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SEMICONTINUITY PROBLEMS IN THE CALCULUS OF VARIATIONS

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1. INTRODUCTION

WE ARE concerned with integral functionals of the form

$$F(\Omega, u) = \int_\Omega f(x, u, Du)\,\mathrm{d}x, \tag{1.1}$$

where Ω is a bounded open set of $\mathbf{R}^n$ and $u \in H^{1,p}(\Omega)$ for some $p \geqslant 1$. Throughout the paper we assume that $f: \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ satisfies the following conditions:

(a) $0 \leqslant f(x, s, \xi) \leqslant g(x, |s|, |\xi|)$, with g increasing with respect to $|s|$ and $|\xi|$, and locally summable in x.
(b) $f(x, s, \xi)$ is measurable with respect to x, upper semicontinuous with respect to ξ, and continuous with respect to s uniformly as ξ varies on each bounded set of $\mathbf{R}^n$. (1.2)

Two meaningful cases in which (1.2b) is satisfied are the following: (i) $f = f(x, \xi)$ is measurable in x and upper semicontinuous in ξ; (ii) f is a Caratheodory function, i.e. measurable in x and continuous in (s, ξ).

Several authors have studied (and proved, under suitable hypothesis) the sequential lower semicontinuity (s.l.s.) of $F(\Omega, \cdot)$ in the weak topology of $H^{1,p}(\Omega)$. A well known theorem of Serrin [1] assures the s.l.s. of F under the assumption that f is a non negative continuous function, convex in ξ. More recently some improvements of Serrin's theorem have been given by De Giorgi [2], Berkowitz [3], Cesari [4], Ioffe [5], Olech [6] by considering the s.l.s. of the functional

$$(u, v) \in L^p(\Omega) \times [L^q(\Omega)]^n \to \int_\Omega f(x, u, v)\,\mathrm{d}x \tag{1.3}$$

in the product topology of the $L^p(\Omega)$ topology and the $[L^q(\Omega)]^n$ weak topology. Moreover in [5], [6] (see also Ball's result [7]) both necessary and sufficient conditions (such as convexity with respect to ξ) are given for the s.l.s. of (1.3).

In Section 2 of this paper we prove that the convexity of f with respect to ξ is a necessary condition for the s.l.s. of F. This result was first discovered by Tonelli ([8], Chapter X, Section 1) in the case $f \in C^2$, $n = 1$; it was proved by Caccioppoli and Scorza Dragoni [9] for $f \in C^1$, $n = 2$

and for any n by McShane [10] and later by Morrey ([11], Theorems 4.4.2 and 4.4.3) in the case that f is a C^0 function of vector valued arguments. Recently Ekeland and Temam ([12], Chapter X) have proved the necessity of convexity if f is a Caratheodory function independent of s. In Theorem 2.4 we give a generalization of these results in the case that f satisfies (2.1).

In Section 3 we consider the following problem: if f is not convex with respect to ξ, which is the greatest functional (s.l.s. in the weak topology of $H^{1,p}(\Omega)$) which is less than or equal to $F(\Omega, \cdot)$? We list some cases in which it is possible to characterize this functional as the integral

$$\int_\Omega f^{**}(x, u, Du)\mathrm{d}x, \tag{1.4}$$

where $f^{**}(x, s, \xi)$ is the greatest function (convex in ξ) which is less than or equal to f. Some unsolved problems remain when f^{**} is not a Caratheodory function.

We note a few references concerning the above discussed problem: some results of Ekeland and Temam ([12], Chapter X) and Theorem 2 of authors' note [13], which we extend in Section 3 of this paper; a similar problem considered by Serrin [1] (see also [14], [15], [16]) in the space $BV(\Omega)$ of functions with bounded variation; the problem considered in [17] with the $L^p(\Omega)$ norm topology instead of the $H^{1,p}(\Omega)$ weak topology; a survey of these and other related results given in [18]; theorem 3H of Rockafellar [19] about integrals of the type of (1.3).

In Section 4 we apply the results of Section 3 to the relaxation (in the sense of Ekeland and Temam [12]) of some variational problems. Relaxation means that, starting from a minimum problem for F which lacks a solution, a second minimum problem (the relaxed or generalized problem) is formulated involving an integral of the type of (1.4), with the same infimum and whose optimal solutions are the limit points of the sequences minimizing the first problem.

We study the relaxation for the Dirichlet and Neumann problems, for the obstacle problem and the relaxation for the convex set of functions with a prescribed bound on the Lipschitz constant.

2. NECESSARY CONDITIONS FOR THE SEMICONTINUITY

In this section we prove that a necessary condition for the sequentially lower semicontinuity (s.l.s.) in the weak topology of $H^{1,p}(\Omega)$† of the integral (1.1) is the convexity of $f(x, s, \xi)$ with respect to ξ, under the following assumptions on $f: \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$

$$\left.\begin{array}{l}\text{(a)}\ \ |f(x,s,\xi)| \leqslant g(x,|s|,|\xi|),\ g \text{ like in (1.2a).}\\ \text{(b)}\ \ f(x,s,\xi) \text{ measurable in } x \text{ and continuous in } (s,\xi).\end{array}\right\} \tag{2.1}$$

LEMMA 2.1. Let u_k be a sequence converging to u in $w^* - H^{1,\infty}$ such that $\|Du_k\|_{L^\infty} \leqslant r$. Then, for any $\varepsilon > 0$ there exists v_k converging to u in $w^* - H^{1,\infty}$ such that $v_k = u$ on $\partial\Omega$, $\|Dv_k\|_{L^\infty} \leqslant r + \varepsilon$ and

$$\lim_k [F(\Omega, u_k) - F(\Omega, v_k)] = 0. \tag{2.2}$$

† By $H^{1,p}(\Omega)$ ($1 \leqslant p \leqslant \infty$, Ω open set in $\mathbf{R}^n$) we indicate the usual Sobolev space

$$H^{1,p}(\Omega) = \{u \in L^p(\Omega): Du \in [L^p(\Omega)]^n\}$$

equipped with the usual norm. We indicate by $w - H^{1,p}$ ($w^* - H^{1,\infty}$) the weak topology in $H^{1,p}(\Omega)$ (the weak* topology in $H^{1,\infty}(\Omega)$).

Proof. Set $\delta_k = \varepsilon^{-1} \| u_k - u \|_{L^\infty}$ and $\Omega_k = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \delta_k\}$. Let us define v_k on $\Omega_k \cup \partial\Omega$ such that $v_k = u_k$ on Ω_k, $v_k = u$ on $\partial\Omega$. Clearly v_k is a Lipschitz function, in fact if $x \in \Omega_k$, $y \in \partial\Omega$:

$$|v_k(x) - v_k(y)| \leqslant r|x - y| + \varepsilon\delta_k < (r + \varepsilon)|x - y|.$$

Using McShane's lemma ([12], Chapter X, Section 1.1) v_k can be extended on Ω so that $\| Dv_k \|_{L^\infty} \leqslant r + \varepsilon$.

For $x \in \Omega$, $v_k(x) \to v(x)$ and also, by the preceding bound, $v_k \rightharpoonup u$ in $w^* - H^{1,\infty}$. From the relation

$$|F(\Omega, u_k) - F(\Omega, v_k)| \leqslant 2 \int_{\Omega \setminus \Omega_k} g(x, s, r + \varepsilon)\, \mathrm{d}x$$

with $s = \max\{\sup_k \| u_k \|_{L^\infty}, \sup_k \| v_k \|_{L^\infty}\}$ we deduce (2.2) as $\lim_k |\Omega \setminus \Omega_k| = 0$.

