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Asymptotic Growth for the Parabolic Equation of Prescribed Mean Curvature

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1. INTRODUCTION

We consider the behavior for large times of the solution of the Dirichlet problem for the parabolic equation of prescribed mean curvature

$$u_t = A(u) + h(x),$$

where (1.1)

$$A(u) = \nabla \cdot g(u) \qquad \text{with} \quad g(u) = (1 + |\nabla u|^2)^{-1/2}\, \nabla u,$$

on a given spatial domain Ω, with given initial values at $t = 0$, and with given time independent boundary values for u on the boundary $\partial\Omega$.

The solution $u(x, t)$ to which we refer is the unique "pseudosolution" (henceforth we will usually just say "solution") of the Dirichlet problem, as introduced by Lichnewsky and Temam in [12]; this is the limit as $\varepsilon \to 0$ of the classical solutions of the regularized Dirichlet problem with an added $\varepsilon\, \Delta u$ term in the equation. Such (pseudo)solutions may "detach" from their desired boundary values in finite time at some or all points of $\partial\Omega$ and assume their desired boundary values only in a generalized sense; this, however, is not the pathology which principally concerns us.

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If $h(x)$ is sufficiently small, it is to be expected that the parabolic solution will tend to a stationary (pseudo)solution of the corresponding elliptic Dirichlet problem; Lichnewsky and Temam have in fact shown that this occurs for the special case $h(x) \equiv 0$. It is well known, however, that this equation cannot attain stationary solutions on all Ω if h is too large (more specifically, if the integral of h on any proper subset G is $\geqslant \text{mis}\, \partial G$, or $\leqslant -\text{mis}\, \partial G$).

We have several conjectures regarding the asymptotic behavior as $t \to +\infty$ of parabolic solutions when $h(x)$ *is* too large; we believe that the solution develops a "rising elliptic cap" on a certain geometrically identifiable "maximal subset" Ω^*, rising with an identifiable asymptotic growth rate c^*. We present these conjectures in a loose form for the case of a general Ω and $h(x)$ in Section 2. However, we have succeeded in proving only some of these conjectures, and those only in the very restricted setting of radial symmetry for Ω and $h(x)$.

In Section 3 we present the maximum principle for the parabolic equation (1.1), and use it with certain "extremal subsolutions" to investigate "detachment" from the boundary values, failure of stationary solutions, and other pathologies of the pseudosolutions of Eq. (1.1).

Sections 4, 5, and 6 present the self-contained results which we have been able to obtain in the radially symmetric setting. All our results require additional hypotheses on h sufficient to force concavity on our $u(\cdot, t)$ on all Ω. Hence one has bounded gradients and uniform parabolicity, and thus smooth solutions, in the interior. Section 7 discusses one of the technical hypotheses used to ensure concavity.

Finally, in Section 8 we present tables and graphs of some numerical computations which show very clearly the development of the rising elliptic cap on Ω^* and its asymptotic speed c^*. Moreoever, in these computations it becomes clear that u_t is approaching a limit $v(x)$ on *all* Ω; we therefore conjecture briefly about the nature of this limiting speed $v(x)$ in general.

The elliptic equation of prescribed mean curvature has been well studied for some time. Serrin [16] considers classical solutions, with conditions on h and the curvature of $\partial\Omega$ to ensure that the solution exists and continuously assumes the boundary data for arbitrary Dirichlet boundary values. Temam [18], Bombieri and Giusti [1], Miranda [14], Giaquinta [5], Gerhardt [3], and Giusti [7] consider variational (pseudo)solutions of the Dirichlet problem which may assume the boundary data only in a generalized sense. The book of Gilberg and Trudinger ([6, Chapter 15]) is a good reference for these topics. Giusti [8] considers certain extremal solutions (with infinite normal derivative on $\partial\Omega$) which shall be of especial interest to us in the present paper.

The first consideration of the corresponding parabolic equation seems to have been by Iannelli and Vergara Caffarelli [9], in the case $h(x) = 0$.

Lichnewsky and Temam [12] have introduced the pseudosolutions, mentioned previously, in the general case of $h(x, t)$ with first partials in L^2_{loc}. As mentioned, these weak solutions may assume the Dirichlet boundary data only in a generalized sense. Gerhardt [4] has further studied these pseudosolutions with other (Neumann) boundary conditions.

2. Conjectures in General

Equation (1.1) can be interpreted as the equation of heat flow with temperature u, unit heat capacity, and with source function $h(x)$ and conductivity $(1+|\nabla u|^2)^{-1/2}$. The important point is that the flux function g saturates with norm one when $|\nabla u| \to \infty$; thus this flux can serve to remove (or add) heat through the boundary of any subset G at a rate of at most $\operatorname{mis} \partial G$ (the $n-1$ dimensional measure of ∂G). In fact, applying the divergence theorem over any subset G we have

$$\int_G u_t\, dx = \int_{\partial G} g \cdot \nu\, d\sigma + \int_G h(x)\, dx, \tag{2.1}$$

where ν is the unit outward normal vector. Thus

$$-\operatorname{mis} \partial G + \int_G h(x)\, dx \leqslant \int_G u_t\, dx \leqslant \operatorname{mis} \partial G + \int_G h(x)\, dx. \tag{2.2}$$

Here equality below (or above) is assumed if and only if $g \cdot \nu = -1$ (or $g \cdot \nu = +1$) on ∂G (henceforth we will usually write "$\partial u/\partial \nu = -\infty$" when in fact we mean $g \cdot \nu = -1$ instead; except in pathological situations the two are equivalent). Hence, if the integral of h on G is greater than $\operatorname{mis} \partial G$ (or less than $-\operatorname{mis} \partial G$), then the total heat in G must continue to grow (or decrease) at a nonzero rate and u cannot attain a stationary solution on G.

It therefore seems that a necessary and sufficient condition for existence of a (classical, with bounded gradient in the interior) stationary solution ought to be that

$$-\operatorname{mis} \partial G < \int_G h(x)\, dx < \operatorname{mis} \partial G, \tag{2.3}$$

on every proper subset G of Ω. In fact, this has been proved by Giusti [8] when $h(x)$ is Lipschitz continuous. When Ω is an "extremal set" for $h(x)$, that is equality above in (2.3) holds for Ω and inequality for every proper subset G, then he shows that there exists an "extremal elliptic solution" $w(x)$ on Ω; it is a classical smooth solution in the interior, has $\partial w/\partial \nu = -\infty$ (or $\partial w/\partial \nu = +\infty$) everywhere on $\partial \Omega$, and is unique to within an additive constant.

Dividing (2.2) through by the measure of G and letting $\mathrm{MV}(u, G)$ denote the mean value of u on G, we have

$$(\mathrm{MV}(u, G))_t \geqslant -\frac{\operatorname{mis} \partial G}{\operatorname{mis} G} + \mathrm{MV}(h, G) \equiv \mathrm{MR}(h, G). \tag{2.4}$$

Thus $\mathrm{MR}(h, G)$ denotes the minimum rate at which the mean value of u on G could be increasing, with equality if and only if $\partial u/\partial v = -\infty$ on all ∂G.

Now consider a fixed nonnegative (for the sake of simplicity) and smooth $h(x)$ on Ω, which is too large somewhere for (2.3) to be satisfied, and let $u(x, t)$ be the solution (i.e., pseudosolution) of the Dirichlet problem for given time independent boundary values. We believe that there exists a set Ω^* on which u asymptotically grows fastest, all at the same asymptotic rate c^*. Because u is growing fastest on this set, $\partial u/\partial v$ should tend to $-\infty$ on all $\partial\Omega^*$; thus we can identify that c^* should be equal $\mathrm{MR}(h, \Omega^*)$, the minimum rate for Ω^*.

Because the mean value of u, $\mathrm{MV}(u, G)$, on any other set G is growing at least as fast as $\mathrm{MR}(h, G)$, we should have that

$$\begin{aligned} \mathrm{MR}(h, G) &\leqslant \limsup_{t \to +\infty} (\mathrm{MV}(u, G))_t \leqslant c^* \\ &= \limsup_{t \to +\infty} (\mathrm{MV}(u, \Omega^*))_t = \mathrm{MR}(h, \Omega^*). \end{aligned} \tag{2.5}$$

Thus we can characterize Ω^* geometrically as a set which maximizes $\mathrm{MR}(h, G)$ over all subsets G. Let us assume hypotheses on $h(x)$ which force existence of a *unique* such set.

Since u_t tends to c^* on Ω^*, u should asymptotically satisfy the elliptic problem

$$\begin{aligned} A(w^*) + h(x) - c^* &= 0 \qquad \text{in } \Omega^*, \\ \partial w^*/\partial v &= -\infty \qquad \text{on } \partial\Omega^*. \end{aligned} \tag{2.6}$$

Now on Ω^* the function $h(x) - c^*$ satisfies the criterion of Giusti for existence of the unique (to within an additive constant) extremal solution w^* of (2.6). In fact, by the definition of c^* and Ω^*, we have

$$\int_G (h(x) - c^*)\, dx \leqslant \operatorname{mis} \partial G \tag{2.7}$$

for every subset G of Ω^*, and equality only for $G = \Omega^*$. We use the inequality $\operatorname{mis} \partial(\Omega^* \backslash G) \leqslant \operatorname{mis} \partial\Omega^* + \operatorname{mis} \partial G$ to verify the other inequality

$$\int_G (h(x) - c^*)\, dx > \operatorname{mis} \partial G; \tag{2.8}$$

we obtain

$$\begin{aligned} c^* \operatorname{mis}(\Omega^*\backslash G) &> -\operatorname{mis} \partial(\Omega^*\backslash G) + \int_{\Omega^*\backslash G} h(x)\,dx, \\ &\geqslant -\operatorname{mis} \partial\Omega^* - \operatorname{mis} \partial G + \int_{\Omega^*\backslash G} h(x)\,dx, \end{aligned} \tag{2.9}$$

and, since $c^* = \mathrm{MR}(h, \Omega^*)$, we get (2.8) as desired.

Therefore there exists a unique solution w^* of (2.6). We normalize by specifying that $w^*(x_0) = 0$ at some fixed interior point x_0 in Ω^*. The parabolic solution u should asymptotically assume the shape of w^* on Ω^*, and thus resemble a "rising elliptic cap" on Ω^*, rising with limiting speed c^*; that is, we expect that $u(x, t) - u(x_0, t)$ converges, as $t \to +\infty$, to $w^*(x)$ uniformly on compact subsets of Ω^*.

This asymptotic behavior should be independent of addition of constants to $h(x)$ and independent of the particular initial and boundary Dirichlet data.

These conjectures, if true, would give a better understanding of the transition from existence to nonexistence of stationary solutions on all Ω, as h is increased; we summarize these conjectures below.

We let c^* be defined by

$$c^* = c^*(h) = \sup\{\mathrm{MR}(h, G) \colon G \subset \Omega\}. \tag{2.10}$$

If $c^* < 0$ then there exists a unique pseudosolution $w(x)$ to the elliptic Dirichlet problem (as shown in [5] and [8]) and the parabolic solution $u(x, t)$ should tend to this $w(x)$.

If $c^* > 0$ then all our parabolic solutions should form a rising elliptic cap on Ω^*, rising with limiting speed c^*.

If $c^* = 0$ there are two cases: If the extremal elliptic solution w^* on Ω^* is unbounded, then $u(x, t)$ should rise indefinitely as $t \to \infty$, but with asymptotic speed zero and with u assuming the shape of w^* on Ω^*. If w^* is bounded, then $w^*(x) + K$, for some constant K, should be the pseudosolution of the elliptic Dirichlet problem, and $u(x, t)$ should tend to $w^*(x) + K$.

