JOURNAL OF DIFFERENTIAL EQUATIONS **105**, 296–333 (1993)

Regularity for Elliptic Equations with General Growth Conditions

PAOLO MARCELLINI

Dipartimento di Matematica "U. Dini," Università di Firenze, Viale Morgagni 67/A, 50134 Firenze, Italy

Received July 9, 1991; revised October 28, 1991

Contents.

1. INTRODUCTION

To give an idea of the elliptic problems (involving partial differential equations and integral functionals) we study in this paper, as well as of the notations and the nomenclature we use, we consider a list of integrals of the calculus of variations with *non-standard growth conditions.* We begin with an example of an integral with *anisotropic growth conditions*

$$F_1(v)=\int_\Omega \left\{(1+|Dv|^2)^{p/2}+\sum_{i=1}^n |v_{x_i}|^{q_i}\right\} dx, \tag{1.1}$$

where Ω is an open set of $\mathbb{R}^n$, $n\geqslant 2$, and $p, q_i>1$ for $i=1, 2, ..., n$. The following

$$F_2(v)=\int_\Omega (1+|Dv|^2)^{\alpha(x)/2}\,dx, \qquad \text{with} \quad 1<p\leqslant\alpha(x)\leqslant q, \tag{1.2}$$

is an example of a functional which satisfies the *p, q-growth condition.* Also F_1 has a growth of type p, q, with $q=\max\{p; q_1; q_2; ...; q_n\}$. The functional

$$F_3(v)=\int_\Omega f(x, v, Dv)\,dx, \quad \text{with } f(x, s, \xi)\sim|\xi|^p\log(1+|\xi|) \text{ as } |\xi|\to+\infty, \tag{1.3}$$

0022-0396/93 $5.00

satisfies the p, q-growth condition with q arbitrarily close to p. When in similar situations we have $p=q$, then we use the classical well known terminology of *natural growth conditions.*

Finally, if the growth with respect to the gradient Du is not bounded by a power and, for example, is an *expopnential growth,* like in

$$F_4(v)=\int_\Omega f(x, v, Dv)\, dx, \qquad \text{with } \ f(x, s, \xi) \sim \exp|\xi|^\alpha \text{ as } |\xi| \to +\infty, \tag{1.4}$$

for some $\alpha>0$, then we have a problem with *general growth conditions.*

In this paper we consider a class of elliptic partial differential equations in divergence form, including Euler's equations of the integral-functionals described previously, of the type

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} a^i(x, u, Du) = b(x, u, Du), \qquad x \in \Omega, \tag{1.5}$$

where the vector field $(a^i(x, s, \xi))$ is locally Lipschitz-continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and satisfies the *general growth* (or *ellipticity*) *conditions*

$$mg_1(|\xi|)\, |\lambda|^2 \leqslant \sum_{i,j} a^i_{\xi_j}(x, s, \xi)\, \lambda_i \lambda_j, \qquad |a^i_{\xi_j}(x, s, \xi)| \leqslant Mg_2(|\xi|), \forall i, j, \tag{1.6}$$

for some positive constants m, M and functions $g_1, g_2\colon [0, +\infty) \to (0, +\infty)$ related by

$$g_2(\sqrt{n}\, t) \cdot (1+t^2) \leqslant \text{const} \left\{ \int_0^t (g_1(s))^{1/2}\, ds \right\}^{2^*}, \qquad \forall t \geqslant 1, \tag{1.7}$$

where 2^* is the Sobolev exponent ($2^*=2n/(n-2)$ if $n>2$ and 2^* is any real number greater than 2 if $n=2$). Of course, we could scale the functions g_1 and g_2 and take $m=M=1$; in the previous form we have a simpler notation later, for example, when we consider in particular $g_1=g_2$. The factor $\sqrt{n}$ in the argument of g_2 is unnecessary if g_2 satisfies the so called Δ_2 *condition* ($g_2(2t) \leqslant \text{const} \cdot g_2(t)$, $\forall t \geqslant 0$; see [19]), but it is relevant for variational problems with exponential growth, like in (1.4); in this paper we do not assume the Δ_2 condition.

It is easy to see that the functional F_2 in (1.2) satisfies (1.6) and (1.7), with $g_1(t)=(1+t^2)^{(p-2)/2}$ and $g_2(t)=(1+t^2)^{(q-2)/2}$, if and only if $q/p \leqslant 2^*/2$. Note that this restriction on the ratio q/p is satisfied in any dimension in the case of the functional F_3 in (1.3); note also that $q/p \leqslant 2^*/2$ is not a restriction if $n=2$. Of course, integrals of the calculus of variations with natural growth conditions (i.e., $q=p$) satisfy (1.6), (1.7) in any dimension.

In this paper we show that (1.6), (1.7) is sufficient (roughly speaking; precise statements can be found in the next sections) for local regularity (local Lipschitz continuity and then, as a consequence, $C^{1,\alpha}_{\mathrm{loc}}$ and C^{∞} regularity) of weak solutions to the differential equation (1.5).

In particular, we succeed in proving local Lipschitz continuity and $C^{1,\alpha}_{\mathrm{loc}}$ regularity of minimizers of integral-functionals with exponential "slow" growth; precisely, we obtain regularity of minimizers of integrals of the type F_4 in (1.4), of growth $\exp|\xi|^{\alpha}$, by assuming that $\alpha < 2\log[n/(n-2)]/\log n$ (in particular, no restrictions on α if $n=2$).

Isotropic and p, q-growth with $q/p < 2^*/2$ are covered by the regularity results of this paper. A class of uniformly elliptic equations satisfying (1.6), with

$$g_1(t) = g_2(t) \overset{\text{def}}{=} g(t) \qquad \text{and} \qquad 0 \leqslant g'(t) \leqslant \text{const}\,\frac{g(t)}{t}, \ \forall t > 0, \qquad (1.8)$$

enters in our regularity theory (in fact a more general class of problems is considered in Section 7); a non-trivial example of this type is (see [11, 36]; see also the example in the same spirit in [25])

$$F_5(v) = \int_{\Omega} \Phi(|Dv|)\,dx, \qquad \text{where} \quad \Phi(t) = t^{a + b\sin(\log\log t)}, \ \forall t \geqslant e. \qquad (1.9)$$

If $a, b \in \mathbb{R}^+$ and $a > 1 + b\sqrt{2}$, then the function $\Phi(t)$ is convex (for $t \geqslant e$ and, of course, it can be extended to the interval $[0, e]$ with the desired properties), F_5 satisfies the ellipticity conditions (1.6), (1.8) and thus the regularity results of Section 7 apply to the minimizers of this integral. Note that F_5 satisfies a p, q-growth condition with ratio q/p arbitrarily large in dependence of a and b. More general situations can also be treated by our method (see the functionals F_6 and F_7 in (7.4)).

