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THE STORED-ENERGY FOR SOME DISCONTINUOUS DEFORMATIONS IN NONLINEAR ELASTICITY

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Dedicated to Ennio De Giorgi on his sixtieth birthday

Sommario. In questo lavoro viene applicata all'elasticità non lineare un'idea che Ennio De Giorgi ha dimostrato essere feconda in contesti diversi, ad esempio nello studio delle superfici minime o della Γ-convergenza. Si tratta della cosiddetta "estensione dello spazio ambiente": è opportuno ricercare a priori superfici cartesiane di area minima nella classe BV delle funzioni a variazione limitata, piuttosto che tra le funzioni di classe C^1, come pure in problemi di Γ-convergenza o di esistenza del minimo di integrali del calcolo delle variazioni è opportuno estendere lo spazio ambiente C^1 fino ad uno spazio di Sobolev $H^{1,p}$, od anche perfino ad L^p.

L'idea di base, già sperimentata da De Giorgi e da altri, ha dato buoni frutti. Il resto non so. L'una e l'altro gli sono dedicati con affetto in occasione del suo 60^o compleanno.

1. Introduction. We consider an elastic body that occupies a bounded open set $\Omega \subset \mathbf{R}^n$ in a reference configuration. We denote by $u(x)$ a deformation of Ω; that is, a particle $x \in \Omega$ is displaced to $u(x) \in \mathbf{R}^n$. By Du we denote the $n \times n$ matrix of the deformation gradient. We are concerned with *hyperelastic* materials having stored-energy function $W(\xi), \xi \in \mathbf{R}^{n \times n}$. That is, the total stored-energy E is given by

$$E(u) = \int_\Omega W(Du(x))dx. \tag{1.1}$$

If the material is frame-indifferent and isotropic, the energy function W can be represented by

$$W(\xi) = \phi(v_1, \dots, v_n), \tag{1.2}$$

where $v_1, \dots, v_n$ are the eigenvalues of the symmetric matrix $(\xi^T \xi)^{1/2}$ assuming that $\det \xi$ (the determinant of ξ) is positive.

One of the most interesting problems in this field is to find appropriate (both from the mathematical and the physical point of view) assumptions on the behaviour of W; that is, to describe the largest number of properties of the stored-energy that are common to a given class of materials and that are useful in a mathematical approach.

Some models have been proposed and have been studied in *incompressible elasticity*, in which the deformation $u(x)$ is subjected to the pointwise constraint $\det Du(x) = 1$. There are also papers that relate the proposed theoretical expression of the stored-energy with measures in experiments (see the references in Ball [1], [2]).

In the incompressible case, the stored energy $\hat{W}(\xi)$, defined for ξ with $\det \xi = 1$, can be extended to every ξ with $\det \xi > 0$, by setting

$$W_1(\xi) = \hat{W}\left(\frac{\xi}{(\det \xi)^{1/n}}\right); \tag{1.3}$$

this kind of extension has been studied in a recent paper by Charrier, Dacorogna, Hanouzet, Laborde [6].

Up to now the most considered form of stored-energy for *compressible materials* is of the type:

$$W(\xi) = W_1(\xi) + g(\det \xi), \tag{1.4}$$

where g is a given real function.

Odgen in [8] proposed for rubberlike solids a stored-energy of the form (1.4); more precisely, he proposed an expression in terms of v_i of the form

$$W(\xi) = \phi(v_1, \ldots, v_n) = \sum_{i,j} a_j v_i^{\alpha_j} + g(v_1 \cdot v_2 \ldots v_n). \tag{1.5}$$

Some of the exponents α_j can be negative; of course the product $v_1 \cdot v_2 \ldots v_n$ is the determinant of the matrix ξ.

The contribution to the energy, corresponding to the function g, expressed by the integral

$$\int_\Omega g(\det Du(x))dx, \tag{1.6}$$

takes into account the part of the energy that depends on changes in volume. We assume that the energy goes to $+\infty$ if we expand the solid ($\det Du \to +\infty$) or if we compress the solid to a point ($\det Du \to 0$). More precisely, we assume that $g = g(t)$, defined for $t > 0$, goes to $+\infty$ both as $t \to +\infty$ and as $t \to 0^+$. We assume also that g is smooth, so that g has a minimum, say at $t = 1$.

In the following we quote some considerations by Odgen [18]. First, there are physical reasons to think that $g(t)$ is decreasing for $t < 1$ and increasing for $t > 1$. Secondly, an inequality considered by Odgen is:

$$(tg')' = g' + tg'' > 0 \quad , \quad \forall t > 0. \tag{1.7}$$

For $t < 1$, since $g' \leq 0$, we have $g'' > 0$, thus g is convex in $(0, 1)$. The conclusion is not the same for $t > 1$, where the above inequality can also be satisfied if g is concave.

