Non convex integrals of the Calculus of Variations

Paolo MARCELLINI

Istituto Matematico "U. Dini", Università di Firenze

Viale Morgagni 67/A, 50134 FIRENZE, Italy

Contents

1. Introduction

In these notes of a 1989-CIME Course on *Methods of Nonconvex Analysis* we consider problems related to some integrals of the Calculus of Variations whose integrands are not convex. We present some results for integrals which either lack completely of any convexity condition, or that satisfy the so called *quasiconvexity condition* in the sense of Morrey.

In section 2 we recall the relations between convexity of the integrand and lower semicontinuity of the integrals. In sections 3 and 4 we consider integrals of the Calculus of Variations respectively with non convex integrands and with quasiconvex integrands; we present in the simpliest way some ideas of the theories, some results and references on these subjects.

In section 5 we prove new existence theorems for a class of (non coercive) non convex one-dimensional integrals of the Calculus of Variations that can be

applied, in particular, to the classical *Newton's problem*. Finally, in section 6, we exhibit an explicit nonconvex functional, arising from nonlinear elasticity, for which existence of minimizers is still open.

Newton's problem of motion in a resisting medium can be considered the first problem in the Calculus of Variations since the introduction of integrals and derivatives. It was published by Newton in 1685; more than three centuries ago and some years before *the brachystochrone problem*, that is one of the most popular classical problems of the Calculus of Variations, that was solved by Johann Bernoulli in 1696. Newton formulated the problem to find the shape of a surface of revolution (whose profile is a function u=u(x), like in figure 1), moving in a resistent medium in the direction of its axis of symmetry, so that it offers the least resistence (for example the profile of a missile, or of a submarine).

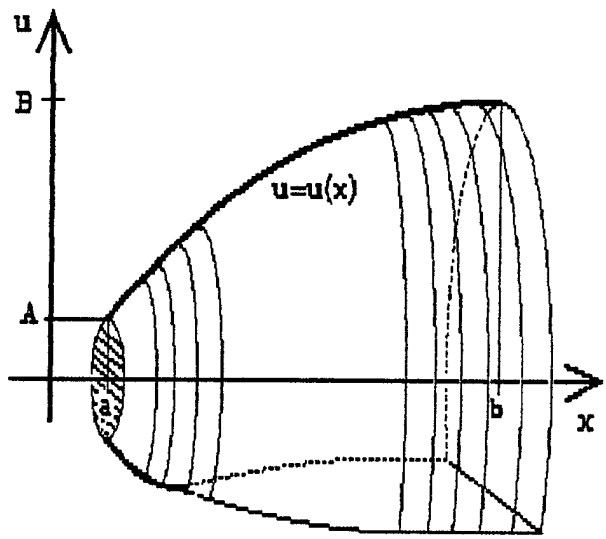

FIGURE 1

By assuming that the resistence is proportional to the normal component of the element of area of the surface, the problem is reduced to minimize the integral (see for example the chapter IX of the book of Tonelli [147], or section 1.2 of Goldstine [85]):

$$F(u) = 2\pi \int_a^b u \frac{(u')^3}{1+(u')^2} \, dx \tag{1.1}$$

in the class of absolutely continuous functions $u=u(x)$ whith given boundary values $u(a)=0$, $u(b)=B$ $(B\geq 0)$ and with the constraint on the derivative $u'(x)\geq 0$, for a.e. x in $[a,b]$.

It seems ([85], section 1.2) that the contemporaries did not believed to the solution proposed by Newton. In any case the proofs that followed ware not simple (for example Tonelli, in section 140 of [147], gives 13 pages of proof) and usually they are related to the integral in parametric form

$$F(u_1,u_2) = 2\pi \int_0^1 u_2 \frac{(u_2')^3}{(u_1')^2+(u_2')^2}\, dt \quad . \tag{1.2}$$

In principle (but we will show that it is not necessary), one of the reasons to prefer the integral (1.2) to (1.1), arises from the fact that the physical solution may detach from the boundary datum $u(a)=0$, i.e. the surface of revolution may be flat at $x=a$ ($u(a)$ can be strictly greather than 0) and thus, in principle, it may be easier to describe the solution by a parametric curve, rather than by a cartesian function (see section 5 for a cartesian approach to the problem).

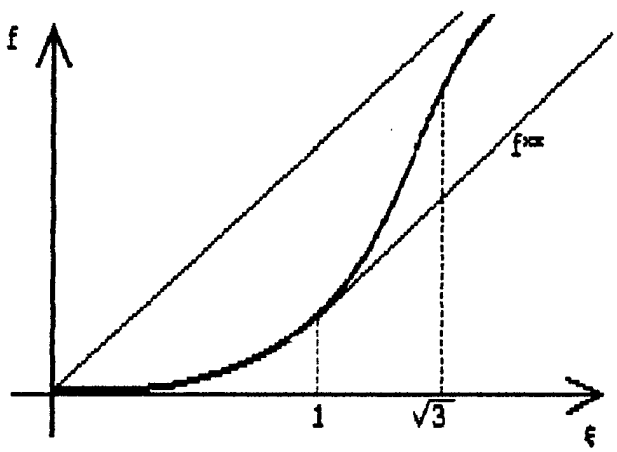

FIGURE 2

However, the patology that mainly interest us in the contest of this CIME Course, is the lack of convexity of the integrand with respect to the derivative $u'=\xi$ (an other relevant patology is the lack of coercivity). The graph of the integrand

$$f(u,u') = f(s,\xi) = 2\pi\, s\, \frac{\xi^3}{1+\xi^2} \quad , \tag{1.3}$$

is represented in figure 2, for fixed $s>0$. The function $f(s,\xi)$ is not convex for every $\xi \geq 0$ and it has an inflection point at $\xi=\sqrt{3}$.

We think that Newton's problem well motivates the study of integrals of the Calculus of Variations with non convex integrands (non convex with respect to the gradient of the unknown solution). In section 5 we present an existence theorem that can be applied to Newton's problem too.

2. Convexity and weak lower semicontinuity

Let us consider functions u defined on a bounded open set Ω of $\mathbb{R}^n$ with values in $\mathbb{R}^N$ ($n\geq 1$; $N\geq 1$). Roughly speaking convexity of $f(x,s,\xi)$ with respect to ξ is a sufficient condition for the lower semicontinuity, in the weak topology of $W^{1,p}(\Omega;\mathbb{R}^N)$, of the integral functional (Du is the gradient of u):

$$F(u) = \int_\Omega f(x,u,Du)\, dx \ . \tag{2.1}$$

In fact it is also sufficient for the lower semicontinuity in the product topology of the strong L^p norm-topology and the (sequential) weak topology of L^q, for some $p,q\geq 1$, of

$$G(u,v) = \int_\Omega f(x,u,v)\, dx \ . \tag{2.2}$$

Precise statements, with sufficient conditions for the weak lower

semicontinuity of F and G, can be found in the original first work by Tonelli [146] and then in the papers by Serrin [136], De Giorgi [61], Berkowitz [23], Cesari [41], Ioffe [89], Olech [126].

We can schematize the relations between convexity and lower semicontinuity (l.s.c) in the following way:

$$f(x,s,\xi) \textit{ convex in } \xi \;\Rightarrow\; G(u,v) \textit{ l.s.c.} \;\Rightarrow\; F(u) \textit{ l.s.c.} \;\overset{?}{\Rightarrow}\; f(x,s,\xi) \textit{ convex in } \xi \;.$$

Convexity of $f(x,s,\xi)$ with respect to ξ is also necessary for the lower secontinuity of $G(u,v)$, while it turns out to be necessary for the lower semicontinuity of $F(u)$ only if either n or N are equal to 1 (in the general case $n\geq 1$, $N\geq 1$ one is lead to introduce the quasiconvexity condition by Morrey [119]; see also section 4 of this paper).

The necessity of convexity (when $N=1$) was first discovered by Tonelli ([146], chapter X, section 1) in the case $f\in C^2$; then by Caccioppoli and Scorza Dragoni [35] for $f\in C^1$; by McShane [115] and Morrey ([119], theorems 4.4.2 and 4.4.3) for $f\in C^0$; by Ekeland and Temam ([67], chapter X) for f Carathéodory independent of s; by Marcellini and Sbordone [108] for a general Carathéodory function f. About necessary conditions for the case $n,N\geq 1$, we quote Morrey [118], [119] and Acerbi and Fusco [3].

The following theorem can be found in [108] (in the theorem 2.4 of [108] it is cosidered the case $n\geq 1$ and $N=1$, the other $n=1$ and $N\geq 1$ being similar):

THEOREM 2.1 - *Let* $f(x,s,\xi)$ *be a Carathéodory function satisfying the growth condition*

$$|f(x,s,\xi)| \leq g(x,|s|,|\xi|) \tag{2.3}$$

with g *increasing with respect to* $|s|$ *and* $|\xi|$ *and locally integrable with respect to* x. *Then, if* $F(u)$ *in* (2.1) *is sequentially lower semicontinuous in*

the weak topology of $W^{1,p}(\Omega;\mathbb{R}^N)$ *for some* $p \in [1,+\infty]$ *(weak* if* $p=+\infty$*) and if either* $n=1$ *or* $N=1$, *then* $f(x,s,\xi)$ *is convex with respect to* ξ.

To give an idea of the proof of theorem 2.1 we begin, for general $n,N \geq 1$, with the following result:

LEMMA 2.2 - *Assume (for simplicity) that* $f=f(\xi)$ *does not depend on* (x,s). *If the corresponding integral* $F(u)$ *is sequentially lower semi-continuous in the weak topology of* $W^{1,p}$ *then*

$$\int_\Omega f(\xi + D\varphi(x))\,dx \geq \int_\Omega f(\xi)\,dx = f(\xi)\,|\Omega| \tag{2.4}$$

for every $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$. *Here* $|\Omega|$ *is the measure of* $\Omega \subset \mathbb{R}^n$ *and* n,N *are greather or equal than one.*

Proof - Let Y be an n-dimensional cube containing Ω. We can extend $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ to Y by defining $\varphi = 0$ in $Y-\Omega$. Then we can extend φ to $\mathbb{R}^n$ by periodicity. In this way φ turns out to be an Y-periodic function on $\mathbb{R}^n$. The sequence φ_k, defined by

$$\varphi_k(x) = \frac{1}{k}\,\varphi(kx), \qquad \forall\ k \in N, \quad \forall\ x \in \mathbb{R}^n,$$

converges to zero, uniformly in $\mathbb{R}^n$ (consequence of $|\varphi_k(x)| \leq (1/k)\|\varphi\|_{L^\infty}$). Since the gradient $D\varphi_k(x) = D\varphi(kx)$ is bounded in $W^{1,\infty}$ uniformly with respect to $k \in N$ ($\|D\varphi_k\|_{L^\infty} = \|D\varphi\|_{L^\infty}$), then φ_k converges to zero also in the weak* topology of $W^{1,\infty}(\Omega;\mathbb{R}^N)$. Then, for every $\xi \in \mathbb{R}^{nN}$, the sequence $\langle\xi,x\rangle + \varphi_k(x)$ converges to $\langle\xi,x\rangle$. By the lower semicontinuity assumption we have

$$\int_\Omega f(\xi)\,dx \leq \liminf_{k\to\infty} \int_\Omega f(\xi + D\varphi_k(x))\,dx \quad . \tag{2.5}$$

Since $D\varphi_k(x) = D\varphi(kx)$, by the changement of coordinates $x' = kx$ and by the fact that φ is Y-periodic (so that the integral over kY is equal k^n times the integral over Y), we get

$$\begin{aligned}\int_\Omega f(\xi + D\varphi_k(x))\,dx &= \int_\Omega f(\xi + D\varphi(kx))\,dx = \frac{1}{k^n}\int_{k\Omega} f(\xi + D\varphi(x'))\,dx' \\ &= \frac{1}{k^n}\int_{kY} f(\xi + D\varphi(x))\,dx = \int_Y f(\xi + D\varphi(x))\,dx = \int_\Omega f(\xi + D\varphi(x))\,dx\ .\end{aligned} \tag{2.6}$$

From (2.5), (2.6), we get the conclusion (2.4).

Sketch of the proof of theorem 2.1 - Take for example $n=1$ and $N\geq 1$ and, like in lemma 2.2, $f=f(\xi)$. Let $\Omega = (0,1)$. Let $\xi = \lambda\xi_1 + (1-\lambda)\xi_2$, with $\xi_1,\xi_2 \in \mathbb{R}^N$ and $\lambda \in (0,1)$. Let us consider a function $u \in W^{1,\infty}((0,1);\mathbb{R}^N)$ with gradient u' satisfying

$$u'(x) = \begin{cases} \xi_1 & \text{if} \quad x \in [0,\lambda] \\ \xi_2 & \text{if} \quad x \in [\lambda,1] \end{cases} ;$$

then

$$u(1) = u(0) + \int_0^1 u'(x)\,dx = u(0) + \int_0^\lambda \xi_1\,dx + \int_\lambda^1 \xi_2\,dx = u(0) + \xi\ .$$

If we define $\varphi(x) = u(x) - u(0) - \xi x$, then $\varphi(0) = \varphi(1) = 0$; thus, by (2.4):

$$f(\xi) \leq \int_0^1 f(\xi + D\varphi(x))\,dx = \int_0^\lambda f(\xi_1)\,dx + \int_\lambda^1 f(\xi_2)\,dx = \lambda f(\xi_1) + (1-\lambda)f(\xi_2)\ .$$

This completes the proof for the case $f=f(\xi)$, $n=1$ and $N\geq 1$. A similar argument holds if $n\geq 1$ and $N=1$ (see [108] for general $f=f(x,s,\xi)$ too). On the contrary, if both n and N are greather than one, then the above argument holds only if ξ_1 and ξ_2 satisfy a compatibility condition: the $n\times N$ matrix $\xi_2 - \xi_1$

must be of rank one. In this case one get the *rank-one convexity*, as stated in section 4 (proposition 4.2).

If the integral F in (2.1) is not lower semicontinuous, then it is useful to consider the *relaxed* functional $\bar{F}$ (in the weak topology of $W^{1,p}$) defined on $W^{1,p}$ by

$$(2.7) \qquad \bar{F}(u) = \sup \{G(u) : \ G \leq F, \ G \ \textit{weakly l.s.c. in} \ W^{1,p}\} .$$

If $f(x,s,\xi) \geq m|\xi|^p$ for $p>1$ and $m>0$, then the relaxed functional $\bar{F}$ in $W^{1,p}$ can obtained also through the formula

$$(2.8) \qquad \bar{F}(u) = \inf_{\{u_k\}} \left\{ \liminf_{k\to\infty} F(u_k) : \ u_k \ \textit{weakly converges to} \ u \ \textit{in} \ W^{1,p} \right\} .$$

An elementary, but important property is that, under some growth conditions on f, the infimum of F on $u_0 + W_0^{1,p}(\Omega)$ (u_0 is the boundary datum and it is fixed in $W^{1,p}(\Omega)$) is equal to the infimum of $\bar{F}$ on the same set. The advantage to consider $\bar{F}$ is that, being the supremum of a family of lower semicontinuous functionals, it is lower semicontinuous in the weak topology of $W^{1,p}$.

