u,\Omega) &\leqslant \liminf_{h\to+\infty} \int_\Omega |\det Du_h(x)|\,dx \\
&= \int_{B_1} |\det Dw(x)|\,dx + \liminf_{h\to+\infty} \int_{B_{\varrho_h}\setminus B_{\varrho_h(1-\sigma_h)}} |\det Du_h(x)|\,dx \\
&\quad + \int_\Omega |\det Du(x)|\,dx \leqslant \varepsilon + \mathrm{TV}^{\mathrm{s}}(v,B_1) \\
&\quad + \liminf_{h\to+\infty} \int_{B_{\varrho_h}\setminus B_{\varrho_h(1-\sigma_h)}} |\det Du_h(x)|\,dx + \int_\Omega |\det Du(x)|\,dx. \qquad (28)
\end{aligned}$$

We evaluate the last integral in the right-hand side. For $\varrho_h(1-\sigma_h) < \varrho < \varrho_h$, we have

$$\begin{aligned}
\det Du_h &= \frac{1}{\varrho}\begin{vmatrix} \frac{\partial u_h^1(\varrho,\vartheta)}{\partial\varrho} & \frac{\partial u_h^1(\varrho,\vartheta)}{\partial\vartheta} \\ \frac{\partial u_h^2(\varrho,\vartheta)}{\partial\varrho} & \frac{\partial u_h^2(\varrho,\vartheta)}{\partial\vartheta} \end{vmatrix} \\
&= \frac{1}{\varrho}\begin{vmatrix} \eta_h'(\varrho)[v^1(\vartheta) - u^1(\varrho_h,\vartheta)] & \eta_h(\varrho)\frac{\partial v^1(\vartheta)}{\partial\vartheta} + [1-\eta_h(\varrho)]\frac{\partial u_\vartheta^1(\varrho_h,\vartheta)}{\partial\vartheta} \\ \eta_h'(\varrho)[v^2(\vartheta) - u^2(\varrho_h,\vartheta)] & \eta_h(\varrho)\frac{\partial v^2(\vartheta)}{\partial\vartheta} + [1-\eta_h(\varrho)]\frac{\partial u^2(\varrho_h,\vartheta)}{\partial\vartheta} \end{vmatrix},
\end{aligned}$$

and thus, since $|\eta_h'(\varrho)| \leqslant \frac{c_1}{\sigma_h\varrho_h}$ for some constant c_1, we have

$$\begin{aligned}
\int_{B_{\varrho_h}\setminus B_{\varrho_h(1-\sigma_h)}} |\det Du_h|\,dx &\leqslant \frac{c_1}{\sigma_h\varrho_h} \int_{\varrho_h(1-\sigma_h)}^{\varrho_h} d\varrho \int_0^{2\pi} |v(\vartheta) - u(\varrho_h,\vartheta)| \\
&\quad \cdot \left\{ \left|\frac{\partial v(\vartheta)}{\partial\vartheta}\right| + \left|\frac{\partial u(\varrho_h,\vartheta)}{\partial\vartheta}\right| \right\} d\vartheta \\
&\leqslant c_2 \sup\{|v(\vartheta) - u(\varrho_h,\vartheta)| : \vartheta\in[0,2\pi]\} \\
&\quad \int_0^{2\pi} \left\{ \left|\frac{\partial v(\vartheta)}{\partial\vartheta}\right| + \left|\frac{\partial u(\varrho_h,\vartheta)}{\partial\vartheta}\right| \right\} d\vartheta.
\end{aligned}$$

By (27), there exists a new constant c_3 such that

$$\int_{B_{\varrho_h}\setminus B_{\varrho_h(1-\sigma_h)}} |\det Du_h|\,dx \leqslant c_3 \sup\{|v(\vartheta) - u(\varrho,\vartheta)| : \vartheta\in[0,2\pi]\},$$

and thus, by assumption (6) and by (28), letting $\varepsilon\to 0$ we obtain

$$\mathrm{TV}(u,\Omega) \leqslant \mathrm{TV}^{\mathrm{s}}(u,\Omega) \leqslant \int_\Omega |\det Du|\,dx + \mathrm{TV}(v,B_1).$$

This upper bound, together with the lower bound of Step 1, yields the conclusion. □

3. Maps with values on the "eight" curve

In this section we consider maps with values on the "*eight*" *curve*. The "*eight*" *curve* in $\mathbb{R}^2$ is the union γ of the two circles γ^+, γ^-, of radius 1 with centers at $(1,0)$ and at $(-1,0)$, respectively. Some explicit examples of such maps are given below. In the next theorem we present *two estimates* yielding an upper bound and a lower bound.

Theorem 12 (The "eight" curve). *Let $\gamma = \gamma^+ \cup \gamma^- \subset \mathbb{R}^2$ be the union of the two circles of radius 1 with centers at $(1,0)$ and at $(-1,0)$. Let $v : [0,2\pi] \to \gamma$ be a Lipschitz-continuous curve such that $v(0) = v(2\pi)$. Let $(I_j)_{j\in\mathbb{N}}$ be a sequence of disjoint open intervals (possibly empty) of $[0,2\pi]$ such that the image $v(I_j)$ is contained either in γ^+ or in γ^-, and $v(\vartheta) = (0,0)$ when $\vartheta \notin \bigcup_{j\in\mathbb{N}} I_j$. Then, if $u(x) := v(x/|x|)$, the following upper estimate holds*

$$\mathrm{TV}(u, B_1) \leqslant \frac{1}{2} \sum_{j\in\mathbb{N}} \left| \int_{I_j} \{v^1(\vartheta) v_\vartheta^2(\vartheta) - v^2(\vartheta) v_\vartheta^1(\vartheta)\}\, d\vartheta \right|. \tag{29}$$

Moreover, denoting by I_j^+ any previous interval I_j such that $v(I_j) \subset \gamma^+$, and by I_k^- any previous interval I_k such that $v(I_k) \subset \gamma^-$, we have the following lower estimate

$$\mathrm{TV}(u, B_1) \geqslant \frac{1}{2} \left\{ \left| \sum_{j\in\mathbb{N}} \int_{I_j^+} \{v^1 v_\vartheta^2 - v^2 v_\vartheta^1\}\, d\vartheta \right| + \left| \sum_{k\in\mathbb{N}} \int_{I_k^-} \{v^1 v_\vartheta^2 - v^2 v_\vartheta^1\}\, d\vartheta \right| \right\}. \tag{30}$$

Remark 13. If the curve $v : [0,2\pi] \to \gamma = \gamma^+ \cup \gamma^-$ admits only two intervals I_1^+ and I_2^- such that $v(I_1^+) \subset \gamma^+, v(I_2^-) \subset \gamma^-$, then the above estimates for $\mathrm{TV}(u, B_1)$ are in fact equalities. The same happens if the intervals are three, say I_1^+, I_2^- and I_3^+. In fact, this case can be reduced to the previous one by periodicity. If the intervals are four, say I_1^+, I_2^-, I_3^+ and I_4^-, then we may have a gap between the lower bound and the upper bound stated in Theorem 12, unless the integral of $v^1 v_\vartheta^2 - v^2 v_\vartheta^1$ has the same sign, respectively, in I_1^+, I_3^+ and in I_2^-, I_4^-. These considerations are exploited in the study some of the examples below.