In relation tȯ the concavity of g we have two possible situations, schematized in figures 1 and 2.

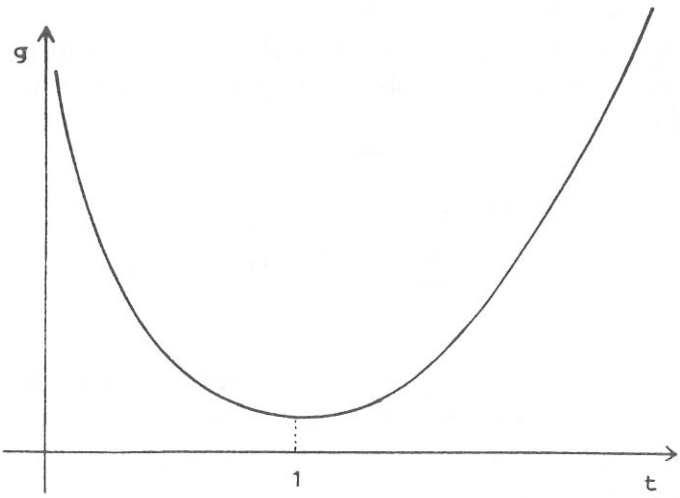

Fig.1. Graph of a convex g.

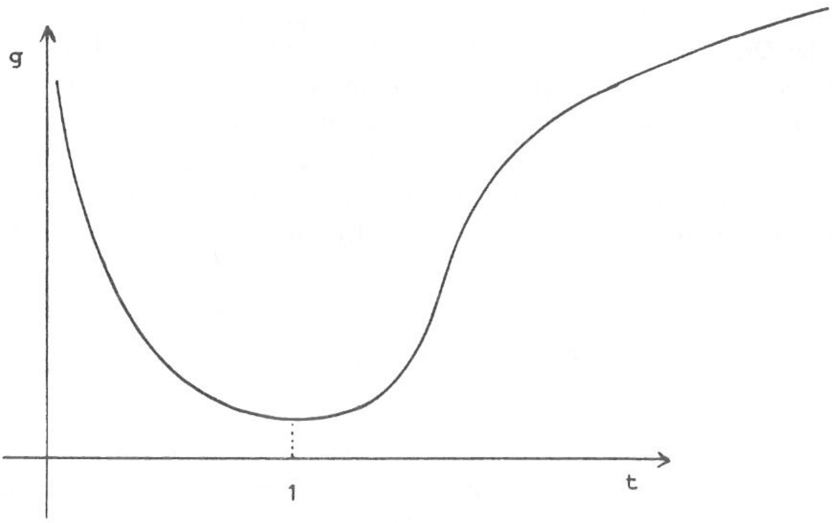

Fig.2. Graph of a non convex g.

Finally let us quote from Odgen [18], pages 572, 573: *"Little has been determined experimentally about situations in which* $t > 1$"; *"It seems likely (though it is not certain) that, so long as the material remains elastic, an increase in all-round tension would always be required to effect a volume increase; we would then have* $g'' > 0$ *for all* t (> 0). *Once the* elastic limit *is reached the possibility of* g'' *being less than zero cannot be ruled out"*.

From a mathematical point of view, in particular from the point of view of the existence of an energy minimum, the situation schematized in figure 1 is easier for at least two reasons.

First, the convexity of g plays a role in the existence of equilibrium solutions. In fact Ball [1] pointed out the importance in this theory of the assumption of *quasi-convexity* (see (5.2)) of $W(\xi)$, and recently Ball and Murat ([4], lemma 4.3) proved that quasiconvexity of a function of the type

$$W(\xi) = |\xi|^p + g(\det \xi) \tag{1.8}$$

implies convexity of g, if $1 < p < 2n$.

A second reason to say that the situation schematized in figure 1 is easier for existence of minima, depends on the coercivity condition:

$$g(t) \geq \text{ constant } \cdot t^r \quad , \quad t > 0 \tag{1.9}$$

for an exponent $r > 1$. Such an r may exist in the case of figure 1, but it cannot exist in the case of figure 2, if g is concave for large t.

The validity of (1.9) for some $r > 1$ is one of the assumptions in the well known existence theorem by Ball [1]. Recently, new existence results for some integrals of elasticity have been obtained by Ball and Murat ([4], theorem 6.1) and by Marcellini ([14], theorem 1).

Here we present a *mathematical argument* to show that the case $r = 1$ in (1.9) is one of the most relevant in this theory. In some sense, $r = 1$ is an "intermediate" case between the scheme considered in figure 1 (with g convex) and the scheme in figure 2 (with g concave for large t), since, if $r = 1$, then g is "linear at infinity".