Lemma 2.2. Let $F(\Omega, \cdot)$ be $w - H^{1,p}$ s.l.s. on $u_0 + H_0^{1,p}(\Omega)$ for a fixed $u_0 \in H^{1,p}(\Omega)$. Then, for any $\Omega' \subseteq \Omega$, $F(\Omega', \cdot)$ is $w^* - H^{1,\infty}$ s.l.s.

Proof. Let $\Omega' \subset\subset \Omega$ and $\varphi \in C_0^1(\Omega)$ such that $\varphi = 1$ on Ω'. If $u_k \in H^{1,\infty}(\Omega)$ converges to u in $w^* - H^{1,\infty}(\Omega')$, let $v_k \in H^{1,\infty}(\Omega)$ satisfy on Ω' the conditions of previous lemma and coincide with u on $\Omega \setminus \Omega'$. Then, as the sequence $w_k = \varphi v_k + (1 - \varphi)u_0 \in u_0 + H_0^{1,p}(\Omega)$ converges in $w - H^{1,p}$ to $w = \varphi u + (1 - \varphi)u_0$, $w_k = w$ on $\Omega \setminus \Omega'$, $w_k = v_k$ and $w = u$ on Ω', we have by the s.l.s. of F

$$F(\Omega', u) + F(\Omega \setminus \Omega', w) \leqslant \liminf_k \{F(\Omega', v_k) + F(\Omega \setminus \Omega', w)\},$$

and by (2.2)

$$F(\Omega', u) \leqslant \liminf_k F(\Omega', v_k) = \liminf_k F(\Omega', u_k).$$

This proves the lemma in the case $\Omega' \subset\subset \Omega$. In general, if $\Omega'' \subset\subset \Omega' \subseteq \Omega$ we deduce from the preceding

$$F(\Omega'', u) \leqslant \liminf_k F(\Omega'', u_k) \leqslant \liminf_k F(\Omega', u_k),$$

and the result, passing to the limit as $\Omega'' \nearrow \Omega'$.

Let us now introduce some notations. Set $\Omega =]0, a[^n$ and for a fixed $\lambda \in]0, 1[\cap Q$ let us define $\chi(t) = 1$ if $0 < t < \lambda a$, $\chi(t) = 0$ if $\lambda a \leqslant t \leqslant a$, and extend it to $\mathbf{R}$ by periodicity; then set

$$\Omega_k = \{x \equiv (x_i) \in \Omega : \chi(kx_1) = 1\}. \tag{2.3}$$

Lemma 2.3. Let Ω and Ω_k be as above. If ψ_k converges to ψ in $L^1(\Omega)$, we have

$$\lim_k \int_{\Omega_k} \psi_k(x)\, \mathrm{d}x = \lambda \int_\Omega \psi(x)\, \mathrm{d}x. \tag{2.4}$$

Proof. For l, m integers $0 < l < m$, let $\lambda = l/m$ and $\Omega_k^{(0)}$ be the sets in (2.3) with $\lambda = 1/m$. We can assume the existence of the limit

$$\lim_k \int_{\Omega_k^{(0)}} \psi_k(x)\, \mathrm{d}x = L.$$

For $j = 1, 2, \ldots, m-1$ set $y_k^{(j)} = (ja/mk, 0, \ldots, 0)$ and $\Omega_k^{(j)} = \Omega_k^{(0)} + y_k^{(j)}$ so that

$$\bigcup_{j=0}^{l-1} \overline{\Omega_k^{(j)}} = \overline{\Omega_k}, \qquad \bigcup_{j=0}^{m-1} \overline{\Omega_k^{(j)}} = \overline{\Omega}.$$

If we put $\psi = 0$ out of Ω, we obtain

$$\left| \int_{\Omega_k^{(0)}} \psi_k(x)\,dx - \int_{\Omega_k^{(j)}} \psi_k(x)\,dx \right| = \left| \int_{\Omega_k^{(0)}} \{\psi_k(x) - \psi_k(x + y_k^{(j)})\}\,dx \right|$$

$$\leqslant 2\|\psi_k - \psi\|_{L_1} + \int_\Omega |\psi(x) - \psi(x + y_k^{(j)})|\,dx,$$

and this tends to zero as $k \to +\infty$ since translations are L^1-continuous. Then, passing to the limit in the relation

$$\sum_{j=0}^{m-1} \int_{\Omega_k^{(j)}} \psi_k(x)\,dx = \int_\Omega \psi_k(x)\,dx,$$

we obtain $mL = \int_\Omega \psi(x)\,dx$ and also

$$\lim_k \int_{\Omega_k} \psi_k(x)\,dx = \lim_k \sum_{j=0}^{l-1} \int_{\Omega_k^{(j)}} \psi_k(x)\,dx = 1/m \int_\Omega \psi(x)\,dx,$$

which coincides with (2.4) since the sequence $\int_{\Omega_k} \psi_k(x)\,dx$ is relatively compact.

We establish now the principal result of this section

THEOREM 2.4. Let Ω be a bounded open set in $\mathbf{R}^n$, $p \in [1, \infty]$, $u_0 \in H^{1,p}(\Omega)$, and $F(\Omega, .)$ as in (1.1) with f satisfying (2.1) be a s.l.s. functional on $u_0 + H_0^{1,p}(\Omega)$ in the weak (weak* if $p = \infty$) topology of $H^{1,p}(\Omega)$. Then, for almost every $x \in \Omega$, for every $s \in \mathbf{R}$, $f(x, s, .)$ is convex.

Proof. Let $\xi \equiv (\xi_i)$, $\eta \equiv (\eta_i) \in \mathbf{R}^n$ with $\xi \neq \eta$; in particular $\xi_1 \neq \eta_1$. For λ rational let us define the function (extended by periodicity to $\mathbf{R}$)

$$\varphi(t) = \begin{cases} \xi_1 & 0 < t < \lambda \\ \eta_1 & \lambda \leqslant t \leqslant 1. \end{cases}$$

For any integer k set

$$\Phi_k(t) = \int_0^t \varphi(k\tau)\,d\tau.$$

Define now $u_k(x_1, \ldots, x_n) = \Phi_k\left(x_1 + \sum_{i=2}^n a_i x_i\right) + \sum_{i=2}^n b_i x_i$, where a_i, b_i satisfy

$$\xi_1 a_i + b_i = \xi_i, \quad \eta_1 a_i + b_i = \eta_i \qquad \forall i = 2, \ldots, n, \tag{2.5}$$

so that $(u_k)_{x_1}$ has the only values ξ_1 and η_1.

If $\Omega' \subset \Omega$ is an open cube with sides 2^{-j} (j integer) parallel to the hyperplane $x_1 + \sum_{i=2}^n a_i x_i = 0$, let Ω_k be the subset of Ω' in which $(u_k)_{x_1} = \xi_1$. From (2.5) we deduce $(u_k)_{x_i} = \xi_i$ in Ω_k and $(u_k)_{x_i} = \eta_i$ in $\Omega'\backslash\Omega_k$; so from Lemma 2.3 we deduce

$$\lim_k \int_{\Omega'} (u_k)_{x_i}\varphi \, dx = \lim_k \left\{ \int_{\Omega_k} \xi_i \varphi \, dx + \int_{\Omega'\backslash\Omega_k} \eta_i \varphi \, dx \right\} = \int_{\Omega'} \{\lambda \xi_i + (1-\lambda)\eta_i\} \varphi \, dx, \forall \varphi \in L^1(\Omega').$$

Since $u_k(0) = 0$, we deduce the uniform convergence in Ω' of u_k to $u(x) = \sum_{i=1}^n \{\lambda\xi_i + (1-\lambda)\eta_i\} x_i$ and also $u_k \rightharpoonup u$ in $w^* - H^{1,\infty}(\Omega')$; so that $\psi_k(x) = f(x, u_k(x), \xi)$ converges to $\psi(x) = f(x, u(x), \xi)$ a.e. and in $L^1(\Omega')$using (2.1a), and similarly for $\psi_k'(x) = f(x, u_k(x), \eta)$.