Let us denote by γ the image of the "*eight*" *curve*, i.e., the union of the two circles γ^+ and γ^- of radius 1, with center at $(1,0)$ and at $(-1,0)$, respectively, Below we will use some elementary representation formulas for γ^+ and γ^-. Precisely, for γ^+ we will use the representation formulas

$$\gamma^+ := \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1^2 + \xi_2^2 - 2\xi_1 = 0\}, \tag{31}$$

$$\xi \in \gamma^+ \setminus (0,0) \;\Leftrightarrow\; \begin{cases} \xi_1 = 2\cos^2 \mathrm{Arg}\, \xi, \\ \xi_2 = 2\cos \mathrm{Arg}\, \xi \cdot \sin \mathrm{Arg}\, \xi. \end{cases} \tag{32}$$

With the aim to prove Theorem 12, we start with some preliminary results concerning a map w with values in the circle γ^+. We give a first lemma without proof.

Lemma 14. *Let $w:[0,2\pi]\to\gamma^+$ be a Lipschitz-continuous curve such that $w(0)=(2,0)$. The real function*

$$R(\vartheta):=\begin{cases}0 & \text{if } w(\vartheta)=(0,0),\\ \frac{w^1(\vartheta)w_\vartheta^2(\vartheta)-w^2(\vartheta)w_\vartheta^1(\vartheta)}{|w(\vartheta)|^2} & \text{if } w(\vartheta)\neq(0,0),\end{cases} \tag{33}$$

is bounded in $[0,2\pi]$ by a constant depending only on the Lipschitz constant of w. Moreover, setting

$$A_w(\vartheta):=\int_0^\vartheta \frac{w^1(t)w_\vartheta^2(t)-w^2(t)w_\vartheta^1(t)}{|w(t)|^2}\,dt, \tag{34}$$

then for every $\alpha,\beta\in[0,2\pi]$ such that $w(\alpha)\neq(0,0)$ and $w(\beta)\neq(0,0)$, there exists $k\in\mathbb{Z}$ such that

$$A_w(\beta)-A_w(\alpha)=\operatorname{Arg}w(\beta)-\operatorname{Arg}w(\alpha)+k\pi. \tag{35}$$

The proof of this lemma is simple and is left to the reader. The next result is similar to the *umbrella* Lemma 6, with the main difference that here the starting point of the *umbrella-stick* is placed at a boundary point of the circle γ^+. The proof is similar to the proof of Lemma 6, and therefore we do not give the details.

Lemma 15 (The "umbrella" lemma for the "eight" curve). *Let $w:[0,2\pi]\to\gamma^+$ be a Lipschitz-continuous curve. Assume that there exist $\alpha,\beta\in[0,2\pi],\alpha<\beta$, such that $A_w(\alpha)=A_w(\beta)$. Then for every $\varepsilon>0$ there exists a Lipschitz-continuous map $\tilde w:S(\alpha,\beta)\to\mathbb{R}^2$ satisfying the boundary conditions*

$$\begin{cases}\tilde w(1,\vartheta)=w(\vartheta) & \forall\vartheta\in[\alpha,\beta],\\ \tilde w(\varrho,\alpha)=\varrho w(\alpha) & \forall\varrho\in[0,1],\\ \tilde w(\varrho,\beta)=\varrho w(\beta) & \forall\varrho\in[0,1],\end{cases}$$

and such that $\int_{S(\alpha,\beta)}|\det D\tilde w(x)|\,dx<\varepsilon$.

Lemma 16. *Let $w:[0,2\pi]\to\gamma^+$ be a Lipschitz-continuous map. If $\alpha,\beta\in[0,2\pi],\alpha<\beta$, are such that $A_w(\alpha)=A_w(\beta)$, and if the function $A_w(\vartheta)$ is piecewise strictly monotone in $[\alpha,\beta]$ (with a finite number of monotonicity intervals), then*

$$\int_\alpha^\beta\{w^1(\vartheta)w_\vartheta^2(\vartheta)-w^2(\vartheta)w_\vartheta^1(\vartheta)\}\,d\vartheta=0.$$

Proof. This result can be proved following an argument just as that of Lemma 8. □

Lemma 17. *Let $u:[0,2\pi]\to\gamma=\gamma^+\cup\gamma^-$ be a Lipschitz-continuous map. Assume that there exist N disjoint open intervals $I_j\subset[0,2\pi]$ such that $u(I_j)$ is contained either in γ^+ or in γ^- for every $j=1,2,\dots,N$, and $u(\vartheta)=(0,0)$ when $\vartheta\notin\bigcup_{j=1}^N I_j$. Assume, in addition, that the function*

$$\vartheta\to u^1(\vartheta)u_\vartheta^2(\vartheta)-u^2(\vartheta)u_\vartheta^1(\vartheta) \tag{36}$$

has piecewise constant sign in $[0,2\pi]$. Then, for every $\varepsilon>0$, there exists a Lipschitz-continuous map $\tilde w:B_1\to\mathbb{R}^2$ satisfying the boundary condition $\tilde w(1,\vartheta)=u(\vartheta)$ for every $\vartheta\in[0,2\pi]$, and such that

$$\int_{B_1}|\det D\tilde w(x)|\,dx<\varepsilon+\frac{1}{2}\sum_{j=1}^N\left|\int_{I_j}\{u^1(\vartheta)u_\vartheta^2(\vartheta)-u^2(\vartheta)u_\vartheta^1(\vartheta)\}\,d\vartheta\right|. \tag{37}$$