The regularity theory for elliptic equations and for integrals of the calculus of variations with non-standard growth conditions was first considered and studied in [29, 30]. Examples of singular weak solutions were given in [13, 28] (see also [16, 30]); of course, the non-standard growth in these examples violates (1.6), (1.7). A condition, sharper than (1.6), (1.7), was introduced in [4] to study the special anisotropic case of the type of F_1; further results for anisotropic functionals and equations were given in [12, 16, 18, 35] and in [1], where, for the first time, partial regularity for elliptic anisotropic systems was considered. A number of other papers deal with non-standard growth conditions; for example [2, 3, 7, 22–24, 32, 37].

Regularity of weak solutions under natural growth conditions has an older story; a main reference is the book [20]. More recent Lipschitz continuity results can be found in [6, 9, 10, 27, 38]. The uniformly elliptic case (1.6), (1.8) was considered in Section 4 of [34] and recently in [25].

Gradient estimates for classical solutions of non-uniformly elliptic equations were given in [17, 21, 33, 34], with particular interest in the mean curvature equation and in its generalizations. The paper [34], by Simon, especially contains some results that can be compared with ours, with two main differences: first, Simon assumes that the solutions are smooth and then he proves gradient bounds, while here we prove that the weak solutions are smooth in the process of proving the gradients bounds. Secondly, Simon's assumptions are more general when the modulus of ellipticity $g_1(|\xi|)$ in (1.6) goes to zero as $|\xi| \to +\infty$ (like in the mean curvature equation), while we assume that $g_1(|\xi|)$ is an increasing function of $|\xi|$; on the contrary, our assumptions are more general when $g_1(|\xi|) \to +\infty$ as $|\xi| \to +\infty$, to allow us to consider, for example, exponential growth or behavior of the type of F_6 or F_7 in (7.4), which do not enter in Simon's theory.

Finally let us mention a recent regularity result in [26], that is related to a particular functional with exponential growth, precisely to the integral over Ω of $\exp(|Du|^2)$. To obtain the regularity of the minimizers, the author used this nice trick: due to the peculiar properties of the considered integral, the Euler's equation in classical from simply reduced to $\Delta u + 2\sum u_{x_i} u_{x_j} u_{x_i x_j} = 0$ without exponential factors. Then he applied to the classical solutions of this equation the a priori uniform local bound for the gradient, as given in Part 2 of Section 5 of [33], or in [34].

This paper is a first approach to the regularity of weak solutions of P.D.E. and of minimizers of integrals of the C.o.V. under conditions general enough to include, for example, the uniformly elliptic case, the case of p, q-growth, and the exponential growth. We emphasize that we give a first approach; in fact, many interesting problems still remain to be solved; for example: (a) find a sharp condition on the relation between the maximum and the minimum eigenvalue of the matrix $(a^i_{\xi_j})$; in particular, is the presence of the term $\sqrt{n}$ in the main assumption (1.7) really necessary for Lipschitz regularity? (b) The L^∞ bound of the gradient Du is obtained in Section 2 in dependence of the L^1-norm of $g_2(|Du|) \cdot (1 + |Du|^2)$; can we obtain similar results under weaker summability conditions, for example, with $g_1(|Du|) \cdot (1 + |Du|^2) \in L^1_{\text{loc}}$? Note that in Section 4 we propose a first answer to this question. (c) The examples of singular minimers in [13, 16, 28, 30], are related to equations and integrals independent of x and u; the solutions are discontinuous on a line, so that they are discontinuous at the boundary too. By introducing in the equations the dependence on x and u too, is it possible to find examples of solutions that are singular only on isolated interior points?

The Introduction and the references were revised after the referee's report, mainly as to what concerns the studies on gradient estimates for non-uniformly elliptic equations done in the 1970s. The author thanks the referee for his useful advice.

2. Local Lipschitz Continuity

In this section we consider the equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x, u, Du) = b(x, u, Du), \qquad x \in \Omega, \tag{2.1}$$

and we assume that $a^i(x, s, \xi)$, for $i = 1, 2, ..., n$, are locally Lipschitz continuous functions in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ (Ω is an open subset of $\mathbb{R}^n$, for some $n \geqslant 2$).

We consider functions $g_1, g_2\colon [0, +\infty) \to (0, +\infty)$ sastisfying, for some constants $c > 0$, the conditions:

(i) $g_1(t)$, $g_2(t)$ are increasing, strictly positive and $g_1 \leqslant g_2$ in $[0, +\infty)$;

(ii) the function $G(t) = g_2(t) \cdot (1 + t^2)$ is convex in $[0, +\infty)$;

(iii) $$g_2(\sqrt{n}\, t) \cdot (1 + t^2) \leqslant c \left\{ \int_0^t (g_1(s))^{1/2}\, ds \right\}^{2^*}, \forall t \geqslant 1; \tag{2.2}$$

where $2^* = 2n/(n-2)$ if $n > 2$, while, if $n = 2$, then 2^* is any fixed real number greater than 2.

Remark 2.1. Property (2.2)(i) can be related to the fact that in the special case where, for example, $g_1(t)$ is the second derivative of the function $t \to (1 + t^2)^{p/2}$, then it corresponds to the condition $p \geqslant 2$. The convexity (and, as a consequence of (i), the monotonicity) of the function $G(t)$ in (ii) is related to the functional sets defined in (2.8), (2.9); the regularity theory we propose does not change if we assume that $G(t)$ is convex only for $t \geqslant t_0$, for some t_0. Finally, property (iii) was discussed in the Introduction (see also Section 7).

Remark 2.2. Since $g_2\colon [0, +\infty) \to (0, +\infty)$ is positive and increasing, then $g_2(t) \geqslant g_2(0) > 0$, for all $t \in [0, +\infty)$, and thus

$$1 + g_2(t) \leqslant \frac{g_2(0) + 1}{g_2(0)} \cdot g_2(t), \qquad \forall t \in [0, +\infty). \tag{2.3}$$

Similarly $1 + g_1(t) \leqslant \text{const} \cdot g_1(t)$, $\forall t \in [0, +\infty)$.

About the derivatives of the coefficients a^i with respect to ξ, we assume that, for every $s_0 > 0$, there are positive constants m, M such that, for every $\xi, \lambda \in \mathbb{R}^n$, for a.e. $x \in \Omega$ and for every $s \in [-s_0, s_0]$,

$$\sum_{i,j} a^i_{\xi_j}(x, s, \xi)\, \lambda_i \lambda_j \geqslant m g_1(|\xi|)\, |\lambda|^2; \tag{2.4}$$

$$\begin{gathered} |a^i_{\xi_j}(x, s, \xi)| \leqslant M g_2(|\xi|), \\ |a^i_{\xi_j}(x, s, \xi) - a^j_{\xi_i}(x, s, \xi)| \leqslant M[g_1(|\xi|) g_2(|\xi|)]^{1/2}, \forall i, j. \end{gathered} \tag{2.5}$$

About the derivatives of the coefficients a^i with respect to x and s we assume that, for every $\xi \in \mathbb{R}^n$, for $|s| \leqslant s_0$, and for a.e. $x \in \Omega$,

$$\begin{gathered} |a^i_{x_k}(x, s, \xi)| \leqslant M(1 + |\xi|)[g_1(|\xi|)\, g_2(|\xi|)]^{1/2}, \\ |a^i_s(x, s, \xi)| \leqslant M[g_1(|\xi|)\, g_2(|\xi|)]^{1/2}, \forall i, k. \end{gathered} \tag{2.6}$$

