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Bounds for the Third Membrane Eigenvalue

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1. INTRODUCTION

We are concerned with the eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in} \quad \Omega, \qquad u = 0 \qquad \text{on} \qquad \partial\Omega, \tag{1}$$

where Ω is a bounded open set in the (x, y)-plane, $\partial\Omega$ is the boundary of Ω and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator. We denote by λ_n, $n = 1, 2,...,$ the nondecreasing sequence of eigenvalues and by u_n ($\| u_n \|_{L^2} = 1$) the sequence of the corresponding eigenfunctions.

In 1956 Payne *et al.* [4] discovered that there exists an upper bound for λ_n (in terms of λ_i, $i = 1,..., n - 1$) independent of the shape of Ω. For example, they proved that for any n and for any Ω

$$\lambda_{n+1} \leqslant 3\lambda_n . \tag{2}$$

There have been improvements in (2), especially for small values of n. With $n = 1$ Brands [1] proved that $\lambda_2 \leqslant 2.687\ \lambda_1$ and later De Vries [2] obtained $\lambda_2 \leqslant 2.658\ \lambda_1$. We recall that if Ω is a disk then $\lambda_2/\lambda_1 = 2.538...$. It has been conjectured [4] that for all plane domains the maximum ratio of successive eigenvalues occurs for a disk with $n = 1$.

Here we are interested in the inequalities

$$\lambda_3 \leqslant A\lambda_1 , \qquad \lambda_2 + \lambda_3 \leqslant B\lambda_1 . \tag{3}$$

Taking into account the known values of λ_1, λ_2, λ_3 for the disk and the rectangle, we must have

$$A > 3.181, \qquad B > 5.077.$$

In the quoted paper, Payne *et al.* proved that (3) holds with constants

$$A = 4.334,^1 \qquad B = 6. \tag{4}$$

[1] In [4] it is explicitly shown that $\lambda_3 \leqslant 5\ \lambda_1$ holds; but $\lambda_3/\lambda_1 \leqslant 13/3$ easily follows from the other inequalities proved there: $\lambda_3 \leqslant \lambda_1 + 2\lambda_2$ and $\lambda_3 \leqslant 6\lambda_1 - \lambda_2$.

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Later Brands [1] showed that (3) holds with

$$A = 4.098, \qquad B = 5.646, \tag{5}$$

and recently Hile and Protter [3] have improved these constants, obtaining

$$A = 4.015, \qquad B = 5.622. \tag{6}$$

The aim of this paper is to prove that (3) holds with constants

$$A = 3.917, \qquad B = 5.596. \tag{7}$$

2. Proof of (3), (7)

Let us choose the coordinates so that

$$\int_\Omega x u_1{}^2 = \int_\Omega y u_1{}^2 = \int_\Omega y u_1 u_2 = 0; \tag{8}$$

in fact with a rotation we can have $\int_\Omega y u_1 u_2 = 0$ and then with a translation (which does not change $\int_\Omega y u_1 u_2 = 0$) the first two equalities in (8) can be satisfied.

As in [4] for $i = 1, 2$ we introduce the trial functions

$$\phi_i = x u_i - \sum_{j=1}^{2} a_{ij} u_j, \qquad \text{where} \quad a_{ij} = \int_\Omega x u_i u_j .$$

Since each ϕ_i is orthogonal to u_1 and u_2, we have

$$\lambda_3 \leqslant \frac{-\int_\Omega \phi_i \, \Delta \phi_i}{\int_\Omega \phi_i{}^2} \qquad \text{and} \qquad \int_\Omega \phi_i{}^2 = \int_\Omega x u_i \phi_i .$$

Using the relations

$$-\int_\Omega \phi_i \, \Delta\phi_i = \lambda_i \int_\Omega x u_i \phi_i - 2 \int_\Omega (u_i)_x \phi_i ,$$

we get

$$(\lambda_3 - \lambda_i) \int_\Omega \phi_i{}^2 \leqslant -2 \int_\Omega (u_i)_x \phi_i . \tag{9}$$