Proof. Fix $j\in\{1,2,\dots,N\}$ and assume that $u(I_j)\subset\gamma^+$. We follow the method of proof of Lemma 9, using Lemma 15 in place of Lemma 6, and Lemma 16 in place of Lemma 8. Setting $I_j:=(\alpha_j,\beta_j)$, we construct a Lipschitz-continuous map $\tilde w_j:S(\alpha_j,\beta_j)\to\mathbb{R}^2$

$$\begin{cases}\tilde w_i(1,\vartheta)=u(\vartheta) & \forall\vartheta\in[\alpha_j,\beta_j],\\ \tilde w_i(\varrho,\alpha_j)=\varrho\cdot u(\alpha_j)=(0,0) & \forall\varrho\in[0,1],\\ \tilde w_i(\varrho,\beta_j)=\varrho\cdot u(\beta_j)=(0,0) & \forall\varrho\in[0,1],\end{cases} \tag{38}$$

and the estimate

$$\int_{S(\alpha_j,\beta_j)}|\det D\tilde w_i(x)|\,dx<\frac{\varepsilon}{N}+\frac{1}{2}\left|\int_{I_j}\{u^1(\vartheta)u_\vartheta^2(\vartheta)-u^2(\vartheta)u_\vartheta^1(\vartheta)\}\,d\vartheta\right|. \tag{39}$$

A similar conclusion holds if, instead, we have $u(I_j)\subset\gamma^-$. Then the result follows by taking $\tilde w:B_1\to\mathbb{R}^2$ defined by

$$\tilde w(\varrho,\vartheta):=\begin{cases}\tilde w_j(\varrho,\vartheta) & \forall\vartheta\in(\alpha_j,\beta_j)=I_j,\\ (0,0) & \forall\vartheta\notin\bigcup_{j=1}^N I_j.\end{cases}$$ □

Proof of Theorem 12. *Step* 1 (*lower bound—first part*). Let $v:[0,2\pi]\to\gamma=\gamma^+\cup\gamma^-$ be a Lipschitz-continuous map. With $u(x):=v(x/|x|)$ then $u\in L^\infty(B_1;\mathbb{R}^2)\cap W^{1,p}(B_1;\mathbb{R}^2)\cap W^{1,\infty}_{\mathrm{loc}}(B_1\backslash\{0\};\mathbb{R}^2)$ for every $p\in(1,2)$. By (48) (lower bound obtained

in Lemma 22 in the general n—dimensional case) we have

$$\mathrm{TV}(u, B_1) \geqslant \left| \int_{B_1} \det D\tilde{u}(x)\, dx \right|,$$

where $\tilde{u} : B_1 \to \mathbb{R}^2$ is any Lipschitz-continuous map which assumes the boundary value $\tilde{u} = u$ on ∂B_1 (e.g., $\tilde{u}(x) = |x| v(x/|x|) = |x| u(x)$ for $x \in B_1 \setminus \{0\}$ and $\tilde{u}(0) = 0$). By formula (19) of Lemma 10 (valid on B_r for every $r \in (0, 1]$), since $\tilde{u} = u$, $\partial \tilde{u}/\partial \vartheta = \partial u/\partial \vartheta$ on ∂B_1, we have

$$\mathrm{TV}(u, B_1) \geqslant \left| \frac{1}{2} \int_0^{2\pi} \{ v^1(\vartheta) v^2_\vartheta(\vartheta) - v^2(\vartheta) v^1_\vartheta(\vartheta) \}\, d\vartheta \right|. \tag{40}$$

As in the statement of Theorem 12, we denote by $(I_j)_{j \in \mathbb{N}}$ a sequence of disjoint open intervals in $[0, 2\pi]$ (possibly empty) such that the image $v(I_j)$ is contained either in γ^+ or in γ^-, and $v(\vartheta) = (0, 0)$ when $\vartheta \notin \bigcup_{j \in \mathbb{N}} I_j$. Then we can write (40) equivalently as

$$\mathrm{TV}(u, B_1) \geqslant \left| \frac{1}{2} \sum_{j \in \mathbb{N}} \int_{I_j} \{ v^1(\vartheta) v^2_\vartheta(\vartheta) - v^2(\vartheta) v^1_\vartheta(\vartheta) \}\, d\vartheta \right|. \tag{41}$$

Step 2 (*lower bound—second part*). Let $\delta > 0$ and let $\{u_h = u_h^1, u_h^2)\}_{h \in \mathbb{N}}$ be a sequence in $W^{1,2}(B_1; \mathbb{R}^2)$ converging to u in the weak topology of $W^{1,2}(B_1; \mathbb{R}^2), p \in (1, 2)$ and such that

$$\mathrm{TV}(u, B_1) + \delta \geqslant \lim_{h \to +\infty} \int_{B_1} |\det Du_h(x)|\, dx.$$

Consider $u_h^+ := (|u_h^1|, u_h^2) \in W^{1,2}(B_1; \mathbb{R}^2)$. Clearly $\{u_h\}$ converges to $u^+ = (|u^1|, u^2)$ in the weak topology of $W^{1,p}(B_1; \mathbb{R}^2)$ as $h \to +\infty$. Since $|\det Du_h^+(x)| = |\det Du_h(x)|$ for almost every $x \in B_1$, we obtain

$$\mathrm{TV}(u, B_1) + \delta \geqslant \lim_{h \to +\infty} \int_{B_1} |\det Du_h(x)|\, dx = \lim_{h \to +\infty} \int_{B_1} |\det Du_h^+(x)|\, dx \geqslant \mathrm{TV}(u^+, B_1).$$

The total variation of the map $u^+ : B_1 \to \gamma^+$ may be obtained using formula (9), with $u^+ = (|u^1|, u^2) = (|v^1|, v^2)$. Therefore, as $\delta \to 0^+$ we have

$$\mathrm{TV}(u, B_1) \geqslant \mathrm{TV}(u^+, B_1) = \frac{1}{2} \left| \int_0^{2\pi} \left\{ |v^1| \frac{dv^2}{d\vartheta} - v^2 \frac{d|v^1|}{d\vartheta} \right\} d\vartheta \right|.$$