About the right hand side $b(x, s, \xi)$ we assume that there exist a bounded Carathéodory function α and a locally Lipschitz continuous function β such that, for every $\xi \in \mathbb{R}^n$, for $|s| \leqslant s_0$ and for a.e. $x \in \Omega$,

$$b(x, s, \xi) = \alpha(x, s, \xi) + \beta(x, s, \xi) \qquad \text{with} \quad |\alpha(x, s, \xi)| \leqslant M$$

and

$$\begin{gathered} |\beta_{x_k}(x, s, \xi)| \leqslant M(1 + |\xi|)\, g_2(|\xi|), \forall k; \qquad |\beta_s(x, s, \xi)| \leqslant M g_2(|\xi|); \\ |\beta_{\xi_j}(x, s, \xi)| \leqslant M[g_1(|\xi|)\, g_2(|\xi|)]^{1/2}, \quad \forall j. \end{gathered} \tag{2.7}$$

Related to the function $G(t) = g_2(t) \cdot (1 + t^2)$ we define the functional sets

$$W_0^{1,G}(\Omega) = \left\{ v \in W_0^{1,2}(\Omega) : \int_\Omega G(|Dv|)\, dx < +\infty \right\}; \tag{2.8}$$

$$W_{\text{loc}}^{1,G}(\Omega) = \left\{ v \in W_{\text{loc}}^{1,2}(\Omega) : \int_{\Omega'} G(|Dv|)\, dx < +\infty, \forall \Omega' \subset\subset \Omega \right\}. \tag{2.9}$$

Under the previous assumptions, by a *weak solution of class* $W_{\text{loc}}^{1,G}(\Omega)$ *to the equation* (2.1) we mean a function $u \in W_{\text{loc}}^{1,G}(\Omega)$ such that, for every $\Omega' \subset\subset \Omega$,

$$\int_\Omega \left\{ \sum_{i=1}^n a^i(x, u, Du)\, \phi_{x_i} + b(x, u, Du)\phi \right\} dx = 0, \qquad \forall \phi \in W_0^{1,G}(\Omega'). \tag{2.10}$$

The heuristic reason of such a definition relies on the fact that Euler's equation of integral-functionals of the calculus of variations has this form when a minimizer u belongs to $W^{1,G,\sigma}_{\mathrm{loc}}(\Omega)$ for some $\sigma > 1$ (see Section 5). Here, for the regularity theory, we will utilize only the weaker assumption $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$.

In this section we will prove the following:

THEOREM 2.3. *Let the previous assumptions* (2.2)–(2.7) *hold. Let* $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$ *be a weak locally bounded solution to Eq.* (2.1), *with the property that there exists* $\beta > 0$ *such that the integral on the right hand side of the following estimate* (2.11) *is finite. Then u is of class* $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ *and, for every* ρ, R $(0 < \rho < R \leqslant \rho + 1)$, *there is a constant c such that*

$$\sup\{|Du(x)| : x \in B_\rho\} \leqslant c \left\{ \int_{B_R} g_2(|Du|)(1 + |Du|^2)^{1+\beta/2}\, dx \right\}^{1/\beta}, \tag{2.11}$$

where B_ρ, B_R *are balls compactly contained in* Ω, *of radii respectively* ρ, R *and with the same center.*

THEOREM 2.4. *Let the previous assumptions* (2.2)–(2.7) *hold, with* (2.2)(iii) *replaced by the strongest condition*

$$g_2(\sqrt{n}\, t) \cdot (1 + t^2)^{1+\beta/2} \leqslant c \left\{ \int_0^t (g_1(s))^{1/2}\, ds \right\}^{2^*}, \qquad \forall t \geqslant 1, \tag{2.12}$$

for some $\beta > 0$. *Then every weak locally bounded solution* $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$ *to Eq.* (2.1) *is of class* $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ *and, for every* ρ, R $(0 < \rho < R \leqslant \rho + 1)$, *there is a constant c such that*

$$\sup\{|Du(x)| : x \in B_\rho\} \leqslant c \left\{ \int_{B_R} g_2(|Du|)(1 + |Du|^2)\, dx \right\}^{2^*/(2\beta)}. \tag{2.13}$$

In a standard way, for example, as in [20, Sect. 6 of Chap. 4] or as in [15, Sect. 8 of Chap. V] (see also Theorem D of [29]), by the positivity of g_1 (see Remark 2.2), from Theorems 2.3 and 2.4 we deduce the following:

COROLLARY 2.5. *Let either the assumptions of Theorem* 2.3 *or the assumptions of Theorem* 2.4 *hold. Let us assume also that for* $i = 1, 2, \ldots, n$, $a^i \in C^{k,\alpha}_{\mathrm{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ *and* $b \in C^{k-1,\alpha}_{\mathrm{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ *for some* $k \geqslant 1$. *If* $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$ *is a weak locally bounded solution to Eq.* (2.1), *then* $u \in C^{k+1,\alpha}_{\mathrm{loc}}(\Omega)$.

Remark 2.6. In the previous results 2.3, 2.4, 2.5 we have considered weak *locally bounded* solutions. The *a priori* boundedness is used in the proofs only to consider the coefficients $a^i(x, s, \xi)$ and $b(x, s, \xi)$ in the assumptions (2.4), (2.5), (2.6), (2.7) for $|s| \leqslant s_0$. If these coefficients do not depend explicitly on s (or, more generally, if the constants m, M in the assumptions are independent of $s \in \mathbb{R}$), then the conclusions of Theorems 2.3, 2.4, and 2.5 hold regardless of the *a priori* boundedness of the weak solutions.

The rest of this section is devoted to the proof of Theorems 2.3 and 2.4. To this aim we consider a real function $\Phi = \Phi(t)$, defined for $t \in \mathbb{R}$, satisfying the properties

$$\begin{aligned} &\Phi \in C^1(\mathbb{R}); \qquad \Phi' \text{ nonnegative and bounded in } \mathbb{R}; \\ &\Phi \text{ convex in } [0, +\infty); \qquad \Phi(-t) = -\Phi(t),\ \forall t \in \mathbb{R}. \end{aligned} \tag{2.14}$$

Since $\Phi(0) = 0$, by its convexity ($\Phi(s) \geqslant \Phi(t) + \Phi'(t)(s-t)$, with $s = 0$ and $t \geqslant 0$) it follows that Φ satisfies also the property

$$|\Phi(t)| \leqslant \Phi'(t) \cdot |t|, \qquad \forall t \in \mathbb{R}. \tag{2.15}$$

Fixed $k \in \{1, 2, ..., n\}$ we denote by e_k the unit coordinate vector in the x_k direction and we define the difference quotient Δ_h in the direction e_k (we do not denote explicitly the dependence on k) by $\Delta_h v(x) = [v(x + he_k) - v(x)]/h$. If v is defined a.e. in Ω, then the function $\Delta_h v$ is defined a.e. in $\Omega_h = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < h\}$. Most of the properties of the difference quotient that we will use in this paper are well known; the interested reader can see, for example, [5] (see also Lemma 2.7 of [30]) and Section 3 of this paper.