A relevant problem related to $\bar{F}$ is to know if it is a functional of integral form; the so called "integral representation" of $\bar{F}$, posed in our context for the weak topology of $W^{1,p}$, can be posed for other topologies too, like for example the norm-L^p topology. Some references are: Serrin [136], Rockafellar [132], Ferro [70], Buttazzo and Dal Maso [33], Carbone and Sbordone [37], Marcellini [99] and most of the quoted papers on Γ-convergence ; in any case a good reference on relaxation in the weak topology of $W^{1,p}$ remains the book by Ekeland and Temam [67].

In the following we quote a theorem of relaxation, proved by Ekeland and Temam ([67], chapter X) for $f=f(x,\xi)$ independent of s and by Marcellini and

Sbordone ([108], corollary 3.12) in the general case. We mention also a corresponding relaxation theorem by Acerbi and Fusco [3] in the quasiconvex case.

THEOREM 2.3 - *Let* $f(x,s,\xi)$ *be a Carathéodory function satisfying the growth condition* (2.3) *and the coercivity condition*

$$f(x,s,\xi) \geq m|\xi|^{\alpha} - k \tag{2.9}$$

for some $m>0$, $k\geq 0$ *and* $\alpha>1$. *Then the functional* $\bar{F}$ *(relaxed in the weak topology of* $W^{1,p}$ *; here* $N=1$), *can be represented by the integral*

$$\bar{F}(u) = \int_{\Omega} f^{**}(x,u,Du)\,dx \quad , \qquad \forall\ u \in W^{1,\infty}(\Omega)\ , \tag{2.10}$$

where, as usual, $f^{**}(x,s,\xi)$ *is the greathest function less than or equal to* $f(x,s,\xi)$ *and convex with respect to* ξ.

3. Non convex integrals

In this section we consider the integral $F(u)$ in (2.1), when either n or N are equal to one, with integrand $f(x,s,\xi)$ possibly not convex with respect to ξ. We consider also the integral $\bar{F}$ in (2.10), with integrand f^{**} .

We already said in section 2 that, under growth conditions for f such as

$$m|\xi|^{p} - k \leq f(x,s,\xi) \leq M(1 + |\xi|^{p}) \tag{3.1}$$

for some $p>1$ and $M\geq m>0$, then the infimum of F on $u_0 + W_0^{1,p}(\Omega)$ is equal to the infimum of $\bar{F}$ on the same linear space. Let us assume that F has a minimizer u; then

$$F(u) = \inf F = \inf \bar{F} \leq \bar{F}(u) \leq F(u)\ ;$$

therefore u is a minimizer for $\bar{F}$ too, and $\bar{F}(u) = F(u)$, i.e.:

$$\int_\Omega \left\{ f(x,u,Du) - f^{**}(x,u,Du) \right\} dx = 0 .$$

Since the integrand $f - f^{**}$ in nonnegative, it follows that

$$f(x,u(x),Du(x)) = f^{**}(x,u(x),Du(x)), \quad \text{a.e. in } \Omega. \tag{3.2}$$

This is a simple but important fact: if the original integral functional F has a minimizer u, then it must be a minimizer of $\bar{F}$ too, and the identity (3.2) necessarily holds. Thus, to look for minimizers of F, we can see if there is at least a minimizer of $\bar{F}$ that is also a minimizer of F. Let us see how to apply these considerations to one of the simplest problems in the Calculus of Variations, related to $f=f(\xi)$, with $n=N=1$:

$$\inf \left\{ \int_0^1 f(u')\, dx \ : \ u(0)=0,\ u(1)=\xi_0 \right\} . \tag{3.3}$$

Let us assume that

$$\lim_{\xi \to \pm\infty} \frac{f(\xi)}{|\xi|} = +\infty . \tag{3.4}$$

Then f^{**} is a proper convex function (i.e. it does not assume the value $-\infty$). By Jensen's inequality (see for example (4.4) in this paper), valid for the convex function f^{**}, we have

$$\bar{F}(u) = \int_0^1 f^{**}(u'(x))\, dx \geq f^{**}\left(\int_0^1 u'(x)\, dx \right) = f^{**}(\xi_0) . \tag{3.5}$$

(recall that $u(1)-u(0)= \xi_0$). Since (3.5) holds with the equality sign if u' is identically equal to ξ_0, then $u_0(x)= \xi_0 x$ is a minimizer for $\bar{F}$.

Of course, if $f(\xi_0)= f^{**}(\xi_0)$, then $u_0(x)$ is a minimizer for (3.3) too.

Otherwise, by (3.4), there are ξ_1, ξ_2 and $\lambda \in (0,1)$ such that

$$f(\xi_1) = f^{**}(\xi_1)\,, \qquad f(\xi_2) = f^{**}(\xi_2)\ ; \tag{3.6}$$

$$\xi_0 = \lambda\xi_1 + (1-\lambda)\xi_2\ ; \quad f^{**}(\xi_0) = \lambda f^{**}(\xi_1) + (1-\lambda)f^{**}(\xi_2)\ . \tag{3.7}$$

Then the function $\bar{u} \in W^{1,\infty}(0,1)$ defined by the conditions: $\bar{u}(0)=0$ and

$$\bar{u}'(x) = \xi_1 \quad \text{if} \quad x \in [0,\lambda], \qquad \bar{u}'(x) = \xi_2 \quad \text{if} \quad x \in (\lambda,1],$$

clearly is a solution of problem (3.3); in fact

$$F(\bar{u}) = \int_0^\lambda f(\xi_1)\,dx + \int_\lambda^1 f(\xi_2)\,dx = \int_0^\lambda f^{**}(\xi_1)\,dx + \int_\lambda^1 f^{**}(\xi_2)\,dx =$$

$$= \lambda f^{**}(\xi_1) + (1-\lambda)f^{**}(\xi_2) = f^{**}(\xi_0) = \bar{F}(u_0) \le \bar{F}(u) \le F(u)\ , \ \forall\ u.$$

We have proved the following result (the part of the statement on the uniqueness and on the regularity it is easy to verify):

PROPOSITION 3.1 - *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying* (3.4). *Then the variational problem* (3.3) *has a solution in the class* $W^{1,\infty}(0,1)$, *for every* $\xi_0 \in \mathbb{R}$. *Moreover, if* $f(\xi_0) \neq f^{**}(\xi_0)$, *then the solution is not unique and no solution is of class* $C^1(0,1)$.

With an argument similar to that one used to prove proposition 3.1, it is possible to obtain the following existence theorem for non convex integrands $f=f(x.\xi)$, in the case $n=1$, $N\ge 1$:

THEOREM 3.2 - *Let* $f(x,\xi)$ *be a Carathéodory function for* $x \in [a,b]$ and $\xi \in \mathbb{R}^N$, *satisfying the growth conditions*

$$f(x,\xi) \ge |\xi|^p\ , \qquad f(\cdot,\xi) \in L^1(a,b)\ , \tag{3.8}$$

for some $p > 1$ *and* $m > 0$. *Then, for every fixed boundary value* u_0, *the integral*

$$F(u) = \int_a^b f(x,u'(x))\,dx \tag{3.9}$$

has a minimizer in $u_0 + W_0^{1,p}((a,b);\mathbb{R}^N)$.

Theorem 3.2 has been proved by Olech [125]; results of similar type have been given by Cesari [42], Aubert and Tahraoui [10], [11] and Marcellini [100] (see also [148], [92], [65]). About the minimization of integral-functional involving non convex integrands we refer also to Marcellini [101], [102], Mascolo and Schianchi [112], [113], [114], Aubert and Tahraoui [12], Tahraoui [141], Raymond [128], [129], [130], Kinderlherer and Mascolo [90], Mascolo [110], [111], Cellina and Colombo [40]. We refer also to chapter 16 of the book by Cesari [43] and to chapter 5 of the book by Dacorogna [54].

In the remainder of this section we state some of the existence results proved in the papers quoted above. Before, let us mention that, if $f(x,s,\xi)$ depends explicitly on s, then the related Dirichlet minimization problem may lack the solution. For example, the coercive integral

$$F(u) = \int_0^1 \left\{ \left[(u')^2 - 1\right]^2 + u^2 \right\} dx \tag{3.10}$$

has not a minimizer in the class of functions $u \in W^{1,4}(0,1)$ such that $u(0)=u(1)=0$. In fact, for every u, $F(u) \geq 0$, $F(u) \neq 0$ ($F(u)=0$ would imply at the same time that $u' = \pm 1$ and that $u=0$ for a.e. $x \in [0,1]$) and also $\inf\{F(u)\}=0$ (in fact it is possible to construct explicitly minimizing sequences u_k such that

$$u_k' = \pm 1 \quad a.e. \quad in \quad [0,1] \qquad and \qquad |u_k(x)| \leq \frac{1}{k} \quad for\ all \quad x \in [0,1] \quad).$$

The following theorems 3.3 and 3.4, due to Raymond (both consequence of theorem 6.11 of [129]), give existence of solutions, in the case $n=N=1$, to the variational (non convex) problem

$$\min \left\{ \int_a^b f(x,u,u')\,dx \ : \quad u \in u_0 + W_0^{1,p}(a,b) \right\}, \tag{3.11}$$

where u_0 is a fixed boundary datum and f is a smooth function that satisfies, for some $p>1$ and $M \geq m > 0$, the growth conditions

$$m|\xi|^p \leq f(x,s,\xi) \leq M(1 + |s|^p + |\xi|^p) \ ; \tag{3.12}$$

$$|f_s(x.s,\xi)| \leq M(1 + f(x,s,\xi)) \ . \tag{3.13}$$

As before f^{**} is the greathest function, convex with respect to ξ, less than or equal to f.

THEOREM 3.3 (Raymond [129]) - *Let* f^{**} *be a function of class* $C^2([a,b],\mathbb{R}^2)$. *Under the assumptions* (3.12), (3.13), *if*

$$f_s^{**} - f_{\xi x}^{**} - \xi f_{\xi s}^{**} \neq 0 \ , \quad \forall \ (x,s,\xi), \tag{3.14}$$

then problem (3.11) *has a solution.*

THEOREM 3.4 (Raymond [129]) - *Let* f *be a function of class* $C^4([a,b],\mathbb{R}^2)$. *Under the assumptions* (3.12),(3.13) (*and under some other theonical conditions states in* (3.3) *and* (5.4) *of* [129]), *if*

$$f_{ss}^{**} - f_{\xi sx}^{**} - \xi f_{\xi ss}^{**} \leq 0 \ , \quad \forall \ (x,s,\xi), \tag{3.15}$$

then problem (3.11) *has a solution.*

The following theorem, due to Cellina and Colombo [40], is the most general result (up to now) relative to the case $n=1$ and $N\geq 1$. The proof in [40] is based on Liapunov's theorem on the range of vector valued measures (see the lecture by Olech in this CIME Course).

THEOREM 3.5 (Cellina-Colombo [40]) - *Let* $p>1$. *The variational problem*

$$\min \left\{ \int_a^b \{f(x,u') + g(x,u)\}\, dx : \quad u \in u_0 + W_0^{1,p}((a,b);\mathbb{R}^N) \right\}$$

has a solution if we assume that $f: [a,b]\times\mathbb{R}^N \to \mathbb{R}\cup\{+\infty\}$ *and* $g: [a,b]\times\mathbb{R}^N \to \mathbb{R}$ *are Carathéodory functions such that*

(3.16) $g(x,s)$ *is concave with respect to* $s \in \mathbb{R}^N$;

(3.17) $g(x,s) \geq -M(1 + |s|^p)$ *for some* $M\geq 0$;

(3.18) $f(x,\xi) \geq m(1 + |\xi|^p)$ *for some* $m>0$;

(3.19) *the quotient* M/m *is strictly smaller than the best constant in Sobolev's inequality for functions of* $W_0^{1,p}$.

We end this section by considering functions $u: \mathbb{R}^n \to \mathbb{R}$ (i.e. $n\geq 1$ and $N=1$). In this context there are existence results ([112], [113], [114]; [12]; [130]; [90]; [110], [111]) and nonexistence and uniqueness results ([101], [102]).

Let us state a uniqueness theorem for convex, but not strictly convex, integrals and, as a consequence, a nonexistence result. Note the assumption $n\geq 2$ in theorem 3.6 and in corollary 3.7.

THEOREM 3.6 ([102]) - *Let* $g:[0,+\infty)\to[0,+\infty)$ *be a convex function such that* $g(t)>g(0)$ *for every* $t>0$. *Let* Ω *be a convex bounded open set of* $\mathbb{R}^n$ *with* $n\geq 2$. *If there exists a solution* u *of class* $C^n(\overline{\Omega})$ *to the problem*

$$\min \left\{ F(u) = \int_\Omega g(|Du|)\, dx \quad : \quad u \in u_0 + W_0^{1,\infty}(\Omega) \right\} \tag{3.20}$$

then u *is the unique minimizer in* $u_0 + W_0^{1,\infty}(\Omega)$.

COROLLARY 3.7 ([101]) - *Let* $g: [0,+\infty) \to [0,+\infty)$ *be a function of class* C^1 *and let* g^{**} *be the greatest convex function in* $[0,+\infty)$ *less than or equal to* g. *Let* ξ_0 *be a vector in* $\mathbb{R}^n$, *with* $n \geq 2$, *such that* $g(|\xi_0|) > g^{**}(|\xi_0|)$ *and* $(g^{**})'(|\xi_0|) > 0$ ($(g^{**})'$ *is the derivative of* g^{**}). *Then the following variational problem lacks the solution*

$$\min \left\{ F(u) = \int_\Omega g(|Du|)\, dx \quad : \quad u \in (\xi_0,x) + W_0^{1,\infty}(\Omega) \right\} . \tag{3.21}$$

The following is one of the existence results given by Mascolo and Schianchi; it is the theorem 2.2 of [114] (see also [90], [110], [111],[112],[113]).

THEOREM 3.8 (Mascolo-Schianchi [114]) - *Let* f *be a function of class* $C^2(\overline{\Omega}\times\mathbb{R}^n)$ *satisfying the growth conditions:*

$$m(|\xi|^2 - 1) \leq f(x,s,\xi) \leq M(1 + |\xi|^2) , \tag{3.22}$$

$$|f_{x\xi}| \leq M(1 + |\xi|) , \qquad |f_{\xi\xi}| \leq M , \tag{3.23}$$

for all $(x,\xi) \in \overline{\Omega}\times\mathbb{R}^n$. *Let us assume also that the set* $K(x) = \{ \xi \in \mathbb{R}^n : f^{**}(x,\xi) < f(x,\xi) \}$ *is connected and bounded in* $\mathbb{R}^n$ *and that, for every* $x\in\overline{\Omega}$,

$$f^{**}(x,\xi) = \sum_{i=1}^n m_i(x)\xi_i + q(x) , \qquad \forall\ \xi \equiv (\xi_i) \in K(x) ; \tag{3.24}$$

$$m_i \in C^1(\overline{\Omega}) \quad \textit{and} \quad \text{meas}\{ x\in\Omega : \sum_{i=1}^n (m_i)_{x_i} = 0 \} = 0 . \tag{3.25}$$

Let $u_0 \in W^{1,2}(\Omega)$. *Then the following variational problem has a solution:*

$$\min \left\{ F(u) = \int_\Omega f(x,Du)\, dx \quad : \quad u \in u_0 + W_0^{1,2}(\Omega) \right\} . \tag{3.26}$$

then u *is the unique minimizer in* $u_0 + W_0^{1,\infty}(\Omega)$.