Of particular interest is the case $h =$ constant. In this case the subset Ω^* on which the cap forms should minimize the ratio $\operatorname{mis} \partial G/\operatorname{mis} G$ over all subsets G of Ω. Thus numerical solution of the parabolic problem should give one a practical approach to numerical solution of this isoperimetric problem. This same problem arises in other contexts, for example in certain problems of elastic failure for uniformly loaded elastic plates; see [10, 13, 17].

3. Maximum Principle and Detachment from Boundary Values

We begin with the following form of the maximum principle, essentially due to Lichnewsky and Temam [12]:

Lemma 3.1. *Let Ω be a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let T be any positive number. Let $u(x,t)$, $v(x,t)$ be functions such that the quantities u_t, v_t, $A(u)$, $A(v)$ are defined on any open subset G compactly contained in Ω, possibly in a weak sense. Let us assume that u_t, v_t are bounded, that $v-u$ is bounded from above, and that u, v, ∇u, ∇v are defined on $\partial\Omega$ as the limit (possibly infinite) of the respective values in Ω.*
Then the conditions:

$$\begin{aligned}
&\text{(i)} \quad v_t - A(v) \leqslant u_t - A(u) \text{ in } \Omega \times (0,T);\\
&\text{(ii)} \quad v(x,0) \leqslant u(x,0) \text{ in } \Omega;\\
&\text{(iii)} \quad \text{for every } x \in \partial\Omega \text{ and } t \in (0,T) \text{ we}\\
&\qquad \text{have either } v \leqslant u \text{ or } g(v)\cdot\nu \leqslant g(u)\cdot\nu\\
&\qquad (\nu \text{ is the unit outward normal to } \partial\Omega);
\end{aligned} \tag{3.1}$$

imply that $v \leqslant u$ in $\bar{\Omega} \times [0,T]$.

Proof. Let $(v-u)^+ = \max\{v-u; 0\}$, $\Omega^+ = \Omega^+(t) = \{x \in \Omega \colon v-u > 0\}$ and let $G \Subset \Omega$ be an open set with smooth boundary. Since $(v-u)^+$ is a smooth function with respect to x (for example, $(v-u)^+$ is Lipschitz continuous if u and v are such), integration by parts holds in G, so from (i) we get

$$\begin{aligned}
\int_G \frac{1}{2}\frac{\partial}{\partial t}[(v-u)^+]^2\,dx &= \int_G (v_t-u_t)(v-u)^+\,dx\\
&= \int_G [A(v)-A(u)](v-u)^+\,dx\\
&= -\int_{G\cap\Omega^+} [g(v)-g(u)]\cdot\nabla(v-u)\,dx\\
&\quad + \int_{\partial G} ([g(v)-g(u)]\cdot\nu)(v-u)^+\,d\sigma.
\end{aligned} \tag{3.2}$$

First we use the fact that $g(\xi) = \nabla f(\xi)$ for the function $f(\xi) = (1+|\xi|^2)^{1/2}$. Because the function f is convex we have that $[\nabla f(\xi_1) - \nabla f(\xi_2)]\cdot(\xi_1-\xi_2) \geqslant 0$ for all ξ_1, ξ_2 in $\mathbb{R}^n$; hence the integrand is $\geqslant 0$ inside the integral on $G \cap \Omega^+$. Later we let G tend to Ω and use the bounded convergence theorem (recall that $\|g\| \leqslant 1$ and that $(v-u)^+$ is bounded) to see that the integral on ∂G tends to the integral on $\partial\Omega$. Hence, by boundary condition (iii) on $\partial\Omega$, we obtain that

$$\frac{d}{dt}\|(v-u)^+\|^2_{L^2(\Omega)} \leqslant 0. \tag{3.3}$$

Since $(v-u)^+=0$ for $t=0$, we must have $(v-u)^+=0$ for all $t\geqslant 0$.

Note that by this maximum principle the extremal Neumann boundary condition $\partial v/\partial\nu=-\infty$ (i.e., more precisely $g(v)\cdot\nu=-1$) is an extremal boundary condition for solutions (or subsolutions) of the parabolic equation. That is, if on some subset G we have a function $v(x,t)$ satisfying

$$\begin{aligned} &v_t\leqslant A(v)+h \qquad &&\text{in} \quad G\times(0,T),\\ &\partial v/\partial\nu=-\infty &&\text{on } \partial G\times(0,T),\end{aligned} \tag{3.4}$$

then $v\leqslant u$ in $G\times(0,T)$ for any solution (or supersolution u of the parabolic equation (1.1) whose initial values are greater, i.e., $u(x,0)\geqslant v(x,0)$. In particular, the extremal elliptic solution $w^*(x)$ solving (2.6) in Ω^* gives the "extremal subsolution"

$$v(x,t)=w^*(x)+c^*t-K, \tag{3.5}$$

which can be used as a minorant for any solution $u(x,t)$ of (1.1) in Ω^* (with a proper choice of the constant K).

We now use these extremal subsolutions to show several examples of "detachment" from the boundary values for pseudosolutions of the parabolic Dirichlet problem. For simplicity we restrict outselves to 1-dimension on the interval $(-1,+1)$. Let us consider the Dirichlet problem with linear initial values,

$$\begin{aligned} &\text{(i)} \quad u_t=[(1+u_x^2)^{-1/2}u_x]_x+h(x) \text{ in } (-1,1),\\ &\text{(ii)} \quad u(-1,t)=a,\ u(+1,t)=b \text{ for } t\geqslant 0,\\ &\text{(iii)} \quad u \text{ linear at } t=0.\end{aligned} \tag{3.6}$$

By the maximum principle, the pseudosolution $u(x,t)$ is non-decreasing with respect to t when $h(x)$ is nonnegative. Also Lichnewsky and Temam have shown that $u(x,t)$, at a boundary point, either assumes the desired boundary value $\phi(x)$ continuously or else $\partial u/\partial\nu$ is $\pm\infty$ with the sign of the discontinuity of $\phi(x)-u(x,t)$; see also our Proposition 6.2.

EXAMPLE 3.2. We give first an example where h is sufficiently small that there exists a stationary solution w, but for which the parabolic solution u detaches from one of the boundary values in finite time.

Let $h(x)=$ any positive constant H less than the critical value 1 (thus c^*

in (2.10) is <0). All stationary solutions w of (3.6i) are circles of radius $1/H$ (i.e., curvature $A(w) = -H$). If $b-a>0$ is sufficiently small there exists such a circular function w which assumes both boundary values. If $b-a>0$ is larger than this, however, such a circle spanning the two boundary values fails to exist; instead let us show that, as $t \to +\infty$, $u(x,t)$ converges uniformly to $w(x)$, where $w(x)$ is the circular function (of curvature $-H$) satisfying the boundary conditions

$$\frac{\partial w}{\partial v}(-1) = -\infty \qquad (\text{i.e., } w_x(-1) = +\infty), \qquad w(1) = b. \tag{3.7}$$

By the maximum principle, this function w bounds u from above (using the fact that $u(x,t)$ at $x = \pm 1$ must either assume its desired boundary values or lie above them with $\partial u/\partial v = -\infty$). Similarly, we obtain a rising subsolution to bound u from below; let $w^\delta(x)$ $(0<\delta<H)$ be the circular function of radius $1/(H-\delta)$ (i.e., curvature $A(w^\delta) = -(H-\delta)$) satisfying the conditions

$$\frac{\partial w^\delta}{\delta v}(-1) = -\infty, \qquad \max w^\delta(x) = a \ (= \min u(x,0)). \tag{3.8}$$

Then the function $v^\delta(x,t) = w^\delta + \delta t$ is a solution of the parabolic equation (3.6i), rising with speed δ, which, by the maximum principle and the boundry behavior of u, must lie below $u(x,t)$ until that time t^δ that v^δ rises to satisfy $v^\delta(1, t^\delta) = b$ at the right-hand boundary. Thus by the monotonicity with respect to t, $u(x,t)$ is trapped for all $t \geqslant t^\delta$ between the circle $w(x)$ above and the nearly equal circle $v^\delta(x, t^\delta)$ below.

EXAMPLE 3.3. We next give an example where there exists no stationary solution and the parabolic solution detaches from all its boundary values in finite time.

Let $h(x) \equiv H = \text{constant} > 1$ in (3.6); thus $c^* = H - 1 > 0$ and the circular function of radius 1, $w^*(x) = (1-x^2)^{1/2}$, is the extremal elliptic solution on all $\Omega^* = (-1, +1)$ mentioned in (2.6). The corresponding rising elliptic cap $v^*(x,t) = w^*(x) + c^*t - K$ gives a rising subsolution which bounds $u(x,t)$ from below, thus forcing it to detach from both its boundary values in finite time.

EXAMPLE 3.4. We now give an example where there exists no stationary solution on all Ω, but the parabolic solution detaches from none of its boundary values.

Choose $0 < x_1 < x_2 < 1$ and let $h(x)$ be a smooth nonnegative function which is $\equiv H_1 = \text{constant} > 1/x_1$ for $|x| \leqslant x_1$, and is $\equiv H_2 = 0$ for $x_2 \leqslant |x| \leqslant 1$. Let the boundary values a and b both be zero. Let w be the circular function of radius x_1 on $|x| \leqslant x_1$, i.e., $w(x) = (x_1^2 - x^2)^{1/2}$, then $v(x, t) = w(x) + (H_1 - 1/x_1)\, t - x_1$ is a rising extremal subsolution on $|x| < x_1$ which, by the maximum principle, bounds $u(x, t)$ from below on $|x| \leqslant x_1$. On $x_2 \leqslant |x| \leqslant 1$, however, we can construct a supersolution maintaining the zero boundary balues at $x = \pm 1$, which bounds $u(x, t)$ from above; ideally we would like to use the solution v of the problem

$$\begin{aligned} &\text{(i)} \quad v_t - A(v) = h(x) = 0 \text{ in } x_2 < |x| < 1, \\ &\text{(ii)} \quad v = 0 \text{ at } x = \pm 1. \\ &\text{(iii)} \quad \partial v/\partial \nu = +\infty \text{ at } x = \pm x_2. \end{aligned} \tag{3.9}$$

Instead we will construct a supersolution for this problem. Let $x_3 \in (x_2, 1)$, and let $v_1(x, t)$ be the circular function of center x_3, radius $x_3 - x_2$, with center rising with speed $(x_3 - x_2)^{-1}$, i.e.,

$$v_1(x, t) = -((x_3 - x_2)^2 - (x - x_3)^2)^{1/2} + (x_3 - x_2)^{-1}\, t + K, \tag{3.10}$$

where the choice $K = x_3 - x_2$ makes this $v_1 \geqslant 0$ at $t = 0$. This v_1 is a solution of Eq. (3.9i) in $x_2 < x < x_3$ and it also satisfies the infinite slope condition (3.9iii) at $x = x_2$. Now consider the linear function $v_2(x, t)$ which passes through the desired zero boundary condition at $x = 1$ and is tangent to the rising circular function v_2 at some moving point $x_4(t)$, $x_2 < x_4(t) < x_3$. The combined C^1 function v

$$\begin{aligned} v(x, t) &= v_1(x, t) \qquad \text{in} \quad x_2 \leqslant x \leqslant x_4(t), \\ &= v_2(x, t) \qquad \text{in} \quad x_4(t) \leqslant x \leqslant 1, \end{aligned} \tag{3.11}$$

is a supersolution on all $x_2 \leqslant x \leqslant 1$. Extending it by symmetry to negative x, we have a supersolution which bounds $u(x, t)$ above on all $x_2 \leqslant |x| \leqslant 1$; $u(x, t)$ therefore must not detach from its zero boundary values at $x = \pm 1$.