Let $\Omega' \subset\subset \Omega$. Let η be a function of class $C_0^1(\Omega')$ such that $0 \leqslant \eta \leqslant 1$. If h is sufficiently small it is well defined in Ω the function

$$\phi = \varepsilon \Delta_{-h}(\eta^2 \Phi(\Delta_h u)), \tag{2.16}$$

for $\varepsilon \in \mathbb{R}$. Let us prove that, for every fixed h, there exists $\varepsilon > 0$ such that $\phi \in W^{1,G}_{\mathrm{loc}}(\Omega)$; to this aim we use the notations and the results of Lemma 3.3: if $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$ then $\Delta_h u$, that is a linear combination of the two functions $x \to u(x)$, $x \to u(x + he_k)$, belongs to $W^{1,G,\tau_1}_{\mathrm{loc}}(\Omega)$ for some τ_1, by property (i) of Lemma 3.3. Then $\Phi(\Delta_h u) \in W^{1,G,\tau_2}_{\mathrm{loc}}(\Omega)$, by (iii), $\eta^2\Phi(\Delta_h u) \in W^{1,G,\tau_3}_{\mathrm{loc}}(\Omega)$ by (ii), and $\Delta_{-h}(\eta^2\Phi(\Delta_h u)) \in W^{1,G,\tau_4}_{\mathrm{loc}}(\Omega)$ by (i). Thus $\phi \in W^{1,G}_{\mathrm{loc}}(\Omega)$ for $\varepsilon \leqslant 1/\tau_4$. Finally $\phi \in W^{1,G}_0(\Omega'')$ for some open set $\Omega'' \subset\subset \Omega$, if h is sufficiently small.

We note here that we will use the conditions (2.4), (2.5), (2.6), (2.7) on the derivatives of $a^i(x, s, \xi)$ and on $b(x, s, \xi)$, with s_0 equal to the essential supremum of u on Ω''.

By using ϕ as a test function in the weak form (2.10) of our equation, with simple computations we obtain (note that we have simplified both sides of the equation by the factor ε)

$$\int_\Omega \sum_{i=1}^n \Delta_h a^i(x, u, Du)(\eta^2\Phi' \Delta_h u_{x_i} + 2\eta\eta_{x_i}\Phi)\, dx = \int_\Omega b(x, u, Du)\, \Delta_{-h}(\eta^2\Phi)\, dx. \tag{2.17}$$

Let us compute $\Delta_h a^i(x, u, Du)$:

$$\begin{aligned}\Delta_h a^i(x, u, Du) &= \frac{1}{h}\int_0^1 \frac{d}{dt} a^i(x + the_k, u + th\,\Delta_h u, Du + th\,\Delta_h Du)\, dt \\ &= \int_0^1 \left(a^i_{x_k} + a^i_s\, \Delta_h u + \sum_{j=1}^n a^i_{\xi_j}\, \Delta_h u_{x_j}\right) dt. \end{aligned}\tag{2.18}$$

From (2.17) we deduce that

$$\int_\Omega \int_0^1 \eta^2\Phi' \sum_{i,j} a^i_{\xi_j}\, \Delta_h u_{x_i}\, \Delta_h u_{x_j}\, dx\, dt \tag{2.19}$$

$$= -\int_\Omega \int_0^1 \eta^2\Phi' \sum_{i=1}^n (a^i_{x_k} + a^i_s\, \Delta_h u)\, \Delta_h u_{x_i}\, dx\, dt \tag{2.20}$$

$$-\int_\Omega \int_0^1 2\eta\Phi \sum_{i=1}^n \left(a^i_{x_k} + a^i_s\, \Delta_h u + \sum_{j=1}^n a^i_{\xi_j}\, \Delta_h u_{x_j}\right) \eta_{x_i}\, dx\, dt \tag{2.21}$$

$$+\int_\Omega b(x, u, Du)\, \Delta_{-h}(\eta^2\Phi)\, dx. \tag{2.22}$$

Let us estimate separately the terms in the right hand side. Let us start with (2.20); by the assumption (2.6) and by the inequality $|ab| \leqslant \varepsilon a^2 + b^2/(4\varepsilon)$, valid for every $a, b \in \mathbb{R}$ and every $\varepsilon > 0$, we have

$$\begin{aligned}&\left|\int_\Omega \int_0^1 \eta^2\Phi' \sum_{i=1}^n (a^i_{x_k} + a^i_s\, \Delta_h u)\, \Delta_h u_{x_i}\, dx\, dt\right| \\ &\quad \leqslant M\sqrt{n} \int_\Omega \int_0^1 \eta^2\Phi'(g_1 \cdot g_2)^{1/2}\, (1 + |Du + th\Delta_h\, Du| + |\Delta_h u|) \\ &\quad \times |\Delta_h\, Du|\, dx\, dt \\ &\quad \leqslant \varepsilon M\sqrt{n} \int_\Omega \int_0^1 \eta^2\Phi' \cdot g_1 \cdot |\Delta_h\, Du|^2\, dx\, dt \\ &\quad + \frac{M\sqrt{n}}{4\varepsilon} \int_\Omega \int_0^1 \eta^2\Phi' \cdot g_2 \cdot (1 + |Du + th\Delta_h\, Du| + |\Delta_h u|)^2\, dx\, dt. \end{aligned}\tag{2.23}$$

About the term (2.21), we have the following estimates (2.24) and (2.25), the first of them being a consequence of the assumption (2.6) and of the property of Φ in (2.15) ($|\Phi(t)| \leqslant \Phi'(t) \cdot |t|$):

$$\left| \int_\Omega \int_0^1 \eta \Phi \sum_{i=1}^n (a^i_{x_k} + a^i_s \Delta_h u)\, \eta_{x_i}\, dx\, dt \right|$$

$$\leqslant \sqrt{n}\, M \int_\Omega \int_0^1 \eta\, |D\eta|\, |\Phi|\, (g_1 \cdot g_2)^{1/2}\, (1 + |Du + th\Delta_h\, Du| + |\Delta_h u|)\, dx\, dt$$

$$\leqslant \sqrt{n}\, M \int_\Omega \int_0^1 \eta\, |D\eta|\, \Phi' g_2 \cdot (1 + |Du + th\Delta_h\, Du| + |\Delta_h u|)^2\, dx\, dt. \quad (2.24)$$