Then, using Lemma 2.3 and, by Lemma 2.2, the semicontinuity of $F(\Omega', u)$, we have

$$\begin{aligned}\int_{\Omega'} f(x, u, \lambda\xi + (1-\lambda)\eta) \, dx &\leqslant \liminf_k \left\{ \int_{\Omega_k} f(x, u_k, \xi) \, dx + \int_{\Omega \ \Omega_k} f(x, u_k, \eta) \, dx \right\} \\ &= \lambda \int_{\Omega'} f(x, u, \xi) \, dx + (1-\lambda) \int_{\Omega'} f(x, u, \eta) \, dx. \qquad (2.6)\end{aligned}$$

Using the semicontinuity of the first term of (2.6) we pass from λ rational to any $\lambda \in (0, 1)$.

Moreover, replacing u_k by $u_k + s$ we obtain

$$\int_{\Omega'} f(x, u+s, \lambda\xi + (1-\lambda)\eta) \, dx \leqslant \lambda \int_{\Omega'} f(x, u+s, \xi) \, dx + (1-\lambda) \int_{\Omega'} f(x, u+s, \eta) \, dx \qquad (2.7)$$

for any $\lambda \in (0, 1)$, $s \in \mathbf{R}$, Ω'.

Let x be a Lebesgue point for the functions in (2.7) as s varies in Q, dividing by meas Ω' and letting $\Omega' \to x$ we find

$$f(x, u+s, \lambda\xi + (1-\lambda)\eta) \leqslant \lambda f(x, u+s, \xi) + (1-\lambda) f(x, u+s, \eta)$$

which holds also for any $s \in \mathbf{R}$, by the continuity of f in s.

3. CHARACTERIZATION OF THE S.L.S. ENVELOPE

The aim of this section is the characterization of the s.l.s. envelope of the functional $F(\Omega, .)$ in (1.1) with respect to the weak topology in $H^{1,p}(\Omega)$. In this section f verifies (1.2).

For any bounded open set $\Omega \subseteq \mathbf{R}^n$, $u \in H^{1,\infty}(\Omega)$: $\|Du\|_{L^\infty} \leqslant r$ let us define the functionals

$$\bar{F}(r, \Omega, u) = \inf\left\{ \liminf_k F(\Omega, u_h) : u_k \xrightarrow{w^* - H^{1,\infty}} u, \quad \|Du_k\|_{L^\infty} \leqslant r \right\}, \qquad (3.1)$$

$$\bar{F}_0(r, \Omega, u) = \inf\left\{ \liminf_k F(\Omega, u_k) : u_k - u \xrightarrow{w^* - H^{1,\infty}} 0, \ \|Du_k\|_{L_\infty} \leqslant r \right\}. \qquad (3.2)$$

Remark. Since the $w^* - H^{1,\infty}$ topology is metrizable on bounded sets in $H^{1,\infty}(\Omega)$, the infima in (3.1), (3.2) are minima and $\bar{F}(r, \Omega, .)$, $\bar{F}_0(r, \Omega, .)$ are the s.l.s. envelope of $F(\Omega, .)$ in $w^* - H^{1,\infty}$ on the set $\{u \in H^{1,\infty}(\Omega): \|Du\|_{L^\infty} \leqslant r\}$.

LEMMA 3.1. If $\Omega, \Omega_1, \Omega_2$, are bounded open sets of $\mathbf{R}^n$ satisfying $\Omega_1 \cap \Omega_2 = \varnothing, \overline{\Omega_1 \cup \Omega_2} = \bar{\Omega}$, then $\forall u \in H^{1,\infty}(\Omega)$, $r \geqslant \|Du\|_{L^\infty}$, we have

$$\bar{F}(r, \Omega, u) \geqslant \bar{F}(r, \Omega_1, u) + \bar{F}(r, \Omega_2, u), \qquad (3.3)$$

$$\bar{F}_0(r, \Omega, u) \leqslant \bar{F}_0(r, \Omega_1, u) + \bar{F}_0(r, \Omega_2, u). \tag{3.4}$$

Proof. Let u_k converge to u in $w^* - H^{1,\infty}$ and $\lim_k F(\Omega, u_k) = \bar{F}(r, \Omega, u)$. We have

$$\begin{aligned}\bar{F}(r, \Omega, u) &= \lim_k \{F(\Omega_1, u_k) + F(\Omega_2, u_k)\} \\ &\geqslant \liminf_k F(\Omega_1, u_k) + \liminf_k F(\Omega_2, u_k) \\ &\geqslant \bar{F}(r, \Omega_1, u) + \bar{F}(r, \Omega_2, u).\end{aligned}$$

This proves (3.3), and (3.4) is proved similarly.

A simple consequence of Lemma 2.1 is the following

LEMMA 3.2. If Ω is a bounded open set in $\mathbf{R}^n$, $u \in H^{1,\infty}(\Omega)$, $r' > r \geqslant \|Du\|_{L^\infty}$, then:

$$\bar{F}(r, \Omega, u) \geqslant \bar{F}_0(r', \Omega, u).$$

For Ω bounded open set in $\mathbf{R}^n$, $u \in H^{1,\infty}(\Omega)$, $r \geqslant \|Du\|_{L^\infty}$, let us define

$$\Phi(r, \Omega, u) = \sup_{r' > r} \bar{F}(r', \Omega, u) = \sup_{r' > r} \bar{F}_0(r', \Omega, u). \tag{3.5}$$

Remark. The equality of the suprema in (3.5) follows from Lemma 3.2 and the fact that $\bar{F} \leqslant \bar{F}_0$. Part of what follows could be simplified assuming f Caratheodory. In fact in this case $\Phi = \bar{F} = \bar{F}_0$ as a consequence of Lemma 4.5.

We shall study the dependence of Φ on Ω, u and r.

LEMMA 3.3. For any $u \in H^{1,\infty}(\Omega)$ and $r \geqslant \|Du\|_{L^\infty}$ there exists a measurable function h such that $0 \leqslant h(x) \leqslant f(x, u(x), Du(x))$ a.e. and

$$\Phi(r, \Omega, u) = \int_\Omega h(x)\, \mathrm{d}x, \tag{3.6}$$

for any Ω bounded open set in $\mathbf{R}^n$.

Proof. Using (3.3), (3.4), (3.5), if $\Omega_1 \cap \Omega_2 = \varnothing$ and $\overline{\Omega_1 \cup \Omega_2} = \bar{\Omega}$ we have

$$\Phi(r, \Omega, u) = \Phi(r, \Omega_1, u) + \Phi(r, \Omega_2, u). \tag{3.7}$$

Let us denote by $\mathscr{R}$ the ring of the finite unions of all the cubes $\{x \equiv (x_i): x_i^0 < x_i \leqslant x_i^0 + a\}$ and, fixed u and r, for $\Omega \in \mathscr{R}$ $\mu(\Omega) = \Phi(r, \Omega \backslash \partial\Omega, u)$. The set function μ is non negative, finitely additive by (3.7) and absolutely continuous, i.e.

$$0 \leqslant \mu(\Omega) \leqslant \int_{\Omega\backslash\partial\Omega} f(x, u(x), Du(x))\, dx, \qquad \forall \Omega \in \mathscr{R}.$$

Then μ is a measure on $\mathscr{R}$ (see [20] Section 9, Theorem F) which can be extended to the family of all Lebesgue measurable sets (see [20] Section 13, Theorems A, B); the Radon–Nykodim theorem assures that μ has a density $h(x)$ satisfying $0 \leqslant h(x) \leqslant f(x, u(x), Du(x))$. So we have (3.6) for any $\Omega \in \mathscr{R}$ and also for any Ω bounded open set in $\mathbf{R}^n$, since μ and $\Phi(r, ., u)$ are absolutely continuous.