Recall that $v^1(\vartheta) > 0$ if $\vartheta \in I_j$, where $v(I_j) \subset \gamma^+$ (and analogously $v^1(\vartheta) < 0$ if $v(I_j) \subset \gamma^-$, while $v^1(\vartheta) = 0$ if $\vartheta \notin \bigcup_{j \in \mathbb{N}} I_j$). Again, as in the statement of Theorem 12, we denote by I_j^+, with the $+$ sign, any interval I_j such that $v(I_j) \subset \gamma^+$, and by I_k^- any interval

I_k such that $v(I_k)\subset\gamma^-$. Thus,

$$\mathrm{TV}(u,B_1)\geqslant\frac{1}{2}\left|\sum_{j\in\mathbb{N}}\int_{I_j^+}\{v^1v_\vartheta^2-v^2v_\vartheta^1\}\,d\vartheta-\sum_{k\in\mathbb{N}}\int_{I_k^-}\{v^1v_\vartheta^2-v^2v_\vartheta^1\}\,d\vartheta\right|. \tag{42}$$

Step 3 (*lower bound—conclusion*). Using the results of the previous Steps 1 and 2, in particular (41) and (42), we have

$$\mathrm{TV}(u,B_1)\geqslant\frac{1}{2}\max_{\pm}\left|\sum_{j\in\mathbb{N}}\int_{I_j^+}\{v^1v_\vartheta^2-v^2v_\vartheta^1\}\,d\vartheta\pm\sum_{k\in\mathbb{N}}\int_{I_k^-}\{v^1v_\vartheta^2-v^2v_\vartheta^1\}\,d\vartheta\right|.$$

Since $\max_\pm|a\pm b|=|a|+|b|$, we finally obtain the lower bound (30).

Step 4 (*upper bound*). Assume first that $u:[0,2\pi]\to\gamma=\gamma^+\cup\gamma^-$ is a Lipschitz-continuous map satisfying the further assumptions of Lemma 17. In Particular, we assume that there exist N disjoint open intervals $I_j\subset[0,2\pi]$ such that $u(I_j)$ is contained either in γ^+ or in γ^- for every $j=1,2,\ldots,N$, and $u(\vartheta)=(0,0)$ when $\vartheta\notin\bigcup_{j=1}^N I_j$. We also assume that the function $\vartheta\to u^1(\vartheta)u_\vartheta^2(\vartheta)-u^2(\vartheta)u_\vartheta^1(\vartheta)$ has piecewise constant sign in $[0,2\pi]$. Then, for every $\varepsilon>0$, there exists a Lipschitz-continuous map $\tilde{w}:B_1\to\mathbb{R}^2$ satisfying the boundary condition $\tilde{w}(1,\vartheta)=u(\vartheta)$ for $\vartheta\in[0,2\pi]$, and (37). For every $h\in\mathbb{N}$ we define

$$u_h(\varrho,\vartheta):=\begin{cases}u(\vartheta) & \text{if } 1/h\leqslant\varrho\leqslant1,\\ \tilde{w}(\varrho_h,\vartheta) & \text{if } 0\leqslant\varrho\leqslant1/h.\end{cases}$$

As in Step 2 of the Proof of Theorem 1, u_h converges to u strongly in $W^{1,p}(B_1:\mathbb{R}^2)$ for every $p\in(1,2)$, as $h\to+\infty$. Finally, by (37),

$$\begin{aligned}\int_{B_1}|\det Du_h(x)|\,dx&=\int_{B_{1/h}}|h^2\det Dw(\varrho h,\vartheta)|\,dx=\int_{B_1}|\det D\tilde{w}(\varrho,\vartheta)|\,dx\\&<\varepsilon+\frac{1}{2}\sum_{j=1}^N\left|\int_{I_j}\{u^1(\vartheta)u_\vartheta^2(\vartheta)-u^2(\vartheta)u_\vartheta^1(\vartheta)\}\,d\vartheta\right|,\end{aligned}$$

and thus we obtain the conclusion (29) in this case, i.e.,

$$\mathrm{TV}(u,B_1)\leqslant\mathrm{TV}^{\mathrm{s}}(u,B_1)\leqslant\frac{1}{2}\sum_{j=1}^N\left|\int_{I_j}\{u^1u_\vartheta^2-u^2u_\vartheta^1\}\,d\vartheta\right|.$$

Step 5 (*upper bound again*). Consider first the case where $u:[0,2\pi]\to\gamma$ satisfies the conditions of the previous Step 4, with the possible additional assumption that the function $\vartheta\to u^1(\vartheta)u_\vartheta^2(\vartheta)-u^2(\vartheta)u_\vartheta^1(\vartheta)$ has piecewise constant sign in $[0,2\pi]$. Assume further that there exist N disjoint open intervals $I_j\subset[0,2\pi]$ such that $u(I_j)$ is contained either in γ^+ or in γ^- for every $j=1,2,\ldots,N$, and $u(\vartheta)=(0,0)$ when $\vartheta\notin\bigcup_{j=1}^N I_j$. We proceed in a way similar to that of Step 3 of the proof of Theorem 1.

We consider one of such intervals I_j such that $u(I_j)\subset\gamma^+$ and, without loss of generality, we can assume that $u(0)=(2,0)\in\gamma^+$. Note that the map $u(\vartheta)$ can be represented in the form

$$u(\vartheta)=2\cos A_w(\vartheta)(\cos A_w(\vartheta),\sin A_w(\vartheta))$$

for $\vartheta\in I_j$. As in Step 3 of the proof of Theorem 1, we may find a sequence $(u_{j,k})_{k\in\mathbb{N}}$, with $u_{j,k}: I_j\to\mathbb{R}^2$, such that, as $k\to+\infty$,

$$\begin{cases} u_{j,k}\to u & \text{in } C^0(\overline{I_j}),\\ \frac{du_{j,k}}{d\vartheta}\to\frac{du}{d\vartheta} & \text{strongly in } L^q(I_j)\ \forall q\geqslant 1.\end{cases}$$

Moreover $u_{j,k}(\vartheta)=u(\vartheta)$ for $\vartheta\in\partial I_j$, and $u^1_{j,k}\,du^2_{j,k}/d\vartheta-u^2_{j,k}\,du^1_{j,k}/d\vartheta$ has piecewise constant sign in I_j. Then the map $u_{j,k}(\vartheta)$ satisfies all the assumptions of the previous Step 4. We define

$$u_k(\vartheta):=\begin{cases} u_{j,k}(\vartheta) & \text{if } \vartheta\in I_j,\\ (0,0) & \text{if } \vartheta\notin\bigcup_j I_j.\end{cases}$$