COROLLARY 3.7 ([101]) - *Let* $g: [0,+\infty) \to [0,+\infty)$ *be a function of class* C^1 *and let* g^{**} *be the greatest convex function in* $[0,+\infty)$ *less than or equal to* g. *Let* ξ_0 *be a vector in* $\mathbb{R}^n$, *with* $n \geq 2$, *such that* $g(|\xi_0|) > g^{**}(|\xi_0|)$ *and* $(g^{**})'(|\xi_0|) > 0$ ($(g^{**})'$ *is the derivative of* g^{**}). *Then the following variational problem lacks the solution*

$$\min \left\{ F(u) = \int_\Omega g(|Du|)\, dx \quad : \quad u \in (\xi_0,x) + W_0^{1,\infty}(\Omega) \right\} . \tag{3.21}$$

The following is one of the existence results given by Mascolo and Schianchi; it is the theorem 2.2 of [114] (see also [90], [110], [111],[112],[113]).

THEOREM 3.8 (Mascolo-Schianchi [114]) - *Let* f *be a function of class* $C^2(\overline{\Omega}\times\mathbb{R}^n)$ *satisfying the growth conditions:*

$$m(|\xi|^2 - 1) \leq f(x,s,\xi) \leq M(1 + |\xi|^2) , \tag{3.22}$$

$$|f_{x\xi}| \leq M(1 + |\xi|) , \qquad |f_{\xi\xi}| \leq M , \tag{3.23}$$

for all $(x,\xi) \in \overline{\Omega}\times\mathbb{R}^n$. *Let us assume also that the set* $K(x) = \{ \xi \in \mathbb{R}^n: f^{**}(x,\xi) < f(x,\xi) \}$ *is connected and bounded in* $\mathbb{R}^n$ *and that, for every* $x\in\overline{\Omega}$,

$$f^{**}(x,\xi) = \sum_{i=1}^n m_i(x)\xi_i + q(x) , \qquad \forall\ \xi \equiv (\xi_i) \in K(x) ; \tag{3.24}$$

$$m_i \in C^1(\overline{\Omega}) \quad \textit{and} \quad \text{meas}\{ x\in\Omega : \sum_{i=1}^n (m_i)_{x_i} = 0 \} = 0 . \tag{3.25}$$

Let $u_0 \in W^{1,2}(\Omega)$. *Then the following variational problem has a solution:*

$$\min \left\{ F(u) = \int_\Omega f(x,Du)\, dx \quad : \quad u \in u_0 + W_0^{1,2}(\Omega) \right\} . \tag{3.26}$$

REMARK 3.9 - The function $f(\xi) = g(|\xi|)$ in corollary 3.7 is not convex in $\mathbb{R}^n$, if $g(|\xi_0|) > g^{**}(|\xi_0|)$. In this case the set $K = \{\xi \in \mathbb{R}^n: f(\xi) > f^{**}(\xi) = g^{**}(|\xi|)\}$ has radial symmetry and f^{**} is not linear on K (its graph is not a plane if $(g^{**})'(|\xi_0|) > 0$, but it is a cone on every connected component). This suggests that assumption (3.24) in theorem 3.8 is a necessary condition. On the contrary, (3.25) should not be necessary.

4. Quasiconvex integrals

In this section we consider again integrals of the Calculus of Variations of the type

$$F(u) = \int_\Omega f(x,u,Du)\, dx \quad , \tag{4.1}$$

where $f = f(x,s,\xi)$ is a Carathéodory function, Ω is a bounded open set of $\mathbb{R}^n$ and $u \in W^{1,p}(\Omega;\mathbb{R}^N)$, for some $p \in [1,+\infty]$, is a vector-valued function defined in $\Omega \in \mathbb{R}^n$ with values in $\mathbb{R}^N$ ($n \geq 1$ and $N \geq 1$).

Here we consider functions f that are quasiconvex with respect to ξ. We anticipate that quasiconvexity is different from convexity when both n and N are greater then one, and that it reduces itself to the usual convexity if either n or N are equal to one. We say that $f(x,s,\xi)$ is *quasiconvex* with respect to ξ in Morrey's sense (see [118], [119]; see also lemma 2.2 of this paper) if

$$\int_\Omega f(x_0,s_0,\xi + D\varphi(x))\, dx \quad \geq \quad f(x_0,s_0,\xi)\, |\Omega| \tag{4.2}$$

for a.e. $x_0 \in \Omega$, for every $s_0 \in \mathbb{R}^N$ and for every $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$.

As we will state more precisely later, roughly speaking quasiconvexity of f is equivalent to the lower semicontinuity of the integral F (in the weak topology of $W^{1,p}$, for some p). This mathematical condition has been first considered by Morrey [118] in 1952; then Ball [15] in 1977 pointed out its interest and

applicability to nonlinear elasticity. The book by Morrey [119] (expecially section 4.4) and the paper by Ball [15] are the first references for quasiconvexity; a good reference is also the recent book by Dacorogna [54].

Part of the interest on quasiconvexity relies on the following semicontinuity result (due to Morrey [118] for f continuos; see [2], [3] for f Carathéodory):

THEOREM 4.1 - *Let* $f=f(x,s,\xi)$ *be a Carathéodory function such that*

$$|f(x,s,\xi)| \leq g(x,|s|,|\xi|) , \tag{4.3}$$

where g *is a function locally integrable with respect to* x *and increasing with respect to* $|s|$ *and* $|\xi|$. *Then the integral* $F(u)$ *in* (4.1) *is sequentially lower semicontinuous in the weak* topology of* $W^{1,\infty}(\Omega;\mathbb{R}^N)$ *if and only if* $f(x,s,\xi)$ *is quasiconvex with respect to* ξ.

Note that x and s play the role of parameters in the definition (4.2); thus, in speaking of properties of f, we can omit to denote explicitly the dependence on x and s and we can use the notation $f=f(\xi)$.

If $f=f(\xi)$ is convex then it is quasiconvex too; this fact is a consequence of *Jensen's inequality* (valid for every convex function $f: \mathbb{R}^m \to \mathbb{R}$) :

$$f\Big(\frac{1}{|\Omega|} \int_\Omega v(x)\, dx \Big) \leq \frac{1}{|\Omega|} \int_\Omega f(v(x))\, dx , \qquad \forall\ v \in L^1(\Omega;\mathbb{R}^m) \tag{4.4}$$

($|\Omega|$ is the measure of Ω). By taking $m=nN$ and $v(x)=\xi+D\varphi(x)$, if $f:\mathbb{R}^{nN}\to\mathbb{R}$ is convex, then, for every $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ we have

$$\frac{1}{|\Omega|} \int_\Omega f(\xi+D\varphi(x))\, dx \geq f\Big(\frac{1}{|\Omega|} \int_\Omega (\xi + D\varphi(x))\, dx \Big) = f(\xi) ,$$

the integral of $D\varphi$ on the set Ω being zero, since φ is equal to zero on the boundary $\partial\Omega$.

That quasiconvexity is equivalent to convexity if either n or N are equal to 1 is a consequence, as a particular case, of the following proposition 4.2, due to Morrey. Recall that $\xi,\lambda \in \mathbb{R}^{nN}$ are $n\times N$ matrices and that, if either n or N are equal to 1, then automatically any $\lambda\neq 0$ it is a rank-1 matrix.

PROPOSITION 4.2 - *If* $f: \mathbb{R}^{nN} \to \mathbb{R}$ *is quasiconvex, then the function* $g(t)=f(\xi + t\lambda)$ *is convex on* $\mathbb{R}$ *for every* $\xi\in\mathbb{R}^{nN}$ *and for every matrix* $\lambda\in\mathbb{R}^{nN}$ *of rank-1* (in this case f is said to be *rank-1 convex*).

If $f \in C^2(\mathbb{R}^{nN})$ then rank-1 convexity is equivalent to the *Legendre-Hadamard condition* (see also the *strong ellipticity condition* by Nirenberg [123]) :

$$\sum f_{\xi^i_\alpha \xi^j_\beta} \lambda_\alpha \lambda_\beta \eta^i \eta^j \geq 0 , \qquad \forall \lambda \in \mathbb{R}^n , \quad \forall \eta \in \mathbb{R}^N . \tag{4.5}$$

An important example of quasiconvex function $f(\xi)$ that is not convex is given, for $n=N=2$, by

$$f(\xi) = \det \xi = \xi_1\xi_4 - \xi_2\xi_3 \qquad \left(\textit{here} \quad \xi = \begin{bmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{bmatrix} \right) . \tag{4.6}$$

The prove that the function $f(\xi)$ in (4.6) is quasiconvex is based on the following identity, valid for every $u \equiv (u^1,u^2) \in C^2(\Omega;\mathbb{R}^2)$:

$$f(Du) = \det Du = u^1_{x_1}u^2_{x_2} - u^1_{x_2}u^2_{x_1} = \left(u^1u^2_{x_2}\right)_{x_1} - \left(u^1u^2_{x_1}\right)_{x_2} .$$

Since $f(Du)$ is a divergence, the integral of $f(Du)$ over Ω can be reduced to a surface integral on the boundary $\partial\Omega$; thus it depends only on the boundary values of $u(x)=\langle\xi,x\rangle + \varphi(x)$, that are the same as the boundary values of $\langle\xi,x\rangle$, if φ is equal to zero on $\partial\Omega$. Therefore

$$\int_\Omega f(\xi + D\varphi(x))\, dx \quad = \quad \int_\Omega f(\xi)\, dx \quad = \quad f(\xi)\, |\Omega| \, .$$

In two dimensions ($n=N=2$), by mean of $\det \xi = \xi_1\xi_4 - \xi_2\xi_3$, we can characterize the notion of quasiconvexity for quadratic forms (see Terpstra [145], D.Serre [135], Marcellini [103]):

THEOREM 4.3 - *Let* (a_{ij}) *be a* 4×4 *real matrix. The quadratic form* $f(\xi) = \sum_{i,j=1}^{4} a_{ij}\xi_i\xi_j$ *is quasiconvex in* $\mathbb{R}^4$ *if and only if there exists a real number* λ *such that the new quadratic form* $f(\xi) - \lambda \det \xi$ *is positive semidefinite in* $\mathbb{R}^4$.

The fact that functions of the type $f(Du) = \det Du$ play a role in this context is not casual; they have important applications in nonlinear elasticity (see the references at the end of this section) and in differential geometry. For example, the area of a paramatric 2-dimensional regular surface in $\mathbb{R}^3$, represented in parametric form by $u \equiv (u^1(x), u^2(x), u^3(x))$ with $x \equiv (x_1, x_2) \in \Omega \subset \mathbb{R}^2$, is given by the integral

$$\int_\Omega \sqrt{\left[\frac{\partial(u^2,u^3)}{\partial(x_1,x_2)}\right]^2 + \left[\frac{\partial(u^3,u^1)}{\partial(x_1,x_2)}\right]^2 + \left[\frac{\partial(u^1,u^2)}{\partial(x_1,x_2)}\right]^2} \; dx \; . \tag{4.7}$$

Up to now the deepest lower semicontinuity result for quasiconvex integrals is the following:

THEOREM 4.4 - *Let* $f = f(x,s,\xi)$ *be a Carathéodory function, quasiconvex with respect to* ξ *and satisfying the growth conditions*

$$-m(1 + |s|^r + |\xi|^r) \le f(x,s,\xi) \le M(x,s)\,(1 + |\xi|^q) \tag{4.8}$$

where $M(x,s)$ *is a Carathéodory function,* $m \in \mathbb{R}$ *and* $1 \leq r < q$. *Then the integral* $F(u)$ *in* (4.1) *is sequentially lower semicontinuous in the weak topology of* $W^{1,q}(\Omega;\mathbb{R}^N)$.

Theorem 4.4 has been proved by Acerbi and Fusco [3], by assuming slightly more restrictive growth conditions than (4.8). In the form presented here it can be found in [104]. It extends analogous semicontinuity results by Morrey [118],[119] and Meyers [116], who assumed a type of uniform continuity of f with respect to (x,s). None of the known proofs of theorem 4.4 are elementary and it would be interesting to find a simpler way to get the conclusion. In particular, the proof in [104] follows a procedure introduced in [109] and uses a representation formula by Dacorogna [53], the variational principle by Ekeland [66] (see also Ekeland's lecture in this book) and a higher summability result by Giaquinta and Giusti [78].

We note explicitly that the assumption $r<q$ is necessary, in the sense that there exists a counterexample by Murat and Tartar (see section 4.1 of [121]; see also Ball and Murat [21]) in the case $n=N=q=r=2$. Note also that this fact is typical of the quasiconvex case and it is in contrast with the convex case, where the condition $r=q$ is enough for the weak lower semicontinuity in $W^{1,q}$ (see for example [23], [41], [89], [126]).

An other important difference with respect to the convex case is the following: il $f(x,s,\xi)$ is a Carathéodory integrand, nonnegative and convex with respect to ξ, then the corresponding integral (4.1) is sequentially lower semicontinuous in the weak topology of $W^{1,1}$; on the contrary, theorem 4.4 states that (even under the further assumption that $f(x,s,\xi)$ is nonnegative) the integral (4.1) is sequentially lower semicontinuous in the weak topology of $W^{1,q}$, where q is the growth-exponent in (4.8).

Under (4.8), it is still not closed the problem to know whether the integral $F(u)$ is sequentially weakly lower semicontinuous in $W^{1,p}$, for $p<q$. Related to this problem of lower semicontinuity there is a problem of definition of the integral

$F(u)$ when $u \in W^{1,p}(\Omega;\mathbb{R}^N)$, but $u \notin W^{1,q}(\Omega;\mathbb{R}^N)$; we do not enter into details on this point and we refer to [105]. We quote below a semicontinuity result (theorem 2.1 of [105]; see also [104] and [36]) for *some* p smaller than q; we emphasize that the semicontinuity problem for *general* $p \in [1,q)$ is still open.

THEOREM 4.5 [105] - *Let* $f = f(x,\xi)$ *be a quasiconvex function satisfying:*

$$0 \leq f(x,\xi) \leq M\left[1 + |\xi|^q\right] \tag{4.9}$$

$$f(x,t\xi) \leq M\left[1 + f(x,\xi)\right] \qquad , \ \forall\, t \in [0,1] \tag{4.10}$$

$$\left|f(x_1,\xi) - f(x_2,\xi)\right| \leq \lambda(|x_1 - x_2|)\left[1 + f(x,\xi)\right] \tag{4.11}$$

where $M \geq 0$, $q > 1$, *and* $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ *is a modulus of continuity* ($\lambda = \lambda(t)$ *is increasing for* $t > 0$ *and* $\lambda(t) \to 0$ *as* $t \to 0^+$). *Then*

$$\int_\Omega f(x,Du)\,dx \leq \liminf_{k\to\infty} \int_\Omega f(x,Du_k)\,dx \ ,$$

for every u, $u_k \in C^1(\Omega;\mathbb{R}^N)$, *such that* u_k *converges to* u *in the weak topology of* $W^{1,p}(\Omega;\mathbb{R}^N)$, *for* $p > qn/(n+1)$.