Let us indicate by w the oscillation of f, i.e.

$$w(x, r, s, \delta) = \sup\{|f(x, s_1, \xi) - f(x, s_2, \xi)|:|\xi| \leqslant r, |s_i| \leqslant s, |s_1 - s_2| \leqslant \delta\};$$

since $0 \leqslant w(x, r, s, \delta) \leqslant 2g(x, s, r)$, then w is integrable with respect to x and the integral extended on every open set Ω tends to zero as $\delta \to 0^+$.

LEMMA 3.4. If u_1, $u_2 \in H^{1,\infty}(\Omega)$, $\|Du_1\|_{L^\infty} \leqslant r$, $Du_1 = Du_2$, $\|u_i\|_{L^\infty} \leqslant s$, then for $r' > r$, $s' > s$ one has

$$|\Phi(r, \Omega, u_1) - \Phi(r, \Omega, u_2)| \leqslant \int_\Omega w(x, r', s', \|u_1 - u_2\|_{L^\infty})\, dx. \tag{3.8}$$

Proof. Let u_k converge to u_1 in $w^* - H^{1,\infty}$ with $\|Du_k\|_{L^\infty} \leqslant r''(r < r'' < r')$ and $\lim F(\Omega, u_k) = \bar{F}(r, \Omega, u)$. Set $v_k = u_k - u_1 + u_2$; as v_k converges to u_2 in $w^* - H^{1,\infty}$ we have $\|v_k\|_{L^\infty} \leqslant s'$ for k large, so that

$$\bar{F}(r'', \Omega, u_2) - \bar{F}(r'', \Omega, u_1) \leqslant \liminf_k \{F(\Omega, v_k) - F(\Omega, u_k)\} \leqslant \int_\Omega w(x, r', s', \|u_1 - u_2\|_{L^\infty})\, dx.$$

By charging u_1 with u_2 and passing to the limit as $r'' \to r^+$ we obtain the result.

For any r fixed we denote by $\mathscr{B}_r$ the set of all polynomials u on $\mathbf{R}^n$ with degree less than or equal to one and rotational coefficients such that $|Du| \leqslant r$. Let A be the set of $x \in \mathbf{R}^n$ which are Lebesgue points for any function $h = h_u$ associated with $u \in \mathscr{B}_r$ by Lemma 3.3. The set

$$B = \{(x, s, \xi) \in A \times \mathbf{R} \times \mathbf{R}^n : \exists u \in \mathscr{B}_r : u(x) = s, Du = \xi\}$$

is dense on $\{(x, s, \xi) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n : |\xi| \leqslant r\}$; we define on it the function

$$\varphi(x, s, \xi) = h_u(x), \tag{3.9}$$

where u is the polynomial in $\mathscr{B}_r$ such that $u(x) = s$, $Du = \xi$ and $h = h_u$ is the function associated to u by Lemma 3.3.

LEMMA 3.5. The function φ in (3.9) is convex in ξ.

Proof. Fixed (x_0, s, ξ), $(x_0, s, \eta) \in B$ with $\xi \neq \eta$ set $v(x) = \langle \xi, x - x_0 \rangle + s$. Let λ, Ω, Ω_k, $\Omega_k^{(j)}$, $y_k^{(j)}$ be like in the proof of Lemma 2.3 in the previous section, and set $v_k^{(j)}(x) = v(x + y_k^{(j)})$. From the relation $\Phi(r, \Omega_k^{(j)}, v) = \Phi(r, \Omega_k^{(0)}, v_k^{(j)})$ and (3.8) it follows for some (r', s')

$$\limsup_k |\Phi(r, \Omega_k^{(0)}, v) - \Phi(r, \Omega_k^{(j)}, v)| \leqslant \limsup_k \int_\Omega w(x, r', s', \|v - v_k\|_{L^\infty})\, dx = 0,$$

and, as in Lemma 2.3, $\lim_k \Phi(r, \Omega_k, v) = \lambda \Phi(r, \Omega, v)$.

Similarly if $w(x) = \langle \eta, x - x_0 \rangle + s$, we deduce

$$\lim_k \Phi(r, \Omega \backslash \Omega_k, w) = (1 - \lambda)\, \Phi(r, \Omega, w).$$

Let u_k be defined as in the proof of Theorem 2.4 with $u_k(x_0) = s$; then by (3.8) we have

$$\begin{aligned}\limsup_k \Phi(r, \Omega, u_k) &\leqslant \limsup_k \{\Phi(r, \Omega_k, v) + \Phi(r, \Omega\backslash\Omega_k, w) \\ &\quad + |\Phi(r, \Omega_k, u_k) - \Phi(r, \Omega_k, v)| + |\Phi(r, \Omega\backslash\Omega_k, u_k) - \Phi(r, \Omega\backslash\Omega_k, w)|\} \\ &\leqslant \lambda\Phi(r, \Omega, v) + (1-\lambda)\Phi(r, \Omega, w) + 2\int_\Omega w(x, r', s', \delta_\Omega|\xi - \eta|)\, dx,\end{aligned}$$

where δ_Ω is the side of Ω.

For $\varepsilon > 0$ fixed we can choose δ_Ω small enough in order to obtain, by the semicontinuity of Φ, the relation

$$\Phi(r, \Omega, u) \leqslant \lambda\Phi(r, \Omega, v) + (1-\lambda)\Phi(r, \Omega, w) + \varepsilon|\Omega|,$$

with

$$u(\cdot) = \langle \lambda\xi + (1-\lambda)\eta, \cdot - x_0) + s = w^* - H^{1,\infty} \lim_k u_k(\cdot).$$

Dividing the two sides of last inequality by $|\Omega|$, passing to the limit as $\Omega \to x_0$, by Lemma 3.3 we get

$$h_u(x_0) \leqslant \lambda h_v(x_0) + (1-\lambda)h_w(x_0) + \varepsilon,$$

and the result as $\varepsilon \to 0$.

In order to explicit the dependence of Φ on u, let us introduce the functions f^{**} and f_r^{**} (see [12], [21]):

$f^{**}(x, s, \xi)$ is the greatest function less than f on $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$ convex in ξ. (3.10)

$f_r^{**}(x, s, \xi)$ is the greatest function less than f on $\mathbf{R}^n \times \mathbf{R} \times \{|\xi| \leqslant r\}$ convex in ξ. (3.11)

We will use the following:

LEMMA 3.6. f^{**} is measurable; f_r^{**} is a Caratheodory function for $|\xi| < r$; moreover $\lim_{r\to\infty} f_r^{**} = f^{**}$.

Proof. It is well known that

$$f^{**}(x, s, \xi) = \sup\{\langle \xi^*, \xi\rangle - f^*(x, s, \xi^*) : \xi^* \in \mathbf{R}^n\},$$

where

$$f^*(x, s, \xi^*) = \sup\{\langle \xi^*, \xi\rangle - f(x, s, \xi) : \xi \in \mathbf{R}^n\}.$$

Then f^{**} and f_r^{**} are measurable in (x, s). In order to prove that f_r^{**} is Caratheodory it is enough to prove its continuity in s uniformly for $|\xi| \leqslant r$, and utilize the continuity in ξ for $|\xi| < r$.