By using again the property of Φ in (2.15) and by Lemma 3.5 we have also

$$\left| \int_\Omega \int_0^1 \eta \Phi \sum_{i,j} a^i_{\xi_j} \Delta_h u_{x_j} \eta_{x_i}\, dx\, dt \right|$$

$$\leqslant \int_\Omega \int_0^1 \eta \Phi'\, |\Delta_h u| \left| \sum_{i,j} a^i_{\xi_j} \Delta_h u_{x_j} \eta_{x_i} \right| dx\, dt$$

$$\leqslant c_1 \int_\Omega \int_0^1 \left(\eta^2 \Phi' \sum_{i,j} a^i_{\xi_j} \Delta_h u_{x_i} \Delta_h u_{x_j} \right)^{1/2}$$

$$\cdot (\Phi' g_2 \cdot |\Delta_h u|^2 \cdot |D\eta|^2)^{1/2}\, dx\, dt$$

$$\leqslant \varepsilon c_1 \int_\Omega \int_0^1 \eta^2 \Phi' \sum_{i,j} a^i_{\xi_j} \Delta_h u_{x_i} \Delta_h u_{x_j}\, dx\, dt$$

$$+ \frac{c_1}{4\varepsilon} \int_\Omega \int_0^1 |D\eta|^2 \cdot \Phi' g_2 \cdot |\Delta_h u|^2\, dx\, dt. \quad (2.25)$$

To estimate the term (2.22) we recall that $b = \alpha + \beta$. We obtain the following inequalities (2.26) and (2.27), the first being a consequence of (2.7) and of Lemma 3.1, with $\Omega' = \mathrm{supp}(\Delta_{-h}(\eta^2 \Phi))$

$$\left|\int_\Omega \alpha(x, u, Du)\, \Delta_{-h}(\eta^2 \Phi)\, dx\right|$$

$$\leqslant M \int_\Omega |\Delta_{-h}(\eta^2\Phi)|\, dx \leqslant M \int_\Omega \left|\frac{\partial}{\partial x_k}(\eta^2\Phi)\right| dx$$

$$\leqslant M \int_\Omega (2\eta\, |\eta_{x_k}|\, |\Phi| + \eta^2 \Phi'\, |\Delta_h u_{x_k}|)\, dx$$

$$\leqslant M \left\{ \int_\Omega \eta\, |D_\eta|\, \Phi' \cdot (1 + |\Delta_h u|^2)\, dx \right.$$

$$\left. + \varepsilon \int_\Omega \eta^2 \Phi'\, |\Delta_h u_{x_k}|^2\, dx + \frac{1}{4\varepsilon} \int_\Omega \eta^2 \Phi'\, dx \right\}, \tag{2.26}$$

the second, derived similarly to (2.24), (2.25), being a consequence of (2.7) and (2.15)

$$\left|\int_\Omega \beta(x, u, Du)\, \Delta_{-h}(\eta^2\Phi)\, dx\right|$$

$$= \left| \int_\Omega \int_0^1 \eta^2 \Phi \left(\beta_{x_k} + \beta_s\, \Delta_h u + \sum_{j=1}^n \beta_{\xi_j}\, \Delta_h u_{x_j} \right) dx\, dt \right|$$

$$\leqslant \int_\Omega \int_0^1 \eta^2\, |\Phi|\, (|\beta_{x_k}| + |\beta_s| \cdot |\Delta_h u|)\, dx\, dt$$

$$+ \int_\Omega \int_0^1 \eta^2\, |\Phi| \left(\sum_{j=1}^n |\beta_{\xi_j}| \cdot |\Delta_h u_{x_j}| \right) dx\, dt$$

$$\leqslant M \int_\Omega \int_0^1 \eta^2 \Phi' g_2 \cdot (1 + |Du + th\Delta_h\, Du| + |\Delta_h u|)^2\, dx\, dt$$

$$+ \sqrt{n}\, M \int_\Omega \int_0^1 \eta^2 \Phi'\, |\Delta_h u|\, (g_1 g_2)^{1/2} \cdot |\Delta_h\, Du|\, dx\, dt$$

$$\leqslant M \int_\Omega \int_0^1 \eta^2 \Phi' \cdot g_2 \cdot (1 + |Du + th\Delta_h\, Du| + |\Delta_h u|)^2\, dx\, dt$$

$$+ \varepsilon \sqrt{n}\, M \int_\Omega \int_0^1 \eta^2 \Phi' \cdot g_1 \cdot |\Delta_h\, Du|^2\, dx\, dt$$

$$+ \frac{\sqrt{n}\, M}{4\varepsilon} \int_\Omega \int_0^1 \eta^2 \Phi' \cdot g_2 \cdot |\Delta_h u|^2\, dx\, dt. \tag{2.27}$$

Finally, to estimate the left hand side of (2.19) we use the ellipiticity assumption (2.4):

$$\int_{\Omega}\int_0^1 \eta^2\Phi' \sum_{i,j} a^i_{\xi_j}\, \Delta_h u_{x_i}\, \Delta_h u_{x_j}\, dx\, dt$$

$$\geqslant m \int_{\Omega}\int_0^1 \eta^2\Phi' \cdot g_1 \cdot |\Delta_h Du|^2\, dx\, dt. \tag{2.28}$$

By the relations from (2.19) to (2.28), by choosing ε sufficiently small, we deduce that there is a positive constant c_2 such that the following estimate holds (note in particular that the last integral in (2.26) goes on the right hand side of (2.29), since g_2 is strictly positive):

$$\int_{\Omega}\int_0^1 \eta^2\Phi' g_1 \cdot |\Delta_h Du|^2 \quad dx\, dt$$

$$\leqslant c_2 \int_{\Omega}\int_0^1 (\eta^2 + |D\eta|^2)\, \Phi' \cdot g_2$$

$$\cdot (1 + |Du + th\Delta_h Du|^2 + |\Delta_h u|^2)\, dx\, dt. \tag{2.29}$$

We recall that, originated by (2.18), the argument of the functions g_1 and g_2 is $|Du + th\Delta_h Du|$, i.e., $g_1 = g_1(|Du + th\Delta_h Du|)$, $g_2 = g_2(|Du + th\Delta_h Du|)$, while (see (2.16)) the argument of Φ' is $\Delta_h u$, i.e., $\Phi' = \Phi'(\Delta_h u) = \Phi'(|\Delta_h u|)$ (in fact Φ' is even).

Since $g_1 : [0, +\infty) \to (0, +\infty)$ is strictly positive and increasing, there is a positive constant c_3 such that $g_1(t) \geqslant c_3$, $\forall t \in [0, +\infty)$ (in fact $c_3 = g_1(0)$). Now, let us first consider the case $\Phi(t) = t$; if in the previous estimate (2.29) we take $\eta = 1$ on an open subset $\Omega' \subset\subset \Omega$. with $\operatorname{supp}(\eta) \subset\subset \Omega'' \subset\subset \Omega$, we deduce that

$$\int_{\Omega'} |\Delta_h Du|^2\, dx\, dt \leqslant c_4 \int_{\operatorname{supp}(\eta)} \int_0^1 g_2(|Du + th\Delta_h Du|)$$

$$\cdot (1 + |Du + th\Delta_h Du|^2 + |\Delta_h u|^2)\, dx\, dt. \tag{2.30}$$

By Lemma 3.4(i) and by Lemma 3.1 we obtain

$$\int_{\Omega'} |\Delta_h Du|^2\, dx \leqslant 2c_4 \int_{\operatorname{supp}(\eta)} \int_0^1 g_2(|Du + th\Delta_h Du|)$$

$$\cdot (1 + |Du + th\Delta_h Du|^2)\, dx\, dt$$

$$+ c_4 \int_{\Omega''} g_2(|u_{x_k}|) \cdot (1 + |u_{x_k}|)^2\, dx. \tag{2.31}$$

Since, for $t \in [0, 1]$, we have

$$|Du + th\Delta_h Du| = |(1-t)\, Du(x) + t\, Du(x + he_k)| \leqslant (1-t)\, |Du(x)| + t\, |Du(x + he_k)|, \tag{2.32}$$

then, by the convexity of $G(t) = g_2(t) \cdot (1 + t^2)$, with the change of variable $x' = x + he_k$, we obtain

$$\int_{\Omega'} |\Delta_h Du|^2\, dx \leqslant 3c_4 \int_{\Omega''} g_2(|Du|) \cdot (1 + |Du|^2)\, dx. \tag{2.33}$$

By assumption, since $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$, then the right hand side is finite; thus the left hand side is bounded by a constant independent of h. By a well known property of the difference quotient we obtain that Du has first partial derivatives in $L^2(\Omega')$, i.e., $u \in W^{2,2}(\Omega')$.