Without loss of generality we can assume that $-2 \int_\Omega (u_2)_x \phi_2$ is positive because, if not, by (9) we should have $\lambda_3 = \lambda_2$ so that our results should be a consequence of the known bounds for λ_2/λ_1. Therefore, by applying Schwarz's inequality, we have

$$-2 \int_\Omega (u_2)_x \phi_2 \leqslant \frac{4 \int_\Omega (u_2)_x^2 \int_\Omega \phi_2^2}{-2 \int_\Omega (u_2)_x \phi_2} ,$$

which, combined with (9), gives

$$\lambda_3 - \lambda_2 \leqslant \frac{4 \int_\Omega (u_2)_x^2}{-2 \int_\Omega (u_2)_x \phi_2} . \tag{10}$$

Now the denominator of (10) is

$$\begin{aligned} -2 \int_\Omega (u_2)_x \phi_2 &= -2 \int_\Omega x(u_2)_x u_2 + 2 \sum_{j=1}^{2} a_{2j} \int_\Omega (u_2)_x u_j \\ &= -\int_\Omega x(u_2{}^2)_x + 2a_{21} \int_\Omega (u_2)_x u_1 + a_{22} \int_\Omega (u_2{}^2)_x \\ &= 1 + 2 \int_\Omega x u_1 u_2 \int_\Omega (u_2)_x u_1 . \end{aligned} \tag{11}$$

Moreover

$$\begin{aligned} \int_\Omega x u_1 u_2 &= -\frac{1}{\lambda_2} \int_\Omega x u_1 \, \Delta u_2 \\ &= \frac{1}{\lambda_2} \int_\Omega (u_1 + x(u_1)_x)(u_2)_x + x(u_1)_y (u_2)_y \\ &= -\frac{1}{\lambda_2} \int_\Omega (2(u_1)_x + x \, \Delta u_1) \, u_2 \\ &= -\frac{2}{\lambda_2} \int_\Omega (u_1)_x u_2 + \frac{\lambda_1}{\lambda_2} \int_\Omega x u_1 u_2 , \end{aligned}$$

from which we derive

$$(\lambda_2 - \lambda_1) \int_\Omega x u_1 u_2 = -2 \int_\Omega (u_1)_x u_2 = 2 \int_\Omega (u_2)_x u_1 . \tag{12}$$

So from (10), (11) and (12) we get

$$\lambda_3 - \lambda_2 \leqslant \frac{4 \int_\Omega (u_2)_x^2}{1 + (\lambda_2 - \lambda_1)(\int_\Omega x u_1 u_2)^2} . \tag{13}$$

Now we start again from (9) with $i = 1$. We use (12) and the analogue of (11) (on interchanging 1 and 2):

$$\begin{aligned} (\lambda_3 - \lambda_1) \int_\Omega \phi_1{}^2 &\leqslant -2 \int_\Omega (u_1)_x \phi_1 \\ &= 1 + 2 \int_\Omega x u_1 u_2 \int_\Omega (u_1)_x u_2 \\ &= 1 - (\lambda_2 - \lambda_1) \left(\int_\Omega x u_1 u_2 \right)^2 . \end{aligned} \tag{14}$$

Since $a_{11} = \int_\Omega xu_1^2 = 0$, we have $\phi_1 = xu_1 - u_2 \int_\Omega xu_1u_2$, so that

$$\int_\Omega \phi_1^2 = \int_\Omega x^2u_1^2 - \left(\int_\Omega xu_1u_2\right)^2.$$

So from the above relations we derive

$$(\lambda_3 - \lambda_1) \int_\Omega x^2u_1^2 \leqslant 1 + (\lambda_3 - \lambda_2) \left(\int_\Omega xu_1u_2\right)^2. \tag{15}$$

We have a similar, but simpler, situation if x is replaced by y since $\int_\Omega yu_1u_2 = 0$. The analogues of (13) and (15) are, respectively,

$$\lambda_3 - \lambda_2 \leqslant 4 \int_\Omega (u_2)_y^2, \tag{16}$$

$$(\lambda_3 - \lambda_1) \int_\Omega y^2u_1^2 \leqslant 1. \tag{17}$$

From (13) and (16) we deduce

$$2 + (\lambda_2 - \lambda_1) \left(\int_\Omega xu_1u_2\right)^2 \leqslant \frac{4\lambda_2}{\lambda_3 - \lambda_2}$$