Clearly the maps $u_k(\vartheta)$ converge to u in the strong topology of $W^{1,p}(B_1;\mathbb{R}^2)$ for every $p\in[1,2)$, as $k\to+\infty$, and from Step 4 we obtain the upper bound (29) under our assumptions, i.e.,

$$\begin{aligned}\mathrm{TV}(u,B_1)&\leqslant \mathrm{TV}^{\mathrm{s}}(u,B_1)\leqslant\liminf_{k\to+\infty}\mathrm{TV}^{\mathrm{s}}(u_k,B_1)\\ &=\lim_{k\to+\infty}\frac{1}{2}\sum_{j=1}^{N}\left|\int_{I_j}\left\{u^1_{j,k}\frac{du^2_{j,k}}{d\vartheta}-u^2_{j,k}\frac{du^1_{j,k}}{d\vartheta}\right\}d\vartheta\right|\\ &=\frac{1}{2}\sum_{j=1}^{N}\left|\int_{I_j}\left\{u^1\frac{du^2}{d\vartheta}-u^2\frac{du^1}{d\vartheta}\right\}d\vartheta\right|.\end{aligned}\tag{43}$$

Finally, when the intervals I_j are infinitely many, the upper bound (29) is deduced from the previous case of finitely many intervals I_j $(j=1,2,\ldots,N)$, approximating u by

$$u_N(\vartheta):=\begin{cases} u(\vartheta) & \text{if } \vartheta\in\bigcup_{j=1}^{N} I_j,\\ (0,0) & \text{if } \vartheta\notin\bigcup_{j=1}^{N} I_j,\end{cases}$$

Indeed, applying (43) to each u_N and passing to the limit as $N\to+\infty$, we obtain

$$\begin{aligned}\mathrm{TV}(u,B_1)&\leqslant \mathrm{TV}^{\mathrm{s}}(u,B_1)\leqslant \liminf_{N\to+\infty}\mathrm{TV}^{\mathrm{s}}(u_N,B_1)\\ &\leqslant \lim_{N\to+\infty}\frac{1}{2}\sum_{j=1}^{N}\left|\int_{I_j}\left\{u^1\frac{du^2}{d\vartheta}-u^2\frac{du^1}{d\vartheta}\right\}d\vartheta\right|\\ &=\frac{1}{2}\sum_{j=1}^{\infty}\left|\int_{I_j}\left\{u^1\frac{du^2}{d\vartheta}-u^2\frac{du^1}{d\vartheta}\right\}d\vartheta\right|. \qquad \square\end{aligned}$$

As an application of the estimates (29) and (30) we propose some examples related to the "*eight*" *curve*. Notice that these examples illustrate cases where $\mathrm{TV}(u,B_1)$ differs from the total variation $|\mathrm{Det}\,Du|(B_1)$ of the distributional determinant.

Example 18. Let $h,k\in\mathbb{Z}$, and let $v:[0,2\pi]\to\gamma$ be the curve whose image turns $|h|$ times in γ^- and $|k|$ times in γ^+, according to the parametric representation

$$v(\vartheta):=\begin{cases}(-1,0)+(\cos 2h\vartheta,\sin 2h\vartheta) & \text{if } 0\leqslant\vartheta\leqslant\pi,\\ (1,0)-(\cos 2k\vartheta,\sin 2k\vartheta) & \text{if } \pi\leqslant\vartheta\leqslant 2\pi.\end{cases}$$

Since

$$v^1v^2_\vartheta-v^2v^1_\vartheta=\begin{cases}2h(1-\cos 2h\vartheta) & \text{if } 0<\vartheta<\pi,\\ 2k(\cos 2k\vartheta-1) & \text{if } \pi<\vartheta<2\pi,\end{cases}$$

then, with $u(x):=v(x/|x|)$, by the representation formulas (29), (30), we have

$$\begin{cases}\mathrm{TV}(u,B_1)=(|h|+|k|)\pi,\\ |\mathrm{Det}\,Du|(B_1)=|h-k|\pi,\end{cases}\quad \forall h,\ k\in\mathbb{Z}. \tag{44}$$

Example 19. We consider the map

$$v(\vartheta):=\begin{cases}(-1,0)+(\cos 2\vartheta,\sin 2\vartheta) & \text{if } 0\leqslant\vartheta\leqslant\pi,\\ (1,0)\pm(-\cos 2\vartheta,\sin 2\vartheta) & \text{if } \pi\leqslant\vartheta\leqslant 2\pi,\end{cases} \tag{45}$$

and we extend it by periodicity from $[0,2\pi]$ to $\mathbb{R}$. Then we define $v_h(\vartheta):=v(h\vartheta)$, for a given parameter $h\in\mathbb{Z}$. The image of v_h is contained in γ^+ and γ^- in correspondence with two sets of disjoint open intervals of $[0,2\pi]$ which, with the notations introduced above, we denote by I_j^+ and I_k^-, respectively. Then $v(I_j)\subset\gamma^+$ and $v(I_k)\subset\gamma^-$. With

$u_h(x) := v_h(x/|x|)$, by (29) and (30) we obtain

$$\begin{aligned}\mathrm{TV}(u_h, B_1) &= \frac{1}{2}\left|\sum_j \int_{I_j^+} \left\{ v_h^1 \frac{\partial v_h^2}{\partial \vartheta} - v_h^2 \frac{\partial v_h^1}{\partial \vartheta} \right\} d\vartheta \right| \\ &\quad + \frac{1}{2}\left|\sum_k \int_{I_k^-} \left\{ v_h^1 \frac{\partial v_h^2}{\partial \vartheta} - v_h^2 \frac{\partial v_h^1}{\partial \vartheta} \right\} d\vartheta \right| \\ &= \frac{1}{2}\left|\sum_j \int_{I_j^+} 2h(1-\cos 2h\vartheta)\, d\vartheta \right| \\ &\quad + \frac{1}{2}\left|\sum_k \int_{I_k^-} 2h(\cos 2h\vartheta - 1)\, d\vartheta \right| = 2|h|\pi.\end{aligned}$$

In this situation we have

$$\begin{cases} \mathrm{TV}(u_h, B_1) = 2|h|\pi, \\ |\mathrm{Det}\, Du_h|(B_1) = 0, \end{cases} \quad \forall h \in \mathbb{Z}.$$