The mathematical argument is based on studying an hyperelastic material for which the phenomenon of *cavitation* may occur. We describe the phenomenon of cavitation in the next section. We present an approach substantially different from Ball's approach on cavitation [2]. We will show that, under our approach, cavitation may occur only if g is not convex and if it is linear at infinity. Moreover, if g is linear at $+\infty$ and cavitation occurs, then the slope $g'(+\infty)$ should be equal to the radial component of the Cauchy stress at the surface of the cavity. Thus we propose an indirect method to find experimentally the behaviour of the function $g(t)$ for large values of t.

On reading this paper a referee pointed out to me the references [27], [28], [29], where it is studied the "Blatz-Ko material". The energy function W, proposed by Blatz and Ko for $n = 3$, with respect to $t = \det Du$ is of the form (see formula (2.21) of [29])

$$W(..,..,t) = 2\beta t^{-1} + (1-\beta)(t^{-2} + 2t). \tag{1.10}$$

The parameter β, determinated experimentally under three distinct homogeneous deformations, takes the values:

(1.11) $(a)\quad \beta = 0.13 \qquad (b)\quad \beta = 0.07 \qquad (c)\quad \beta = -0.19.$

In our context it is very interesting to notice that $g(t)$ is linear at infinity and, in the cases (a), (b), g is convex like in figure 1, while in the case (c) g is not convex and behaves like the function in figure 2.

2. The phenomenon of cavitation. The phenomenon of cavitation has been first studied by Ball [2]. The reader interested in cavitation is also referred to the papers by Stuart [25], [26], Podio Guidugli, Vergara Caffarelli, Virga [19], [20], Sivaloganathan [23], [24].

The idea of cavitation is as follows: we consider a body that occupies the unit ball $\Omega = \{|x| < 1\}$ of $\mathbf{R}^n$, with $n \geq 2$. We expand the body with deformation $u(x) = \lambda x$ at $|x| = 1$, for some $\lambda > 1$ (that is, we impose the boundary condition that the deformed surface of the body is a sphere of radius λ). We expect that, if λ is too large, then for some materials a hole forms inside the body.

To describe mathematically the phenomenon of cavitation, following Ball [2], we consider radial deformations

$$u(x) = v(r)\frac{x}{r} \quad , \qquad \text{with} \quad r = |x|, \tag{2.1}$$

where $v(r)$ is a function defined for $r \in [0,1]$, such that $v \geq 0$, $v' \geq 0$.

A computation shows that the eigenvalues v_i of $(Du^T Du)^{1/2}$ are

$$v_1 = v'(r) \quad , \quad v_i = \frac{v(r)}{r} \quad \text{for} \quad i = 2, \dots, n. \tag{2.2}$$

Thus the determinant of Du, i.e. the product of the v_i for $i = 1, \dots, n$, is equal to

$$\det Du = v'(r) \left(\frac{v(r)}{r} \right)^{n-1} . \tag{2.3}$$

By using polar coordinates, the stored-energy (1.1), (1.2) takes the form:

$$E = \omega_n \int_o^1 r^{n-1} \phi \left(v', \frac{v}{r}, \dots, \frac{v}{r} \right) dr, \tag{2.4}$$

where, as usual, ω_n is the $(n-1)$-measure of the surface of the unit sphere in $\mathbf{R}^n$.

The transformation $u(x)$ defined in (2.1) is a map of the unit sphere to the sphere of radius $v(1)$. If $v(0) > 0$, then in this deformation a *cavity* forms at the center, with radius $v(0)$. In this case it is easy to see that u belongs to the Sobolev space $H^{1,p}(\Omega; \mathbf{R}^n)$ for every $p < n$, but u does not belong to $H^{1,n}(\Omega; \mathbf{R}^n)$.

If $v(0) > 0$, then $u(x) = v(r)x/r$ is a singular transformation at $x = 0$, and the corresponding energy needs to be defined carefully. We propose a definition in the next section.

3. Definition of the stored-energy for discontinuous deformations. From this point we follow a different approach from Ball's approach in [2]. For fixed $p > 1$, we consider the set

$$\mathcal{A} = \{ v \in H^{1,p}_{loc}(0,1) : v(0) = 0, \quad v \geq 0, \quad v' \geq 0 \quad \text{a.e.} \}; \tag{3.1}$$

where by the notation "$v \geq 0$" we mean $v(r) \geq 0$ for every $r \in (0,1)$; and by "$v' \geq 0$ a.e." we mean $v'(r) \geq 0$ for almost every $r \in (0,1)$. We recall that every function $v \in H^{1,p}_{loc}(0,1)$ is continuous in $(0,1)$.

Since we consider functions $v = v(r)$ that are increasing with respect to r, we can define v at the endpoints of $(0,1)$, by defining, for example at $x = 0$:

$$v(0) = \inf\{v(r) : r \in (0,1)\} = \lim_{r \to 0+} v(r). \tag{3.2}$$

Thus the value $v(0)$ in (3.1) is defined by (3.2).