Now we go to the limit as $h \to 0$ in (2.29) (we recall that $\Phi = \Phi(|\Delta_h u|)$ and that $g = g(|Du + th\Delta_h Du|)$). First we observe that since Φ is Lipschitz continuous, the right hand side can be estimated, similarly to (2.33) for the case $\Phi(t) = t$, by a quantity independent of h. Thus we can go to the limit as $h \to 0$ in the right hand side by Lebesgue's Theorem, since $\Delta_h u$ converges almost everywhere to $\partial u/\partial x_k$ and u is continuous a.e.; we go to the limit in the left hand side too, since the integral is lower semicontinuous. We obtain

$$\int_\Omega \eta^2 \Phi'(|u_{x_k}|) \cdot g_1(|Du|) \cdot |Du_{x_k}|^2\, dx \leqslant 2c_2 \int_\Omega (\eta^2 + |D\eta|^2)\, \Phi'(|u_{x_k}|) \cdot g_2(|Du|) \cdot (1 + |Du|^2)\, dx. \tag{2.34}$$

Now we rewrite (2.34) with Φ' identically equal to 1 and we sum up with (2.34):

$$\int_\Omega \eta^2 (1 + \Phi'(|u_{x_k}|)) \cdot g_1(|Du|) \cdot |Du_{x_k}|^2\, dx \leqslant 2c_2 \int_\Omega (\eta^2 + |D\eta|^2)(1 + \Phi'(|u_{x_k}|)) \cdot g_2(|Du|) \cdot (1 + |Du|^2)\, dx. \tag{2.35}$$

We recall that, like in (2.14), Φ is an odd function of class $C^1(\mathbb{R})$, convex for positive values of its argument, whose derivative Φ' is nonnegative and bounded in $\mathbb{R}$. If we consider a more general Φ, with derivative Φ' not bounded, we can approximate Φ by a sequence of functions Φ_r, each of them being equal to Φ in the interval $[-r, r]$, and then extended linearly in $\mathbb{R}$ as a function of class $C^1(\mathbb{R})$. We insert Φ_r in (2.35) and we go the limit as $r \to +\infty$ by the monotone convergence theorem. We obtain the validity of (2.35) for every Φ not necessarily with bounded derivative, i.e., $\Phi \in C^1([0, +\infty))$, $\Phi(0) = 0$, and

$$\Phi' \text{ nonnegative an increasing in } [0, +\infty). \tag{2.36}$$

Let us define

$$G(t) = 1 + \int_0^t \{(1 + \Phi'(s)) \cdot g_1(s)\}^{1/2} \, ds; \tag{2.37}$$

then we have

$$|D(\eta G(|u_{x_k}|))|^2 \leqslant 2\eta^2(1 + \Phi'(|u_{x_k}|)) \, g_1(|u_{x_k}|) \, |Du_{x_k}|^2 + 2 \, |D\eta|^2 \, [G(|u_{x_k}|)]^2. \tag{2.38}$$

Since g_1 and Φ' are increasing and since g_2 $(\geqslant g_1)$ is positive, there exists a constant c_5 such that

$$[G(t)]^2 \leqslant [1 + \{(1 + \Phi'(t)) \cdot g_1(t)\}^{1/2} \cdot t]^2 \leqslant c_5(1 + \Phi'(t)) \cdot g_2(t) \cdot (1 + t^2). \tag{2.39}$$

We combine (2.38) and (2.39) with (2.35); thus, by Sobolev's inequality, there exist constants c_6 and c_7 such that

$$\begin{aligned} &\left\{ \int_\Omega [\eta G(|u_{x_k}|)]^{2^*} \, dx \right\}^{2/2^*} \\ &\qquad \leqslant c_6 \int_\Omega |D(\eta G(|u_{x_k}|))|^2 \, dx \\ &\qquad \leqslant c_7 \int_\Omega (\eta^2 + |D\eta|^2)(1 + \Phi'(|u_{x_k}|)) \cdot g_2(|Du|)(1 + |Du|^2) \, dx, \end{aligned} \tag{2.40}$$

where $2^* = 2n/(n-2)$ if $n > 2$, while 2^* is any fixed real number greater than 2, if $n = 2$.

Now we sum up with respect to $k = 1, 2, ..., n$; we use the inequality $\sum t_k^p \leqslant (\sum t_k)^p$ with $p = 2^*/2$ and Minkowski's inequality, again with exponent $2^*/2$:

$$
\begin{aligned}
&\left\{\int_\Omega \sum_{k=1}^{n} [\eta G(|u_{x_k}|)]^{2^*}\,dx\right\}^{2/2^*}\\
&\quad\leqslant \left\{\int_\Omega \left[\sum_{k=1}^{n} (\eta G(|u_{x_k}|))^2\right]^{2^*/2} dx\right\}^{2/2^*}\\
&\quad\leqslant \sum_{k=1}^{n} \left\{\int_\Omega [\eta G(|u_{x_k}|)]^{2^*}\,dx\right\}^{2/2^*}\\
&\quad\leqslant c_7 \sum_{k=1}^{n} \int_\Omega (\eta^2 + |D\eta|^2)(1+\Phi'(|u_{x_k}|))\\
&\quad\quad\cdot g_2(|Du|)(1+|Du|^2)\,dx. \qquad (2.41)
\end{aligned}
$$

Let us consider $\Phi(t)=t^{1+\beta}$ for $t\geqslant 0$, where $\beta>0$ is like in the statement of Theorem 2.3; by using the inequality $\sqrt{2(a+b)}\geqslant\sqrt{a}+\sqrt{b}$, it is easy to see that there is a constant c_8 (independent of β) such that

$$
\begin{aligned}
\frac{1}{t}\int_0^t (1+\Phi'(s))^{1/2}\,ds &\geqslant \frac{1}{t\sqrt{2}}\int_0^t \{1+(\Phi'(s))^{1/2}\}\,ds\\
&\frac{c_8}{\sqrt{1+\beta}}(1+t^{\beta/2}), \qquad \forall t>0. \qquad (2.42)
\end{aligned}
$$

We recall the definition of G in (2.37) with this particular Φ, we use Lemma 3.4(v), assumption (2.2) (iii), and (2.42). For some constant c_9 we obtain

$$
\begin{aligned}
\{G(t)\}^{2^*} &\geqslant \left\{1+\frac{1}{t}\int_0^t (1+\Phi'(s))^{1/2}\,ds\cdot\int_0^t (g_1(s))^{1/2}\,ds\right\}^{2^*}\\
&\geqslant c_9\,\frac{1+t^{\beta(2^*/2)}}{(1+\beta)^{2^*/2}}\cdot g_2(\sqrt{n}\,t)(1+t^2) \qquad (2.43)
\end{aligned}
$$

for every $t\geqslant 1$; moreover, if $0\leqslant t\leqslant 1$, then the right hand side is bounded (with the constants c_9 independent of β), while the left hand side is greater than or equal to 1. Thus (2.43) holds for every $t\geqslant 0$.