The interest of quasiconvexity in the Calculus of Variations has been pointed out by Morrey [118] in 1952 and later in his deep and important (but also difficult to read) book [119] (for quasiconvexity see especially section 4.4). In 1977 J.Ball gave a big impulse to a new study of this condition by showing, in his celebrated paper [15], how quasiconvexity can be interpreted and used in the context of nonlinear elasticity. Since the paper by Ball, many other contributions come out (only a few of them have been quoted in this section). The interested reader can see the references at the end of this paper (surely incomplete, like unfortunately often it happens in these cases) and, like already said, the recent book [54].

5. New existence theorems for non convex integrals

To give an idea of our approach to the existence of minimizers for a class of one-dimensional non convex integrals, we consider again Newton's problem of motion in a resisting medium. In the class of absolutely continuous functions $u(x)$ with boundary values $u(a)=0$, $u(b)=B$ (≥ 0), satisfying the constraint $u'(x)\geq 0$ a.e. in $[a,b]$ (see figure 3), the integral to minimize is

$$F(u) = 2\pi \int_a^b u \frac{(u')^3}{1+(u')^2}\, dx \ . \tag{5.1}$$

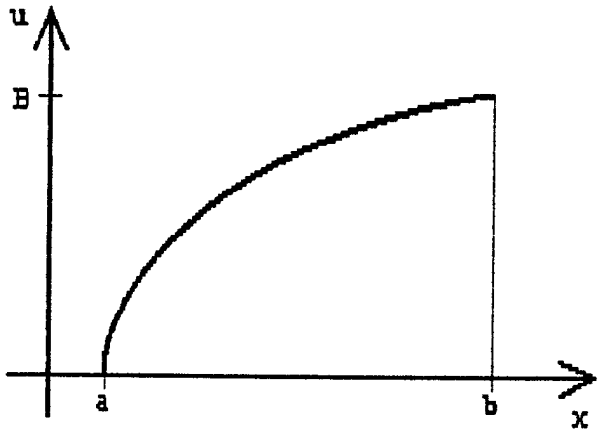

FIGURE 3

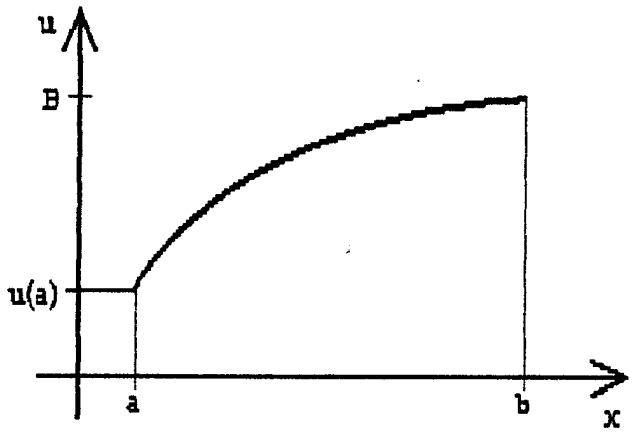

FIGURE 4

In principle, expecially from the physical point of view, it is interesting to consider also functions $u(x)$ that assume a positive value $u(a)$ at $x=a$, as in figure 4. This corresponds, in the application, to have a surface of revolution that is flat at $x=a$.

Newton's model is based on the assumption that the resistence of the medium is proportional (and the constant of proportionality is assumed to be equal to 1) to the normal component of the element of the area of the surface. Therefore, in general, the total resistence $\bar{F}$, corresponding to u in figure 4, according to Newton's model it should be given by

$$\bar{F}(u) = \pi\,[u(a)]^2 + 2\pi \int_a^b u \frac{(u')^3}{1+(u')^2}\, dx \ . \tag{5.2}$$

The value of $\overline{F}$ in (5.2), deduced from physical considerations, can be obtained by a mathematical argument too. In fact it turns out to be the value that we get if we extend F "by lower semicontinuity". The scheme is the following: let F be the integral

$$F(u) = \int_a^b f(x,u,u')\,dx \tag{5.3}$$

defined for u belonging to

$$\mathcal{W}_p = \left\{u \in W^{1,p}(a,b):\ u(a)=A,\ u(b)=B,\ u' \geq 0 \text{ a.e}\right\} \tag{5.4}$$

where $A<B$ and $p \geq 1$. The closure of $\mathcal{W}_p$ in the (either strong or weak) topology of $W^{1,p}_{loc}(a,b)$ is given by

$$\overline{\mathcal{W}}_p = \left\{u \in W^{1,p}_{loc}(a,b):\ u(a) \geq A,\ u(b) \leq B,\ u' \geq 0 \text{ a.e}\right\} \tag{5.5}$$

(here, by definition, the values $u(a)$ and $u(b)$ are respectively the infimum and the supremum of $\{u(x):\ x \in (a,b)\}$). The extension of F "by lower semicontinuity" from $\mathcal{W}_p$ to $\overline{\mathcal{W}}_p$ is the functional $\overline{F}$ defined for $u \in \overline{\mathcal{W}}_p$ by

$$\overline{F}(u) = \inf_{\{u_k\}} \left\{\liminf_{k\to\infty} F(u_k):\ (u_k) \subset \mathcal{W}_p,\ u_k \textit{ weakly converges to } u \textit{ in } W^{1,p}_{loc}\right\}. \tag{5.6}$$

In the different context of the change of the functional set from $\mathcal{W}_p$ to $\overline{\mathcal{W}}_p$, the definition in (5.6) is similar to the definition (2.8) of a relaxed functional.

For our applications, we shall consider Carathéodory functions $f(x,s,\xi)$ that satisfy the conditions (please, excuse the abuse of notation of the letter a as lefth endpoint of the interval $[a,b]$ and as function $a(x,s)$):

(5.7) *there exist* $K \geq 0$, *a convex function* $h(\xi)$ *and continuous functions* $a(x,s)$, $b(x,s)$ *such that, for every* $x \in [a,b]$, $s \in \mathbb{R}$ *and* $\xi \geq 0$,

(a) $a(x,s)h(\xi) - K \leq f(x,s,\xi) \leq a(x,s)h(\xi) + b(x,s)$

(b) $h(\xi) \geq 0$; $\quad a(x,s) \geq 0$.

THEOREM 5.1 ([29], theorem 2.4) - *Let* $f = f(x,s,\xi)$ *be a Carathéodory function, convex with respect to* $\xi \geq 0$ *and satisfying* (5.7). *Then, for every* $u \in \overline{\mathcal{W}}_p$, $\overline{F}(u)$ *can be represented by*

$$\overline{F}(u) = \int_a^b f(x,u,u')\,dx + \widetilde{h}\left\{\int_A^{u(a)} a(a,s)ds + \int_{u(b)}^B a(b,s)ds\right\} \tag{5.8}$$

where $\widetilde{h} = \lim_{\xi \to +\infty} \frac{h(\xi)}{\xi}$.

REMARK 5.2 - In principle $\widetilde{h}$ may assume also the value $+\infty$; in this case, if $a(x,s)$ is positive at $x=a$ and $x=b$ for a.e. $s \in \mathbb{R}$, then $\overline{F}$ is finite if and only if u assumes the boundary values $u(a)=A$, $u(b)=B$. Therefore, if $\lim_{\xi \to +\infty} h(\xi)/\xi = +\infty$, then the minimization of $\overline{F}$ on $\overline{\mathcal{W}}_p$ is equivalent to the minimization of F on $\mathcal{W}_p$.

REMARK 5.3 - In Newton's problen, the function $f(s,\xi) = 2\pi s\xi^3/(1+\xi^2)$ is not convex with respect to $\xi \in [0,+\infty)$; however we will apply theorem 5.1 to the integral functional defined through the convexified function $f^{**}(s,\xi)$. In any case in Newton's problen $a(x,s)=2\pi s$, $\widetilde{h} = \lim_{\xi \to +\infty} \left[\xi^3/(1+\xi^2)\right]/\xi = 1$ and $p=1$; thus the extended functional $\overline{F}$ in (5.8), for $u \in W^{1,1}(a,b)$ such that $u(a) \geq 0$ and $u(b)=B$, agrees with (5.2).

We consider f^{**}, the greathest function convex with respect to $\xi \geq 0$ and less than or equal to f. We assume that f^{**} admits continuous partial derivatives f^{**}_s, $f^{**}_{\xi x}$, $f^{**}_{\xi s}$. We assume also that the convex function $h(\xi)$ in (5.7) is defined for every $\xi \in \mathbb{R}$ and that:

(5.9) *there exist an exponent* $p \geq 1$, *a function* $M: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *and a positive constant* L, *such that, for every* $\delta, r > 0$:

(a) $|f_s^{**}(x,s,\xi)| \leq M(\delta,r)\left[1+|\xi|^p\right]$, $\forall\ (x,s,\xi) \in [a+\delta, b-\delta] \times [-r,r] \times \mathbb{R}$

(b) $h(\xi) \leq L\left[1+|\xi|^p\right]$, $\forall\ \xi \in \mathbb{R}$

We are ready to give an existence result for the variational problem

$$\min\left\{\bar{F}(u) : \quad u \in W^{1,p}_{loc}(a,b),\ u(a) \geq A,\ u(b) \leq B,\ u' \geq 0 \text{ a.e}\right\}, \tag{5.10}$$

with $\bar{F}$ given by (5.8). We emphasize that we don't make any convexity nor coercivity assumptions on f.

THEOREM 5.4 - *Let* $f(x,s,\xi)$ *satisfy* (5.7) *and* (5.9). *Let us assume that, for every* $(x,s) \in (a,b) \times \mathbb{R}$ *and* $\xi \geq 0$, *the function*

$$\varphi(x,s,\xi) = f_s^{**} - f_{\xi x}^{**} - \xi f_{\xi s}^{**} \tag{5.11}$$

has a definite sign (either $\varphi \geq 0$ *or* $\varphi \leq 0$*). Then the variational problem* (5.10) *has a solution* u_0, *which belongs to* $W^{1,\infty}_{loc}(a,b)$ *and satisfies the estimate*

$$|u_0'(x)| \leq \frac{2}{\delta} \max\{\ |A|\ ;\ |B|\ \}, \qquad \forall\ x \in [a+\delta, b-\delta]\ ,\ \forall\ \delta \in (0, \tfrac{b-a}{2})\ . \tag{5.12}$$

Moreover, if $\varphi \geq 0$, *then* u_0 *assumes the boundary value* $u_0(a) = A$, *while, if* $\varphi \leq 0$, *then* $u_0(b) = B$. *Finally, if* $\tilde{h} = \lim_{\xi \to +\infty} h(\xi)/\xi = +\infty$ *(and* $a(x,\cdot)$ *is positive a.e. at* $x=a$ *and* $x=b$*) then at the same time* $u_0(a) = A$, $u_0(b) = B$ *and* u_0 *minimizes* F *too.*

REMARK 5.5 - We shall prove theorem 5.4 following a method developed by Botteron and Marcellini [29] for the convex case. The condition that φ in (5.11) has a definite sign is similar to (and it improves) the assumption $\varphi \neq 0$, first considered by Raymond (see theorem 6.11 of [129]; see also theorem 3.3 in this paper).

REMARK 5.6 - If f has the special form

$$f(x,s,\xi) = a(x,s)g(\xi) + b(x,s) \tag{5.13}$$

(or, in the case of a general f, if ξ_0 below is independent of x,s), then the assumption that φ in (5.11) has a definite sign, can be replaced by the more direct condition that

$$\varphi_0(x,s,\xi) = f_s - f_{\xi x} - \xi f_{\xi s} \tag{5.14}$$

has a definite sign. In fact, if $f(x,s,\xi) > f^{**}(x,s,\xi)$, since f^{**} is linear (with respect to ξ) in a neighbourhood $\mathcal{U}$ of (x,s,ξ), there exists ξ_0 such that

$$f^{**}(x,s,\xi) = f(x,s,\xi_0) + f_\xi(x,s,\xi_0)(\xi-\xi_0)\,, \qquad \forall\ (x,s,\xi)\in\mathcal{U}.$$

In $\mathcal{U}$ then we have

$$\varphi(x,s,\xi) = f_s^{**} - f_{\xi x}^{**} - \xi f_{\xi s}^{**} = f_s(x,s,\xi_0) + f_{\xi s}(x,s,\xi_0)(\xi-\xi_0) - \\ - f_{\xi x}(x,s,\xi_0) - \xi f_{\xi s}(x,s,\xi_0) = \varphi_0(x,s,\xi_0)\,.$$

Let us also notice that, if $f(x,s,\xi)$ has the special form (5.13), then in theorem 5.4 it is sufficient to assume that $g(\xi)$, $a(x,s)$, $b(x,s)$ are functions of class C^1.

Proof of theorem 5.4 - First we extend f^{**} to $\xi<0$ by

$$f^{**}(x,s,\xi) = f^{**}(x,s,0) + f_\xi^{**}(x,s,0)\xi \qquad (\xi<0).$$

With this position, as in remark 5.6 (with f replaced by f^{**}), we can see that the structure condition (that $\varphi(x,s,\xi)$ in (5.11) has a definite sign) is satisfied also for $\xi<0$, and thus for every (x,s,ξ).

For every $\epsilon\in(0,1]$ and $k>0$ we introduce the notations

$$g^{\epsilon,k}(x,s,\xi) = \alpha_k * f^{**}(x,s,\xi) + k(\xi^-)^q + \epsilon(1+\xi^2)^{q/2}, \tag{5.15}$$

where $q=\max\{p;3\}$, $\xi^- = -\min\{\xi;0\}$, $\alpha=\alpha(\xi)$ is a positive mollifier with compact

support in $[-1,1]$ and finally $\alpha_k(\xi)=k\alpha(k\xi)$. The variational problem

$$(5.16) \qquad \min\left\{G^{\epsilon,k}(u) = \int_a^b g^{\epsilon,k}(x,u,u')\,dx \;:\; u\in W^{1,q}(a,b),\ u(a)=A,\ u(b)=B\right\},$$

related to the convex and coercive integral $G^{\epsilon,k}(u)$, has a solution $u_{\epsilon,k}\in W^{1,q}(a,b)$. We shall use the regularity properties of $u_{\epsilon,k}$ stated in the following lemma:

LEMMA 5.7 - *For every* $\epsilon\in(0,1]$ *and* $k>0$ $u_{\epsilon,k}$ *is of class* $C^2[a,b]$ *and satisfies the Euler's equation in strong form:*

$$(5.17) \qquad \frac{d}{dx}\left[g_\xi^{\epsilon,k}(x,u_{\epsilon,k},u'_{\epsilon,k})\right] = g_s^{\epsilon,k}(x,u_{\epsilon,k},u'_{\epsilon,k}) .$$

Moreover, for every fixed $\epsilon\in(0,1]$, $\|u'_{\epsilon,k}\|_{L^\infty(a,b)}$ *is bounded uniformly with respect to* k.