To this aim, fixed $x \in \mathbf{R}^n$, $s_0 \in \mathbf{R}$, $\varepsilon > 0$, using (1.2b), there exists $\delta > 0$ such that $|f(x, s, \xi) - f(x, s_0, \xi)| < \varepsilon$ for $|\xi| \leqslant r$ and s such that $|s - s_0| < \delta$; so that for these s, ξ we obtain

$$f^{**}(x, s, \xi) - \varepsilon \leqslant f(x, s, \xi) - \varepsilon < f(x, s_0, \xi),$$

As $f^{**}(x, s, \xi) - \varepsilon$ is convex in ξ we have $f^{**}(x, s, \xi) - \varepsilon \leqslant f^{**}(x, s_0, \xi)$ and, changing s in s_0, the assertion.

Let us now set $\psi(x, s, \xi) = \lim_{r\to\infty} f_r^{**}(x, s, \xi)$; this is convex in ξ and it is less than f since $f_r^{**}(x, s, \xi) \leqslant f(x, s, \xi)$ for $|\xi| \leqslant r$, and so $\psi \leqslant f^{**}$. The opposite inequality follows from $f_r^{**}(x, s, \xi) \geqslant f^{**}(x, s, \xi)$ for $r \geqslant |\xi|$.

In the following lemma we continue the study of the dependence of Φ on u.

LEMMA 3.7. If $r'' < r < r'$, for any bounded open set $\Omega \subset \mathbf{R}^n$, with a Lipschitz boundary and for any $u \in H^{1,\infty}(\Omega)$ such that $\|Du\|_{L^\infty} \leqslant r''$, we have

$$\int_\Omega f_{r'}^{**}(x, u, Du)\,dx \leqslant \Phi(r, \Omega, u) \leqslant \int_\Omega f_{r''}^{**}(x, u, Du)\,dx. \tag{3.12}$$

Proof. If $(x_0, s_1, \xi), (x_0, s_2, \xi) \in B$ with $|s_i| \leqslant s$, setting in (3.8) $u_i(x) = \langle \xi, x - x_0 \rangle + s_i$, $i = 1, 2$, dividing by $|\Omega|$ and passing to the limit as $\Omega \to x_0$ we obtain

$$|\varphi(x_0, s_1, \xi) - \varphi(x_0, s_2, \xi)| \leqslant w(x, r', s', |s_1 - s_2|)$$

for any $r' > r$ and $s' > s$. So, $\varphi(x, \cdot, \xi)$ is uniformly continuous on bounded sets, and, as $\varphi(x, s, \cdot)$ is locally Lipschitz in $|\xi| < r$, φ may be extended to $A \times \mathbf{R} \times \{|\xi| < r\}$ by continuity. We continue to indicate it by φ. By Lemma 3.3 φ is less than f in $|\xi| < r$, and so φ, being a convex function, is less than $f_{r''}^{**}$. Let $u \in \mathscr{B}_{r''}$ with $|Du| \leqslant r''$; then $(x, u(x), Du(x)) \in B$ for any $x \in A$, and

$$\Phi(r, \Omega, u) = \int_\Omega h_u(x)\,dx = \int_\Omega \varphi(x, u, Du)\,dx \leqslant \int_\Omega f_{r''}^{**}(x, u, Du)\,dx.$$

The additivity of $\Phi(r, \cdot, u)$ and the semicontinuity of $\Phi(r, \Omega, \cdot)$ imply

$$\Phi(r, \Omega, u) \leqslant \int_\Omega f_{r''}^{**}(x, u, Du)\,dx, \tag{3.13}$$

for any piecewise affine u. Moreover Proposition 2.9 of Chapter X in [12] imply that for any $u \in H^{1,\infty}(\Omega)$ there exists a sequence u_k of piecewise affine functions uniformly converging to u, such that Du_k converges to Du a.e. and $\|Du_k\|_{L^\infty} \leqslant \|Du\|_{L^\infty}$. Using these approximating functions we get (3.13) for any $u \in H^{1,\infty}(\Omega)$ (with $\|Du\|_{L^\infty} \leqslant r''$).

In order to prove the first inequality in (3.12) it suffices to remark that, $f_{r'}^{**}$ being convex in ξ and Caratheodory in $|\xi| < r'$, the functional $u \to \int_\Omega f_{r'}^{**}(x, u, Du)\,dx$ is s.l.s. in $w^* - H^{1,\infty}(\Omega)$ on the set of all the functions u such that $\|Du\|_{L^\infty} \leqslant r'''$ with $r < r''' < r'$; so this functional being less than F, it is also less than $\bar{F}(r''', \Omega, u)$ and *a fortiori* than $\Phi(r, \Omega, u)$.

We can now prove the following

THEOREM 3.8. Let f satisfy (1.2); then, for any bounded open set Ω with Lipschitz boundary and for $u \in H^{1,\infty}(\Omega)$ we have

$$\inf\left\{\liminf_k \int_\Omega f(x, u_k, Du_k)\,dx : u_k \to u \text{ in } w^* - H^{1,\infty}(\Omega)\right\} \tag{3.14a}$$

$$= \inf\left\{\liminf_k \int_\Omega f(x, u_k, Du_k)\,dx : u_k - u \to 0 \text{ in } w^* - H_0^{1,\infty}(\Omega)\right\} \tag{3.14b}$$

$$= \int_\Omega f^{**}(x, u, Du)\,dx. \tag{3.14c}$$

Proof. By (3.5), by previous lemma and the monotonicity in r of the functions $\bar{F}, \bar{F}_0, \Phi, f_r^{**}$ it follows the existence and the equalities of the following limits as $r \to +\infty$:

$$\lim_r \bar{F}(r, \Omega, u) = \lim_r \bar{F}_0(r, \Omega, u) = \lim_r \Phi(r, \Omega, u) = \lim_r \int_\Omega f_r^{**}(x, u, Du)\,dx. \tag{3.15}$$

If we indicate as $\Psi(\Omega, u)$ the functional in (3.14a), we find $\Psi(\Omega, u) \leqslant \bar{F}(r, \Omega, u)$ for any r, and so $\Psi(\Omega, u) \leqslant \lim_r \bar{F}(r, \Omega, u)$. Vice versa, fixed $u \in H^{1,\infty}(\Omega)$ and $\varepsilon > 0$ it is possible to find a sequence u_k converging to u in $w^* - H^{1,\infty}$ such that $\Psi(\Omega, u) + \varepsilon \geqslant \liminf_k F(\Omega, u_k)$; if $r \geqslant \|Du_k\|_{L^\infty}$ we have by consequence $\Psi(\Omega, u) + \varepsilon \geqslant \bar{F}(r, \Omega, u)$; and so

$$\Psi(\Omega, u) = \lim_r \bar{F}(r, \Omega, u). \tag{3.16}$$

Similarly, denoting by $\Psi_0(\Omega, u)$ the functional in (3.14b) we find

$$\Psi_0(\Omega, u) = \lim_r \bar{F}_0(r, \Omega, u). \tag{3.17}$$

Finally, the Lebesgue theorem and Lemma 3.6 imply

$$\int_\Omega f^{**}(x, u, Du)\,dx = \lim_r \int_\Omega f_r^{**}(x, u, Du)\,dx. \tag{3.18}$$

Let us see now as in some cases it is possible to solve the initial problem of this section, using previous theorem. Let us note first that the functionals in (3.14a) and (3.14b) are not generally s.l.s., as the weak* topology of $H^{1,\infty}(\Omega)$ is not metrizable; similarly the integral in (3.14c) in general is not s.l.s. as $f^{**}(x, s, \xi)$ is not Caratheodory, i.e. is not continuous in s neither l.s. in s. We consider now some examples.