It is easy to see that the set $\mathcal{A}$ is dense, with respect to the strong topology of $H^{1,p}_{loc}(0,1)$, in the set $\bar{A}$ defined by

$$\bar{A} = \{v \in H^{1,p}_{loc}(0,1) : v \geq 0, \quad v' \geq 0 \quad \text{a.e.}\}. \tag{3.3}$$

By its convexity, the set $\bar{A}$ is closed both in the strong and in the weak topology of $H^{1,p}_{loc}(0,1)$.

With abuse of notation we denote the energy either by $E(u)$ or by $E(v)$, where $u(x)$ and $v(r)$ are associated by (2.1). The integral E in (2.4) is well defined in $\mathcal{A}$, since the integrand is assumed to be positive; in fact E is the supremum (with respect to a, b) of the corresponding integrals on subintervals $[a,b] \subset (0,1)$.

We extend E from $\mathcal{A}$ to $\bar{A}$. We denote the extension by F. The idea is to define the energy $F(v)$ for $v \in \bar{A}$ by continuity:

$$F(v) = \lim_k E(v_k), \tag{3.4}$$

where v_k is a sequence in $\mathcal{A}$ that converges (we will consider either the strong or the weak topology) to v. To be sure that the definition is independent of the particular sequence v_k, we proceed as follows.

As usual we use the letter s to denote the strong topology, and the letter w to denote the weak topology of $H^{1,p}_{loc}(0,1)$. For every $v \in \bar{A}$ we define:

$$F_s(v) = \inf\{\liminf_k E(v_k) : v_k \in A, v_k \xrightarrow{s} v\}, \tag{3.5}$$

$$F_w(v) = \inf\{\liminf_k E(v_k) : v_k \in A, v_k \xrightarrow{w} v\}.$$

The scheme of the above definitions is classical. It was introduced by Lebesgue in his thesis [11], and then considered again by De Giorgi, Giusti, Miranda (see e.g. [10], [16]), Serrin [22], and recently by many others (see for example [5], [7], [9]). In this context this scheme was introduced by the author in [13].

It is easy to see that F_s is lower semicontinuous in the strong topology of $H^{1,p}_{loc}(0,1)$ and that (under coercivity conditions) F_w is lower semicontinuous in the weak topology. It is less easy to derive representation formulas for F_s and F_w; we consider this problem in the next section.

4. Representation formulas for the stored-energy. With the aim of giving a characterization of F_s, F_w, we state our assumptions on the integrand ϕ in (2.4).

$$\phi(\xi,\eta,\ldots,\eta) \quad \textit{is a continuous function for} \quad \xi>0 \quad \textit{and} \quad \eta>0. \tag{4.1}$$

(4.2) *There exist an exponent* $q<n$, *some positive constants* c, ξ_o *and a convex function* $h:[0,+\infty)\to[0,+\infty)$ *such that :*

$$\phi(\xi,\eta,\ldots,\eta) \geq h(\xi\eta^{n-1}), \quad \forall\xi\geq 0, \quad \forall\eta\geq 0; \tag{4.2a}$$

$$\phi(\xi,\xi,\ldots,\xi) \leq c(1+\xi^p)+h(\xi^n), \quad \forall\xi\geq\xi_o. \tag{4.2b}$$

Note that we do not require that $\phi=\phi(\xi,\eta,...,\eta)$ is convex with respect to ξ, neither do we require that ϕ is bounded from above as $\xi\to 0^+$.

Assumptions (4.1), (4.2) are very general and natural in the theory of nonlinear elasticity by Ball [1], [2]. Of course they can be satisfied by integrands of the type (1.8) with $p<n$ and g as in figures 1 or 2.

Theorem 1. *Let E, F_s, F_w be defined respectively by (2.4), (3.5), (3.6). Under assumptions (4.1), (4.2) the following representation formula holds:*

$$F_s(v)=\omega_n\int_o^1 r^{n-1}\phi\left(v',\frac{v}{r},\ldots,\frac{v}{r}\right)dr+\tilde{h}\frac{\omega_n}{n}[v(0)]^n, \tag{4.3}$$

for every $v \in \bar{A}$, where the constant $\tilde{h} \in [0, +\infty]$ is given by

$$\tilde{h} = \lim_{t \to +\infty} h(t)/t = \lim_{t \to +\infty} h'(t). \tag{4.4}$$

Moreover, if $\phi = \phi(\xi, \eta, ..., \eta)$ is convex with respect to ξ, the above representation formula holds also for F_w, i.e. $F_w(v) = F_s(v)$ for every $v \in \bar{A}$.