EXAMPLE 3.5. A slight variation of Example 3.4 shows that the pseudosolution u can even build up an internal discontinuity in finite time if h is allowed to have a bounded discontinuity (this of course contrasts sharply with the smoothing properties of uniformly parabolic equations).

Let $0 < x_0 < 1$, let $h(x) \equiv H_1 > 1/x_0$ in $|x| < x_0$, let $h(x) \equiv H_2$ in $x_0 \leqslant |x| < 1$ and (purely for simplicity) let H_2 be very slightly negative. Here $c^* = H_1 - 1/x_0$ and $\Omega^* = \{|x| < x_0\}$. Consider the function

$$\begin{aligned} v(x, t) &= (x_0^2 - x^2)^{1/2} + c^* t && \text{in} \quad |x| < x_0, \\ &= -\{H_2^{-2} - [x - (x_0 + H_2^{-1}]^2\}^{1/2} && \text{in} \quad x_0 \leqslant x \leqslant 1. \\ &= -\{H_2^{-2} - [x + (x_0 + H_2^{-1})]^2\}^{1/2} && \text{in} \quad -1 \leqslant x \leqslant -x_0. \end{aligned} \tag{3.12}$$

Thus v in $|x| < x_0$ is a rising circular cap, concave with radius x_0, rising with speed $c^* > 0$, with $\partial v/\partial \nu = -\infty$ at $|x| = x_0$; in $x_0 \leqslant |x| \leqslant 1$ it is a stationary circular cap, convex with radius $1/H_2$. At time $t = 0$ this function patches together continuously at $x = \pm_0$ with infinite slope, but at $t > 0$ it has developed a jump discontinuity of magnitude c^*t. It seems probable (but we have not done the details) that this v is the pseudosolution of the parabolic Dirichlet problem for its given initial values at $t = 0$, and its given (constant) boundary values at $x = \pm 1$ (that is, v^ε is the limit of the solutions u^ε of the corresponding Dirichlet problem for the regularized equation). Now consider the pseudosolution u (the limit in some sense of the regularized solution u^ε) for the Dirichlet problem with *zero* initial and boundary data ($a = b = 0$ in (3.6ii)). The solutions u^ε are sandwiched between the solutions $v^\varepsilon \pm K$, where K is chosen sufficiently large to sandwich the initial data. Thus in the limit (in whatever sense this limit may exist) the pseudosolution u would have to be sandwiched between $v \pm K$; hence u would have to develop a discontinuity at $|x| = x_0$ in finite time.

Alternatively one can show (giving complete details) that if there exists a pseudosolution $u(x, t)$, regular enough in all $|x| < x_0$ and in $x_0 < |x| < 1$ for the maximum principle (Lemma 3.1) to hold there, then u must develop a discontinuity at $x = \pm x_0$. One merely uses (as in Example 3.4) the function $v(x, t) + K$ as a supersolution bounding $u(x, t)$ above on $x_0 < |x| < 1$, and $v(x, t) - K$ as a subsolution bounding $u(x, t)$ below on $|x| < x_0$.

In more than 1 dimension it is easy to make both H_1 and H_2 strictly positive in both Examples 3.4 and 3.5 because there then exists a stationary supersolution $w(x)$ on the annulus $x_0 < |x| < 1$ with the boundary conditions $w = 0$ at $|x| = 1$ and $\partial w/\partial \nu = +\infty$ at $|x| = x_0$.

4. Existence of a Smooth Concave Pseudosolution

In this and in the following sections we assume that Ω and h have radially symmetric structure, i.e., Ω is the n-dimensional sphere B_R with center at the origin and radius R and h is a function of $r = |x| = (\sum x_i^2)^{1/2}$.

The problem we consider is

$$\begin{aligned} &\text{(i)} \quad u_t = A(h) + h && \text{in } \Omega \times (0, +\infty), \\ &\text{(ii)} \quad u(x, 0) = 0 && \text{on } \Omega, \\ &\text{(iii)} \quad u(x, t) = 0 && \text{on } \partial\Omega \times [0, +\infty). \end{aligned} \tag{4.1}$$

On h we assume the following hypotheses (for a discussion of the hypotheses see Section 7):

$$
\begin{aligned}
&\text{(i)}\quad h(r) \text{ is a nonnegative, } C^{1,\alpha}\text{, concave function for}\\
&\qquad 0 \leqslant r \leqslant R, \text{ with the derivative } h_r = 0 \text{ at } r = 0;\\
&\text{(ii)}\quad \text{moreover, if } n > 1,\ h \text{ satisfies also}\\
&\qquad h(r)r \leqslant h(R)R \text{ for every } r \text{ in } [0, R].
\end{aligned}
\tag{4.2}
$$

We will consider spherically symmetric functions $u(x) = u(r)$ (please excuse the abuse of notation), for which the following formulas hold:

$$
\begin{aligned}
A(u) &= \frac{u_{rr}}{(1+u_r^2)^{3/2}} + \frac{n-1}{r}\,\frac{u_r}{(1+u_r^2)^{1/2}}\\
&= \frac{1}{r^{n-1}}\,\frac{d}{dr}\left(r^{n-1}\frac{u_r}{(1+u_r^2)^{1/2}}\right);\\
\Delta u &= u_{rr} + \frac{n-1}{r}\,u_r\\
&= \frac{1}{r^{n-1}}\,\frac{d}{dr}\,(r^{n-1}u_r).
\end{aligned}
\tag{4.3}
$$

We shall obtain the existence of a "pseudosolution" to problem (4.1) as the limit, as $\varepsilon \to 0$, of the solution u^ε of the corresponding uniformly parabolic regularized Dirichlet problem

$$
\begin{aligned}
&\text{(i)}\quad u_t^\varepsilon = A(u^\varepsilon) + \varepsilon\,\Delta u^\varepsilon + h &&\text{in } \Omega \times (0, +\infty),\\
&\text{(ii)}\quad u^\varepsilon(x, 0) = 0 &&\text{on } \Omega,\\
&\text{(iii)}\quad u^\varepsilon(x, t) = 0 &&\text{on } \partial\Omega \times [0, +\infty).
\end{aligned}
\tag{4.4}
$$

We now state the main result of this section. The proof will then follow through several lemmas.

THEOREM 4.1. *Under hypotheses* (4.2) *the classical solutions* $u^\varepsilon(x, t)$ *of problem* (4.4) *are pointwise (monotonically nondecreasing) convergent in* $\Omega \times [0, \infty)$, *as* $\varepsilon \to 0$, *to a smooth function* $u(x, t)$ (*if* h *has Hölder continuous derivatives through order* K, $K \geqslant 1$, *with exponent* α, *then derivatives of the form* $\partial^{l+m}u/\partial t^l\,\partial x^m$, $2l + m \leqslant K + 2$, *are Hölder continuous with exponent* α *in* x *and* $\alpha/2$ *in* t) *that satisfies the differential equation* (4.1i). *Moreover, if for some* T *problem* (4.1) *has a classical solution* $u^0(x, t)$ *for all times* $t \leqslant T$ (*i.e., assuming the boundary values continuously*), *then* $u = u^0$ *in* $\bar{\Omega} \times [0, T]$.

LEMMA 4.2. *For every* $\varepsilon > 0$ *problem* (4.4) *has a unique solution* $u^\varepsilon(x, t)$, *which is smooth (as in Theorem* 4.1), *radially symmetric in* x *and nondecreasing with respect to* t.

Proof. Since (4.4i) is a uniformly parabolic equation, the existence of a unique smooth solution of (4.4) follows from the results of Ladyzenskaja, Solonnikov and Uralceva ([11, Theorem 6.1, Chapter V]). The radial symmetry follows from the uniqueness. The monotonicity with respect to t follows easily from the maximum principle (we mean the classical formof the maximum principle, which, in our case, could be proved as done for Lemma 3.1) and the nonnegativity of h.

LEMMA 4.3. *For every* $\varepsilon > 0$ *and* $t > 0$, $u^\varepsilon(\cdot, t)$ *is a concave function in* $\bar{\Omega}$.

Proof. Let $v(r) = u^\varepsilon(x, t)$ (we do not denote explicitly the dependence on ε and t). Let $\bar{u}$ be the concave hull of $u^\varepsilon(\cdot, t)$ and let $\bar{v}(r) = \bar{u}(x)$. We will now prove that $\bar{u}$ is a subsolution to (4.4); thus by the maximum principle it will follow that $\bar{u} \leqslant u^\varepsilon$, and thus that $\bar{u} = u^\varepsilon$.

First we extend v and $\bar{v}$ to be odd functions on $[-R, R]$. It is clear that $\bar{v}$ is a C^1 function, that its second derivative exists, with the exception of at most a denumerable set of points (where $\bar{v}$ detaches from v), and that $\bar{v}_{rr}$ equals zero if $v < \bar{v}$ and equals v_{rr} otherwise. It is sufficient to ignore the denumerable set of points where $\bar{v}$ detaches from v in showing that $\bar{v}$ satisfies the appropriate differential inequality for a subsolution.

Let r be a given point where $v(r) < \bar{v}(r)$. Since $\bar{v}$ is linear near r, there exist $\lambda \in (0, 1)$ and two contact points $r_1 < r < r_2$, such that

$$\begin{aligned} &\text{(i)} \quad r = \lambda r_1 + (1-\lambda) r_2, \\ &\text{(ii)} \quad v(r) = \lambda v(x_1) + (1-\lambda)\, v(x_2), \\ &\text{(iii)} \quad \bar{v}_t(x) = \lambda v_t(r_1) + (1-\lambda)\, v_t(r_2). \end{aligned} \tag{4.5}$$

We distinguish the cases $r_1 \leqslant 0 < r_2 \leqslant R$; $0 < r_1 < r_2 < R$; $0 < r_1 < r_2 = R$; the remaining cases can be reduced to these by symmetry.

If $r_1 \leqslant 0 < r_2 \leqslant R$ then one may choose $r_1 = -r_2$ by symmetry and we have $v_r(r_1) = v_r(r_2) = 0$. Moreover $r_2 \neq R$, since for $t > 0$, $\bar{v}$ is positive somewhere. By the concavity of v at the contact point r_k, $k = 1, 2$, we have $v_{rr}(r_k) \leqslant 0$. Thus, using formula (4.3), the differential equation (4.4i) for $v = u^\varepsilon$, and (4.5iii), we obtain

$$\begin{aligned} \bar{v}_t(r) &\leqslant \lambda h(r_1) + (1-\lambda)\, h(r_2) \leqslant h(r) \\ &= A(\bar{u}(r)) + \varepsilon\, \Delta \bar{u}(r) + h(r). \end{aligned} \tag{4.6}$$

If $0 < r_1 < r_2 < R$ we have $\bar{v}_r(r) = v_r(r_1) = v_r(r_2) \leqslant 0$ for all $t_1 \leqslant r \leqslant r_2$ (let us denote this common slope by $\bar{v}_r$). Moreover $v_{rr}(r_k) \leqslant 0$. Thus, from (4.3), and (4.4i) we have

$$v_t(r_k) \leqslant \frac{n-1}{r_k} \{(1+\bar{v}_r^2)^{-1/2} + \varepsilon\} \bar{v}_r + h(r_k). \tag{4.7}$$

Now, using the concavity of the functions $-1/r$ and $h(r)$, we get

$$\begin{aligned}
\bar{v}_t(r) &= \lambda v_t(r_1) + (1-\lambda) v_t(r_2) \\
&= (n-1)\left(\frac{\lambda}{r_1} + \frac{1-\lambda}{r_2}\right) \{(1+\bar{v}_r^2)^{-1/2} + \varepsilon\} \bar{v}_r + \lambda h(r_1) + (1-\lambda) h(r_2) \\
&\leqslant \frac{n-1}{r} \{(1+\bar{v}_r^2)^{-1/2} + \varepsilon\} \bar{v}_r + h(r) \\
&= A(\bar{u}(r)) + \varepsilon \Delta \bar{u}(r) + h(r),
\end{aligned} \tag{4.8}$$

as desired.