We postpone the proof of lemma 5.7 and first we conclude the proof of theorem 5.4. Since $u_{\epsilon,k}\in C^2$, we can write Euler's equation (5.17) in the form

$$g_{\xi\xi}^{\epsilon,k}\,u''_{\epsilon,k} = g_s^{\epsilon,k} - \left[g_{\xi x}^{\epsilon,k} + g_{\xi s}^{\epsilon,k}u'_{\epsilon,k}\right] = \alpha_k * f_s^{**} - \left[\alpha_k * f_{\xi x}^{**} + u'_{\epsilon,k}\,\alpha_k * f_{\xi s}^{**}\right] .$$

If we denote $L(r)=\sup\left\{|f_{\xi s}^{**}(x,s,\xi)|:\ x\in[a,b],\ |s|\le r,\ |\xi|\le r\right\}$, then, for such values of x,s,ξ, we have

$$\left|\xi\alpha_k*f_{\xi s}^{**} - \alpha_k*\left[\xi f_{\xi s}^{**}\right]\right| = \left|\xi\int_{\mathbb{R}}\alpha_k(t)\,f_{\xi s}^{**}(x,s,\xi-t)\,dt - \int_{\mathbb{R}}\alpha_k(t)(\xi-t)f_{\xi s}^{**}(x,s,\xi-t)dt\right|$$

$$\le L(r+1)\int_{\mathbb{R}}\alpha_k(t)|t|dt = \frac{L(r+1)}{k}\int_{\mathbb{R}}\alpha(t)|t|dt \le \frac{L(r+1)}{k}\int_{\mathbb{R}}\alpha(t)dt = \frac{L(r+1)}{k} .$$

For $r\ge \sup_{k>0}\left\{\|u'_{\epsilon,k}\|_{L^\infty(a,b)}\,;\,\|u_{\epsilon,k}\|_{L^\infty(a,b)}\right\}$ we obtain

$$(5.18)\qquad \left|g^{\epsilon,k}_{\xi\xi}\, u''_{\epsilon,k} - \alpha_k * \left(f^{**}_s - f^{**}_{\xi x} - u'_{\epsilon,k}\, f^{**}_{\xi s}\right)\right| \le \frac{L(r+1)}{k} .$$

Now we introduce the further assumption that $\varphi(x,s,\xi)$ in (5.11) has a *strict* definite sign for every x,s,ξ (we will see at the end of the proof how to remove this assumption).

By the addendum $\epsilon(1+\xi^2)^{q/2}$ we have $g^{\epsilon,k}_{\xi\xi}>0$. Since α_k is a positive mollifier and since $\|u'_{\epsilon,k}\|_{L^\infty(a,b)}$, $\|u_{\epsilon,k}\|_{L^\infty(a,b)}$ are bounded uniformly with respect to k, for k sufficiently large the sign of $u''_{\epsilon,k}$ in (5.18) is the same as the sign of $\varphi(x,u_{\epsilon,k},u'_{\epsilon,k}) = f^{**}_s - f^{**}_{\xi x} - u'_{\epsilon,k}\, f^{**}_{\xi s}$. Therefore $u_{\epsilon,k}$ is either convex or concave in [a,b]. Let us consider the case $\varphi>0$, so that $u_{\epsilon,k}$ is convex. Since the difference quotient of $u_{\epsilon,k}$ is increasing, for every $x\in(a,b)$ we have

$$(5.19)\qquad \frac{u_{\epsilon,k}(a) - u_{\epsilon,k}(x)}{a - x} \le u'_{\epsilon,k}(x) \le \frac{u_{\epsilon,k}(b) - u_{\epsilon,k}(x)}{b - x} .$$

Now, if $a+\delta\le x\le b-\delta$ $(\delta\in(0,\frac{b-a}{2}))$, we obtain

$$(5.20)\qquad |u'_{\epsilon,k}(x)| \le \frac{2}{\delta}\|u_{\epsilon,k}\|_{L^\infty(a,b)}, \qquad \forall\ x\in[a+\delta,b-\delta] .$$

For $\epsilon\in(0,1]$ fixed, $u_{\epsilon,k}$ is bounded in $W^{1,\infty}(a,b)$ uniformly with respect to k (see lemma 5.7; here it is sufficient to know that $u_{\epsilon,k}$ is bounded in $W^{1,q}(a,b)$); as $k\to+\infty$, up to a subsequence, $u_{\epsilon,k}$ converges to a function u_ϵ in the weak* topology of $W^{1,\infty}(a,b)$ and in the strong topology of $L^\infty(a,b)$. Thus, by (5.20) (and by the lower semicontinuity of $k\to\|u'_{\epsilon,k}\|_{L^\infty(a+\delta,b-\delta)}$) :

$$(5.21)\qquad |u'_\epsilon(x)| \le \frac{2}{\delta}\|u_\epsilon\|_{L^\infty(a,b)}, \qquad \forall\ x\in[a+\delta,b-\delta],\ \forall\ \delta\in(0,\tfrac{b-a}{2}) .$$

Since $k\,\|[u'_{\epsilon,k}]^-\|^q_{L^q(a,b)}$ is bounded (it follows from the following formula (5.30)), the negative part of $u'_{\epsilon,k}$ converges to zero strongly in $L^q(a.b)$ and thus $u'_\epsilon \ge 0$ a.e. in [a,b]. Therefore $u_\epsilon(x)$ is increasing in [a,b] and it is bounded in

$L^\infty(a,b)$ independently of ϵ $(A=u_\epsilon(a)\leq u_\epsilon(x)\leq u_\epsilon(b)=B$ for every $x\in[a,b])$.

By (5.21) as $\epsilon\to 0$, up to a subsequence, u_ϵ converges in the weak* topology of $W^{1,\infty}_{loc}(a,b)$ to a function $u_0\in W^{1,\infty}_{loc}(a,b)$ that satisfies the estimate (5.12). This function u_0 is the minimizer we are looking for. With the aim to prove this fact, let us show first that u_ϵ is a minimizer of the variational problem

$$\min\left\{ \int_a^b g^\epsilon(x,u,u')\,dx \;:\; u\in W^{1,q}(a,b),\ u(a)=A,\ u(b)=B,\ u'\geq 0 \text{ a.e.}\right\}, \tag{5.22}$$

where

$$g^\epsilon(x,s,\xi) = f^{**}(x,s,\xi) + \epsilon(1+\xi^2)^{q/2} . \tag{5.23}$$

In fact, since $u'_{\epsilon,k}$ is bounded in $L^\infty(a+\delta,b-\delta)$ uniformly with respect to k and since $\alpha_k * f^{**}$ converges to f^{**} uniformly on bounded sets of $[a,b]\times\mathbb{R}\times\mathbb{R}$, then we have

$$\lim_{k\to+\infty} \int_{a+\delta}^{b-\delta} \left\{ \alpha_k * f^{**}(x,u_{\epsilon,k},u'_{\epsilon,k}) - f^{**}(x,u_{\epsilon,k},u'_{\epsilon,k}) \right\} dx = 0 .$$

Therefore, by the lower semicontinuity of the integral, for every $v\in W^{1,q}(a,b)$ such that $v(a)=A$, $v(b)=B$ and $v'\geq 0$ a.e. in $[a,b]$, we obtain

$$\int_{a+\delta}^{b-\delta} g^\epsilon(x,u_\epsilon,u'_\epsilon)\,dx \leq \liminf_{k\to\infty} \int_{a+\delta}^{b-\delta} g^\epsilon(x,u_{\epsilon,k},u'_{\epsilon,k})\,dx =$$

$$= \liminf_{k\to\infty} \int_{a+\delta}^{b-\delta} \left\{ \alpha_k * f^{**}(x,u_{\epsilon,k},u'_{\epsilon,k}) + \epsilon(1 + u'^2_{\epsilon,k})^{q/2} \right\} dx$$

$$\leq \liminf_{k\to\infty} G^{\epsilon,k}(u_{\epsilon,k}) \leq \liminf_{k\to\infty} G^{\epsilon,k}(v)$$

$$= \lim_{k\to\infty} \int_a^b \left\{ \alpha_k * f^{**}(x,v,v') + \epsilon(1 + v'^2)^{q/2} \right\} dx = \int_a^b g^\epsilon(x,v,v')\,dx .$$

As $\delta\to 0$, by the monotone convergence theorem, we see that u_ϵ is solution to (5.22).

Before to go to the limit as $\epsilon \to 0$, we recall the notations $\mathcal{W}_p$, $\overline{\mathcal{W}}_p$ respectively in (5.4), (5.5) and, for every $u \in \overline{\mathcal{W}}_p$, similarly to (5.6) we define

$$\overline{G}(u) = \inf_{\{u_k\}} \left\{ \liminf_{k \to \infty} \int_a^b f^{**}(x,u_k,u_k')dx : (u_k) \subset \mathcal{W}_p, \ u_k \text{ weakly converges to } u \text{ in } W^{1,p}_{loc} \right\}.$$

By theorem 5.1 we can represent $\overline{G}(u)$ in the form

$$\overline{G}(u) = \int_a^b f^{**}(x,u,u')\,dx + \widetilde{h}\left\{ \int_A^{u(a)} a(a,s)ds + \int_{u(b)}^B a(b,s)ds \right\}. \tag{5.24}$$

By the definition of $\overline{G}$, for every $v \in \mathcal{W}_q = \mathcal{W}_p \cap W^{1,q}(a,b)$ we have

$$\begin{aligned} \overline{G}(u_0) &\le \liminf_{\epsilon \to 0} \int_a^b f^{**}(x,u_\epsilon,u_\epsilon')\,dx \le \liminf_{\epsilon \to 0} \int_a^b g^\epsilon(x,u_\epsilon,u_\epsilon')\,dx \\ &\le \liminf_{\epsilon \to 0} \int_a^b g^\epsilon(x,v,v')\,dx = \int_a^b f^{**}(x,v,v')\,dx = \overline{G}(v). \end{aligned}$$

Therefore $\overline{G}(u_0) \le \overline{G}(v)$ for every $v \in \mathcal{W}_q$. Again by the definition of $\overline{G}$ and by its continuity in $W^{1,p}(a,b)$ we get also $\overline{G}(u_0) \le \overline{G}(v)$ for every $v \in \overline{\mathcal{W}}_p$.

Thus u_0 minimizes (5.24). We will obtain that u_0 minimizes (5.8) too, by showing that

$$f(x,u_0(x),u_0'(x)) = f^{**}(x,u_0(x),u_0'(x)), \qquad \text{a.e. in } [a,b]. \tag{5.25}$$

We know that u_0 is either convex or concave in $[a,b]$. Therefore, the set $I=\{x \in [a,b]: u_0'(x)=0\}$ is an interval (possibly empty) and on this set (5.25) holds, since $f(x,s,0) = f^{**}(x,s,0)$ (recall that f^{**} is the greathest convex function less than or equal to f for $\xi \ge 0$). If, for example, u_0 is concave in $[a,b]$ and $I=[x_0,b]$ for some $x_0 \in [a,b]$, then $u_0'(x) \ge u_0'(x_1) > 0$ for all $x \le x_1 < x_0$ and, in the set (a,x_1), u_0 solves Euler's equation in weak form and also (see for example [119], theorem 1.10.1) in the form

$$(5.26) \qquad f_\xi^{**}(x,u_0(x),u_0'(x)) = \text{constant} + \int_a^x f_s^{**}(t,u_0(t),u_0'(t))\,dt \ .$$

Since $u_0'(x)$ is monotone, then it is continuous almost everywhere in (a,x_1). Now let us assume that at a point $x\in(a,x_1)$, of continuity of $u_0'(x)$, we have $f(x,u_0(x),u_0'(x))\neq f^{**}(x,u_0(x),u_0'(x))$. We use the fact that f^{**} is linear in a neighbourhood of $\xi=u_0'(x)$ and thus f_ξ^{**} is independent of ξ; we take the derivative at x in both sides of (5.26) and we obtain

$$f_{\xi x}^{**}(x,u_0(x),u_0'(x)) + u_0'(x)f_{\xi s}^{**}(x,u_0(x),u_0'(x)) = f_s^{**}(x,u_0(x),u_0'(x)) \ ;$$

this contrasts with our main assumption that φ in (5.11) is different from zero if $\xi>0$. Therefore (5.25) holds and u_0 minimizes (5.10).

About the boundary values of u_0, if the variational problem is coercive, in the sense that $\widetilde{h}=\lim_{\xi\to+\infty} h(\xi)/\xi = +\infty$ and $a(x,s)$ is positive a.e. at $x=a$ and $x=b$, then $\overline{F}(u_0)<+\infty$ implies that $u(a)=A$ and $u(b)=B$ (see also remark 5.2).

Finally, if $\varphi>0$ (respectively $\varphi<0$), then, for every $\epsilon>0$, u_ϵ is increasing and convex (respectively concave) in $[a,b]$, it assumes the boundary values $u_\epsilon(a)=A$, $u_\epsilon(b)=B$ and pointwise converges to u_0 in (a,b). Then u_0 assumes at least one of the boundary values (as in the statement of theorem 5.4) according to the next lemma 5.8.

This completes the proof of theorem 5.4 under the further assumption that $\varphi(x,s,\xi)$ in (5.11) has a strict definite sign. We obtain the proof in the general case by approximation through, for example, the integrand

$$(5.27) \qquad f^\epsilon(x,s,\xi) = f(x,s,\xi) + \epsilon\, e^{\pm s} \ ,$$

where $\epsilon\in(0,1]$ and the sign $\pm$ is chosen in dependence on the sign of φ (sign $+$ if $\varphi\geq 0$). Then

$$(5.28) \qquad \varphi^{\epsilon}(x,s,\xi) = f_s^{**} - f_{\xi x}^{**} - \xi f_{\xi s}^{**} \pm \epsilon\, e^{\pm s} = \varphi(x,s,\xi) \pm \epsilon\, e^{\pm s}$$

has a strict definite sign. By the previous part of the proof, for every ϵ, there exists u_ϵ that minimizes the functional (5.8), with f replaced by f^{ϵ}. The minimizer u_ϵ either is convex in $[a,b]$, if $\varphi \geq 0$, or is concave if $\varphi \leq 0$. Moreover $A \leq u^{\epsilon}(x) \leq B$, for all $x \in [a,b]$. According to the next lemma 5.9, we can extract a subsequence, that we still denote by u_ϵ, that converges to a function u_0 in the norm topology of $W^{1,q}_{loc}(a,b)$. The functional (5.8) to minimize is lower semicontinuous with respect to the *strong* topology of $W^{1,q}_{loc}(a,b)$ (and it is even continuous in $W^{1,q}(a,b)$ for $q \geq p$, where p is the exponent in (5.9)). Similarly to the first part of the proof, we can go to the limit as $\epsilon \to 0$ and we obtain that u_0 is a solution to our variational problem.

Proof of lemma 5.7 - By the assumptions (5.7a), (5.9b), for $|s| \leq r$ we have

$$-K \leq f^{**}(x,s,\xi) \leq f(x,s,\xi) \leq a(x,s)\left[1+|\xi|^p\right] + b(x,s) \leq M(r)\left[1+|\xi|^p\right],$$

where $M(r) = \max\{a(x,s) + |b(x.s)|: x \in [a,b],\ s \in [-r,r]\}$; then we deduce (see formula (2.11) in [104]) that, for some constant c_1,

$$(5.29) \qquad \left|f_\xi^{**}(x,s,\xi)\right| \leq c_1\, M(r)\left[1+|\xi|^{p-1}\right], \qquad \forall\ (x,s,\xi):\ |s| \leq r.$$

It follows that $g^{\epsilon,k}$ in (5.15) satisfies natural growth conditions on the derivatives $g_\xi^{\epsilon,k}$ and $g_s^{\epsilon,k}$ (by taking into account the assumption (5.9a) too). Then, since $g^{\epsilon,k} \in C^2$, by a classical argument (see Morrey [119], theorem 1.10.1) $u_{\epsilon,k} \in C^2$ and it solves Euler's equation (5.17).