Note that in principle F_s and F_w in (3.5), (3.6) depend on p; but, under the assumptions of the above theorem, F_s and F_w are actually independent of p.

Before giving the proof of theorem 1 we state in the following lemma 2 a known result about convex functions.

For a convex function $h : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ the inequality of convexity can be written by:

$$\frac{1}{\sum \lambda_i} \sum \lambda_i h(\xi_i) \geq h\left(\frac{\sum \lambda_i \xi_i}{\sum \lambda_i}\right), \tag{4.5}$$

where $\xi_i \in \mathbf{R}$ and $\lambda_i \geq 0$, with $\sum \lambda_i \neq 0$. By approximating L^1-functions $\lambda(r), \xi(r)$ by step functions, each of them assuming a finite number of values λ_i, ξ_i, we easily obtain the following form of Jensen's inequality (the usual Jensen's inequality is obtained for $\lambda(r)$ = constant):

Lemma 2. *Let $h : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ be a convex function and let λ, ξ be L^1 functions, with $\lambda \geq 0, \lambda \not\equiv 0$.*

$$\frac{\int_a^b \lambda(r) h(\xi(r)) dr}{\int_a^b \lambda(r) dr} \geq h\left(\frac{\int_a^b \lambda(r)\xi(r) dr}{\int_a^b \lambda(r) dr}\right). \tag{4.6}$$

Proof of Theorem 1. If $v(0) = 0$ formula (4.3), i.e. $F_s = E$, follows from the lower semicontinuity of E in the strong topology of $H^{1,p}_{loc}(0,1)$ (by Fatou's lemma) and, if $\phi(\xi, \eta, ..., \eta)$ is convex with respect ξ, then $F_w = F_s = E$ follows from the lower semicontinuity of E in the weak topology of $H^{1,p}_{loc}(0,1)$.

Thus let us consider $v(0) > 0$ and let v_k be a sequence in $\mathcal{A}$ that converges to v in the strong topology of $H^{1,p}_{loc}(0,1)$. We extract a subsequence of v_k, that we still denote by v_k, with the properties: (i) the subsequence of real numbers $E(v_k)$ admits a limit and this limit is equal to the limes inferior of the original sequence; (ii) $v_k(r)$ converges to $v(r)$ for every $r \in (0,1)$.

Let M be an upper bound for $v_k(1/2)$, so that

$$v_k(1/2) \leq M \quad , \quad \forall k. \tag{4.7}$$

For every $r \in (0,1)$ we have

$$\lim_k v_k(r) = v(r) \geq v(0); \tag{4.8}$$

thus, for every natural ν, by choosing $r = 1/\nu$, there exists k_ν such that

$$v_{k_\nu}(1/\nu) > v(0) - 1/\nu. \tag{4.9}$$

If we define $w_\nu(r) = (v(0)\nu - 1)r$ and if ν is sufficiently large ($\nu > (2M+1)/v(0)$), by (4.7) and (4.9) we obtain

$$w_{k_\nu}(1/2) > M \geq v_{k_\nu}(1/2) \quad , \quad w_{k_\nu}(1/\nu) < v_{k_\nu}(1/\nu). \tag{4.10}$$

Therefore, if ν is sufficiently large, there exists $r_\nu \in (1/\nu, 1/2)$ such that $v_{k_\nu}(r_\nu) = w_{k_\nu}(r_\nu) = (v(0)\nu - 1)r_\nu$; moreover the following relations hold:

$$\lim_\nu r_\nu = 0 \quad ; \quad \liminf_\nu v_{k_\nu}(r_\nu) \geq v(0). \tag{4.11}$$

In fact r_ν converges to zero since, by (4.7), we have

$$\frac{1}{\nu} < r_\nu = \frac{v_{k_\nu}(r_\nu)}{v(0)\nu - 1} \leq \frac{v_{k_\nu}(1/2)}{v(0)\nu - 1} \leq \frac{M}{v(0)\nu - 1}; \tag{4.12}$$

while the second relation in (4.11) holds by (4.9), since $v_{k_\nu}(r_\nu) \geq v_{k_\nu}(1/\nu) > v(0) - 1/\nu$.