If $0 < r_1 < r_2 = R$, then the second contact point is on the boundary and there we have no information about the trangency of $\bar{v}$ to v or about the negativity of v_{rr} (and it is at this point what we will have to use hypothesis (4.2ii)); hence (4.7) holds only at r_1 and not at $r_2 = R$. Since $v_t(R) = 0$ and $\lambda = (R-r)/(R-r_1)$, using (4.5iii) and the fact that $\bar{v}_{rr}(r) = 0$, we obtain

$$\begin{aligned}
&\bar{v}_t(r) - (A(\bar{u}) + \varepsilon \Delta \bar{u})(r) \\
&\quad = \lambda v_t(r_1) - (A(\bar{u}) + \varepsilon \Delta \bar{u})(r) \\
&\quad = \frac{R-r}{R-r_1} v_t(r_1) - \frac{n-1}{r} \{(1+\bar{v}_r^2)^{-1/2} + \varepsilon\} \bar{v}_r \equiv f(r).
\end{aligned} \tag{4.9}$$

Note that this $f(r)$ is a convex function of r. Thus it is $\leqslant$ the concave function $h(r)$ on the whole interval $r_1 \leqslant r \leqslant R$ (as desired for the proof) if it is $\leqslant h(r)$ at the endpoints. At the endpoint r_1 we get $f(r_1) \leqslant h(r_1)$ immediately from (4.7). At the endpoint R we have

$$\begin{aligned}
f(R) &= -\frac{r_1}{R} \frac{n-1}{r_1} \{(1+\bar{v}_r^2)^{-1/2} + \varepsilon\} \bar{v}_r \\
&\leqslant \frac{r_1}{R} (h(r_1) - v_t(r_1)) \\
&\leqslant \frac{r_1}{R} h(r_1) \leqslant h(R).
\end{aligned} \tag{4.10}$$

Here, when $n > 1$, we have used inequality (4.7) for $K = 1$ and the fact that $v_t(r_1) \geqslant 0$, together with the assumption (4.2ii) that $h(r_1)r_1 \leqslant h(R)R$. When $n = 1$, it is clear in (4.9) that $f(R) = 0 \leqslant h(R)$, without the use of hypothesis (4.2ii).

It remains to consider the points r where $v_{rr}(r) = \bar{v}_{rr}(r)$. If $v_t(r) = \bar{v}_t(r)$ then $\bar{v}$ is locally a solution of the equation; otherwise, if $v_t(r) < \bar{v}_t(r)$, we are again in the above considered case for which (4.5) holds.

LEMMA 4.4. *The solution u^ε of (4.4) are nondecreasing with respect to ε and converge, as $\varepsilon \to 0$, to a function $u(x, t)$, concave with respect to x and nondecreasing with respect to t. Moreover $u^\varepsilon(\cdot, t)$ converges to $u(\cdot, t)$ in $H^{1,p}_{\mathrm{loc}}(\Omega)$ for any p in $[1, +\infty)$, and $u^\varepsilon_t(x, \cdot)$ converges to $u_t(x, \cdot)$ in the weak* topology of $L^\infty([0, \infty))$.*

Proof. Since $u^\varepsilon(\cdot, t)$ is concave, $\Delta u^\varepsilon \leqslant 0$; thus, for $0 < \bar{\varepsilon} < \varepsilon$, we have

$$\begin{aligned} u^\varepsilon_t &= A(u^\varepsilon) + \varepsilon\, \Delta u^\varepsilon + h \\ &\leqslant A(u^\varepsilon) + \bar{\varepsilon}\, \Delta u^\varepsilon + h. \end{aligned} \tag{4.11}$$

By the maximum principle we get $u^\varepsilon \leqslant u^{\bar{\varepsilon}}$. From the concavity of $u^\varepsilon(\cdot, t)$ we also derive that

$$0 \leqslant u^\varepsilon_t = A(u^\varepsilon) + \varepsilon\, \Delta u^\varepsilon + h \leqslant h, \tag{4.12}$$

and thus $0 \leqslant u^\varepsilon(x, t) \leqslant th(|x|)$. Therefore u^ε, being bounded and nondecreasing in ε, has a finite limit, say $u(x, t)$, that is concave and radially symmetric in x, and nondecreasing in t.

Since u^ε_t is bounded independently of ε, $u^\varepsilon(x, \cdot)$ converges to $u(x, \cdot)$ in the weak* topology of $L^\infty([0, \infty))$. Moreover, for fixed t, the net $u^\varepsilon(\cdot, t)$ by its concavity and equiboundness is equilipschitzian on any compact subset $\bar{G}$ of Ω (see, for example, Section 2.3 of Chapter I of [2]); thus $u^\varepsilon(\cdot, t)$ converges to $u(\cdot, t)$ in the weak* topology of $H^{1,\infty}(G)$. Let us use the notation $u^\varepsilon(r, t) = u^\varepsilon(x, t)$ and $u(r, t) = u(x, t)$ (again excuse the abuse of notation). For fixed t, u^ε_r is a nonincreasing function of r that converges to u_t in $w^* - L^\infty_{\mathrm{loc}}([0, R))$. We want to show that u^ε_r converges, as $\varepsilon \to 0$, almost everywhere to u_r. To this aim, if $r_0 \in (0, R)$ is a Lebesgue point of u_r, for sufficiently small $\delta > 0$ we have

$$\int_{r_0}^{r_0+\delta} u^\varepsilon_r(s, t)\, ds \leqslant u^\varepsilon_r(r_0, t)\, \delta \leqslant \int_{r_0-\delta}^{r_0} u^\varepsilon_r(s, t)\, ds. \tag{4.13}$$

As $\varepsilon \to 0$ we have

$$\frac{1}{\delta}\int_{r_0}^{r_0+\delta} u_r(s,t)\,ds \leqslant \liminf_{\varepsilon\to 0} u_r^\varepsilon(r_0,t) \leqslant \limsup_{\varepsilon\to 0} u_r^\varepsilon(r_0,t) \leqslant \frac{1}{\delta}\int_{r_0-\delta}^{r_0} u_r(s,t)\,ds. \tag{4.14}$$

As $\delta \to 0$ we obtain the pointwise convergence of u_r^ε to u_r. Now, with the use of the Lebesgue dominated convergence theorem we infer the convergence of u_r^ε to u_r in $L^p_{\mathrm{loc}}([0,R))$.

Proof of Theorem 4.1. Let T be fixed. We multiply both sides of differential equation (4.4i) by a test function $\phi(x,t)$, with $\{x: \phi \neq 0\} \Subset \Omega$, and we integrate over $\Omega \times [0,T]$. We obtain

$$\int_\Omega dx \int_0^T u_t^\varepsilon \phi\,dt = -\int_0^T dt \int_\Omega \{(g(u^\varepsilon) + \varepsilon\,\nabla u^\varepsilon)\cdot\nabla\phi - h\phi\}\,dx. \tag{4.15}$$

We can go to the limit as $\varepsilon \to 0$ using the results of Lemma 4.4. We get that u is a weak solution of the equation $u_t = A(u) + h$. But this equation stays uniformly parabolic on compact subsets since $|\nabla u|$ stays bounded there. That is, for any open sphere B_r with center at the origin and radius $r < R$, we can define a uniformly elliptic operator $\tilde{A}(u) = \nabla \cdot \tilde{g}(u)$ such that $\tilde{g}(\xi) = g(\xi)$ for $|\xi| \leqslant \sup\{|\nabla u|: |x| \leqslant r, 0 \leqslant t \leqslant T\}$. By known existence results (see Theorems 6.1 and 6.2 of Chapter V of [11]), there exists a smooth (as in the statement of Theorem 4.1) function $\tilde{u}(x,t)$ that satisfies $\tilde{u}_t = \tilde{A}(\tilde{u}) + h$ in $B_r \times (0,T)$ and has the same initial and boundary conditions as u (we use the fact that on the boundary of B_r u is constant with respect to x and Lipschitz continuous with respect to t). By the maximum principle for weak solutions we get $\tilde{u} = u$ on $B_r \times [0,T]$. This proves the stated smoothness of u in $\Omega \times [0,\infty)$.

Finally, let us assume that problem (4.1) has a classical solution u^0 in $\bar{\Omega} \times [0,T]$. Since u^ε, for any $\varepsilon > 0$, is a subsolution to problem (4.1) (as a consequence of $\Delta u^\varepsilon \leqslant 0$), by the maximum principle we get $0 \leqslant u^\varepsilon \leqslant u^0$. It follows that $0 \leqslant u \leqslant u^0$ and therefore $u = u^0 = 0$ on $\partial\Omega$. The uniqueness of the solution of problem (4.1) (again a consequence of the maximum principle) implies that $u = u^0$ on $\bar{\Omega} \times [0,T]$.

DEFINITION 4.5. *The function u, the limit in $\Omega \times [0,\infty)$ as $\varepsilon \to 0$ of the u^ε as stated in Theorem* 4.1, *and extended to $\bar{\Omega} \times [0,\infty)$ by continuity, is the pseudosolution of problem* (4.1).

We can extend $u(\cdot, t)$ to $\partial\Omega$ since it is concave with respect to r and thus has a (nonnegative) limit as $r \to R^-$. We have adopted the terminology introduced by Lichnewsky and Temam [12] since out smooth pseudosolution u turns out to be the same as the generalized pseudosolution defined there.

5. Properties of the Elliptic Cap

In the following we consider the radially symmetric solutions of the elliptic equation

$$A(w) + h(r) = 0 \tag{5.1}$$

for as large a ball as possible, where $h(r)$ is as before. Such solutions are unique to within an additive constant of course.

If $c^* \leqslant 0$ (with the exception $c^* = 0$, $n = 1$, $h(R) = 0$) then the solution $w(r)$ will turn out to exist for all $r \leqslant R$ and we can choose the additive constant so as to solve the Dirichlet problem

$$\begin{aligned} A(w) + h(r) &= 0 \qquad \text{in } \Omega = B_R, \\ w &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{5.2}$$

In particular when $c^* = 0$ and $n \geqslant 2$ our hypothesis (4.2ii) guarantees that $w(r)$ is bounded on $r \leqslant R$ even though $w_r = -\infty$ at $r = R$. On the contrary, if $c^* = 0$ but $n = 1$, w is bounded or unbounded in $|0, R|$ as $h(R)$ is > 0 or $= 0$.

If $c^* > 0$ then the solution $w(r)$ of (5.1) will develop infinite gradient and cease to exist at some $r < R$. In this case we consider the equation

$$A(w^*) + h(r) - c^* = 0 \qquad \text{in } \Omega^* = B_{r^*}; \tag{5.3}$$

this solution will develop infinite gradient $(w_r^* = -\infty)$ at a certain $r = r^* \leqslant R$.