For $\epsilon > 0$ fixed, $u_{\epsilon,k}$ is bounded in $W^{1,q}(a,b)$ uniformly with respect to k; to prove this fact, we consider the function $v = v(x)$ with constant slope $v' = (B-A)/(b-a)$. Since $[v']^- = 0$ and $\epsilon \leq 1$, by (5.7), (5.9), (5.15) there are

constants c_2, c_3 such that

$$(5.30)\qquad \epsilon\|u'_{\epsilon,k}\|^q_{L^q(a,b)} + k\,\|[u'_{\epsilon,k}]^-\|^q_{L^q(a,b)} - K(b-a) \le G^{\epsilon,k}(u_{\epsilon,k}) \le$$

$$\le G^{\epsilon,k}(v) \le c_2 \int_a^b \{1 + |v'|^q\}\,dx = c_3 < +\infty \quad .$$

By the embedding theorem, for every $\epsilon>0$, $u_{\epsilon,k}$ is bounded also in $L^\infty(a,b)$. Now we can get a bound for the C^1-norm of $u_{\epsilon,k}$, uniform with respect to k. By integrating both sides of (5.17) and by using the growth condition (5.9a), there are constants c_4, $c_5 = c_5(\epsilon)$, $c_6 = c_6(\epsilon)$ such that

$$(5.31)\qquad \left|g_\xi^{\epsilon,k}(x,u_{\epsilon,k},u'_{\epsilon,k})\right| = \left|c_4 + \int_a^x g_s^{\epsilon,k}(x,u_{\epsilon,k}(t),u'_{\epsilon,k}(t))\,dt\right|$$

$$\le c_5 \int_a^b \{1 + |u_{\epsilon,k}(t)|^q + |u'_{\epsilon,k}(t)|^q\}\,dt \le c_6 \quad .$$

By the term $\epsilon(1+\xi^2)^{q/2}$, since $q\ge 2$, we have $g_{\xi\xi}^{\epsilon,k} > \epsilon q$. Then, for every ξ_1, ξ_2:

$$\left|g_\xi^{\epsilon,k}(x,s,\xi_1) - g_\xi^{\epsilon,k}(x,s,\xi_2)\right| \ge \epsilon q\,|\xi_1 - \xi_2| \quad .$$

Therefore, for $\xi_1 = u'_{\epsilon,k}$, $\xi_2 = 0$ and $s = u_{\epsilon,k}$, by (5.31) and (5.29), there exists a constant $c_7 = c_7(\epsilon)$ such that

$$\epsilon q|u'_{\epsilon,k}| \le \left|g_\xi^{\epsilon,k}(x,u_{\epsilon,k},u'_{\epsilon,k})\right| + \left|g_\xi^{\epsilon,k}(x,u_{\epsilon,k},0)\right| \le c_7 \quad , \quad \forall\ k>0.$$

The following lemmas 5.8 and 5.9 have been used in the proof of theorem 5.4.

LEMMA 5.8 - *Let* $u_\epsilon(x)$ *be a net of real functions defined in* $[a,b]$ *that, as* $\epsilon\to 0$, *converges to* $u_0(x)$ *for every* $x\in(a,b)$. *If, for every* ϵ, $u_\epsilon(x)$ *is increasing and convex (respectively concave) in* $[a,b]$, *then* u_ϵ *converges at*

$x=a$ *(respectively at* $x=b$*) too and*

$$\lim_{\epsilon\to 0} u_\epsilon(a) = u_0(a) \overset{def}{=} \inf\{u_0(x): x\in(a,b)\}$$

$$(\textit{ respectively } \lim_{\epsilon\to 0} u_\epsilon(b) = u_0(b) \overset{def}{=} \sup\{u_0(x): x\in(a,b)\} \quad) \ .$$

Proof - We consider only the convex case. First we extend u_0 at $x=a$ by definig $u_0(a) = \inf\{u_0(x): x\in(a,b)\}$. By its monotoniticy, u_0 turns out to be continuous at $x=a$. For every $x\in(a,b)$, the convexity of u_ϵ gives:

$$u_\epsilon\left(\frac{a+x}{2}\right) \leq \frac{1}{2}\left[u_\epsilon(a) + u_\epsilon(x)\right] \ .$$

We deduce that

$$\liminf_{\epsilon\to 0} u_\epsilon(a) \geq \liminf_{\epsilon\to 0}\left[2\, u_\epsilon\left(\frac{a+x}{2}\right) - u_\epsilon(x)\right] = 2\, u_0\left(\frac{a+x}{2}\right) - u_0(x)$$

and, as $x\to a^+$, $\liminf_{\epsilon\to 0} u_\epsilon(a) \geq u_0(a)$. Since $u_\epsilon(x)$ is increasing, for $x\in(a,b)$ we have

$$\limsup_{\epsilon\to 0} u_\epsilon(a) \leq \limsup_{\epsilon\to 0} u_\epsilon(x) = u_0(x)$$

and, as $x\to a^+$, we get the conclusion $\limsup_{\epsilon\to 0} u_\epsilon(a) \leq u_0(a)$.

LEMMA 5.9 - *Let* u_ϵ *be a net of convex functions in an open convex set* $\Omega\subseteq\mathbb{R}^n$; *let* u_ϵ *be bounded in* $L^\infty(\Omega)$ *uniformly with respect to* ϵ. *Then there exists a subnet of* u_ϵ *that converges in the strong topology of* $W^{1,q}_{loc}(\Omega)$, *for every* $q \in [1,+\infty)$. *The same conclusion holds if* $u_\epsilon=u_\epsilon(x_1,x_2,\ldots,x_n)$ *is separately convex or concave with respect to each compoment* x_i, *for* $i=1,2,\ldots n$.

Proof - The proof is similar if $n>1$ or if $n=1$, the only difference being that, if $n>1$, than it is convenient to work separately with each partial derivative $\partial u_\epsilon/\partial x_i$. Only to simplify the notations, we consider the case $n=1$ and $\Omega=(a,b)$:

since the difference quotient of u_ϵ is increasing , for $a<x_1<x<x_2<b$ we have

$$(5.32) \qquad \frac{u_\epsilon(x_1) - u_\epsilon(x)}{x_1 - x} \leq u_\epsilon'(x) \leq \frac{u_\epsilon(x_2) - u_\epsilon(x)}{x_2 - x} \quad .$$

Thus, like in (5.20), for every $\delta \in (0,\frac{b-a}{2})$ we get a bound for $\|u_\epsilon'\|_{L^\infty(a+\delta,b-\delta)}$, uniform with respect to ϵ. By the Ascoli-Arzelà's theorem we have the relative compactness in $C^0(a,b)$. Let us denote by u_0 the limit function. As $\epsilon\to 0$, we deduce from (5.32) that

$$(5.33) \qquad \frac{u_0(x_1) - u_0(x)}{x_1 - x} \leq \liminf_\epsilon u_\epsilon'(x) \leq \limsup_\epsilon u_\epsilon'(x) \leq \frac{u_0(x_2) - u_0(x)}{x_2 - x} \quad .$$

The function u_0, being convex, is differentiable a.e in (a,b). At every $x\in(a,b)$ where $u_0'(x)$ exists, letting in (5.33) $x_1\to x^-$, $x_2\to x^+$, we get $u_0'(x)=\lim_\epsilon u_\epsilon'(x)$. Since the sequence $u_\epsilon'(x)$ is locally bounded, by the Lebesgue dominated convergence theorem we deduce the convergence of u_ϵ in $W^{1,q}_{loc}$, for every $q<+\infty$.

COROLLARY 5.10 - *Newton's functional*

$$(5.34) \qquad \overline{F}(u) = \pi\,[u(a)]^2 + 2\pi \int_a^b u\,\frac{(u')^3}{1+(u')^2}\,dx$$

has a minimizer u_0 *in the set* $\{u\in W^{1,1}(a,b)\colon u(a)\geq 0,\ u(b)=B,\ u'\geq 0 \text{ a.e}\}$ $(B>0)$; *it is concave in* [a,b] *and satisfies the condition:* $0\leq u_0'(x) \leq 1$ *a.e. in* [a,b].

Proof - We obtain the proof as a consequence of the general theorem 5.4. We recall however that the result of this application is classical (see for example Tonelli [147], section 140). The function $f(s,\xi) = 2\pi s\xi^3/(1+\xi^2)$ satisfies (5.7) with $a(x,s)=2\pi s$, $h(\xi)=g^{**}(\xi)$ being the greathest function convex with respect to $\xi\geq 0$ and less than or equal to $g(\xi) = \xi^3/(1+\xi^2)$; it is easy to see that $h(\xi) =$

$\xi^3/(1+\xi^2)$ for $0\leq\xi\leq 1$ and $h(\xi) = \xi - \frac{1}{2}$ for $\xi>1$. Moreover $f^{**}(s,\xi) =(2\pi s)h(\xi)$ (see the graph of f^{**} in figure 2) and (5.9) holds with $p=1$. Since

$$\varphi_0 = f_s - \xi f_{\xi s} = -\frac{4\pi\,\xi^3}{\left[1+\xi^2\right]^2} < 0\,, \qquad \forall\,\xi>0\,,$$

then, for $\xi>0$, $\varphi = f_s^{**} - \xi f_{\xi s}^{**} < 0$ too (see remark 5.6; however, by a direct computation we can see that $f_s^{**} - \xi f_{\xi s}^{**} = -\pi$ for any $\xi>1$).

Then, by theorem 5.4, there is a function $u_0 \in W^{1,1}_{loc}(a,b)$ (and also $u_0\in W^{1,1}(a,b)$, since $u_0(a)\geq 0$, $u_0(b)\leq B$, $u_0'\geq 0$ a.e) that minimizes $\bar{F}$ in (5.8). That u_0 is also a minimizer of $\bar{F}$ in (5.34) follows from the fact that $u_0(b)=B$ (by theorem 5.4, since $\varphi<0$), $a(x,s)=2\pi s$ and $\tilde{h}=1$. Finally, $u_0'(x) \leq 1$ a.e. in $[a,b]$, since u_0 satisfies (5.25) and $f(s,\xi) = f^{**}(s,\xi)$ if and only if either $s=0$ or $0\leq\xi\leq 1$.

In view of the application to Newton's problem, previously we have considered the constrained problem $u'\geq 0$. Let us mention that, with essentially the same proof (we refer also to [29] for the convex case), we have an existence theorem in the unconstrained case, under the assumptions:

(5.35) *there exist* $p\geq 1$, $L>0$, $K\geq 0$, $a=a(x,s)$, $b=b(x,s)$ *continuous*, $M:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ *and* $h(\xi)$ *convex such that, for every* $\delta,r>0$:

(a) $a(x,s)h(\xi)-K \leq f(x,s,\xi) \leq a(x,s)h(\xi)+b(x,s)$, $\forall x\in[0,1]$, $s\in\mathbb{R}$ *and* $\xi\in\mathbb{R}$

(b) $|\xi| \leq h(\xi) \leq L\left[1+|\xi|^p\right]$, $\forall\,\xi\in\mathbb{R}$

(c) *either* $a=a(x,s)$ *is bounded from below by a positive constant, or* $a=a(s)$ *is independent of* x *and it is positive a.e. in* $\mathbb{R}$

(d) $|f_s^{**}(x,s,\xi)| \leq M(\delta,r)\left[1+|\xi|^p\right]$, $\forall\,(x,s,\xi)\in[a+\delta,b-\delta]\times[-r,r]\times\mathbb{R}$

In general, for functions $u\in W^{1,p}_{loc}(a,b)$ we define the values $u(a)$ and $u(b)$ as the infimum of $\liminf_{k\to+\infty} u(x_k)$, as x_k converges respectively to a^+ and b^-.

In the unconstrained case, the functional to be minimized is the following:

$$(5.36) \qquad \bar{F}(u) = \int_a^b f(x,u,u')\,dx + \tilde{h}\left\{\left|\int_A^{u(a)} a(a,s)ds\right| + \left|\int_{u(b)}^B a(b,s)ds\right|\right\}$$

where $\tilde{h} = \lim_{\xi\to\infty} h(\xi)/\xi$ (only for the sake of simplicity we assume that the limit as $\xi\to+\infty$ is equal to the limit as $\xi\to-\infty$; see [29] for the case of distinct limits).

THEOREM 5.11 - *Let* $f^{**}(x,s,\xi)$ *be the greathest function convex with respect to* ξ *and less than or equal to* $f(x,s,\xi)$. *Let us assume that* f^{**} *admits continuous partial derivatives* f^{**}_s, $f^{**}_{\xi x}$, $f^{**}_{\xi s}$ *and that* (5.35) *is satisfied. Let us assume also that the function* $\varphi(x,s,\xi)$ *in* (5.11) *has a definite sign (either* $\varphi\geq 0$ *or* $\varphi\leq 0$*) for every* $(x,s,\xi)\in(a,b)\times\mathbb{R}^2$. *Then the functional* $\bar{F}$ *in* (5.36) *has a minimizer* u_0 *in* $W^{1,p}_{loc}(a,b)$, *it belongs to* $W^{1,\infty}_{loc}(a,b)\cap L^\infty(a,b)$ *and it satisfies the estimate*

$$(5.37) \qquad \|u_0'\|_{L^\infty(a+\delta,b-\delta)} \leq \frac{2}{\delta}\|u_0\|_{L^\infty(a,b)}, \qquad \forall\ \delta\in(0,\tfrac{b-a}{2}).$$

6. A non convex problem arising from nonlinear elasticity

It seems to be a general principle in Calculus of Variations that, if some singularities of the minimizers form (either in the interior or at the boundary), then these singularities should give contribution to the functional to be minimized. This happens, for example, in classical problems of differential geometry and minimal surfaces (since Lebesgue [96], then De Giorgi, Giusti, Miranda (see [83], [117]), Serrin [136] and many others) and in some recent applications of methods of the Calculus of Variations, for example, in the approach by [107] to *cavitation* in nonlinear elasticity, in the theory of liquid crystals (Bethuel-Brezis-Coron [24]) and in optimal foraging models of behavioural ecology [29]. We notice that, in all these cases, the

energy-integral is extended "by lower semicontinuity", like in (5.6).

By taking the phenomenon of cavitation as a starting point, we conclude these notes by proposing a non convex integral, relevant for applications, for which the existence of minimizers is still an open problem (although we think that the existence theorem 5.4, proved in the previous section, should give a contribution to the solution).

It has been observed sperimentally that some elastic bodies, under linear (homogeneous) deformations of large slope applied at the boundary, brake inside, forming holes (*cavities*) with a certain symmetry. This phenomenon, called *cavitation*, mathematically has been first studied by Ball [16].

