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***L[∞]-Regularity for Variational Problems
with Sharp Non Standard Growth Conditions.***

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L[∞]-Regularity for Variational Problems with Sharp Non Standard Growth Conditions.

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Sunto. — Si prova che le soluzioni di problemi di Dirichlet, relativi ad una classe di equazioni differenziali che include l'esempio (1.1), sono limitate in $\bar{\Omega}$ (se il dato al bordo u_0 è limitato), nell'ipotesi che gli esponenti (q_i) verifichino la limitazione (che sembra ottimale) $\bar{q}^* > q$ in (1.2). Si dà un analogo risultato anche per integrali del calcolo delle variazioni.

I. — Introduction.

Let us consider the following model problem

$$(1.1) \quad \begin{cases} \sum_{i=1}^n \frac{\partial}{\partial x_i} (|u_{x_i}|^{q_i-2} u_{x_i}) = \sum_{i=1}^n \frac{\partial}{\partial x_i} (f_i) & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of \mathbf{R}^n , u is a scalar function, $f_i \in L^\infty(\Omega)$ and $q_i > 1$ for $i = 1, 2, \dots, n$.

We believe that a condition on the exponents (q_i) that should guarantee zero order regularity (local boundedness, Hölder continuity, ...) of the solutions should be

$$(1.2) \quad \bar{q}^* > q,$$

where

$$(1.3) \quad q = \max_i \{q_i\}; \quad p = \min_i \{q_i\}; \quad q_i > 1, \quad \forall i = 1, 2, \dots, n;$$

$$(1.4) \quad \frac{1}{\bar{q}} = \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}; \quad \bar{q}^* = \frac{n\bar{q}}{n-\bar{q}} \quad (\text{if } \bar{q} < n).$$

The condition (1.2) seems sharp in the sense that equations of the type in (1.1) may have unbounded weak solutions ([1], [4]);

in the version of [5], $q_1 = q_2 = \dots = q_{n-1} = q = 2$ and $q_n = q$ can be any real number satisfying, for $n > 3$:

$$(1.5) \quad q > 2 \frac{n-1}{n-3}.$$

We point out that (1.5) is opposite to (1.2), in the sense that it is equivalent to

$$\bar{q}^* < q;$$

however we notice that the counterexample in [1], [4] is related to the local boundedness of weak solutions of the equation in (1.1) and not to the global boundedness of solutions of the Dirichlet problem (1.1); more explicitly, in the example of [1], [4] the function u , being singular on a line, is unbounded not only in the interior of Ω , but also at the boundary $\partial\Omega$.

In this short paper we prove that (1.2) is a sufficient condition for the global L^∞ -boundedness of solutions of some differential problems including (1.1), with u_0 bounded (we limit ourselves to the global boundedness and we do not study here the local boundedness). We give also a similar result for minimizers of some integrals of the calculus of variations.

Our method to get the global boundedness of the solutions is a combination of the original first idea by Stampacchia [7] (see also sec. 7 of ch. 4 and sec. 3 of ch. 5 of [2]) and of a Sobolev-type inequality ([6], [8], [9]).

The regularity problem with non standard growth conditions of the type assumed here has been already considered in [5], in order to get local Lipschitz-continuity and C^∞ -smoothness. Of course the regularity obtained in [5] requires more restrictive assumptions than ours; in particular, the condition $\bar{q}^* > q$ in (1.2) is more general (and in fact it seems optimal) than the corresponding assumption in [5].

2. - L^∞ regularity.

We recall the following Sobolev-type inequality ([6], [8], [9]): *There exists a positive constant c_0 , depending on the set Ω , such that*

$$(2.1) \quad \|v\|_{L^{\bar{q}^*}(\Omega)} \leq c_0 \sum_{i=1}^n \|v_{x_i}\|_{L^{q_i}(\Omega)},$$

for any $v \in C_0^1(\Omega)$, where $q_i > 1$ for $i = 1, 2, \dots, n$, and \bar{q}^* is defined by (1.4).

First let us give a non homogeneous version of the Sobolev inequality in (2.1). Let us recall that Ω is a bounded open set of \mathbf{R}^n .