Let us see in detail these simple properties. First of all we recall that the Dirichlet problem (5.2) has at most one solution, as a consequence of the maximum principle. It follows that, when it exists, the solution must have spherical symmetry. Thus, if $w = w(r)$, $r = |x|$ (again excuse the abuse of notation), Eq. (5.1) can be put in the form

$$\frac{1}{r^{n-1}} \frac{d}{dr} \left(r^{n-1} \frac{w_r}{(1 + w_r^2)^{1/2}} \right) = -h(r), \tag{5.4}$$

and thus we can write an explicit solution formula. The solution $w(r)$ of (5.4), satisfying $w_r(0) = w(0) = 0$, is

$$w(r) = \int_0^r \frac{y(s)\, ds}{(1 - y^2(s))^{1/2}}, \tag{5.5}$$

where

$$y(r) = -r^{1-n} \int_0^r s^{n-1} h(s)\, ds,$$

which exists so long as $|y(r)|$ stays < 1. In order to see where w is defined we will make use of the following proprties of y:

LEMMA 5.1. *The function $y(r)$ defined in (5.5) is nonpositive, nonincreasing, and convex for $0 \leqslant r \leqslant R$. Moreover $y_{rr}(r) > 0$ unless h is constant in $[0, r)$; and, only in the case $n \geqslant 2$, if h is not identically zero in $[0, R]$ then $y_r(R) < 0$.*

Proof. If $n = 1$ the statement is obvious. If $n \geqslant 2$ we have

$$y_r(r) = (n-1)\, r^{-n} \int_0^r s^{n-1} h(s)\, ds - h(r); \tag{5.6}$$

$$\begin{aligned} y_{rr}(r) &= -n(n-1)\, r^{-n-1} \int_0^r s^{n-1} h(s)\, ds + (n-1)\, r^{-1} h(r) - h_r(r) \\ &= (n-1)\, r^{-n-1} \int_0^r s^n h_r(s)\, ds - h_r(r) \\ &= (n-1)\, r^{-n-1} \int_0^r s^n \left\{ h_r(s) - \frac{n+1}{n-1} h_r(r) \right\} ds \\ &\geqslant (n-1)\, r^{-n-1} \int_0^r s^n \{h_r(s) - h_r(r)\}\, ds \geqslant 0. \end{aligned} \tag{5.7}$$

Thus y is convex. From (5.6) with $r = R$ and assumption (4.2ii) we obtain

$$y_r(R) \leqslant (n-1)\, R^{-n} \int_0^R s^{n-2} R h(R)\, ds - h(R) = 0; \tag{5.8}$$

thus, by the convexity of y, we have $y_x(r) \leqslant 0$ for every $0 \leqslant r \leqslant R$.

If h is not constant in $[0, r)$ then $h_r(r) < 0$, and from (5.7) it follows that $y_{rr}(r) > 0$. Finally we observe that in (5.8) the strict inequality holds, unless $h(r)\, r \equiv R h(R)$, i.e., unless $h \equiv 0$ (in fact h is concave and nonsingular at $r = 0$).

As in Section 2 we define c^* as the supremum of $\mathrm{MR}(h, G)$ for all subsets G of $\Omega = B_R$, i.e.,

$$c^* = \sup \left\{ -\frac{\operatorname{mis} \partial G}{\operatorname{mis} G} + \frac{1}{\operatorname{mis} G} \int_G h(|x|)\, dx \colon G \subset \Omega \right\}. \tag{5.9}$$

LEMMA 5.2. *The supremum in* (5.9) *is a maximum and is achieved only when $\bar{G}$ is a closed sphere $\bar{B}_{r^*}$ of center in the origin and radius $r^* \leqslant R$. Moreover*

$$c^* = \max_{0 < r \leqslant R} \left\{ -\frac{n}{r}(1 + y(r)) = -\frac{n}{r} + \frac{n}{r^n} \int_0^r s^{n-1} h(s)\, ds \right\} \tag{5.10}$$

and the maximum in (5.10) *is realized only for $r = r^*$.*

Proof. For any measurable set G contained in $\Omega = B_R$ we can choose a sphere B_r of center at the origin and radius $r \leqslant R$ such that $\operatorname{mis} G = \operatorname{mis} B_r$. Since h is nonincreasing we have

$$\begin{aligned}
\int_G h(|x|(\, dx &= \int_{G \cap B_r} h(|x|)\, dx + \int_{G \backslash B_r} h(|x|)\, dx \\
&\leqslant \int_{G \cap B_r} h(|x|)\, dx + h(r) \operatorname{mis}(G \backslash B_r) \\
&= \int_{G \cap B_r} h(|x|)\, dx + h(r) \operatorname{mis}(B_r \backslash G) \\
&\leqslant \int_{B_r} h(|x|)\, dx.
\end{aligned} \tag{5.11}$$

We have also $\operatorname{mis} \partial G \geqslant \operatorname{mis} \partial B_r$ by the isoparimetric inequality and the equality sign holds only if $\bar{G} = \bar{B}_r$. It follows that supremum (5.9) does not change if we restrict G to vary among such spheres B_r. On letting $G = B_r$ in formula (5.9), we can easily verify that c^* is the supremum as stated in (5.10). Obviously $-n(1 + y(r))/r$ has a maximum in $(0, R]$. With a change of the variable of integration we can write

$$-\frac{n}{r}(1 + y(r)) = -\frac{n}{r} + n \int_0^1 s^{n-1} h(rs)\, ds. \tag{5.12}$$

With this representation we see that $-n(1 + y(r))/r$ is strictly concave; thus has a unique maximum point.

We will use the explicit expression of the solution $w^*(|x|)$ of (5.3). Just as in formula (5.5), w^* formally is given by

$$w^*(r) = \int_0^r \frac{y^*(s)\, ds}{(1 - y^*(s)^2)^{1/2}}, \tag{5.13}$$

where now $y^*(r) = -r^{1-n} \int_0^r s^{n-1} \{h(s) - c^*\}\, ds = y(r) + (c^*/n)r$.

PROPOSITION 5.3. *Dirichlet problem* (5.2) *has a unique* (*bounded*) *solution* (*given, with an appropriate choice of an additive constant, by the function* $w(|x|)$ *defined in* (5.5)), *in each of the following cases*:

(i) $c^* < 0$;

(ii) $c^* = 0$ *and* $n \geqslant 2$;

(iii) $c^* = 0$, $n = 1$ *and* $h(R) > 0$.

On the contrary, Dirichlet problem (5.2) *lacks a solution in the other cases*:

(iv) $c^* = 0$, $n = 1$ *and* $h(R) = 0$;

(v) $c^* > 0$.

Moreover the function w^* *in* (5.13) *is defined for* $r < r^*$ *and satisfies Eq.* (5.3).

Proof. In the case $c^* < 0$, a consequence of (5.10) is that the nonpositive function $y(r)$ is strictly greater than -1 for $0 \leqslant r \leqslant R$. Thus $w(r)$ is well defined on all $[0, R]$ and the function $w(|x|) - w(R)$ satisfies the boundary values for Dirichlet problem (5.2).

If $c^* = 0$, again using (5.10) the minimum of $y(r)$ on the interval $[0, R]$ is -1. Since y is a strictly decreasing function (in fact if h is constant then y is linear, and if h is not constant by Lemma 5.1, y is nonincreasing and strictly convex), we have $y(R) = -1$ and $y(r) > -1$ for $r < R$. Thus $w(r)$ is defined in $[0, R)$. In order to see whether or not $w(|x|)$ tends to a limit $w(R)$ as $|x| \to R$, so that $w(|x|) - w(R)$ solves Dirichlet problem (5.2), we have to see whether the integral in (5.5) that should define $w(R)$ is convergent or not. Actually the integral $w(R)$ is convergent if and only if the function $1 - y^2(r)$ has a zero of first order at $r = r^* = R$, i.e., if and only if $y_r(R) \neq 0$. Thus, if $n \geqslant 2$ case (ii) of our proposition follows from Lemma 5.1; while, if $n = 1$, since $y_r(R) = -h(R)$ Dirichlet problem (5.2) has a solution if and only if $h(R) > 0$, as stated in (iii) and (iv).

If $c^* > 0$ then $y(r^*) < -1$ by (5.10). Thus w is not defined over all B_R and Dirichlet problem (5.2) lacks a solution (we can see that problem (5.2) lacks a solution also from the fact that the inequalities in (2.3) are not satisfied for $G = \Omega^*$, if $c^* > 0$).

We have already shown in Section 2 that Eq. (5.3) has a unique solution in Ω^*. To verify directly that w^* is such a solution we have to show that $|y^*(r)| < 1$ for $r < r^*$. For this purpose let us observe that the unequality (cf. with (5.10)) $c^* \geqslant -n(1 + y(r))/r$ is equivalent to $y^*(r) \equiv y(r) + (c^*/n)\, r \geqslant -1$ and the equality sign holds for the same values of r; thus it holds in both cases only for $r = r^*$. This means that $y^*(r) > -1$ for $r \neq r^*$. Moreover, since $y^*(0) = 0$, by the convexity we infer $y^*(r) < 0$ for $0 < r \leqslant r^*$. Therefore $w^*(|x|)$ is defined for $|x| < r^*$ and satisfies (5.3).

We conclude this section with some remarks. First, we observe that if $c^* \geqslant 0$, $w^*(|x|)$ is the *unique* (up to an additive constant) function that solves Eq. (5.3) in $\Omega^* = B_{r^*}$, since $h(r) - c^*$ satisfies the criterion of Giusti [8] for existence and uniqueness of such a solution without boundary conditions. A second remark is that for many functions h satisfying (4.2) we definitely can have $r^* < R$ (we will exhibit an example in Section 7). In this case, since r^* is the minimum point of $y^*(r) = y(r) + c^* r/n$, we definitely have $c^* > 0$ (in fact y^* is nonincreasing when $c^* \leqslant 0$) and $y_r^*(r^*) = 0$; thus the function $1 - y^*(r)^2$ has at $r = r^*$ a zero of at least second order and therefore the cap w^* is unbounded. The cap $w^* \equiv w$ is unbounded also in case (iv) of the statement of the previous theorem. Finally, we note that w^* is a concave function, since y^* is nonincreasing for $r < r^*$. For the same reason, w, where defined, is a concave function.

6. Behaviour of the Pseudosolution as $t \to +\infty$

We will continue to denote (with abuse of notation) by $u(x, t) = u(r, t)$ the pseudosolution of problem (4.1), and by $u^\varepsilon(x, t) = u^\varepsilon(r, t)$ the solution of the regularized problem (4.4). The following result is a consequence of the maximum principle.

LEMMA 6.1. *$u_{rt} \leqslant 0$ for every $r < R$ and $t > 0$.*

Proof. Let us use the notation $z(r, t) = u_r^\varepsilon(r, t)$ (we do not denote explicitly the dependence on ε). We will show z satisfies a certain parabolic equation and has nonincreasing boundary values. Differentiating in (4.3), (4.4), we obtain

$$\text{(i)} \quad z_t = \frac{d}{dr}\left\{[(1 + z^2)^{-3/2} + \varepsilon]\, z_r + \frac{n-1}{r}\,[(1 + z^2)^{-1/2} + \varepsilon]\, z\right\} + h_r; \tag{6.1}$$

(ii) $z(0, t) = z(r, 0) = 0$;

(iii) $z(R, t)$ nonincreasing in t;

We have used in (6.1iii) the fact that $u^\varepsilon(r,t)$ is nondecreasing with respect to t and that $u^\varepsilon(R,t)=0$. Now we apply the maximum principle for nonlinear uniformly parabolic operators as appears for example in the book of Protter and Weinberger [15, Theorem 12, Chapter 3]). To use the maximum principle it is sufficient to know that, as in our case, the right side of Eq. (6.1i) is continuously differentiable with respect to r, z, z_r, z_{rr} on $(0,R)\times[0,\infty)$. Since $h_r\leqslant 0$, it follows from the maximum principle that $z_t=u^\varepsilon_{rt}\leqslant 0$ in $[0,R]\times[0,\infty)$. We have shown in the proof of Lemma 4.4 that u^ε_r converges almost everywhere to u_r as $\varepsilon\to 0$; hence we can conclude that $u_r(r,t)$ is nonincreasing with respect to t, since u^ε_r is also. Thus u_{rt}, which exists for the C^2 pseudosolution u, is $\leqslant 0$.