According to Ball, the body is schematized by the unit ball $\Omega=\{|x|<1\}$ of $\mathbb{R}^n$, with $n\geq 2$. The body is expanded by a deformation u that, at the boundary $\partial\Omega=\{|x|=1\}$ assumes the value $u(x)=\lambda x$ (that is, the deformed surface of the body is a sphere of radius λ). For some materials it is expected that, if λ is sufficiently large, then a cavity forms inside the body.

Let us consider radial deformations $u: \mathbb{R}^n\to\mathbb{R}^n$ of the type $u(x)= x\,v(r)/r$, where $r=|x|$ and with $v\geq 0$, $v'\geq 0$. It is clear that u maps the unit ball Ω in the ball of $\mathbb{R}^n$ of radius $v(1)$; more precisely, for every $r\in(0,1]$, u maps $\{x\in\Omega: |x|=r\}$ into $\{y\in\mathbb{R}^n: |y|=v(r)\}$. The condition that $u(x)=\lambda x$ on $\partial\Omega$ corresponds to $v(1)=\lambda$. If $v(0)>0$ (like in the previous section, by definition $v(0)$ is the infimum of $v(r)$ for $r>0$), then $u(x)$ is discontinuous at $x=0$ and a ball of center the origin and radius $v(0)$ (a *cavity*) forms inside the image $u(\Omega)$.

The energy integral to be minimized, being in general a functional of u, limitedly to radial deformations can be written as an integral functional of v and, by using polar coordinates, it can be reduced to a simple integral. If u is smooth (in particular, if $v(0)=0$) then the energy integral takes the form (see [16]):

$$F(v) = \omega_n \int_0^1 r^{n-1}\, \Phi\Big(\frac{v}{r}, v'\Big)\, dr \quad , \tag{6.1}$$

where ω_n is the $(n-1)$-measure of the surface of the unit ball in $\mathbb{R}^n$. To study the phenomenon of cavitation it is necessary to extend the energy F to functions v such that $v(0)>0$. As we already said at the beginning of this section, F can be extended "by lower semicontinuity", like in (5.6). Following [107], one is led to minimize the functional

$$(6.2) \qquad \bar{F}(v) = \omega_n \int_0^1 r^{n-1}\, \Phi\Big(\frac{v}{r}, v'\Big)\, dr \;+\; \tilde{h}\, \frac{\omega_n}{n}\, [v(0)]^n ,$$

on the set

$$(6.3) \qquad \bar{W}_p = \Big\{ v \in W^{1,p}_{loc}(0,1):\ v \geq 0,\ v(1)=\lambda,\ v' \geq 0 \ \text{a.e} \Big\} ;$$

here $\Phi=\Phi(s,\xi)$ satisfies assumptions of the type of (5.7), (5.9) and $\tilde{h}$ is defined in a similar way as in the previous section (see more precisely [107]; see also $\tilde{h}$ below in the particular case). Notice that $\Phi(s,\xi)$ is not necessarily convex with respect to ξ.

A simplified model, relevant because it is related to the so called "*Blatz-Ko materials*" (see [25], [88], [94], where in particular the values $\beta=0.13$, $\beta=0.07$, $\beta=-0.19$ are considered), is the following

$$(6.4) \quad \bar{F}(v) = \omega_n \int_0^1 r^{n-1} \Big\{ \Big[v'^p + (n-1)\Big(\frac{v}{r}\Big)^p \Big] + g\Big(v' \Big(\frac{v}{r}\Big)^{n-1} \Big) \Big\} dr \;+\; \tilde{h}\, \frac{\omega_n}{n}\, [v(0)]^n ,$$

where $p<n$, $\tilde{h} = \lim_{t\to+\infty} g^{**}(t)/t = 2(1-\beta)$ and

$$(6.5) \qquad g(t) = \beta\, \frac{2}{t} \;+\; (1-\beta)\Big(2t + \frac{1}{t^2}\Big) \qquad \forall\ t>0, \qquad \text{with } |\beta|<1 .$$

We emphasize that, if β is negative, then $g(t)$ is not convex in $(0,+\infty)$.

Up to now, no existence result seems applicable to the minimization, in the class (6.3), of the functional (6.2) in the general case, neither to the minimization of the simplified version in (6.4), (6.5).

REFERENCES

[1] E.ACERBI-G.BUTTAZZO, *Semicontinuous envelopes of polyconvex integrals*, Proc. R. Soc. Edinb., 96A, (1984), 51-54.

[2] E.ACERBI-G.BUTTAZZO-N.FUSCO, *Semicontinuity and relaxation for integrals depending on vector-valued functions*, J. Math. Pures Appl., 62 (1983), 371-387.

[3] E.ACERBI-N.FUSCO, *Semicontinuity problems in the calculus of variations*, Arch. Rational Mech. Anal., 86 (1984), 125-145.

[4] E.ACERBI-N.FUSCO, *A regularity theorem for minimizers of quasiconvex integrals*, Arch. Rational Mech. Anal., 99 (1987), 261-281.

[5] L.AMBROSIO, *New lower semicontinuity results for integral functionals*, Rend. Accad. Naz. Sci. XL, 11 (1987), 1-42.

[6] S.ANTMAN, *The influence of elasticity on analysis: modern developments*, Bull. Amer. Math. Soc., 9 (1983), 267-291.

[7] G.ANZELLOTTI-G.BUTTAZZO-G.DAL MASO, *Dirichlet problems for demicoercive functionals*, Nonlinear Anal., 10 (1986), 603-613.

[8] H.ATTOUCH, *Variational Convergence of Functional and operators*, Pitman, London, 1984.

[9] G.AUBERT, *A counterexample of a rank one convex function which is not polyconvex in the case n=2*, Proc. R.Soc. Edinb., 106A (1987), 237-240.

[10] G.AUBERT-R.TAHRAOUI, *Théorèmes d'existence pour des problèmes du calcul des variations*, J. Differential Eq., 33 (1979), 1-15.

[11] G.AUBERT-R.TAHRAOUI, *Sur la minimisation d'une fonctionelle non convexe, non différentiable en dimension* 1, Boll. Un. Mat. Ital., 17 (1980), 244-258.

[12] G.AUBERT-R.TAHRAOUI, *Théorèmes d'existence en optimimisation non convexe*, Appl. Anal., 18 (1984), 75-100.

[13] G.AUBERT-R.TAHRAOUI, *Sur la faible fermature de certain ensembles de constraintes en élasticité non-linéaire plane*, Arch. Rational Mech. Anal., 97 (1987), 33-58.

[14] J.M.BALL, *On the calculus of variations and sequentially weakly continuous maps*, Ordinary and Partial Differential Equat., Lecture Notes in Math. n. 564, Springer, 1976, 13-25.

[15] J.M.BALL, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal., 63 (1977), 337-403.

[16] J.M.BALL, *Discontinuous equilibrium solutions and cavitation in nonlinear elasticity*, Phil. Trans. R. Soc. London, 306 (1982), 557-611.

[17] J.M.BALL-J.C.CURRIE-P.J.OLVER, *Null lagrangians, weak continuity and variational problems of arbitrary order*, J. Func. Analysis, 41 (1981), 135-175.

[18] J.M.BALL-R.J.KNOPS-J.E.MARSDEN, *Two examples in nonlinear elasticity, Journée d'analyse non linéaire*, Lecture Notes in Math. 665, Springer, 41-48.

[19] J.M.BALL-J.E.MARSDEN, *Quasiconvexity at the boundary, positivity of the second variation and elastic stability*, Arch. Rational Mech. Anal., 86 (1984), 251-277.

[20] J.M.BALL-V.J.MIZEL, *One dimensional variational problems whose minimizers do not satisfy the Euler-Lagrange equation*, Arch. Rational Mech. Anal., 90 (1985), 325-388.

[21] J.M.BALL-F.MURAT, $W^{1,p}$*quasiconvexity and variational problems for multiple integrals*, J. Funct. Anal., 58 (1984), 225-253.

[22] A.BENSOUSSAN-J.L.LIONS-G.PAPANICOLAOU, *Asymptotic analysis for periodic structures*, North-Holland, Amsterdam, 1978.

[23] L.D.BERKOWITZ, *Lower semicontinuity of integral functionals*, Trans. Am. Math. Soc., 192 (1974), 51-57.

[24] F.BETHUEL-H.BREZIS-J.M.CORON, *Relaxed energies for harmonic maps*, preprint Université Pierre et Marie Curie, 1989.

[25] P.J.BLATZ-W.L.KO, *Application of finite elastic theory to the deformation of rubbery materials*, Trans. Soc. Rheology, 6 (1962), 223-251.

[26] L.BOCCARDO-G.BUTTAZZO, *Quasilinear elliptic equations with discontinuous coefficients*, Atti Accad. Naz. Lincei, 82 (1988), 21-28.

[27] L.BOCCARDO-B.DACOROGNA, *Monotonicity of certain differential operators in divergence form*, Manuscripta Math., 64 (1989), 253-260.

[28] M.BONI, *Un teorema di semicontinuità inferiore*, Rend. Circ. Mat. Palermo, 25 (1976), 53-66.

[29] B.BOTTERON-P.MARCELLINI, *A general approach to the existence of minimizers of one-dimensional non-coercive integrals of the calculus of variations*, preprint Università di Firenze, 1989.

[30] G.BOTTARO-P.OPPEZZI, *Rappresentazione con integrali multipli di funzionali dipendenti da funzioni a valori in uno spazio di Banach*, Ann. Mat. Pura Appl., 139 (1985), 191-225.

[31] H.BREZIS, *Intégrales convexes dans les espaces de Sobolev*, Israel J. Math., 13 (1972), 9-23.

[32] G.BUTTAZZO, *Semicontinuity, relaxation, and integral representation in the calculus of variations*, Research Notes in Math., Longman, to appear.

[33] G.BUTTAZZO-G.DAL MASO, *A characterization of nonlinear functionals on Sobolev spaces which admit an integral representation with a Carathéodory integrand*, J. Math. Pures Appl., 64 (1985), 337-361.

[34] G.BUTTAZZO-G.DAL MASO, *Integral representation and relaxation of local functionals*, Nonlinear Anal., 9 (1985), 515-532.

[35] R.CACCIOPPOLI-G.SCORZA DRAGONI, *Necessità della condizione di Weierstrass per la semicontinuità di un integrale doppio sopra una data superficie*, Memorie Acc. d' Italia, 9 (1938), 251-268.

[36] L.CARBONE-R.DE ARCANGELIS, *Further results on Γ-convergence and lower semicontinuity of integral functionals depending on vector-valued functions*, preprint Università di Salerno, 1989.

[37] L.CARBONE-C.SBORDONE, *Some properties of Γ-limits of integral functionals*, Ann. Mat. Pura Appl., 122 (1979), 1-60.

[38] M.CARRIERO-A.LEACI-E.PASCALI, *On the semicontinuity and the relaxation for integrals with respect to the Lebesgue measure added to integrals with respect to a Radon measure*, Ann. Mat. Pura Appl., 149 (1987), 1-21.

[39] E.CASADIO TARABUSI, *An algebraic characterization of quasi-convex functions*, preprint.

[40] A.CELLINA-G.COLOMBO, *On a classical problem of the calculus of variations without convexity assumptions*, Ann. Inst. Henri Poincaré, Analyse non Linéaire, to appear.

[41] L.CESARI, *Lower semicontinuity and lower closure theorems without seminormality conditions*, Ann. Mat. Pura Appl., 98 (1974), 381-397.

[42] L.CESARI, *An existence theorem without convexity conditions*, SIAM J. Control, 12 (1974), 319-331.

[43] L.CESARI, *Optimization - Theory and applications*, Appl. of Math. 17, Springer-Verlag, 1983.

[44] L.CESARI-P.BRANDI-A.SALVADORI, *Existence theorems concerning simple integrals of the calculus of variations for discontinuous solutions*, Arch. Rational Mech. Anal., 98 (1987), 307-328.

[45] P.CHARRIER-B.DACOROGNA-B.HANOUZET-P.LABORDE, *An existence theorem for slightly compressible material in nonlinear elasticity*, S.I.A.M. Math. Anal., 19 (1988), 70-86.

[46] M.CHIPOT-D.KINDERLEHER, *Equilibrium configurations of crystals*, Arch. Rational Mech. Anal., 103 (1988), 237-277.

[47] P.G.CIARLET, *Mathematical Elasticity, vol. I: Three-dimensional Elasticity*, North-Holland, Amsterdam, 1987.

[48] P.G.CIARLET-J.NEČAS, *Unilateral problems in nonlinear, thee-dimensional elasticity*, Arch. Rational Mech. Anal., 87 (1985), 319-338.

[49] F.H.CLARKE, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[50] J.N.CORVELLEC, *Quelques remarques sur le problème de Lagrange du calcul des variations*, Colloquium Math., 57 (1989), to appear.

[51] B.DACOROGNA, *A relaxation theorem and its applications to the equilibrium of gases*, Arch. Rational Mech. Anal., 77 (1981), 359-386.

[52] B.DACOROGNA, *Weak continuity and weak lower semicontinuity of nonlinear functionals*, Lectures Notes in Math. 922, Springer-Verlag, Berlin, 1982.

[53] B.DACOROGNA, *Quasiconvexity and relaxation of nonconvex problems in the calculus of variations*, J. Func. Analysis, 46 (1982), 102-118.

[54] B.DACOROGNA, *Direct methods in the calculus of variations*, Applied Math. Sciences 78, Springer-Verlag, 1989.

[55] B.DACOROGNA-N.FUSCO, *Semicontinuité des fonctionnelles avec contraites du type "det $\nabla u > 0$"*, Boll. Un. Mat. Ital., 4-B (1985), 179-189.

[56] B.DACOROGNA-P.MARCELLINI, *A counterexample in the vectorial calculus of variations*, Material Instabilities in Continuum Mechanics, edited by J.M.Ball, Calderon Press, Oxford, 1988, 77-83.

[57] B.DACOROGNA-J.MOSER, *On a partial differential equation involving the Jacobian determinant*, Ann. Inst. Henri Poincaré, Analyse non Linéaire, to appear.

[58] G.DAL MASO, *Integral representation on* BV(Ω) *of* Γ*-limits of variational integrals*, Manuscripta Math., 30 (1980), 387-416.

[59] G.DAL MASO-L.MODICA, *A general theory of variational functionals*, Topics in Funct. Analysis 1980-81, Quaderni della Scuola Normale Sup. di Pisa, 1982.

[60] G.DAL MASO-L.MODICA, *Integral functionals determined by their minima*, Rend. Sem. Mat. Univ. Padova, 76 (1986), 255-267.

[61] E.DE GIORGI, *Teoremi di semicontinuità nel calcolo delle variazioni*, Istituto Nazionale di Alta Matematica, Roma, 1968-1969.

[62] E.DE GIORGI, *Sulla convergenza di alcune successioni di integrali del tipo dell'area*, Rendiconti Mat., 8 (1975), 277-294.

[63] E.DE GIORGI, *Some continuity and relaxation problems*, E.De Giorgi Colloquium, Proceedings, Pitman Res. Notes, vol. 125, London (1985), 1-11.

[64] E.DE GIORGI-G.BUTTAZZO-G.DAL MASO, *On the lower semicontinuity of certain integral functionals*, Atti Accad. Naz. Lincei, 74 (1983), 274-282.

[65] I.EKELAND, *Discontinuités de champs hamiltionens et existence de solutions optimales en calcul des variations*, Publications Math. de l' I.H.E.S., 47 (1977), 5-32.