Let us denote by B_R and B_ρ balls compactly contained in Ω, of radii respectively R, ρ ($R-\rho\leqslant 1$), with the same center. Let η be a test function equal to 1 in B_ρ, whose support is contained in B_R, such that $|D\eta|\leqslant 2/(R-\rho)$. From (2.41), (2.43), since $\Phi'(t)=(1+\beta)\,t^\beta$, we deduce that there exists a constant c_{10} such that

$$
\begin{aligned}
&\left\{\int_{B_\rho} \sum_{k=1}^{n} (1+|u_{x_k}|^{\beta(2^*/2)})\,g_2(\sqrt{n}\,|u_{x_k}|)(1+|u_{x_k}|^2)\,dx\right\}^{2/2^*}\\
&\quad\leqslant \frac{c_{10}\cdot(1+\beta)^2}{(R-\rho)^2}\sum_{k=1}^{n}\int_{B_R} (1+|u_{x_k}|^\beta)\cdot g_2(|Du|)(1+|Du|^2)\,dx. \qquad (2.44)
\end{aligned}
$$

We first consider Lemma 3.4(iv) for the function $t \to g_2(t) \cdot (1+t^2)$; then we consider Lemma 3.4(ii) for the functions $t \to 1+t^\beta$ and $t \to g_2(\sqrt{n}\, t) \cdot (1+nt^2)$; we obtain

$$\left[\sum_{k=1}^{n} (1+|u_{x_k}|^\beta)\right] \cdot g_2(|Du|)(1+|Du|^2)$$

$$\leqslant \sum_{k=1}^{n} (1+|u_{x_k}|^\beta) \cdot \sum_{k=1}^{n} g_2(\sqrt{n}\, |u_{x_k}|)(1+n\, |u_{x_k}|^2)$$

$$\leqslant n^2 \cdot \sum_{k=1}^{n} (1+|u_{x_k}|^\beta) \cdot g_2(\sqrt{n}\, |u_{x_k}|)(1+|u_{x_k}|^2). \tag{2.45}$$

By (2.44), (2.45) we can say that there exists a constant c_{11} $(=n^2 \cdot c_{10})$ such that

$$\left\{\int_{B_\rho} \sum_{k=1}^{n} (1+|u_{x_k}|^{\beta(2^*/2)}) \cdot g_2(\sqrt{n}\, |u_{x_k}|)(1+|u_{x_k}|^2)\, dx\right\}^{2/2^*}$$

$$\leqslant \frac{c_{11} \cdot (1+\beta)^2}{(R-\rho)^2} \int_{B_R} \sum_{k=1}^{n} (1+|u_{x_k}|^\beta)$$

$$\cdot g_2(\sqrt{n}\, |u_{x_k}|)(1+|u_{x_k}|^2)\, dx. \tag{2.46}$$

We rewrite the previous estimate with β replaced by $\beta \cdot (2^*/2)^{i-1}$, for $i=1, 2, \ldots$, and with $R=R_{i-1}$, $\rho=R_i$, where, for some $R_0 > \rho_0$,

$$R_i = \rho_0 + \frac{R_0-\rho_0}{2^i}, \qquad \forall i \in \mathbb{N}; \tag{2.47}$$

thus $R-\rho = R_{i-1} - R_i = (R_0-\rho_0) \cdot 2^{-i}$. We iterate (2.46) and we use at the first step (2.44); for every $i \geqslant 1$ we obtain

$$\left\{\int_{B_{R_i}} \sum_{k=1}^{n} (1+|u_{x_k}|^{\beta(2^*/2)^i}) \cdot g_2(\sqrt{n}\, |u_{x_k}|)(1+|u_{x_k}|^2)\, dx\right\}^{(2/2^*)^i}$$

$$\leqslant c_{12} \int_{B_{R_0}} \sum_{k=1}^{n} (1+|u_{x_k}|^\beta) \cdot g_2(|Du|)(1+|Du|^2)\, dx, \tag{2.48}$$

where, since $2/2^* < 1$,

$$c_{12} = \prod_{i=1}^{\infty} \left[\frac{c_{11} \cdot (1 + \beta \cdot (2^*/2)^{i-1})^2 \cdot 4^i}{(R_0 - \rho_0)^2}\right]^{(2/2^*)^i}$$
$$= \exp\left(\sum_{i=1}^{\infty} \left(\frac{2}{2^*}\right)^i \log \frac{c_{11} \cdot (1 + \beta \cdot (2^*/2)^{i-1})^2 \cdot 4^i}{(R_0 - \rho_0)^2}\right)$$
$$\leqslant \exp\left(c_{13} \sum_{i=1}^{\infty} i \left(\frac{2}{2^*}\right)^i\right) < +\infty. \tag{2.49}$$

We go to the limit as $i \to +\infty$ in (2.48); since $g_2 \geqslant c_{14} > 0$ (see Remark 2.2), for every $k = 1, 2, ..., n$, we obtain

$$\sup\{|u_{x_k}|^\beta : x \in B_{\rho_0}\}$$
$$= \lim_{i \to \infty} \left\{\int_{B_{\rho_0}} |u_{x_k}|^{\beta(2^*/2)^i}\, dx\right\}^{(2/2^*)^i}$$
$$\leqslant \lim_{i \to \infty} \left\{\frac{1}{c_{14}} \int_{B_{R_i}} (1 + |u_{x_k}|^{\beta(2^*/2)^i}) \cdot g_2(\sqrt{n}\, |u_{x_k}|)(1 + |u_{x_k}|^2)\, dx\right\}^{(2/2^*)^i}$$
$$\leqslant c_{15} \int_{B_{R_0}} \sum_{k=1}^{n} (1 + |u_{x_k}|^\beta) \cdot g_2(|Du|)(1 + |Du|^2)\, dx. \tag{2.50}$$

We sum up for $k = 1, 2, ..., n$; we obtain the estimate (2.11) and the conclusion of the proof of Theorem 2.3.