From the previous lemma we derive a result that is interesting in itself, since it gives information on the behaviour of the pseudosolution at the boundary (cf. [12]).

PROPOSITION 6.2. *If for some $T>0$, $u(R,T)>0$, then*

$$u_r(R,T)\equiv \lim_{r\to R^-} u_r(r,T)=-\infty. \tag{6.2}$$

(In other words, if the pseudosolution u detaches from the zero boundary data, then the equation is not uniformly parabolic at u near the boundary.)

Proof. We note that limit (6.2) exists, since u is concave in r. Let us define

$$t_0=\sup\{t\leqslant T: u(R,t)=0\}. \tag{6.3}$$

In order to prove the stated result we will show that, if the limit in (6.2) is finite, then t_0 is a maximum and is equal to T, i.e., $u(R,T)=0$. For this purpose we note first that, if the limit in (6.2) is equal to $-L>-\infty$, then

$$0\geqslant u_r(r,t)\geqslant -L, \qquad \forall\, r\leqslant R,\quad \forall\, t\leqslant T, \tag{6.4}$$

since $u_r(r,t)$ is nonincreasing with respect to r and also with respect to t. From (6.4) it follows that $u(r,t)$ converges, as $r\to R^-$, to $u(R,t)$ uniformly with respect to t. Therefore $u(R,t)$ is a continuous function of t, and t_0 in (6.3) is a maximum.

It remains to prove that $t_0=T$. We proceed by contradiction and assume $t_0<T$. We will construct a supersolution (for t near t_0 and r near R) maintaining the zero boundary value at $r=R$, which bounds u from above. Let $v(r)$ be the circular function, concave, of radius $1/h(0)$ ($h(0)$ is the maximum of $h(r)$), i.e., $v(r)$ has curvature equal to $-h(0)$, and such that $v(R)=0$ and $v_r(R)=-2L$. Let us choose a point $r_0<R$ in such a way that v is a decreasing function in $[r_0,R]$ and $v(r_0)>L(R-r_0)$. By construction v satisfies the following properties:

(i) v is a stationary supersolution of (4.4i) for $r_0 < r < R$ and $t < T$; in fact, since both v_r and v_{rr} are nonpositive, we have

$$A(v) + \varepsilon \Delta v + h(r) \leqslant -h(0) + h(r) \leqslant 0 = v_t. \tag{6.5}$$

(ii) (At the initial time t_0.) For $r_0 \leqslant r < R$ we have $v(r) > u^\varepsilon(r, t_0)$; in fact, using the concavity of v and (6.4), we have

$$v(r) > L(R-r) \geqslant u(r, t_0) - u(R, t_0) = u(r, t_0) \geqslant u^\varepsilon(r, t_0). \tag{6.6}$$

(iii) (At the endpoint r_0.) There exists t_1 independent of ε, with $t_0 < t_1 \leqslant T$, such that $v(r_0) > u^\varepsilon(r_0, t)$ for t_1; this follows from the inequality in (6.6), $v(r_0) > u(r_0, t_0)$, the continuity of u, and the fact that $u \geqslant u^\varepsilon$.

(iv) (At the end point R.) For $t \leqslant T$ we have $u^\varepsilon(R, t) = v(R) = 0$.

We use the maximum principle to infer $v(r) \geqslant u^\varepsilon(r, t)$ for $r_0 \leqslant r \leqslant R$ and $t \leqslant t_1$. In the limit as $\varepsilon \to 0$ we obtain $v \geqslant u$ and thus in particular $u(R, t_1) = 0$, which contradicts (6.3).

We are now able to describe the asymptotic behaviour of the pseudosolution u as $t \to +\infty$. The following result proves that, in the special case we are considering, the pseudosolution has the behaviour conjectured in Section 2.

THEOREM 6.3. *Let u be the pseudosolution of the parabolic Dirichlet problem* (4.1) (*with h satisfying* (4.2)) *and let c^* be defined as in* (5.9) (*or in* (2.10)).

In each of the following cases:

(i) $c^* < 0$;

(ii) $c^* = 0$ *and* $n \geqslant 2$;

(iii) $c^* = 0$, $n = 1$, *and* $h(R) > 0$;

the function $u(x, t)$ converges, as $t \to +\infty$, to the solution of the elliptic Dirichlet problem (5.2); *the convergence is in $C^1(\bar{\Omega})$ in case* (i), *and in $C^0(\bar{\Omega}) \cap C^1_{\mathrm{loc}}(\Omega)$ in cases* (ii), (iii). *Moreover in these three cases u is the classical solution of problem* (4.1), *i.e., it assumes continuously the zero boundary data.*

In the other cases:

(iv) $c^* = 0$, $n = 1$, *and* $h(R) = 0$;

(v) $c^* > 0$;

the function u can be represented in the form

$$u(x, t) = u(0, t) + w^*(|x|) + \varepsilon(x, t), \qquad |x| < r^*, \quad t \geqslant 0, \tag{6.7}$$

where w^* is the elliptic cap defined in (5.12) (w^* *solves elliptic equation* (5.3)), *and* $\varepsilon(x, \cdot)$ *goes to zero in* $C^1_{\text{loc}}(\Omega^*)$, *as* $t \to +\infty$. *Moreover* $u_t(x, \cdot)$ *converges to* c^* *in* $C^0_{\text{loc}}(\Omega^*)$ *and* u_r *tends uniformly to* $-\infty$ *for* $|x| \geqslant r^*$.

In the proof of this theorem we will make use of the following lemmas:

LEMMA 6.4. *If* $c^* \geqslant 0$, *for every sufficiently small* $\delta > 0$ *we have*

$$w^*(r) + c^*t \leqslant u(r, t) \leqslant w^*_\delta(r) + (c^* + \delta)t, \tag{6.8}$$

for $t \geqslant 0$ *and* $r < r^*$, w^* *being the elliptic cap defined in* (5.12), *and* w^*_δ *being a solution of*

$$A(w^*_\delta) + h - (c^* + \delta) = 0 \tag{6.9}$$

for as large an r *as possible.*

Proof. The function $\tilde{w}(x, t) = w^*(|x|) + c^*t$ is a subsolution to problem (4.1). In fact $\tilde{w}$ satisfies the differential equation (4.1i) in $B_{r^*} \times (0, \infty)$; moreover $\tilde{w}(x, 0) \leqslant 0$ and $\tilde{w}_r(r^{*-}) = w^*_r(r^{*-}) = -\infty$. By the maximum principle as stated in Lemma 3.1 we have $\tilde{w} \leqslant u$, i.e., the left side of (6.8).

The radially symmetric solution $w_\delta(r)$ of (6.9), such that $w_\delta(0) = 0$, is

$$w_\delta(r) = \int_0^r \frac{y_\delta(s)\, ds}{(1 - y_\delta(s)^2)^{1/2}}, \qquad \text{where} \quad y_\delta(r) = y(r) + (c^* + \delta)\, r/n. \tag{6.10}$$

The function y_δ is convex and satisfies $y_\delta(r) = y_\delta(r) = y^*(r) + \delta r/n \geqslant -1 + \delta r/n$; thus $y_\delta > -1$ for $r \leqslant R$. Moreover, if δ is not too large, $y_\delta < 1$ for $r \leqslant r^*$; therefore w_δ is defined for $r \leqslant r^*$ and is bounded from below. Let us denote by m (< 0) the minimum value of w_δ on the set where it is defined. and let us use the notation

$$w^*_\delta(r) = w_\delta(r) - m; \qquad \tilde{w}_\delta(r, t) = w^*_\delta(r) + (c^* + \delta)t. \tag{6.11}$$

Where it is defined, $\tilde{w}_\delta$ satisfies differential equation (4.1i) and $\tilde{w}_\delta(|x|, 0) \geqslant 0$. Now we have to distinguish two cases: whether there exists $r_0 \in\,]r^*, R]$ such that $y_\delta(r_0) = +1$, or whether such a point does not exist. In the second case $|y_\delta| < 1$ in $[0, R]$, thus w^*_δ and $\tilde{w}_\delta$ are defined and nonnegative for $r \leqslant R$; by Proposition 6.2 either $u(R, t) = 0$ or $u_r(R, t) = -\infty$, thus $u \leqslant \tilde{w}_\delta$ as a consequence of the maximum principle. On the other hand, if $y_\delta(r_0) = +1$ for some $r_0 \geqslant r^*$, then at r_0 we have $(\tilde{w}_\delta)_r = (w_\delta)_r = +\infty$ and thus, once again by the maximum principle on $B_{r_0} \times [0, \infty)$, we get the right side of (6.8).

LEMMA 6.5. *If* $c^* \geqslant 0$ *then* $\lim_{t \to +\infty} u_t(0, t) = c^*$.

Proof. From differential equation (4.1i) and formula (4.3) we get

$$u_t(0, t) = u_{rr}(0, t) + (n-1) \lim_{r \to 0^+} \frac{u_r(r, t)}{r} + h(0)$$
$$= n u_{rr}(0, t) + h(0). \tag{6.12}$$

Since $u_{rt} \leqslant 0$ and $u_r(0, t) = 0$, $u_{rr}(0, t)$ is a nonincreasing function; thus $u_t(0, t)$, as well as $u_{rr}(0, t)$, has a limit as $t \to +\infty$. The limiting value of $u_t(0, t)$ must be c^*, as a consequence of the previous lemma.

LEMMA 6.6. *The limit as $t \to +\infty$ of $u_r(r, t)$ (which exists, since $u_{rt} \leqslant 0$) is finite for every $r < r^*$.*

Proof. We can proceed as in Section 2 to obtain for any ball B_r

$$\mathrm{MV}(u_t, B_r) \geqslant \mathrm{MR}(h, B_r), \tag{6.13}$$

and, if for some r_0, $u_r(r_0, t) \to -\infty$ as $t \to +\infty$, then

$$\lim_{t \to +\infty} \mathrm{MV}(u_t, B_{r_0}) = \mathrm{MR}(h, B_{r_0}). \tag{6.14}$$

The supremum with respect to r of the right side of (6.13) isequal to c^* and is achieved only for $r = r^*$. Since u_t is a nonincreasing function of r, for every $r \leqslant r^*$ we obtain

$$\mathrm{MV}(u_t, B_r) \geqslant \mathrm{MV}(u_t, B_{r^*}) \geqslant \mathrm{MR}(h, B_{r^*}) = c^*. \tag{6.15}$$

If for the sake of contradiction we assume that for some $r_0 < r^*$, $u_r(r_0, t) \to -\infty$ as $t \to +\infty$, then (6.14) holds and we obtain

$$\lim_{t \to +\infty} \mathrm{MV}(u_t, B_{r_0}) = \mathrm{MR}(h, B_{r_0}) < \mathrm{MR}(h, B_{r^*}) = c^*; \tag{6.16}$$

this is in contradiction to (6.15).