(as before we assume $\lambda_3 \neq \lambda_2$), which combined with (15) gives

$$\begin{aligned}(\lambda_3 - \lambda_1) \int_\Omega x^2u_1^2 &\leqslant 1 + \frac{\lambda_3 - \lambda_2}{\lambda_2 - \lambda_1} \left(\frac{4\lambda_2}{\lambda_3 - \lambda_2} - 2\right) \\ &= 1 + \frac{6\lambda_2 - 2\lambda_3}{\lambda_2 - \lambda_1} = \frac{7\lambda_2 - \lambda_1 - 2\lambda_3}{\lambda_2 - \lambda_1}.\end{aligned} \tag{18}$$

Finally, we use the inequality of Brands [1] (cf. also Lemma 4 of [3]):

$$\left(\int_\Omega x^2u_1^2\right)^{-1} + \left(\int_\Omega y^2u_1^2\right)^{-1} \leqslant \frac{\lambda_1}{\lambda_2} (\lambda_1 + 3\lambda_2), \tag{19}$$

which, together with (17) and (18), gives

$$(\lambda_3 - \lambda_1) \left(\frac{\lambda_2 - \lambda_1}{7\lambda_2 - \lambda_1 - 2\lambda_3} + 1\right) \leqslant \frac{\lambda_1}{\lambda_2} (\lambda_1 + 3\lambda_2). \tag{20}$$

Now we note that the above relation is a quadratic inequality with respect to λ_3 and, since by (18), $7\lambda_2 - \lambda_1 - 2\lambda_3$ is positive, the left side of (20) is increasing as λ_3 increases. So (20) gives an upper bound for λ_3 in terms of the lower ratio (with respect to λ_3) of the associated quadratic equation. We can rewrite (20) in the form

$$2\lambda_2\lambda_3^2 - 2(\lambda_1^2 + 3\lambda_1\lambda_2 + 4\lambda_2^2)\, \lambda_3 + 29\lambda_1\lambda_2^2 + 2\lambda_1^2\lambda_2 - \lambda_1^3 \geqslant 0,$$

and, using the notation $\nu = \lambda_2/\lambda_1$, we get

$$\frac{\lambda_3}{\lambda_1} \leqslant \frac{1}{2\nu}(1 + 3\nu + 4\nu^2 - ((1 + 3\nu + 4\nu^2)^2 - 58\nu^3 - 4\nu^2 + 2\nu)^{1/2}). \quad (21)$$

Now we note that, from (14), $1 - (\lambda_2 - \lambda_1)(\int_\Omega x u_1 u_2)^2 \geqslant 0$, so by (15),

$$(\lambda_3 - \lambda_1) \int_\Omega x^2 u_1{}^2 \leqslant 1 + \frac{\lambda_3 - \lambda_2}{\lambda_2 - \lambda_1} = \frac{\lambda_3 - \lambda_1}{\lambda_2 - \lambda_1},$$

and, together with (17) and (19), it follows that

$$(\lambda_2 - \lambda_1) + (\lambda_3 - \lambda_1) \leqslant \frac{\lambda_1}{\lambda_2}(\lambda_1 + 3\lambda_2).$$

Hence

$$\frac{\lambda_3}{\lambda_1} \leqslant 5 - \nu + \frac{1}{\nu}. \quad (22)$$

Let us denote by $f(\nu)$ the right side of (21) and by $g(\nu)$ the right side of (22). It is possible to see that $f(\nu) = g(\nu)$ for $\nu = \nu_0 = 1/20\ (15 + (345)^{1/2}) = 1.6787...$, that f is increasing in $[1, \nu_0]$, and that g is decreasing for $\nu \geqslant 1$. So

$$\begin{aligned}\frac{\lambda_3}{\lambda_1} &\leqslant \max_{\nu\geqslant 1}\{\min(f(\nu); g(\nu))\} = g(\nu_0)\\ &= \frac{7}{3}\nu_0 = \frac{7}{60}(15 + (345)^{1/2}) = 3.91698...,\end{aligned}$$

and for the same reason

$$\frac{\lambda_2 + \lambda_3}{\lambda_1} \leqslant g(\nu_0) + \nu_0 = \frac{1}{6}(15 + (345)^{1/2}) = 5.59569\ldots .$$

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