Example 20. The map $v : [0, 2\pi] \to \gamma$ defined by

$$v(\vartheta) := \begin{cases} (-1,0) + (\cos 4\vartheta, \sin 4\vartheta) & \text{if } 0 \leqslant \vartheta \leqslant \pi/2, \\ (1,0) + (-\cos 4\vartheta, \sin 4\vartheta) & \text{if } \pi/2 \leqslant \vartheta \leqslant \pi, \\ (-1,0) + (\cos 4\vartheta, \sin 4\vartheta) & \text{if } \pi \leqslant \vartheta \leqslant 3\pi/2, \\ (1,0) + (-\cos 4\vartheta, \sin 4\vartheta) & \text{if } 3\pi/2 \leqslant \vartheta \leqslant 2\pi, \end{cases} \tag{46}$$

spans γ^- twice counter-clockwise, and γ^+ twice clockwise. It is a particular case of the previous Example 19 and, with the usual notation $u(x) := v(x/|x|)$, we have $\mathrm{TV}(u, B_1) = 4\pi$ and $|\mathrm{Det}\, Du|(B_1) = 0$.

Consider now the map $\bar{v} : [0, 2\pi] \to \gamma$ defined by

$$\bar{v}(\vartheta) := \begin{cases} (-1,0) + (\cos 4\vartheta, \sin 4\vartheta) & \text{if } 0 \leqslant \vartheta \leqslant \pi/2, \\ (1,0) + (-\cos 4\vartheta, \sin 4\vartheta) & \text{if } \pi/2 \leqslant \vartheta \leqslant \pi, \\ (-1,0) + (\cos 4\vartheta, -\sin 4\vartheta) & \text{if } \pi \leqslant \vartheta \leqslant 3\pi/2, \\ (1,0) + (-\cos 4\vartheta, -\sin 4\vartheta) & \text{if } 3\pi/2 \leqslant \vartheta \leqslant 2\pi, \end{cases} \tag{47}$$

which spans γ^- twice, the first time counter-clockwise and the second time clockwise; while γ^+ is spanned first clockwise and then counter-clockwise. Then again, with $\bar{u}(x) := \bar{v}(x/|x|)$, the estimate (29) yields $\mathrm{TV}(\bar{u}, B_1) \leqslant 4\pi$, while (30) gives $\mathrm{TV}(\bar{u}, B_1) \geqslant 0 = |\mathrm{Det}\, D\bar{u}|(B_1)$. Therefore, this is an example where there is a gap between the estimates (29) and (30).

The last example related to $\bar{v}$ was already considered by Malý [21] and by Giaquinta et al. [18], who proved that the graph of $\bar{u}$ cannot be approximated in area by the graphs of smooth maps.

4. Some lower semicontinuity estimates

In this section, we prove some lower semicontinuity estimates used in the previous sections. Since there are no major technical differences between the 2- and the n-dimensional case, we consider here the general n-dimensional case. We first recall a lower semicontinuity result, valid for *polyconvex* integrands (and for *quasiconvex* integrands as well), related to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^n)$ for some p *below* the critical exponent n. These may be called *nonstandard* lower semicontinuity results, as opposed to the *classical* setting of lower semicontinuity results in the weak topology of $W^{1,p}(\Omega;\mathbb{R}^n)$ when p is equal to the *growth* exponent of the integrand f (see [2,22,24]). We refer to *polyconvex* integrals as in Theorem 21 below, of the type

$$\int_\Omega f(Du)\,dx, \quad \text{with } 0\leqslant f(\xi)\leqslant c(1+|\xi|^p).$$

In the case considered here, the integrand $f(\xi) := |\det \xi| \leqslant n^{-n/2}|Du(x)|^n$ has growth exponent equal to n, while we need to consider the weak topology of $W^{1,p}(\Omega;\mathbb{R}^n)$ for some $p<n$.

Theorem 21 below has been proved by Marcellini [22,23] for $p>n^2/(n+1)$ and by Dacorogna and Marcellini [12] for $p>n-1$ ($p\geqslant 1$ if $n=2$). A limiting case, with $p=n-1$, has been considered under different assumptions by Acerbi and Dal Maso [1], Celada and Dal Maso [11], Dal Maso and Sbordone [13] and by Fusco and Hutchinson [17]. The *relaxation* in this context has been first considered by Fonseca and Marcellini [16].

Precisely, the following theorem holds (we limit ourselves to quote here the polyconvex case, related to maps $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}^m$ with $m=n$). Given a map $u:\Omega\to\mathbb{R}^n$, we denote by $M(Du)$ the vector-valued map

$$M(Du) = (Du, \operatorname{adj}_2 Du, \ldots, \operatorname{adj}_{n-1} Du, \det Du) \in \mathbb{R}^N,$$

where, for $j=2,\ldots,n-1$, $\operatorname{adj}_j Du$ denotes the matrix of all minors $j\times j$ of Du and $N=\sum_{j=1}^n \binom{n}{j}^2$ (in particular $N=5$ if $n=2$).

Theorem 21 (Lower semicontinuity below the critical exponent). *Let Ω be an open set of $\mathbb{R}^n$. Let $g:\mathbb{R}^N\to\mathbb{R}$ be a nonnegative convex function. Then*

$$\liminf_{h\to+\infty}\int_\Omega g(M(Du_h))\,dx \geqslant \int_\Omega g(M(Du))\,dx,$$

for every sequence u_h which converge to u in the weak topology of $W^{1,p}(\Omega;\mathbb{R}^n)$ for some $p>n-1$, with $u,u_h\in W^{1,n}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ for every $h\in\mathbb{N}$.

The first result stated in Lemma 22 gives a lower bound for the total variation. It is a variant of Lemma 5.1 (see also Lemma 2.3) by Marcellini [23], who considered the general *quasiconvex* case with the exponent p below the critical growth exponent n, precisely $n^2/(n+1)<p<n$.

Lemma 22 (Lower bound—first estimate). *Let $u\in L^\infty(\Omega;\mathbb{R}^n)\cap W^{1,p}(\Omega;\mathbb{R}^n)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega\backslash\{0\};\mathbb{R}^n)$ for some $p\in(n-1,n)$. The following estimate holds*

$$\mathrm{TV}(u,\Omega)\geqslant\left|\int_\Omega \det D\tilde{u}(x)\,dx\right|, \tag{48}$$

whenever $\tilde{u}:\Omega\to\mathbb{R}^n$ is a Lipschitz-continuous map which agrees with u on the boundary of Ω, i.e., $\tilde{u}(x)=u(x)$ on $\partial\Omega$.