With the aim of finding a lower bound for F_s in (3.5), we use assumption (4.2a) to obtain

$$\begin{aligned} \liminf_k \; E(v_k) &= \lim_\nu \; E(v_{k_\nu}) \geq \\ &\geq \liminf_\nu \; \omega_n \int_{r_\nu}^1 r^{n-1}\phi\left(v'_{k_\nu}, \frac{v_{k_\nu}}{r}, \dots, \frac{v_{k_\nu}}{r}\right) dr + \\ &+ \liminf_\nu \; \omega_n \int_o^{r_\nu} r^{n-1} h\left(v'_{k_\nu}\left(\frac{v_{k_\nu}}{r}\right)^{n-1}\right) dr. \end{aligned} \tag{4.13}$$

We estimate the second term on the right hand side by using Jensen's inequality (4.6) with $\lambda(r) = r^{n-1}$ and $\xi(r) = v'_{k_\nu}(v_{k_\nu}/r)^{n-1}$. Since $v^n_{k_\nu}/n$ is a primitive of $v'_{k_\nu} v^{n-1}_{k_\nu}$, we have

$$
\begin{aligned}
&\int_o^{r_\nu} r^{n-1} h\left(v'_{k_\nu} \left(\frac{v_{k_\nu}}{r}\right)^{n-1}\right) dr \geq \\
(4.14) \qquad &\geq \frac{r^n_\nu}{n} h\left(\frac{n}{r^n_\nu} \cdot \frac{v^n_{k_\nu}(r_\nu)}{n}\right) \\
&= \frac{v^n_{k_\nu}(r_\nu)}{n} \frac{h(t_\nu)}{t_\nu},
\end{aligned}
$$

where we have posed $t_\nu = (v_{k_\nu}(r_\nu)/r_\nu)^n$. By (4.11) $t_\nu \to +\infty$. By again using (4.11) and definition (4.4), we obtain

$$
\begin{aligned}
(4.15) \qquad &\liminf_\nu \omega_n \int_o^{r_\nu} r^{n-1} h\left(v'_{k_\nu} \left(\frac{v_{k_\nu}}{r}\right)^{n-1}\right) dr \geq \\
&\geq \tilde{h} \frac{\omega_n}{n} [v(0)]^n.
\end{aligned}
$$

Let us go back to (4.13). For every fixed ν_o, if $r_\nu < r_{\nu_o}$, by Fatou's lemma we have:

$$
\begin{aligned}
(4.16) \qquad &\liminf_k E(v_k) \geq \\
&\geq \omega_n \int_{r_{\nu_o}}^1 r^{n-1} \left(v', \frac{v}{r}, \ldots, \frac{v}{r}\right) dr + \tilde{h}\frac{\omega_n}{n}[v(0)]^n.
\end{aligned}
$$

As $\nu_o \to +\infty$ we obtain

$$
(4.17) \qquad F_s(v) \geq \omega_n \int_o^1 r^{n-1} \phi\left(v', \frac{v}{r}, .., \frac{v}{r}\right) dr + \tilde{h}\frac{\omega_n}{n}[v(0)]^n.
$$

To get the opposite inequality we can compare a fixed function $v \in \bar{A}$, having $v(0) > 0$, with the sequence $w_k(r) = kr$. For every $k > v(1/2)$ we can choose $r_k \in (0, 1/2)$ so that $v(r_k) = kr_k$. Since $v(r_k) \leq v(1/2)$, then r_k converges to zero as $k \to +\infty$. let us define v_k by:

$$
(4.18) \qquad v_k(r) = \begin{cases} kr & \text{if } 0 \leq r \leq r_k \\ v(r) & \text{if } r_k < r < 1. \end{cases}
$$

Since $v_k' = v_k/r = k$ for $r \in (0, r_k)$, we have

$$
\begin{aligned}
F_s(v) &\le \limsup_k \omega_n \int_o^1 r^{n-1}\phi\left(v_k', \frac{v_k}{r}, \ldots, \frac{v_k}{r}\right) dr\cdot \\
&\le \omega_n \int_o^1 r^{n-1}\phi\left(v', \frac{v}{r}, \ldots, \frac{v}{r}\right) dr\cdot \qquad (4.19)\\
&\quad \cdot \limsup_k \frac{\omega_n}{n} r_k^n \phi(k, k, \ldots, k).
\end{aligned}
$$

We estimate the last term on the right hand side using assumption (4.2b). Since $kr_k = v(r_k)$, and since $v(r_k)$ converges to $v(0)$, we have

$$
\begin{aligned}
&\limsup_k r_k^n \phi(k, k, ..., k) \le \\
&\le \lim_k r_k^n [c(1 + k^q) + h(k^n)] \qquad (4.20)\\
&= \lim_k k^{-n}[c(1 + k^q) + h(k^n)][v(r_k)]^n = \tilde{h}[v(0)]^n.
\end{aligned}
$$

Here we have used the assumption $q < n$.

Since $v_k(r) = v(r)$ for $r > r_k$ and since $r_k \to 0$, the sequence v_k converges to v in $H^{1,p}_{loc}(0,1)$. Thus the opposite inequality to (4.17) follows from (4.19), (4.20).

The statement relating to F_w follows analogously. The only difference is that (4.16) then follows from the lower semicontinuity of the integral with respect to the weak topology of $H^{1,p}_{loc}(0,1)$ whenever $\phi(\xi, \eta, ..., \eta)$ is convex with respect to ξ.