Example 3.9. For $(s, \xi) \in \mathbf{R} \times \mathbf{R}$ let $f(s, \xi) = (|\xi + 1| + 1)^{|s|}$.

Then, the function

$$f^{**}(s, \xi) = \begin{cases} f(s, \xi) & \text{if } |s| \geqslant 1 \\ 1 & \text{if } |s| < 1 \end{cases}$$

is not l.s. with respect to s. Moreover, if $a < b$, the functional

$$\Psi(u) = \int_a^b f^{**}(u, u')\,dx$$

is not s.l.s. in $H^{1,\infty}$. In fact if $u(x) = 1$, $u_k(x) = 1 - 1/k$, we have $\psi(u) = 2(b - a)$, $\psi(u_k) = b - a$. And so the functionals (3.14a) and (3.14b) are not s.l.s.

For any bounded open set $\Omega \subset \mathbf{R}^n$ let us define on $H^{1,p}(\Omega)$ $(1 \leqslant p \leqslant \infty)$ the functionals

$u \to \bar{F}^{(p)}(\Omega, u)$ is the greatest functional less than $F(\Omega, u)$ on $H^{1,p}(\Omega)$, which is s.l.s. in the weak (weak* if $p = \infty$) topology. (3.19)

$u \to \bar{F}_0^{(p)}(\Omega, u)$ is the greatest functional less than $F(\Omega, u)$ on $u + H_0^{1,p}(\Omega)$, which is s.l.s. in the weak (weak* if $p = \infty$) topology. (3.20)

Moreover set

$$\bar{f}^{**}(x, s, \xi) = \inf\left\{\liminf_k f^{**}(x, s_k, \xi_k) : (s_k, \xi_k) \to (s, \xi)\right\}. \tag{3.21}$$

We deduce the following result from Theorem 3.8:

Corollary 3.10. For any $p \in [1, \infty]$, Ω bounded open set in $\mathbf{R}^n$ with Lipschitz boundary, $u \in H^{1,\infty}(\Omega)$ we have

$$\int_\Omega \bar{f}^{**}(x, u, Du)\,dx \leqslant \bar{F}^{(p)}(\Omega, u) \leqslant \bar{F}_0^{(p)}(\Omega, u) \leqslant \int_\Omega f^{**}(x, u, Du)\,dx. \tag{3.22}$$

Proof. The first inequality follows from the s.l.s. of the functional $u \to \int_\Omega \bar{f}^{**}(x, u, Du)\,dx$ in the weak topology of $H^{1,p}(\Omega)$. The second inequality is trivial, while the third follows from Theorem 3.8, since $\bar{F}_0^{(p)}$ is less than the functional in (3.14b).

Clearly, if for any Ω and $u \in H^{1,\infty}(\Omega)$

$$\int_\Omega \bar{f}^{**}(x, u, Du)\,dx = \int_\Omega f^{**}(x, u, Du)\,dx, \tag{3.23}$$

the preceding result gives the characterization of $\bar{F}^{(p)}$ and $\bar{F}_0^{(p)}$. In the following example we show that the condition $\bar{f}^{**} = f^{**}$ is not necessary for (3.23).

Example 3.11. For $(s, \xi) \in \mathbf{R} \times \mathbf{R}$ let us consider the integrand $f(s, \xi) = (|\xi| + 1)^{|s|}$. We have

$$f^{**}(s, \xi) = \begin{cases} f(s, \xi) & \text{if } |s| \geqslant 1 \\ 1 & \text{if } |s| < 1, \end{cases}$$

$$\bar{f}^{**}(s, \xi) = \begin{cases} f(s, \xi) & \text{if } |s| > 1 \\ 1 & \text{if } |s| \leqslant 1. \end{cases}$$

In this case (3.23) holds because either $\{x : u(x) = 1\}$ has zero measure, or u' vanishes a.e. on this set (see [22], Lemma 7.7).

The previous example shows that it would be interesting to study, similarly to what is done by Ioffe [5] for integrals independent of Du, the notion of measurable equivalence. Here we intend f_1 and f_2 *measurable equivalent* if for any $u \in H^{1,\infty}(\Omega)$ $f_1(x, u(x), Du(x)) = f_2(x, u(x), Du(x))$ a.e.

Let us now give sufficient conditions for the equality $\bar{f}^{**} = f^{**}$.

Corollary 3.12. Let us assume that f verifies (1.2) and one of the following conditions:

(i) $f = f(x, \xi)$.
(ii) $f(x, s, \xi)$ is continuous in s uniformly with respect to ξ.
(iii) $f(x, s, \xi) \geqslant \lambda_1 |\xi|^\alpha - \lambda_2(x)$, with $\alpha > 1$, $\lambda_1 > 0$, $\lambda_2 \in L^1_{\text{loc}}$.

Then f^{**} is a Caratheodory function. Moreover for any Ω with Lipschitz boundary, $u \in H^{1,\infty}(\Omega)$, $p \in [1, \infty]$ we have

$$\bar{F}^{(p)}(\Omega, u) = \bar{F}_0^{(p)}(\Omega, u) = \int_\Omega f^{**}(x, u, Du)\,dx. \tag{3.24}$$

Proof. It suffices to prove that f^{**} is Caratheodory, (3.24) following from Corollary 3.10. In the case (i), that is a particular one of (ii), clearly f^{**} is a Caratheodory function. Case (ii) can be handled as in Lemma 3.6 for f_r^{**}. As to (iii), we observe that for any r there exists $r' > r$ such that $f^{**}(x,s,\xi) = f_r^{**}(x,s,\xi)$ for $|\xi| \leqslant r$; as a consequence f^{**} is Caratheodory like f_r^{**} in $|\xi| < r$.

We conclude this section considering the assumption

$$f(x,s,\xi) \leqslant \Lambda_1(x) + \Lambda_2 |s|^p + \Lambda_3 |\xi|^p, \tag{3.25}$$

with $\Lambda_1 \in L^1$; $\Lambda_2, \Lambda_3 \geqslant 0$, $p \geqslant 1$.

COROLLARY 3.13. Let f satisfy (3.25) and the assumptions of preceding corollary; then (3.24) holds for any $u \in H^{1,p}(\Omega)$.

Proof. The right side in (3.24) is continuous in the strong topology of $H^{1,p}(\Omega)$ (see [23], Theorem 2.1); and so, by approximations with $C^1(\Omega)$ functions and by the semicontinuity of $\bar{F}^{(p)}$, (3.24) implies

$$\bar{F}^{(p)}(\Omega, u) \leqslant \int_\Omega f^{**}(x, u, Du)\,\mathrm{d}x. \tag{3.26}$$

The opposite inequality follows from the s.l.s. in $w - H^{1,p}$ of the second member in (3.26). The same is true for $\bar{F}_0^{(p)}$.