Proof of Theorem 6.3. Let us consider first cases (iv) and (v) and let us define

$$u^*(r, t) = u(r, t) - u(0, t). \tag{6.17}$$

Since $u^*(0, t) = 0$ and $u_r^* = u_r$, using the previous lemma (on the finiteness of the limit of u_r) we can also define for $r < r^*$

$$v^*(r) = \lim_{t \to +\infty} u^*(r, t) = \lim_{t \to +\infty} \int_0^r u_r(s, t)\, ds. \tag{6.18}$$

By Lemma 6.1 (that $u_{rt} = u_{tr} \leqslant 0$) $u_t^*(r, t) = u_t(r, t) - u_t(0, t) \leqslant 0$; thus, for fixed $r_0 < r^*$, we have

$$\int_0^{+\infty} \|u_t^*(\cdot, t)\|_{L^1([0, r_0])}\, dt = \lim_{t \to +\infty} - \int_0^{r_0} u^*(s, t)\, ds$$
$$= \|v^*\|_{L^1([0, r_0])} < +\infty. \tag{6.19}$$

We therefore deduce the existence of a sequence t_k tending to $+\infty$, such that $\|u_t^*(\cdot, t_k)\|_{L^1([0, r_0])}$ goes to zero. For every test function $\phi(r)$ with support in $[0, r_0)$ we get

$$\int_\Omega u_t^*(|x|, t_k)\, \phi\, dx = \int_\Omega \{u_t(|x|, t_k) - u_t(0, t_k)\}\, \phi\, dx$$
$$= -\int_\Omega g(u(|x|, t_k))\, \nabla\phi\, dx + \int_\Omega \{h - u_t(0, t_k)\}\, \phi\, dx. \tag{6.20}$$

Now let k go to $+\infty$. By Lemma 6.5, $u_t(0, t_k) \to c^*$. Since u_r converges to v_r^* for every $r \leqslant r_0$, by Lebesgue's dominated convergence theorem ($|g|$ is $\leqslant 1$) we obtain that v^* is a (locally lipschitzian, becuase of its convexity) weak solution of the 1-dimensional Cauchy problem

$$\begin{aligned} A(v^*) + h - c^* &= 0, \\ v^*(0) = v_r^*(0) &= 0, \end{aligned} \tag{6.21}$$

and therefore, by the uniqueness for this problem, v^* is equal to the elliptic cap w^* given in (5.12).

To complete the proof for cases (iv) and (v) it remains to observe that u^* converges to w^* in $C^1([0, r_0])$ for every $r_0 < r^*$ by Dini's convergence theorem, since $u_r^* = u_r$ monotonically pointwide converges to the continuous function w_r^*. Moreover, since u_r^* is monotone in r and $w_r^*(r^{*-}) = -\infty$, we have that $u_r^*(r, \cdot)$ goes to $-\infty$ uniformly for $r \geqslant r^*$. Finally, since $0 \leqslant u_t \leqslant u_t(0, t)$, it is easy to show that $u_t \to C^*$ in $C^0_{\mathrm{loc}}(\Omega^*)$.

The proof in cases (i), (ii), (iii) is easier. As shown in Section 5, the elliptic Dirichlet problem (5.2) has a classical solution w (assuming the zero boundary value continuously). By the maximum principle (since at the boundary $|x| = R$ either $u(R, t) = 0$ or $u_r(R, t) = -\infty$) w bounds u from above, so that also u assumes the zero boundary value continuously. Moreover u, being nondecreasing with respect to t and bounded, has a finite limit as $t \to +\infty$. If $r < r_0 < R$ we have

$$u(r_0, t) - u(r, t) \leqslant u_r(r, t)(r_0 - r), \tag{6.22}$$

and, since $0 \leqslant u \leqslant w$,

$$u_r(r, t) \geqslant -w(r)(r_0 - r)^{-1}. \tag{6.23}$$

It follows that also $u_r(r, \cdot)$ has a finite limit for each $r < R$. We can proceed as in the first part of the proof (for cases (iii) and (iv)) to see that in fact u converges, as $t \to +\infty$, to w in $C^0(\bar{\Omega}) \cap C^1_{\text{loc}}(\Omega)$. Finally, if $c^* < 0$, u converges to w in $C^1(\bar{\Omega})$, using the fact that in this case w is a function of class $C^1(\bar{\Omega})$.

7. Some Remarks

Let us discuss the hypotheses made on $h(r)$ in Section 4. We are somewhat dissatisfied with our technical hypotheses (4.2i) and (4.2ii) which ensure that the pseudosolution $u(x, t)$ is concave on all Ω and for all t. Only with this concavity property have we been able to give a self-contained proof and to handle the technical difficulties of regularity of the peudosolution, monotonicity of u_r, etc. We use the concavity of h also in the proof of Lemma 5.2 to obtain the uniqueness of Ω^*. But it is important to note that in Section 6 assumption (4.2ii) is not necessary: we can describe the asymptotic growth of $u(x, t)$ with the same arguments of Section 6 only assuming that (4.2i) holds; in this case the proof is not self-contained since the regularity of u and the convergence of u^ε to u are assumed (cf. [4, 12]). The difference between the case that (4.2ii) holds or not is that assumption (4.2ii) eliminates observation of the transition from existence to nonexistence of a stationary solution (as $h(r)$ is increased by an additive constant K, for example, which leaves Ω^* unchanged) in the case of a strictly interior maximal set Ω^*, because as $c^* = c^*(h(r) + K)$ becomes slightly > 0, the solution $u(x, t)$ is expected to grow unboundedly on Ω^* but remains bounded on some other parts of Ω (as in Example 3.4 of Section 3) and hence $u(x, t)$ would not be concave on all Ω. Moreover, only by adding a sufficiently large additive constant K to $h(r)$, thereby forcing c^* to be much > 0, can we be assured that condition (4.2ii) is satisfied.

Let us examine in more detail the relation between hypothesis (4.2) and the concavity of the pseudosolution u. Note first that the nonnegativity and the concavity of h are natural to obtain respectively the monotonicity in t and the concavity in x of $u(x, t)$ for small t; this can be seen using the Taylor formula

$$u(x, t) \cong u(x, 0) + u_t(x, 0)\, t = h(|x|)\, t. \tag{7.1}$$

On the other hand, condition (4.2ii) at the boundary (i.e., $h(r) r \leqslant h(R) R$ when $n > 1$) has been sufficient to guarantee the concavity of $u(\cdot, t)$ for large values of t. Let us show by a simple example that this condition is almost necessary.

We consider the quadratic function

$$h(r) = -ar^2 + b, \qquad a \geqslant 0, \qquad b \geqslant aR^2. \tag{7.2}$$

This h remains nonnegative on $r \leqslant R$. We can easily compute r^* and c^*:

$$\text{If} \quad \left(\frac{n+2}{2a}\right)^{1/3} < R \quad \text{then} \quad r^* = \left(\frac{n+2}{2a}\right)^{1/3}, \quad c^* = b - \frac{3n}{2}\left(\frac{2a}{n+2}\right)^{1/3}; \tag{7.3}$$

$$\text{If} \quad \left(\frac{n+2}{2a}\right)^{1/3} \geqslant R \quad \text{then} \quad r^* = R, \quad c^* = b - \frac{n}{R} - \frac{an}{n+2} R^2. \tag{7.4}$$

If $c^* < 0$ the elliptic cap w exists. In fact, by (5.10) we have $0 \geqslant y(r) > -1$, and thus w is given (up to an additive constant) by (5.5) on all $r \leqslant R$. We can deduce the concavity of w from the relation $w_{rr} = y_r(1 - y^2)^{-3/2}$, on computing the sign of y_r. It is easy to see that y has an internal minimum if

$$b < \frac{3n}{n+2} aR^2. \tag{7.5}$$

Thus, if $c^* < 0$ and the above relation holds (these conditions are not void; for example, they are satisfied if $n = 2$ and $a = b = R = 1$), then the elliptic solution w exists with finite gradient on all $r \leqslant R$, but is not concave. We would expect (since $c^* < 0$) that the solution u of parabolic problem (4.1) would converge, as $t \to +\infty$, to w. But, since w is not concave, u would not be concave in x for large values of t. Of course the given h does not satisfy (4.2ii); in fact $h(r)\, r \leqslant h(R) R$ holds if and only if

$$b \geqslant 3aR^2. \tag{7.6}$$

Thus our sufficient condition (4.2ii) is close to the opposite inequality of (7.5), necessary to the concavity of w. Note also that, for this quadratic $h(r)$, w is always concave when $n = 1$, in accord with our concavity result of Section 4.

With the same family (7.2) of quadratic functions $h(r)$ we can see that r^* definitely can be an interior point of $[0, R]$. In fact, under conditions (7.3), (7.6), h satisfies (4.2) and $r^* < R$. However, as already observed at the beginning of this section, our hypothesis (4.2ii) forces $c^* > 0$ (i.e., excludes the transitional case $c^* = 0$) if $r^* < R$. To prove this we can assume $c^* \leqslant 0$ and obtain, from (7.6) (if $n > 1$) and (7.3), the contradiction

$$b \geqslant 3aR^2 > 3a\left(\frac{n+2}{2a}\right)^{2/3} = 3\left(\frac{n+2}{2}\right)^{2/3} a^{1/3}$$
$$\geqslant 3\left(\frac{n+2}{2}\right)^{2/3} \frac{2}{3}\frac{b}{n}\left(\frac{n+2}{2}\right)^{1/3} = \frac{n+2}{n} b. \tag{7.7}$$

If $n = 1$ we obtain similarly a contradiction starting from $b \geqslant aR^2$. Finally note that, if we change the above inequalities into equalities ($n = 1$, $r^* = R$, $c^* = 0$, $b = aR^2$; i.e., $a = 3/2R^{-3}$, $b = 3/2R^{-1}$) we have an example for which occurs the limiting situation considered in case (iv) of the main Theorem 6.3.

8. Numerical Examples

We now discuss some numerical computations which show very graphically the development of the rising elliptic cap on Ω^* and the detachment from the boundary values. We consider the parabolic equation (1.1) on the $1-d$ interval $\Omega = (-1, 1)$, with zero initial and boundary data, and with $h(x)$ equal the quadratic function $10 - 3x^2$ of (7.2). For this choice of $h(x)$ the maximal c^* and $\Omega^* = (-r^*, r^*)$, defined by Lemma 5.2, are known by (7.3) to be $c^* = 10 - 2^{1/3} - 2^{-2/3} = 8.110118$ and $r^* = 2^{-1/3} = 0.793700$.

We first discretized (1.1) in space only, by the centered difference equation

$$\begin{aligned} \frac{du_i}{dt} &= (g_{i+1/2} - g_{i-1/2})/\Delta x + h_i, \qquad i = 0, 1, \ldots, N, \qquad \text{where} \\ g_{i-1/2} &= [1 + (u_x)_{i-1/2}]^{-1/2}(u_x)_{i-1/2}, \qquad \text{and where} \\ (u_x)_{i-1/2} &= (u_i - u_{i-1})/\Delta x \end{aligned} \tag{8.1}$$

is the "discrete gradient" on the ith cell. Here u_i denotes $u(x_i)$ with equally spaced grid points $x_i = i\,\Delta x$. We used $N + 1 = 51$ grid points on the interval $[0, 1]$, taking advantage of the symmetry at the origin of course by setting $u_{-1} = u_1$ in Eq. (8.1) for $i = 0$.

We then discretized (8.1) in time using Heun's method, a second-order Runge–Kutta method. We used a very tiny $\Delta t = \frac{1}{8}(\Delta x)^2$, which is about one fourth the stability limit for this equation, and double precision arithmetic; hence our solution can be regarded as essentially the exact solution of the system of ODEs (8.1).