By combining theorem 1 with theorem 3.8 of Marcellini and Sbordone [15] (see also [8]) it is possible to prove the following further characterization result when $\phi(\xi, \eta, \ldots, \eta)$ is not convex with respect to ξ.

Theorem 3. *For every $\nu > 0$ let $\phi^{**}(\xi, \eta)$ be the greatest function convex with respect to $\xi > 0$ and less or equal to $\phi(\xi, \eta, \ldots, \eta)$, with ϕ satisfying (4.1), (4.2). Let $\tilde{h}$ be the constant given by (4.4) and let F_w be the functional defined in (3.6). Then, for every $v \in \bar{A}$ we have*

$$
F_w(v) = \omega_n \int_o^1 r^{n-1}\phi^{**}\left(v', \frac{v}{r}, \ldots, \frac{v}{r}\right) dr + \tilde{h}\frac{\omega_n}{n}[v(0)]^n. \qquad (4.21)
$$

5. Interpretation and consequences of the representation formulas. A first consequence of the representation formula (4.3) is stated in the following:

Corollary 4. *Let us assume that (4.1), (4.2) hold and that the energy F_s is represented by (4.3). Then the phenomenon of cavitation may occur only if $h(t)$ is "(sub-)linear at infinity". More precisely, if $h(t) \geq ct^r$ for some $r > 1$ and $c > 0$, then $F_s(v) = +\infty$ for every $v \in \bar{A}$ with $v(0) > 0$.*

A second consequence can be easier described if we consider again the general situation without radial symmetry, with the energy integrand $W(\xi)$ of the form (1.4), that is:

$$W(\xi) = W_1(\xi) + g(\det \xi), \tag{5.1}$$

$W_1(\xi)$ being a quasiconvex function in the sense of the following definition (5.2).

We already pointed out that the linearity of $g(t)$ at ∞ is of interest in our approach. The function $g(t)$ must be convex or concave for large values of t?

If $g(t)$ is convex then it is well known that $W(\xi)$ satisfies the *quasiconvexity condition* by Morrey [17]:

$$\int_\Omega W(\xi + D\phi(x))dx \geq \int_\Omega W(\xi)dx = W(\xi)|\Omega|, \qquad \forall \phi \in C_o^1(\Omega; \mathbf{R}^n). \tag{5.2}$$

J.Ball pointed out that this mathematical condition by Morrey has an interpretation in nonlinear elasticity; we quote from Ball [1], pages 338, 339: "... *for ... a body that admits as a possible displacement a homogeneous strain, we require that this homogeneous strain be an absolute minimizer for the total energy*".

As described in section 2, in cavitation we impose the boundary condition $u(x) = u_o(x)$ at $|x| = 1$, where $u_o(x) = \lambda x$. The displacement u_o is a homogeneous strain. Thus, *under quasiconvexity*, u_o

must be an absolute minimizer and *the phenomenon of cavitation should not occur.*

This is what happens under the present approach; while this fact contrasts sharply with the approach of Ball in [2], where the phenomenon of cavitation occurs with quasiconvex energy integrals.

In fact, under our approach the energy is defined by lower semicontinuity; thus, if inequality (5.2) holds for smooth test functions ϕ, then it holds for the extended energy too.

Therefore, if the stored-energy E in (1.1) is quasiconvex, then $u_o(x) = \lambda x$ is an absolute minimizer among displacements with the same boundary values. In terms of $v(r), v_o(r) = \lambda r$ is an absolute minimizer for E given by (2.4), and thus v_o is an absolute minimizer also for F_s, F_w in (3.5), (3.6).

Of interest is the Euler's first variation of the functional F_s in (4.3). We obtain (for a non formal derivation we can proceed like in theorem 7.3 of Ball [2]) that a minimum of F_s on $\bar{A}$, with the condition $v(1) = \lambda$, formally satisfies the Euler-Lagrange equation:

$$\frac{d}{dr}\left(r^{n-1}\phi_\xi\right) = (n-1)r^{n-2}\phi_\eta, \qquad \forall r \in (0,1), \tag{5.3}$$

and the boundary conditions

$$\text{at} \quad r = 1: \quad v(1) = \lambda, \tag{5.4}$$

$$\begin{aligned} &\text{at} \quad r = 0: \\ &\text{either} \quad v(0) = 0 \quad \text{or} \quad \lim_{r\to 0+}\left(\frac{r}{v(r)}\right)^{n-1}\phi_\xi = \tilde{h}. \end{aligned} \tag{5.5}$$

The expression $(r/v)^{n-1}\phi_\xi$ that appears in (5.5) is called the *radial component of the Cauchy stress.* In principle we could imagine measuring the stress at $r = 0$ of an equilibrium solution with cavity. If $\tilde{h}$ is the value of this measure, then we have an indirect measure of the behaviour of the function $g(t)$ in (1.5) for large values of t; in fact $g(t)$ should behave like in figure 2, with $g(t)/t \to \tilde{h}$ as $t \to +\infty$.