4. RELAXATION

In this section we specialize assumption (1.2). In fact, Ω being a bounded open set in $\mathbf{R}^n$ with Lipschitz boundary, we suppose

$$\left.\begin{array}{l}\text{(a) } \lambda_1|\xi|^p - \lambda_2(x) \leqslant f(x,s,\xi) \leqslant \Lambda_1(x) + \Lambda_2|s|^p + \Lambda_3|\xi|^p, \text{ for some } p > 1;\ \lambda_1, \Lambda_2, \Lambda_3 \\ \text{positive and } \lambda_2,\ \Lambda_1 \in L^1(\Omega). \\ \text{(b) } f(x,s,\xi) \text{ measurable in } x, \text{ upper semicontinuous in } \xi, \text{ and continuous in } s \\ \text{uniformly as } \xi \text{ varies in the bounded sets of } \mathbf{R}^n.\end{array}\right\} \tag{4.1}$$

Let C indicate a closed convex set in $H^{1,p}(\Omega)$. Let us consider the problem

$$\text{to minimize} \int_\Omega f(x,u,Du)\,\mathrm{d}x \quad \text{for} \quad u \in C. \tag{4.2}$$

The generalized or "relaxed" problem (in the sense of Ekeland and Teman [12]) is the following:

$$\text{to minimize } \bar{F}_C(u) \text{ for } u \in C, \tag{4.3}$$

where, for any $u \in C$

$$\bar{F}_C(u) = \inf\left\{\liminf_k \int_\Omega f(x,u_k,Du_k)\,\mathrm{d}x : u_k \xrightarrow{w - H^{1,p}} u,\ u_k \in C\right\}. \tag{4.4}$$

The following proposition justifies the term "generalized" of (4.2) for problem (4.3).

PROPOSITION 4.1. Problem (4.3) has solutions. Moreover

$$\inf\left\{\int_\Omega f(x,u,Du)\,\mathrm{d}x : u \in C\right\} = \min\{\bar{F}_C(u) : u \in C\}, \tag{4.5}$$

and any sequence minimizing (4.2) has the minimum functions of (4.3) as limit points in the weak topology of $H^{1,p}(\Omega)$. Conversely, each minimum function of (4.3) is the $w - H^{1,p}$ limit of a sequence minimizing problem (4.2).

Proof. Condition (4.1a), the s.l.s. of $\bar{F}_C$ in $w - H^{1,p}$ and the reflexivity of $H^{1,p}(\Omega)$ ensure that problem (4.3) has solutions. If m is the infimum in the left side of (4.5), then the constant function equal to m is less than or equal to $\bar{F}_C$, since $\bar{F}_C$ is the greatest s.l.s. functional less than $\int_\Omega f(x, u, Du)$, so that (4.5) follows.

If u_k is a sequence minimizing problem (4.2) and if u is the $w - H^{1,p}$ limit of a subsequence u_{k_r}, then

$$\bar{F}_C(u) \leqslant \liminf_r \int_\Omega f(x, u_{k_r}, Du_{k_r})\,\mathrm{d}x = m.$$

So u is a minimum vector for $\bar{F}_C$. Conversely, since for any $u \in C$ we can choose a sequence u_k converging to u in $w - H^{1,p}$ such that

$$\bar{F}_C(u) = \lim_k \int_\Omega f(x, u_k, Du_k)\,\mathrm{d}x,$$

if $\bar{F}_C(u) = m$, then the corresponding u_k is a sequence minimizing the problem (4.2).

In what follows, using the results of previous section, we will indicate some examples of convex C, which are often considered in recent works on the Calculus of Variations, and for which it is possible to characterize the functional $\bar{F}_C$.

Some of the results which we propose, have been proved with a different method by Ekeland and Temam ([12], Chapter X, Theorem 3.7), in the case that f is a Caratheodory function.

Let us begin with the case that C is a linear manifold in $H^{1,p}(\Omega)$ of the form $C = V + u_0$, with V closed linear subspace of $H^{1,p}(\Omega)$ containing $H_0^{1,p}(\Omega)$, and $u_0 \in H^{1,p}(\Omega)$. We recall that, since (4.1) holds, f^{**} is a Caratheodory function.

THEOREM 4.2. If f satisfies (4.1), then

$$\inf\left\{\int_\Omega f(x, u, Du)\,\mathrm{d}x : u \in V + u_0\right\} = \min\left\{\int_\Omega f^{**}(x, u, Du)\,\mathrm{d}x : u \in V + u_0\right\}.$$

Moreover the minimum functions of the first integral are the limit points in $w - H^{1,p}$ of the sequences minimizing the second integral.

Proof. Let $\bar{F}^{(p)}$ and $\bar{F}_0^{(p)}$ be defined as in (3.19), (3.20) and $\bar{F}_C$ as in (4.4) with $C = V + u_0$. Since $H_0^1(\Omega) \subset V \subset H^1(\Omega)$, for any $u \in V + u_0$ we have $\bar{F}^{(p)}(\Omega, u) \leqslant \bar{F}_C(u) \leqslant \bar{F}_0^{(p)}(\Omega, u)$. Then Corollaries 3.12 and 3.13 imply

$$\bar{F}_C(u) = \int_\Omega f^{**}(x, u, Du)\,\mathrm{d}x, \qquad \forall u \in V + u_0.$$

The assertion follows from Proposition 4.1.

Let us now consider the case of the "obstacle"; i.e.

$$C = \{u \in H_0^{1,p}(\Omega) + u_0 : u \geqslant \psi\}, \tag{4.6}$$

where u_0 and $\psi \in H^{1,p}(\Omega)$. We assume also that $C \cap H^{1,\infty}(\Omega) \neq \varnothing$.

THEOREM 4.3. If f satisfies (4.1) and C is given by (4.6), then

$$\inf\left\{\int_\Omega f(x,u,Du)\,\mathrm{d}x : u \in C\right\} = \min\left\{\int_\Omega f^{**}(x,u,Du)\,\mathrm{d}x : u \in C\right\}.$$

Moreover, the minimum functions of the first integral are the limit points in the weak topology of $H^{1,p}(\Omega)$ of the sequences minimizing the second integral.

Proof. Let us consider the Lipschitz continuous function w defined on Ω by $w(x) = \operatorname{dist}(x, \partial\Omega)$. It is easy to see that the subset of $H^{1,\infty}(\Omega)$

$$C_0 = \{v + \varepsilon w : v \in C \cap H^{1,\infty}(\Omega),\ \varepsilon > 0\}$$

is dense in C with respect to the norm of $H^{1,p}(\Omega)$. By theorem 3.8 for any $u \in C_0$ and $\delta > 0$ there exists a sequence u_k converging to u in $w^* - H^{1,\infty}$ such that

$$\liminf_k \int_\Omega f(x,u_k,Du_k)\,\mathrm{d}x \leqslant \int_\Omega f^{**}(x,u,Du)\,\mathrm{d}x + \delta. \tag{4.7}$$

If E is a compact subset of Ω, then for any $x \in E$ we have $u(x) \geqslant \psi(x) + \varepsilon \operatorname{dist}(E, \partial\Omega)$. As a consequence of u_k uniformly converging to u we deduce $\{x \in \Omega : u_k(x) < \psi(x)\} \subset \Omega \backslash E$ for k large. In particular it follows that $\lim_k \operatorname{meas}\{x \in \Omega : u_k(x) < \psi(x)\} = 0$. Setting $v_k = \max\{u_k, \psi\}$, it is easy to check that v_k converges to u weakly in $H^{1,p}(\Omega)$ and that

$$\liminf_k \int_\Omega f(x,v_k,Dv_k)\,\mathrm{d}x = \liminf_k \int_\Omega f(x,u_k,Du_k)\,\mathrm{d}x.$$

Since $v_k \in C$, by (4.7) we have

$$\bar{F}_C(u) \leqslant \liminf_k \int_\Omega f(x,v_k,Dv_k)\,\mathrm{d}x \leqslant \int_\Omega f^{**}(x,u,Du)\,\mathrm{d}x + \delta,$$

and, as $\delta \to 0$,

$$\bar{F}_C(u) \leqslant \int_\Omega f^{**}(x,u,Du)\,\mathrm{d}x \tag{4.8}$$

for any $u \in C_0$. But, since C_0 is $H^{1,p}(\Omega)$-dense in C, $\bar{F}_C$ is s.l.s. and the right side is $H^{1,p}(\Omega)$-continuous, the above inequality holds for any $u \in C$.