[66] I.EKELAND, *Nonconvex mimimization problems*, Bull. Amer. Math. Soc., 1 (1979), 443-474.

[67] I.EKELAND-R.TEMAM, *Analyse convexe et problèmes variationnels*, Dunod Gauthier-Villars, 1974.

[68] L.C.EVANS, *Quasiconvexity and partial regularity in the calculus of variations*, Arch. Rational Mech. Anal., 95 (1986), 227-252.

[69] L.C.EVANS-R.F.GARIEPY, *Blow-up, compacteness and partial regularity in the calculus of variations*, Indiana U. Math. J., 36 (1987), 361-371.

[70] F.FERRO, *Functionals defined on functions of bounded variation in* $\mathbb{R}^n$ *and the Lebesgue area*, SIAM J.Control Optimization, 16 (1978), 778-789.

[71] N.FUSCO, *Quasi convessità e semicontinuità per integrali multipli di ordine superiore*, Ricerche Mat., 29 (1980), 307-323.

[72] N.FUSCO-J.HUTCHINSON, $C^{1,\alpha}$ *partial regularity of functions minimizing quasiconvex integrals*, Manuscripta Math.,54 (1985), 121-143.

[73] N.FUSCO-J.HUTCHINSON, *Partial regularity in problems motivated by non-linear elasticity*, 1989, preprint.

[74] N.FUSCO-G.MOSCARIELLO, L^2*-lower semicontinuity of functionals of quadratic type*, Ann. Mat. Pura Appl., 129 (1981), 305-326.

[75] A.GAVIOLI, *A lower semicontinuity theorem for the integral of the calculus of variations*, Atti Sem. Mat. Fis. Univ. Modena, 31 (1982), 268-284.

[76] M.GIAQUINTA, *Multiple integrals in the calculus of variations and non-linear elliptic systems*, Annals of Math. Stud. 105, Princeton Univ. Press, 1983.

[77] M.GIAQUINTA, *Quasiconvexity, growth conditions and partial regularity*, P.D.E. and Calculus of Variations, S.Hildebrandt and R.Leis editors, Lecture Notes in Math. 1357, Springer, to appear.

[78] M.GIAQUINTA-E.GIUSTI, *On the regularity of the minima of variational integrals*, Acta Math., 148 (1982), 31-46.

[79] M.GIAQUINTA-E.GIUSTI, *Q-minima*, Ann. Inst. Henri Poincaré, Analyse non Linéaire, 1 (1984), 79-107.

[80] M.GIAQUINTA-G.MODICA, *Partial regularity of minimizers of quasiconvex integrals*, Ann. Inst. Henri Poincaré, Analyse non Linéaire, 3 (1986), 185-208.

[81] M.GIAQUINTA-G.MODICA-J.SOUČEK, *Functionals with linear growth in the calculus of variations* I, Commentationes Math.Univ. Carolinae, 20 (1979), 143-172.

[82] M.GIAQUINTA-G.MODICA-J.SOUČEK, *Cartesian currents, weak diffeomorphisms and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal., 106 (1989), 97-159. *Erratum* in the same journal, to appear.

[83] E.GIUSTI, *Non-parametric minimal surfaces with discontinuous and thin obstacles*, Arch. Rational Mech. Anal., 49 (1972), 41-56.

[84] E.GIUSTI, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Boston, 1984.

[85] H.GOLDSTINE, *A history of the calculus of variations from the 17th through the 19th century*, Studies in History of Math. and Phys. Sc. 5, Springer-Verlag, 1980.

[86] C.HAMBURGER, *A characterization for rank-one-convexity in two dimensions*, Ricerche Mat., 36 (1987), 171-181.

[87] HONG MIN-CHUN, *Existence and partial regularity in the calculus of variations*, Ann. Mat. Pura Appl., 149 (1987), 311-328.

[88] C.O.HORGAN-R.ABEYARATNE, *A bifurcation problem for a compressible nonlinear elastic medium: growth of a microvoid*, J. of Elasticity, 16 (1986), 189-200.

[89] A.D.IOFFE, *On lower semicontinuity of integral functionals* I, SIAM J. Control Optimization, 15 (1977), 521-538.

[90] D.KINDERLEHRER-E.MASCOLO, *Local minima for nonconvex problems*, preprint.

[91] D.KINDERLHERER-G.STAMPACCHIA, *Introduction to variational inequalities and their applications*, Academic Press, New York, 1980.

[92] R.KLOTZLER, *On the existence of optimal processes*, Banach Center Publications, Vol. 1, Warszawa, 1976, 125-130.

[93] R.J.KNOPS-C.A.STUART, *Quasiconvexity and uniqueness of equilibrium solutions in nonlinear elasticity*, Arch. Rational Mech. Anal., 86 (1984), 233-249.

[94] J.K.KNOWLES-E.STERNBERG, *On the ellipticity of the equation of nonlinear elastostatics for a special material*, J. of Elasticity, 5 (1975), 341-361.

[95] R.V.KOHN-G.STRANG, *Optimal design annd relaxation of variational problems* I,II,III, Commun. Pure Appl. Math., 39 (1986), 113-137, 139-182, 353-377.

[96] H.LEBESGUE, *Intégrale, longueur, aire*, Ann. Mat. Pura Appl., 7 (1902), 231-359.

[97] J.L.LIONS-E.MAGENES, *Non-homogeneous boundary value problems and applications* I,II,III, Springer-Verlag, 1972 and 1973.

[98] P.L.LIONS, *Generalized solutions of Hamilton-Jacobi equations*, Pitman, 1982.

[99] P.MARCELLINI, *Some problems of semicontinuity and of Γ-convergence for integrals of the calculus of variations*, Recent Methods in Nonlinear Analisys, edited by De Giorgi, Magenes, Mosco, Pitagora Ed., 1979, 205-222.

[100] P.MARCELLINI, *Alcune osservazioni sull'esistenza del minimo di integrali del calcolo delle variazioni senza ipotesi di convessità*, Rendiconti di Matematica, 13 (1980), 271-281.

[101] P.MARCELLINI, *A relation between existence of minima for nonconvex integrals and uniqueness for non strictly convex integrals of the calculus of variations*, Mathematical Theories of Optimization, Proceedings, edited by J.P.Cecconi and T.Zolezzi, Lecture Notes in Math., vol. 979, Springer, 1983, 216-231.

[102] P.MARCELLINI, *Some remarks on uniqueness in the calculus of variations*, Collège de France Seminar, edited by H.Brézis and J.L.Lions, Research Notes in Math., vol. 84, Pitman, 1983, 148-153.

[103] P.MARCELLINI, *Quasiconvex quadratic forms in two dimensions*, Appl. Math. Optimization, 11 (1984), 183-189.

[104] P.MARCELLINI, *Approximation of quasiconvex functions and lower semicontinuity of multiple integrals*, Manuscripta Math., 51 (1985), 1-28.

[105] P.MARCELLINI, *On the definition and the lower semicontinuity of certain quasiconvex integrals*, Ann. Inst. Henri Poincaré, Analyse non Linéaire, 3 (1986), 391-409.

[106] P.MARCELLINI, *Existence theorems in nonlinear elasticity*, edited by J.B.Hiriart-Urruty, North-Holland, 1986, 241-247.

[107] P.MARCELLINI, *The stored-energy for some discontinuous deformations in nolinear elasticity*, Essays in honor of E.De Giorgi, Vol.II, edited by F.Colombini et al., Birkhäuser, 1989, 767-786.

[108] P.MARCELLINI-C.SBORDONE, *Semicontinuity problems in the calculus of variations*, Nonlinear Anal., 4 (1980), 241-257.

[109] P.MARCELLINI-C.SBORDONE, *On the existence of minima of multiple integrals of the calculus of variations*, J.Math. Pures Appl., 62 (1983), 1-9.

[110] E.MASCOLO, *Some remarks on non-convex problems*, Material Instabilities in Continuum Mechanics, edited by J.M.Ball, Calderon Press, Oxford, 1988, 269-286.

[111] E.MASCOLO, *Local minima of non convex problems*, preprint Università di Firenze, 1989.

[112] E.MASCOLO-R.SCHIANCHI, *Existence theorems for nonconvex problems*, J.Math. Pures Appl., 62 (1983), 349-359.

[113] E.MASCOLO-R.SCHIANCHI, *Nonconvex problems of the calculus of variations*, Nonlinear Anal., 9 (1985), 371-379.

[114] E.MASCOLO-R.SCHIANCHI, *Existence theorems in the calculus of variations*, J. Differential Eq., 67 (1987), 185-198.

[115] B.J.McSHANE, *On the necessary condition of Weierstrass in the multiple integral problem of the calculus of variations*, Ann. Math., 32 (1931), 578-590.

[116] N.MEYERS, *Quasiconvexity and lower semicontinuity of multiple integrals of any order*, Trans. Amer. Math. Soc., 119 (1965), 125-149.

[117] M.MIRANDA, *Un teorema di esistenza e unicità per il problema dell'area minima in n variabili*, Ann. Scuola Norm. Sup. Pisa, 19 (1965), 233-249.

[118] C.B.MORREY, *Quasiconvexity and the lower semicontinuity of multiple integrals*, Pacific J. Math., 2 (1952), 25-53.

[119] C.B.MORREY, *Multiple integrals in the calculus of variations*, D. Grundl. Math. Wiss. 130, Springer-Verlag, 1966.

[120] S.MÜLLER, *Weak continuity of determinants and nonlinear elasticity*, C.R. Acad. Sc. Paris, 307 (1988), 501-506.

[121] F.MURAT, *Compacité par compensation*, II, Proc. Internat. Meeting on Recent Methods in Nonlinear Analysis, De Giorgi, Magenes, Mosco eds., Pitagora, Bologna, (1979), 245-256.

[122] L.NANIA, *On the Γ-convergence for multiple integrals depending on vector valued functions*, Ann. Mat. Pura Appl., 134 (1983), 67-77.

[123] L.NIRENBERG, *Remarks on strongly elliptic partial differential equations*, Comm. Pure Appl. Math., 8 (1955), 648-674.

[124] R.W.ODGEN, *Large deformation isotropic elasticity: on the correlation of theory and experiment for compressible rubberlike solids*, Proc. R. Soc. London A., 328 (1972), 567-583.

[125] C.OLECH, *Integrals of set valued functions and linear optimal control problems*, Colloque sur la théorie mathématique du contrôle optimal, C.B.R.M., Vander Louvain, 1970, 109-125.

[126] C.OLECH, *A characterization of* L^1*-weak lower semicontinuity of integral functionals*, Bull. Polish. Acad. Sci. Math., 25 (1977), 135-142.

[127] P.PODIO GIUDUGLI - G.VERGARA CAFFARELLI - E.G.VIRGA, *Discontinuous energy minimizers in nonlinear elastostatics; an example of J. Ball revisited*, J. Elasticity, 16 (1986), 75-96.

[128] J.P.RAYMOND, *Champs Hamiltoniens, relaxation et existence de solutions en calcul des variations*, J. Differential Eq., 70 (1987), 226-274.

[129] J.P.RAYMOND, *Conditions nécessaires et suffisantes d'existence de solutions en calcul des variations*, Ann. Inst. Henri Poincaré, Analyse non Linéaire, 4 (1987), 169-202.

[130] J.P.RAYMOND, *Théorème d'existence pour des problèmes variationnels non convexes*, Proc. Royal Soc. Edinburgh, 107A (1987), 43-64.

[131] R.T.ROCKAFELLAR, *Convex analysis*, Princeton Univ. Press, Princeton, 1970.

[132] R.T.ROCKAFELLAR, *Integral functionals, normal integrands and measurable selections*, Nonlinear operators and the calculus of variations, edited by Gossez et al., Lecture Notes in Math. 543, Springer, Berlin, 1975, 157-207.

[133] C.SBORDONE, *Su alcune applicazioni di un tipo di convergenza variazionale*, Ann. Scuola Norm. Sup. Pisa, 2 (1975), 617-638.

[134] C.SBORDONE, *Lower semicomtinuity and regularity of minima of variational integrals*, Nonlinear P.D.E. and Appl., Collège de France Seminar, edited by H.Brézis and J.L.Lions, Pitman, 1983, 194-213.

[135] D.SERRE, *Formes quadratiques et calcul des variations*, J. Math. Pures Appl., 62 (1983), 117-196.

[136] J.SERRIN, *On the definition and properties of certain variational integrals*, Trans. Amer. Math. Soc., 101 (1961), 139-167.

[137] J.SIVALOGANATHAN, *A field theory approach to stability of radial equilibria in nonlinear elasticity*, Math. Proc. Cambridge Phil. Soc., 99 (1986), 586-604.

[138] J.SIVALOGANATHAN, *Implications of rank one convexity*, Ann. Inst. Henri Poincaré, Analyse non Linéaire, 5 (1988), 99-118.

[139] C.A.STUART, *Radially symmetric cavitation for hyperelastic materials*, Ann. Inst. Henri Poincaré, Analyse non Linéaire, 2 (1985), 33-66.

[140] C.A.STUART, *Special problems involving uniqueness and multiplicity in hiperelasticity*, preprint.

[141] R.TAHRAOUI, *Théorèmes d'existence en calcul des variations et applications à l'élasticité non linéaire*, C.R.Acad. Sci. Paris, 302 (1986), 495-498.

[142] G.TALENTI, *Calcolo delle variazioni*, Quaderni dell' Unione Matematica Italiana n. 2, Pitagora Ed., 1977.

[143] L.TARTAR, *Compensated compactness*, Heriot-Watt Sympos. 4, Pitman, New York, 1978.

[144] R.TEMAM, *A characterization of quasiconvex function*, Applied Math. Opt., 8 (1982), 287-291.

[145] F.J.TERPSTRA, *Die darstellung biquadratischer formen als summen von quadraten mit anwendung auf die variationsrechnung*, Math. Ann., 116 (1938), 166-180.

[146] L.TONELLI, *Fondamenti di calcolo delle variazioni*, Volume primo, Zanichelli, Bologna, 1921.

[147] L.TONELLI, *Fondamenti di calcolo delle variazioni*, Volume secondo, Zanichelli, Bologna, 1923.

[148] M.VALADIER, *Integration de convexes fermes notamment d'epigraphes inf-convolution continue*, R.I.R.O., 4e année, R-2 (1970), 57-73.

[149] M.VALADIER, *Régularisation s.c.i., relaxation et théorèmes bang-bang*, C.R.Acad. Sci. Paris, 293 (1981), 115-116.

[150] M.VALADIER, *Functions and operators on the space of bounded measures*, Atti Sem. Mat. Fis. Univ. Modena, 34 (1985-86), 353-362.

[151] L.C.YOUNG, *Lectures on the calculus of variations and optimal control theory*, W.B.Saunders Company, Philadelphia, 1969.

[152] V.V.ZHIKOV-S.M.KOZLOV-O.A.OLEINIK-KHA T'EN NGOAN, *Averaging and G-convergence of differential operators*, Uspeki Math. Nauk, 34 (1979), 65-133; Russian Math. Surveys, 34 (1979), 69-147.

[153] T.ZOLEZZI, *On equiwellset minimum problems*, Appl. Math. Optim., 4 (1978), 209-223.