Finally let us observe that the functional F_s in (4.3) can be represented in the form:

$$\begin{aligned} F_s(v) = \omega_n \int_o^1 r^{n-1}\left[\phi\left(v', \frac{v}{r}, \ldots, \frac{v}{r}\right) - \tilde{h}v'\left(\frac{v}{r}\right)^{n-1}\right]dr + \\ + \tilde{h}\frac{\omega_n}{n}[(v(1)]^n, \end{aligned} \tag{5.6}$$

for every $v \in \bar{A}$. Thus in the class of functions $v \in \bar{A}$ such that $v(1) = \lambda$, F_s can be represented in integral form.

On the contrary, in the representation (4.3), the functional F_s is the sum of an integral and a measure concentrated at $r = 0$; notice that this measure is equal to the product of the constant $\tilde{h}$ by the volume of the cavity that forms around the origin. The measure $\tilde{h}(\omega_n/n)[v(0)]^n$ can be interpreted as the energy due to the cavity; this energy is proportional to the volume of the cavity and not to the surface area, like in some standard models.

6. The non radially symmetric case. Let us consider again the general situation without radial symmetry, and a stored-energy of the form

$$(6.1) \qquad E(u) = \int_\Omega W_1(Du, adj\ Du)dx + \int_\Omega g(\det\ Du)dx,$$

where *adj* Du are the adjoints of the $n \times n$ matrix Du. Here W_1 and g are convex functions.

The reader interested in results on the existence of minima is referred to Ball [1], Ball and Murat [4] and Marcellini [14].

Similarly to (3.6), we define:

$$(6.2) \qquad F_w(u) = \inf\{\liminf_k E(u_k) : u_k \in C^1,\ u_k \overset{w}{\to} u \quad \text{in} \quad H^{1,p}\}.$$

We use the following well known result from Ball, Currie and Olver [3], which we quote in loose form: *If* $u \in H^{1,p}(\Omega, \mathbf{R}^n)$ *with* $p > n^2/(n+1)$, *then* Det Du *is well defined as a distribution* (like in [1], [3] we use the notation Det Du, instead of det Du, to remember that the determinant is a distribution).

In the applications to nonlinear elasticity it is natural to impose the restriction that the determinant of the deformation gradient is positive. Since every positive distribution is a measure, *if* $u \in H^{1,p}(\Omega; \mathbf{R}^n)$ *with* $p > n^2/(n+1)$ *and if* Det $Du \geq 0$, *then the distribution* Det Du *is a positive measure.* We can operate on the measure Det Du by the *Lebesgue decomposition* (see for instance [21], theorem 6.9): we denote by $\text{Det}_R Du$ the *regular part* of Det Du, i.e. the part absolutely continuous with respect to the Lebesgue measure, and by $\text{Det}_S\ Du$ the *singular part*, i.e. the complement:

$$(6.3) \qquad \text{Det}_S\ Du = \text{Det}\ Du - \text{Det}_R\ Du.$$

On the basis of the representation results of theorem 1, on the following theorem 5, and on the representation result obtained in section 5 of [13], we formulate the following:

Conjecture. *If* $p > n^2/(n+1)$ *and* Det $Du \geq 0$ *we have*

$$F_w(u) = \int_\Omega W_1(Du, adj\ Du)dx + \tag{6.4}$$
$$+ \int_\Omega g(\mathrm{Det}_R\ Du)dx + \tilde{g} \cdot \text{ Total Variation } (\mathrm{Det}_S Du)$$

for every $u \in H^{1,p}(\Omega;\mathbf{R}^n)$, *where* $\tilde{g}$ *is the limit, as* $t \to +\infty$, *of* $g(t)/t$.

Up to now the above conjecture has been not proved even if the singular part $\mathrm{Det}_S\ Du$ is identically zero. It is known only the case $u \in C^1(\Omega;\mathbf{R}^n)$; in fact the following result has been obtained in [12], [13]:

Theorem 5. *Let* $p > n^2/(n+1)$; *for every* $u \in C^1(\Omega;\mathbf{R}^n)$ *we have* $F_w(u) = E(u)$; *thus formula (6.4) holds on* $C^1(\Omega;\mathbf{R}^n)$. *This means that*

$$E(u) \leq \liminf_k E(u_k), \tag{6.5}$$

for every $u, u_k \in C^1(\Omega;\mathbf{R}^n)$ *such that* u_k *converges to* u *in the weak topology of* $H^{1,p}(\Omega;\mathbf{R}^n)$, *with* $p > n^2/(n+1)$.

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