Now, if $\bar{F}_0^{(p)}$ is defined by (3.20) on C we have $\bar{F}_0^{(p)} \leqslant \bar{F}_C$, since $C \subset H_0^{1,p}(\Omega) + u_0$. This, together with (4.8) and Corollaries 3.12 and 3.13, gives

$$\bar{F}_C(u) = \int_\Omega f^{**}(x,u,Du)\,\mathrm{d}x, \qquad \forall u \in C.$$

The assertion follows from Proposition 4.1.

Finally we give a result of relaxation for the convex set

$$C = \{u \in H_0^{1,\infty}(\Omega) + u_0 : \|Du\|_{L^\infty} \leqslant r\}, \tag{4.9}$$

where $r > 0$ and $u_0 \in H^{1,\infty}(\Omega)$ satisfies $\|Du_0\|_{L^\infty} < r$.

In this last theorem we assume that $f(x,s,\xi)$ is continuous in ξ.

THEOREM 4.4. Let f be a Caratheodory function satisfying (1.2a) and let C be as in (4.9). Then

$$\inf\left\{\int_\Omega f(x,u,Du)\,dx : u \in C\right\} = \min\left\{\int_\Omega f_r^{**}(x,u,Du)\,dx : u \in C\right\},$$

where f_r^{**} is the Caratheodory function defined in (3.11). Moreover the minimum functions of the first integral are the limit points in the weak* topology of $H^{1,\infty}(\Omega)$ of the sequences minimizing the second integral.

We begin with the following

LEMMA 4.5. Let f be a Caratheodory function satisfying (1.2a) and $\bar{F}$, $\bar{F}_0$, Φ be defined by (3.1), (3.2), (3.5). Then for any u such that $\|Du\|_{L^\infty} < r$, we have $\bar{F}(r,\Omega,u) = \bar{F}_0(r,\Omega,u) = \Phi(r,\Omega,u)$, and these functionals are right continuous with respect to r.

Proof. Let us first prove that for $\|Du\|_{L^\infty} < r$ we have

$$\lim_{r'\to r^+} \bar{F}_0(r',\Omega,u) = \bar{F}_0(r,\Omega,u). \tag{4.10}$$

Let us fix $\varepsilon > 0$ and $u \in H^{1,\infty}(\Omega)$ with $\|Du\|_{L^\infty} < r$. For $\eta_1, \eta_2 > 0$ set

$$\Theta(x,\eta_1,\eta_2) = \sup\{|f(x,s_1,\xi_1) - f(x,s_2,\xi_2)| : |s_1 - s_2| \leqslant \eta_1,\ |\xi_1 - \xi_2| \leqslant \eta_2;$$
$$|s_i| \leqslant \|u\|_{L^\infty} + 1;\ |\xi_i| \leqslant r + 1\}.$$

The continuity of f implies that $\Theta(x,\eta_1,\eta_2) \to 0$ as $(\eta_1,\eta_2) \to 0$. Then it is possible to choose $\eta \in]0,1[$ such that

$$\int_\Omega \Theta(x, 2\eta(\|u\|_{L^\infty} + 1), 2\eta(r+1))\,dx < \varepsilon. \tag{4.11}$$

Set $\delta = (r - \|Du\|_{L^\infty})\eta/(1-\eta)$. We can assume that $\delta \in]0,1[$.

Let u_k converge to u in the weak* topology of $H^{1,\infty}(\Omega)$ and satisfy $u_k - u \in H_0^{1,\infty}(\Omega)$, $\|Du\|_{L^\infty} \leqslant r + \delta$. Set $v_k = (1-\eta)u_k + \eta u$. We have easily that $v_k - u$ converges to zero in $w^* - H_0^{1,\infty}$. Moreover

$$\|Dv_k\|_{L^\infty} \leqslant (1-\eta)(r+\delta) + \eta\|Du\|_{L^\infty} = (1-\eta)r + \eta(r - \|Du\|_{L^\infty}) + \eta\|Du\|_{L^\infty} = r.$$

Finally, reminding that $v_k - u_k = \eta(u - u_k)$, from (4.11) we deduce

$$\int_\Omega f(x,v_k,Dv_k)\,dx \leqslant \int_\Omega f(x,u_k,Du_k)\,dx + \varepsilon.$$

So we have proved that $\bar{F}_0(r,\Omega,u) \leqslant \bar{F}_0(r+\delta,\Omega,u) + \varepsilon$. And this, using the fact that $\bar{F}_0$ is increasing with respect to r, gives the relation (4.10). The assertion in the lemma is now a consequence of (4.10), of the relation $\bar{F} \leqslant \bar{F}_0$ and of Lemma 3.2.

Proof of Theorem 4.4. Let us prove that for any u for which $\|Du\|_{L^\infty} \leqslant r$ we have:

$$\bar{F}_0(r,\Omega,u) = \int_\Omega f_r^{**}(x,u,Du)\,dx. \tag{4.12}$$

First we observe that $f_r^{**}(x,s,\xi)$ is a Caratheodory function for $|\xi| \leqslant r$. In fact, according to the proof of Lemma 3.6, f_r^{**} is continuous in s uniformly for $|\xi| \leqslant r$. Moreover it is convex in ξ

and so it is continuous for $|\xi| < r$; the continuity in ξ for $|\xi| = r$ can be obtained as in ([12], Chapter X, Lemma 3.1). Therefore the right side of (4.12) is $w^* - H^{1,\infty}$ s.l.s., so that

$$\bar{F}_0(r, \Omega, u) \geqslant \int_\Omega f_r^{**}(x, u, Du)\,\mathrm{d}x. \tag{4.13}$$

Now we use the previous Lemma 4.5 and the second inequality in (3.12) obtaining for any $\varepsilon > 0$ and u such that $\|Du\|_{L^\infty} \leqslant r$

$$\bar{F}_0(r + \varepsilon, \Omega, u) \leqslant \int_\Omega f_r^{**}(x, u, Du)\,\mathrm{d}x.$$

Using again Lemma 4.5, as $\varepsilon \to 0$ we get the opposite inequality of (4.13) for u such that $\|Du\|_{L^\infty} < r$.

Let us now consider the general case $\|Du\|_{L^\infty} \leqslant r$: the sequence $u_k = (1 - 1/k)u + u_0/k$ converges to u in $H^{1,\infty}(\Omega)$, agrees with u_0 on $\partial\Omega$ and satisfies

$$\|Du_k\|_{L^\infty} \leqslant \|Du\|_{L^\infty} - \frac{1}{k}(\|Du\|_{L^\infty} - \|Du_0\|_{L^\infty}) < r;$$

therefore

$$\bar{F}_0(r, \Omega, u) \leqslant \liminf_k \bar{F}_0(r, \Omega, u_k) \leqslant \liminf_k \int_\Omega f_r^{**}(x, u_k, Du_k)\,\mathrm{d}x = \int_\Omega f_r^{**}(x, u, Du)\,\mathrm{d}x.$$

This implies (4.12). We get the complete proof of our theorem using the proof of Proposition 4.1.

Added in proof. G. Dal Maso sent us an example of a function $f(x, \xi)$ not upper semicontinuous with respect to ξ, such that $\int_\Omega f^{**}(x, Du)\,\mathrm{d}x$ is strictly less than the functional $\bar{F}^{(p)}(\Omega, u)$ defined in (3.19).

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