Since (8.1) is in conservative form, we can sum by parts over any discrete subinterval $G=\{x_m,\dots,x_n\}$ of the grid points and find, analogous to (2.1), that the rate of heat accumulation in G is

$$\frac{d}{dt}\left(\sum_{x_i\in G} u_i\,\Delta x\right)=g_{n+1/2}-g_{m-1/2}+\left(\sum_{x_i\in G_i} h_i\,\Delta x\right). \tag{8.2}$$

Our numerical calculations might be thought of as an attempt to approximate the true solution of the PDE (1.1) (but the degree of approximation would be hard to justify rigorously since our true solution is expected to develop infinite gradients on $\Omega-\Omega^*$ and even to detach discontinuously from its boundary values at $x=\pm 1$). Alternatively, our computations may be considered as tests for the discrete conservation law (8.1) and (8.2) of *discrete conjectures* analogous to those made and proved for the continuous case (that u_i will grow fastest on that maximal discrete subinterval Ω^* which maximizes the "discrete minimal rate" MR(h, G) analogous to (2.4), with speed c_i^{**} equal this maximum value MR(h,Ω^*) as in (2.5), that u_i will develop a "rising elliptic cap" on Ω^*, that the discrete gradient u_x will become infinite at the boundary of Ω^*, that the first interior value u_{49} will in some sense "detach" from the boundary value $u_{50}=0$, etc.)

Table I and Fig. 1 give the values of u_i-u_0 at times 0.5, 1, 2, 10, 20, 40, 45. These show very clearly the detailed evolution of the limiting "elliptic

TABLE I

u_0-u_i; Evolution of the Elliptic Cap on $|x_i|\leqslant 0.78$

x_i	$t=0.5$	$t=1$	$t=2$	$t=10$	$t=20$	$t=40$	$t=45$
0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.10	0.00845	0.00909	0.00935	0.00950	0.00950	0.00951	0.00951
0.20	0.03423	0.03696	0.03811	0.03874	0.03877	0.03879	0.03878
0.30	0.07880	0.08565	0.08859	0.09022	0.09031	0.09034	0.09034
0.40	0.14483	0.15916	0.16557	0.16920	0.16940	0.16947	0.16946
0.50	0.23684	0.26486	0.27839	0.28645	0.28691	0.28704	0.28705
0.60	0.36186	0.41649	0.44684	0.46705	0.46828	0.46863	0.46865
0.70	0.53090	0.64038	0.71963	0.79191	0.79742	0.79902	0.79913
0.74	0.61439	0.75997	0.88301	1.03290	1.04842	1.05317	1.05350
0.76	0.66019	0.82809	0.98280	1.21816	1.25140	1.26247	1.26329
0.78	0.70907	0.90245	1.09746	1.49865	1.59782	1.64104	1.64462
0.80	0.76121	0.98359	1.22890	1.96757	2.43848	3.16458	3.33171
0.82	0.81698	1.07205	1.37847	2.67597	3.99035	6.55354	7.19185
0.84	0.87679	1.16839	1.54671	3.54429	5.83476	10.38304	11.51881
0.90	1.08727	1.51264	2.16264	6.59050	12.00637	22.81685	25.51849
0.96	1.38338	1.98851	2.98444	10.11198	18.87891	36.38640	40.76212
0.98	1.53992	2.28991	3.37522	11.51014	21.46243	41.31132	46.27067
0.00	4.40352	8.52325	16.68283	81.62679	162.73946	324.94865	365.50016

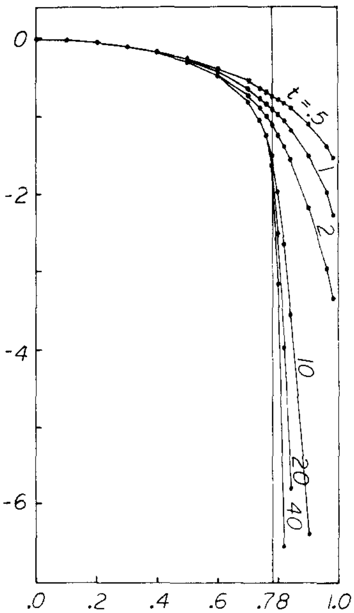

FIG. 1. $u_i - u_0$; detail of the evolving elliptic cap on $|x_i| \leqslant 0.78$.

cap" on $\Omega^* = \{|x_i| \leqslant 0.78\}$. It is quite clear that $u_i - u_0$ reaches a limiting shape on Ω^*, but that for $|x_i| > 0.78$ it continues to decrease with a strictly negative speed and that thus the gradient becomes infinite there. This fact is made even more definite by Table II and Fig. 2, which consider the values of $(d/dt)(u_i)$; these show very clearly that $(d/dt)(u_i)$ approaches a limiting speed v_i everywhere, where v_i is a constant $c_i^{**} \approx c_i^*$ on $|x_i| \leqslant 0.78$, and where v_i is strictly $< c_i^{**}$ on $0.80 \leqslant |x_i| \leqslant 0.98$. Since this limiting speed is much greater at all the interior points $|x_i| \leqslant 0.98$ than its zero value at the boundary, we can loosely say that the solution u_i has "detached" from its zero boundary values at $|x_i| = 1$. Finally, Fig. 3 plots the values of u_i at $t = 2$, 10, 20, and shows all three develoments—the rising elliptic cap on $|x_i| \leqslant 0.78$, the limiting speed v_i (since u_i looks asymptotically like $t \cdot v_i$), and the detachment at $|x_i| = 0.98$ from the boundary values.

In the conjectures and proofs of the previous sections we have concentrated mainly on the behavior and asymptotic growth rate c^* of u on the maximal subset Ω^*. However, it is clear in these particular calculations (done mostly after the previous sections were finished) that the derivative u_t has approached a limiting speed $v(x)$ on *all* Ω, not just on Ω^*. It is thus interesting to conjecture briefly about the nature of this function $v(x)$ in general. Suppose therefore that such a limiting speed $v(x)$ exists on all Ω, and is decently smooth or continuous (with the present convexity hypotheses

TABLE II

$c_i^* - (d/dt)(u_i)$, where $c_i^* = 8.110118$; evolution of the limiting speed v_i equal a constant $c_i^{**} \approx c_i^*$ on $|x_i| \leqslant 0.78$, and $v_i = h_i$ on $0.80 \leqslant |x_i| \leqslant 0.98$

x_i	$t = 0.5$	$t = 1$	$t = 2$	$t = 10$	$t = 20$	$t = 40$	$t = 45$
00	−0.69691	−0.12935	−0.03741	−0.00225	−0.00063	−0.00020	−0.00017
0.10	−0.68002	−0.12807	−0.03724	−0.00225	−0.00063	−0.00020	−0.00017
0.20	−0.62845	−0.12388	−0.0367	−0.00224	−0.00063	−0.00020	−0.00017
0.30	−0.53931	−0.11564	−0.03558	−0.00222	−0.00063	−0.00020	−0.00017
0.40	−0.40722	−0.10072	−0.03338	−0.00219	−0.00063	−0.00020	−0.00017
0.50	−0.22324	−0.0733	−0.02874	−0.00213	−0.00062	−0.00020	−0.00017
0.60	0.0268	−0.02008	−0.01717	−0.00193	−0.00059	−0.00019	−0.00017
0.70	0.36489	0.08962	0.02045	−0.00083	−0.00042	−0.00017	−0.00015
0.74	0.53186	0.16182	0.05792	0.00159	−0.00001	−0.00012	−0.00011
0.76	0.62352	0.2064	0.08662	0.00547	0.00079	−0.00000	−0.00003
0.78	0.72121	0.25742	0.12453	0.01721	0.00444	0.00066	0.00045
0.80	0.8255	0.31542	0.17339	0.05644	0.04028	0.03359	0.03299
0.82	0.93703	0.38081	0.23419	0.13439	0.12858	0.12751	0.12746
0.84	1.05665	0.45387	0.30677	0.22946	0.22731	0.22698	0.22697
0.90	1.47762	0.72139	0.58217	0.54089	0.54026	0.54015	0.54014
0.96	2.06983	1.08092	0.92631	0.87641	0.87519	0.87497	0.87496
0.98	2.38292	1.25064	1.07133	0.99666	0.99285	0.99174	0.99165
1.00	8.11012	8.11012	8.11012	8.11012	8.11012	8.11012	8.11012

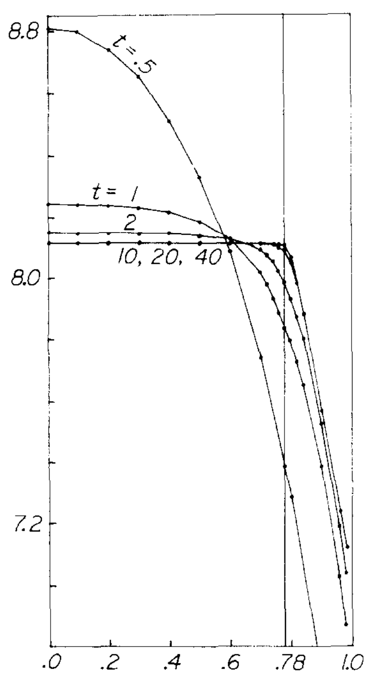

FIG. 2. $(d/dt)(u_i)$; evolution of the limiting speed v_i everywhere.

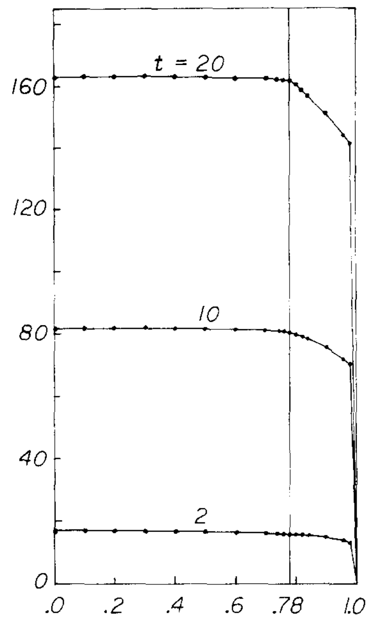

FIG. 3. u_i; development of the rising elliptic cap on $|x_i| \leqslant 0.78$, of the limiting speed v_i everywhere, and detachment from the boundary value at $|x_i| = 0.98$.

(4.2) this should be easy to establish). Now on connected domains where ∇u stays bounded, $|\nabla v|$ must be identically zero. On the other hand, on domains where $|\nabla v| > 0$, ∇u should asymptotically approach $t \nabla v$, and thus g should saturate and approach the unit vector $\nabla v/|\nabla v|$ in the direction of ∇v and ∇u; hence, on such domains v should satisfy the PDE

$$v = \nabla \cdot (\nabla v/|\nabla v|) + h(x). \tag{8.3}$$

For radially symmetric $h(r)$ in n-dimensions, ∇v will be in the radial direction, and hence (8.3) becomes (see 4.3)

$$v = \frac{\pm(n-1)}{r} + h(r), \tag{8.4}$$

where the + or − depends on whether v_r is positive or negative. (In $1-d$ this gives $v(x) = h(x)$, and our present calculations by time $t = 45$ do satisfy $du_i/dt = h(x_i)$ on all $0.80 \leqslant |x_i| \leqslant 0.98$ to within several significant figures.)

For more general $h(r)$, without hypotheses (4.2) of positivity or convexity, we would expect to see $v(r)$ be constant on certain annular domains G^* which locally maximize the minimum rate $\mathrm{MR}(h, G)$ or which minimize the similar "maximum rate" (defined by replacing the − by + in (2.4)) and on which $|\nabla u|$ remains locally bounded, joined by other annular regions on which v satisfies (8.4) with $+$ or $-$ and on which $|\nabla u|$ grows without bound.

For more general domains Ω and $h(x)$ there is much that can be conjectured about the limiting speed $v(x)$ and its equation (8.3); we hope to return to this at a later date.

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