Proof. For fixed $p\in(n-1,n)$, $\delta>0$, consider a sequence $\{u_h\}_{h\in\mathbb{N}}$ in $W^{1,n}(\Omega;\mathbb{R}^n)$ that converges to u in the weak topology of $W^{1,p}(\Omega;\mathbb{R}^n)$, and such that

$$\lim_{h\to+\infty}\int_\Omega|\det Du_h(x)|\,dx\leqslant TV(u,\Omega)+\delta. \tag{49}$$

Let $M:=||u||_{L^\infty(\Omega;\mathbb{R}^n)}\in\mathbb{R}$. Truncate each u_h into $w_h=(w_h^1,w_h^2,\ldots,w_h^n)$ whose components are given by

$$w_h^j(x):=\begin{cases}-M & \text{if } u_h^j(x)\leqslant -M,\\ u_h^j(x) & \text{if } -M\leqslant u_h^j(x)\leqslant M,\\ M & \text{if } u_h^j(x)\geqslant M,\end{cases}$$

for all $j=0,1,\ldots,n$. Clearly $\{u_h\}_{h\in\mathbb{N}}$ still converges to u, as $h\to+\infty$, in the weak topology of $W^{1,p}(\Omega;\mathbb{R}^n)$ and the L^∞-norm $||w_h||_{L^\infty(\Omega;\mathbb{R}^n)}$ is uniformly bounded as $h\in\mathbb{N}$. Moreover, since

$$w_h(x)\neq u_h(x)\Rightarrow\det Dw_h(x)=0,$$

we obtain $|\det Dw_h(x)|\leqslant|\det Du_h(x)|$ for almost every $x\in\Omega$, and

$$\liminf_{h\to+\infty}\int_\Omega|\det Dw_h(x)|\,dx\leqslant\lim_{h\to+\infty}\int_\Omega|\det Du_h(x)|\,dx<TV(u,\Omega)+\delta.$$

Therefore, without loss of generality, passing to a subsequence if necessary, we can assume that the limit relation (49) holds, together with the uniform bound

$$\sup_{h\in\mathbb{N}} ||u_h||_{L^\infty(\Omega;\mathbb{R}^n)} = M < +\infty. \tag{50}$$

Let Ω_0 be an open set compactly contained in Ω and let $R := \operatorname{dist}(\Omega_0, \partial\Omega)/2$, with $0\in\Omega_0$. For every $k\in\mathbb{N}$ set

$$\Omega_i := \left\{x\in\Omega:\ \operatorname{dist}(x,\Omega_0) < \frac{iR}{k}\right\}, \quad \forall i = 1, 2, \ldots, k.$$

For every $i = 1, 2, \ldots, k$, consider a smooth *cut-off* scalar function φ_i with compact support in Ω_i, such that $\varphi_i(x) = 1$ in Ω_{i-1}, $0\leqslant\varphi_i(x)\leqslant 1$ and $|D\varphi_i(x)|\leqslant\frac{k+1}{R}$ for all x. Then, for every $i = 1, 2, \ldots, k$, and for $h\in\mathbb{N}$, define

$$w_{h,i}(x) := (1-\varphi_i(x))u(x) + \varphi_i(x)u_h(x).$$

Then $w_{h,i}(x) = u(x)$ for every $x\in\Omega\backslash\Omega_i$, and in particular for every $x\in\Omega\backslash\Omega_0$. Since $u(x)$ is a smooth map in $\Omega\backslash\Omega_0$ and since $W_{h,i}(x)$ and $\tilde{u}(x)$ are smooth maps in Ω, which coincide with $u(x)$ on the boundary $\partial\Omega$, using the fact that the integral of the Jacobian depends only on the trace at the boundary, we have

$$\begin{aligned}\left|\int_\Omega \det D\tilde{u}(x)\,dx\right| &= \left|\int_\Omega \det Dw_{h,i}(x)\,dx\right| \leqslant \int_\Omega |\det Dw_{h,i}(x)|\,dx\\ &= \int_{\Omega_{i-1}} |\det Du_h(x)|\,dx + \int_{\Omega_i\backslash\Omega_{i-1}} |\det Dw_{h,i}(x)|\,dx\\ &\quad + \int_{\Omega\backslash\Omega_i} |\det Du(x)|\,dx.\end{aligned}$$

Letting $h\to+\infty$, taking into account the limit relation (49), summing up the above relation with respect to $i = 1, 2, \ldots, k$, and dividing both sides by k, we obtain

$$\begin{aligned}\left|\int_\Omega \det D\tilde{u}(x)\,dx\right| &\leqslant \mathrm{TV}(u,\Omega) + \delta\\ &\quad + \frac{1}{k}\limsup_{h\to+\infty}\sum_{i=1}^{k}\int_{\Omega_i\backslash\Omega_{i-1}} |\det Dw_{h,i}(x)|\,dx\\ &\quad + \int_{\Omega\backslash\Omega_0} |\det Du(x)|\,dx.\end{aligned} \tag{51}$$

We estimate the second integral in the right-hand side. To this aim, we recall the following inequality (which, for instance, can be obtained from inequality (2.9) of Marcellini [22])

$$||\det \xi| - |\det \eta|| \leqslant c(1 + |\xi|^{n-1} + |\eta|^{n-1})|\xi - \eta|. \tag{52}$$

As $Dw_{h,i}(x) = D[(I - \varphi_i(x))u(x) + \varphi_i(x)u_h(x)]$, in $\Omega_i \backslash \Omega_{i-1}$ we have

$$\begin{aligned} |Dw_{h,i}(x) - \varphi_i(x)Du_h(x)| &\leqslant |D\varphi_i(x)||u_h(x) - u(x)| \\ &\quad + |1 - \varphi_i(x)||Du(x)| \leqslant \frac{k+1}{R}|u_h(x) - u(x)| + |Du(x)|. \end{aligned}$$

From (52) with $\xi := Dw_{h,i}(x)$ and $\eta := \varphi_i(x)Du_h(x)$ we obtain

$$\begin{aligned} &||\det Dw_{h,i}(x)| - |\det \varphi_i(x)Du_h(x)|| \\ &\quad \leqslant c(1 + |Dw_{h,i}(x)|^{n-1} + |Du_h(x)|^{n-1})\left[\frac{k+1}{R}|u_h(x) - u(x)| + |Du(x)|\right]. \end{aligned}$$