LEMMA 1. — Let p, q_i, q, \bar{q}^* be as in (1.3), (1.4) and let $B \geq 1$. Then the following inequality holds

$$(2.2) \quad \|v\|_{L^{\bar{q}^*}(\Omega)}^q \leq c_0^q n^{q-1} B^{q-p} \sum_{i=1}^n \|v_{x_i}\|_{L^{q_i}(\Omega)}^{q_i},$$

for any $v \in W_0^{1,p}(\Omega)$ such that

$$(2.3) \quad \|v_{x_i}\|_{L^{q_i}(\Omega)} \leq B, \quad \forall i = 1, 2, \dots, n.$$

PROOF. — By (2.3), since $q_i > p$ and $B \geq 1$, for any $i = 1, 2, \dots, n$ we have

$$(2.4) \quad \|v_{x_i}\|_{L^{q_i}(\Omega)}^q \leq B^{q-q_i} \|v_{x_i}\|_{L^{q_i}(\Omega)}^{q_i} \leq B^{q-p} \|v_{x_i}\|_{L^{q_i}(\Omega)}^{q_i}.$$

The convexity of the real function $t \rightarrow t^q$ implies

$$(2.5) \quad \left(\frac{1}{n} \sum_{i=1}^n \|v_{x_i}\|_{L^{q_i}(\Omega)} \right)^q \leq \frac{1}{n} \sum_{i=1}^n \|v_{x_i}\|_{L^{q_i}(\Omega)}^q.$$

Then, from the inequalities in (2.1), (2.5) and (2.4) we deduce that

$$\begin{aligned} (\|v_{L^{\bar{q}^*}(\Omega)}\|)^q &\leq c_0^q \left(\sum_{i=1}^n \|v_{x_i}\|_{L^{q_i}(\Omega)} \right)^q \leq \\ &\leq c_0^q n^{q-1} \sum_{i=1}^n \|v_{x_i}\|_{L^{q_i}(\Omega)}^q \leq c_0^q n^{q-1} B^{q-p} \sum_{i=1}^n \|v_{x_i}\|_{L^{q_i}(\Omega)}^{q_i}. \end{aligned} \quad \square$$

Let us consider the Sobolev spaces

$$(2.6) \quad \begin{cases} W^{1,(q_i)}(\Omega) = \{v \in W^{1,p}(\Omega): v_{x_i} \in L^{q_i}(\Omega)\}, \\ W_0^{1,(q_i)}(\Omega) = W^{1,(q_i)}(\Omega) \cap W_0^{1,p}(\Omega) \end{cases}$$

and let A be a non linear differential operator from $W_0^{1,(q_i)}(\Omega)$ into its dual of the form

$$A(v) = -\operatorname{div}(a(x, v, Dv)),$$

where $a(x, s, \xi) = (a_i(x, s, \xi))$ is a Carathéodory vector valued function on $\Omega \times \mathbf{R} \times \mathbf{R}^n$ such that, for some constants $M \geq m > 0$ and $r < \bar{q}^*$,

$$(2.7) \quad \sum_{i=1}^n a_i(x, s, \xi) \xi_i \geq m \sum_{i=1}^n |\xi_i|^{a_i};$$

$$(2.8) \quad |a_i(x, s, \xi)| \leq M \left(1 + |s|^r + \sum_{j=1}^n |\xi_j|^{a_j} \right)^{(1-1/a_i)}, \quad \forall i = 1, 2, \dots, n.$$

Note that the growth condition (2.8) is satisfied (see Lemma 2.1 of [5]) for example if $a_i = a_i(x, \xi) = (\partial/\partial \xi_i) f(x, \xi)$ and $f(x, \xi)$ is a Carathéodory function, convex with respect to ξ and such that $|f(x, \xi)| \leq c \left(1 + \sum_{i=1}^n |\xi_i|^{a_i} \right)$.

Then, if $f_i \in L^{r_i}(\Omega)$ for $r_i \geq q'_i$ ($q'_i = q_i/(q_i - 1)$), it is possible to modify the classical Leray-Lions [3] method in order to obtain the existence of a solution to the Dirichlet problem, with $u_0 \in W_0^{1,(a_i)}(\Omega)$ fixed:

$$(2.9) \quad \begin{cases} A(u) = - \sum_{i=1}^n \frac{\partial}{\partial x_i} (f_i) & \text{in } \Omega, \\ u - u_0 \in W_0^{1,(a_i)}(\Omega). \end{cases}$$

THEOREM 2. — Let us assume that $u_0 \in L^\infty(\Omega)$ and that

$$(2.10) \quad \frac{\bar{q}^*}{q} \min_i \left\{ 1 - \frac{q'_i}{r_i} \right\} > 1.$$

Then, under (2.7), (2.8), any solution to the Dirichlet problem (2.9) is bounded in $\bar{\Omega}$ and its $L^\infty(\Omega)$ -norm can be estimated in terms of the data. In particular (2.10) is satisfied if $f_i \in L^\infty(\Omega)$ for every $i = 1, 2, \dots, n$, and if $\bar{q}^* > q$, like in (1.2).