Set $M_1 := ||Du||_{L^\infty(\Omega\backslash\Omega_0;\mathbb{R}^{n\times n})} \in \mathbb{R}$. Then, since $p > n-1$, for the second integral in the right-hand side of (51) we have the following bound

$$\begin{aligned} &\int_{\Omega_i\backslash\Omega_{i-1}} |\det Dw_{h,i}(x)|\,dx \\ &\leqslant \int_{\Omega_i\backslash\Omega_{i-1}} |\det \varphi_i(x)Du_h(x)|\,dx \\ &\quad + c\int_{\Omega_i\backslash\Omega_{i-1}} \left\{(1 + |Dw_{h,i}(x)|^{n-1} + |Du_h(x)|^{n-1})\left[\frac{k+1}{R}|u_h(x) - u(x)| + M_1\right]\right\} dx \\ &\leqslant \int_{\Omega_i\backslash\Omega_{i-1}} |\det \varphi_i(x)Du_h(x)|\,dx + c\left\{\int_{\Omega_i\backslash\Omega_{i-1}} (1 + |Dw_{h,i}(x)|^{n-1} + |Du_h(x)|^{n-1})^{\frac{p}{n-1}} dx\right\}^{\frac{n-1}{p}} \\ &\quad \times \left\{\int_{\Omega_i\backslash\Omega_{i-1}} \left[\frac{k}{(R-r)}|u_h(x) - u(x)| + M_1\right]^{\frac{p}{p-(n-1)}} dx\right\}^{\frac{p-(n-1)}{p}}. \end{aligned}$$

The sequences $\{u_h\}_{h\in\mathbb{N}}$ and $\{w_h\}_{h\in\mathbb{N}}$ converge to u in the weak topology of $W^{1,p}(\Omega;\mathbb{R}^n)$ and the L^p-norm of their gradients remains bounded. Up to a subsequence, as $h \to +\infty$, the difference $\{u_h(x) - u(x)\}_{h\in\mathbb{N}}$ converges almost everywhere to zero. By taking into account the uniform bound (50), we can pass

to the limit as $h\to+\infty$ and we obtain

$$\begin{aligned}
&\limsup_{h\to+\infty}\sum_{i=1}^{k}\int_{\Omega_i\setminus\Omega_{i-1}}|\det Dw_{h,i}(x)|\,dx\\
&\quad\leqslant\limsup_{h\to+\infty}\sum_{i=1}^{k}\int_{\Omega_i\setminus\Omega_{i-1}}|\det\varphi_i(x)Du_h(x)|\,dx+c_1\cdot M_1|\Omega_i\setminus\Omega_{i-1}|^{\frac{p-(n-1)}{p}}\\
&\quad\leqslant\limsup_{h\to+\infty}\int_{\Omega\setminus\Omega_0}|\det Du_h(x)|\,dx+c_1kM_1|\Omega\setminus\Omega_0|^{\frac{p-(n-1)}{p}}\\
&\quad=\mathrm{TV}(u,\Omega)+\delta+c_1kM_1|\Omega\setminus\Omega_0|^{\frac{p-(n-1)}{p}}.
\end{aligned}\tag{53}$$

From (51) and (53) we deduce that

$$\begin{aligned}
\left|\int_\Omega\det D\tilde{u}(x)\,dx\right|&\leqslant\mathrm{TV}(u,\Omega)+\delta\\
&\quad+\frac{1}{k}\{\mathrm{TV}(u,\Omega)+\delta+c_1kM_1|\Omega\setminus\Omega_0|^{\frac{p-(n-1)}{p}}\}+\int_{\Omega\setminus\Omega_0}|\det Du(x)|\,dx.
\end{aligned}$$

Letting $k\to+\infty$, $\Omega_0\to\Omega$ and $\delta\to0^+$, we conclude

$$\left|\int_\Omega\det D\tilde{u}(x)\,dx\right|\leqslant\mathrm{TV}(u,\Omega).\qquad\square$$

Lemma 23 (Lower bound—second estimate). *Let u be a function of class $L^\infty(\Omega;\mathbb{R}^n)\cap W^{1,p}(\Omega;\mathbb{R}^n)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega\setminus\{0\};\mathbb{R}^n)$ for some $p\in(n-1,n)$. For every $r>0$ such that $B_r\subset\Omega$ the following estimate holds*

$$\mathrm{TV}(u,\Omega)\geqslant\int_{\Omega\setminus B_r}|\det Du(x)|\,dx+\left|\int_{B_r}\det D\tilde{u}(x)\,dx\right|,\tag{54}$$

where $\tilde{u}:B_r\to\mathbb{R}^n$ is any Lipschitz-continuous map which coincides with u on the boundary of B_r, i.e., $\tilde{u}(x)=u(x)$ on ∂B_r.

Proof. Fix $\delta>0$ and consider a sequence $\{u_h\}_{h\in\mathbb{N}}$ in $W^{1,n}(\Omega;\mathbb{R}^n)$ which converges to u in the weak topology of $W^{1,p}(\Omega;\mathbb{R}^n)$ for some $p\in(n-1,n)$ and such that (49)

holds. For every $r>0$ such that $B_r \subset \Omega$ we have

$$\begin{aligned} \mathrm{TV}(u,\Omega)+\delta &\geqslant \lim_{h\to+\infty}\int_{\Omega}|\det Du_h(x)|\,dx \\ &\geqslant \liminf_{h\to+\infty}\int_{\Omega\setminus B_r}|\det Du_h(x)|\,dx+\liminf_{h\to+\infty}\int_{B_r}|\det Du_h(x)|\,dx \\ &\geqslant \liminf_{h\to+\infty}\int_{\Omega\setminus B_r}|\det Du_h(x)|\,dx+\mathrm{TV}(u,B_r). \end{aligned}$$

We estimate the term $\mathrm{TV}(u,B_r)$ with (48). Moreover, since u,u_h belong to $W^{1,n}(\Omega\setminus B_r;\mathbb{R}^n)$ for every $h\in N$ (and u_h converge to u in the weak topology of $W^{1,p}(\Omega\setminus B_r;\mathbb{R}^n)$ for $p>n-1$), we can apply the lower semicontinuity result below the critical exponent stated in Theorem 21. We reach the conclusion (54) as $\delta\to 0^+$. □

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