PROOF. — Like in the linear case of [7] we use as test function

$$(2.11) \quad w = \operatorname{sign}(u) \cdot \max\{|u| - k, 0\}, \quad \forall k \geq \|u_0\|_{L^\infty(\Omega)}.$$

Since $Dw = Du$ a.e. on the set $A(k) = \{x \in \Omega : |u(x)| > k\}$, then

$$\int_{A(k)} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^{a_i} dx \leq \int_{A(k)} \sum_{i=1}^n |f_i| \left| \frac{\partial u}{\partial x_i} \right| dx \leq \int_{A(k)} \left\{ \varepsilon \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^{a_i} + c_1(\varepsilon) \sum_{i=1}^n |f_i|^{a_i} \right\} dx;$$

here we have used the coercivity assumption (2.7) (note that in this proof we do not use explicitly (2.8)) and Young's inequality, for any $i = 1, 2, \dots, n$, with constants ε and $c_1(\varepsilon)$. By choosing $\varepsilon = m/2$ and by applying Hölder's inequality we get

$$\begin{aligned} \frac{m}{2} \int_{A(k)} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx &\leq c_1 \int_{A(k)} \sum_{i=1}^n |f_i|^{q'_i} dx \leq \\ &\leq c_1 \sum_{i=1}^n \left\{ \int_{A(k)} |f_i|^{r_i} dx \right\}^{q'_i/r_i} \cdot \{\text{meas } A(k)\}^{1-q'_i/r_i} \end{aligned}$$

and finally, for $\gamma = \min_i \{1 - q'_i/r_i\}$, $k \geq k_0$ and k_0 is such that $\text{meas} \{x \in \Omega : |u(x)| > k_0\} \leq 1$:

$$(2.12) \quad \frac{m}{2} \int_{A(k)} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \leq c_1 \left(\sum_{i=1}^n \|f_i\|_{L^{r_i}(\Omega)}^{q'_i} \right) \cdot \{\text{meas } A(k)\}^\gamma.$$

Now we apply Lemma 1 to the function w in (2.11) with $B = \max \{1; \|u_{x_i}\|_{L^{q_i}}, i = 1, 2, \dots, n\}$:

$$(2.13) \quad \left(\int_{\Omega} |w|^{\bar{q}^*} dx \right)^{q/\bar{q}^*} \leq c_0^n n^{q-1} B^{q-p} \int_{\Omega} \sum_{i=1}^n \left| \frac{\partial w}{\partial x_i} \right|^{q_i} dx.$$

From the definition of w and the inequalities (2.12), (2.13) we get, for some constant c_2 ,

$$\left(\int_{A(k)} [(|u| - k)^+]^{\bar{q}^*} dx \right)^{q/\bar{q}^*} \leq c_2 \{\text{meas } A(k)\}^\gamma.$$

Now we follow [7]: if $h > k \geq \max \{k_0; \|u_0\|_{L^\infty(\Omega)}\}$ we easily get

$$(2.14) \quad \text{meas } A(k) \leq c_3 \frac{\{\text{meas } A(k)\}^{\gamma(\bar{q}^*/q)}}{(h - k)^{\bar{q}^*}}.$$

Since $\gamma(\bar{q}^*/q) > 1$ (see (2.10)), then Lemma 4.1 of [7] gives that the real function $k \rightarrow \text{meas } A(k)$ is equal to zero for large values of k , i.e. the conclusion. \square

Now we study the L^∞ -regularity for minimizers of integrals of the calculus of variations of the following type:

$$(2.15) \quad F(v) = \int_{\Omega} f(x, Dv) dx ,$$

where $f(x, \xi)$ is a Carathéodory function such that, for some $m > 0$ and $r > 1$,

$$(2.16) \quad f(x, \xi) \geq m \sum_{i=1}^n |\xi_i|^{q_i} ;$$

$$(2.17) \quad f(\cdot, 0) \in L^r(\Omega) .$$

THEOREM 3. — Let u be a minimizer on $u_0 + W_0^{1,(q_i)}(\Omega)$ of the integral (2.15). If $u_0 \in L^\infty(\Omega)$ and if

$$(2.18) \quad \frac{\bar{q}^*}{q} \left(1 - \frac{1}{r}\right) > 1 ,$$

then $u \in L^\infty(\Omega)$ and its norm can be estimated in terms of the data. In particular the assumption (2.18) is satisfied if $f(\cdot, 0) \in L^\infty(\Omega)$ and $\bar{q}^* > q$, like in (1.2).

PROOF. — We use as test function $v = u - w$, with w defined by (2.11). Then

$$\int_{\Omega} f(x, Du) dx \leq \int_{\Omega} f(x, Dv) dx = \int_{\Omega - A(k)} f(x, Du) dx + \int_{A(k)} f(x, 0) dx .$$

Thus:

$$\int_{A(k)} f(x, Du) dx \leq \int_{A(k)} f(x, 0) dx .$$

By using the assumptions (2.16), (2.17) we get

$$(2.19) \quad m \int_{A(k)} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^{q_i} dx \leq \int_{A(k)} f(x, 0) dx \leq \|f(\cdot, 0)\|_{L^r(\Omega)} \cdot \{\text{meas } A(k)\}^{1-1/r} .$$

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