To prove Theorem 2.4 we consider again (2.41) with $\Phi(t) = t$. Similarly to (2.44) (with the parameter β there considered equal to zero), by using the new assumption (2.12), we obtain

$$\left\{\int_{B_\rho} \sum_{k=1}^{n} g_2(\sqrt{n}\, |u_{x_k}|)(1 + |u_{x_k}|^2)^{1+\beta/2}\, dx\right\}^{2/2^*}$$
$$\leqslant \frac{n \cdot c_{10}}{(R - \rho)^2} \int_{B_R} g_2(|Du|)(1 + |Du|^2)\, dx. \tag{2.51}$$

Then, by Theorem 2.3 and by the fact that $g_2(|Du|) \leqslant \sum_{k=1}^n g_2(\sqrt{n}\, |u_{x_k}|)$ (Lemma 3.4(iv)) we have the conclusion (2.13) with $\rho < R' < R$:

$$\sup\{|Du(x)| : x \in B_\rho\} \leqslant c \left\{\int_{B_{R'}} g_2(|Du|)(1 + |Du|^2)^{1+\beta/2}\, dx\right\}^{1/\beta}$$
$$\leqslant c_{16} \left\{\int_{B_R} g_2(|Du|)(1 + |Du|^2)\, dx\right\}^{2^*/(2\beta)}. \tag{2.52}$$

3. Some Technical Lemmas

In this section we prove some lemmas used in the paper, mainly in the previous section. We begin with some properties of the difference quotient; to this aim we recall that, fixed $k \in \{1, 2, ..., n\}$, we denote by e_k the unit coordinate vector in the x_k direction and we define the difference quotient Δ_h in the direction e_k (we do not denote explicitly the dependence on k) by $\Delta_h v(x) = [v(x+he_k) - v(x)]/h$. If v is a measurable function in Ω, then the function $\Delta_h v$ is defined and measurable in $\Omega_h = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < h\}$.

Lemma 3.1. *Let G: $[0, +\infty) \to [0, +\infty)$ be a convex increasing function. Let v be a measurable function in Ω such that v_{x_k} (its distributional partial derivative with respect to x_k) is a function of class $L^2(\Omega)$ with the property*

$$\int_\Omega G(|v_{x_k}(x)|)\, dx < +\infty; \tag{3.1}$$

then, for every $\Omega' \subset\subset \Omega$, we have

$$\int_{\Omega'} G(|\Delta_h v(x)|)\, dx \leqslant \int_\Omega G(|v_{x_k}(x)|)\, dx, \qquad \forall h < \operatorname{dist}(\Omega', \partial\Omega). \tag{3.2}$$

Proof. We start from the identity

$$\Delta_h v(x) = \frac{v(x+he_k) - v(x)}{h} = \int_0^1 v_{x_k}(x + the_k)\, dt \tag{3.3}$$

and then we use Jensen's inequality for the convex function $t \in \mathbb{R} \to G(|t|)$:

$$G(|\Delta_h v(x)|) = G\left(\left|\int_0^1 v_{x_k}(x+the_k)\, dt\right|\right) \leqslant \int_0^1 G(|v_{x_k}(x+the_k)|)\, dt. \tag{3.4}$$

By integrating over $\Omega' \subset\subset \Omega$, since $(\Omega' + the_k) \subset \Omega$, we obtain the conclusion

$$\begin{aligned}\int_{\Omega'} G(|\Delta_h v(x)|)\, dx &\leqslant \int_{\Omega'} \int_0^1 G(|v_{x_k}(x+the_k)|)\, dx\, dt \\ &= \int_0^1 dt \int_{\Omega' + the_k} G(|v_{x_k}(x')|)\, dx' \leqslant \int_\Omega G(|v_{x_k}(x)|)\, dx. \end{aligned} \tag{3.5}$$

Lemma 3.2. *Let G: $[0, +\infty) \to [0, +\infty)$ be a convex increasing function. Let v be a measurable function in Ω, with compact support in Ω, whose* *partial derivative* v_{x_k} *satisfies* (3.1). *If* Ω *is bounded and we denote by* d *its diameter, then we have the estimate*

$$\int_\Omega G\left(\frac{2}{d}|v(x)|\right)dx \leqslant \int_\Omega G(|v_{x_k}(x)|)\,dx < +\infty. \tag{3.6}$$

Proof. To simplify the notations we assume that $\Omega \subseteq \{(x_1, ..., x_n) : 0 \leqslant x_k \leqslant d\}$. Since v has compact support in Ω, then we have

$$v(x) = \int_0^{x_k} v_{x_k}(x_1, ..., t, ..., x_n)\,dt = \int_d^{x_k} v_{x_k}(x_1, ..., t, ..., x_n)\,dt, \tag{3.7}$$

and thus

$$2\,|v(x)| \leqslant \int_0^d |v_{x_k}(x_1, ..., t, ..., x_n)|\,dt. \tag{3.8}$$

By Jensen's inequality we obtain the conclusion

$$\begin{aligned}\int_\Omega G\left(\frac{2}{d}|v(x)|\right)dx &\leqslant \int_\Omega G\left(\frac{1}{d}\int_0^d |v_{x_k}(x_1, ..., t, ..., x_n)|\,dt\right)dx\\ &\leqslant \frac{1}{d}\int_0^d dx_k \int_\Omega G(|v_{x_k}(x_1, ..., t, ..., x_n)|)\,dx_1 \cdots dt \cdots dx_n\\ &= \int_\Omega G(|v_{x_k}(x)|)\,dx.\end{aligned} \tag{3.9}$$

LEMMA 3.3. *Let* $G\colon [0, +\infty) \to [0, +\infty)$ *be a convex increasing function, let* $\sigma > 0$, *and let us consider the functional set*

$$W^{1,G,\sigma}(\Omega) = \left\{v \in W^{1,1}(\Omega)\colon \int_\Omega G(\sigma\,|Dv|)\,dx < +\infty\right\}. \tag{3.10}$$

Then there exist a positive constant τ *such that* $w \in W^{1,G,\tau}(\Omega)$ *under any one of the following notations and assumptions*:

(i) $v_1, v_2 \in W^{1,G,\sigma}(\Omega)$ *and* w *is a linear combination of* v_1, v_2;

(ii) $v_1 \in W^{1,G,\sigma}(\Omega)$, $v_2 \in W^{1,\infty}(\Omega)$, *and* w *is the product function* $w = v_1 \cdot v_2$;

(iii) $v \in W^{1,G,\sigma}(\Omega)$, Φ *is a Lipschitz continuous function in* $\mathbb{R}$, *and* w *is the composite function* $w = \Phi(v)$.

Proof. (i) If $w=\alpha v_1+\beta v_2$, with $\alpha, \beta \in \mathbb{R}$, then by the convexity of the function $t \to G(\tau(|\alpha|+|\beta|)t)$ we have

$$G(|D(\tau w)|) \leqslant G(\tau(|\alpha|\,|Dv_1|+|\beta|\,|Dv_2|)) \leqslant \frac{|\alpha|\,G(\tau(|\alpha|+|\beta|)\,|Dv_1|)+|\beta|\,G(\tau(|\alpha|+|\beta|)\,|Dv_2|)}{|\alpha|+|\beta|} \tag{3.11}$$

and the conclusion for $\tau=\sigma/(|\alpha|+|\beta|)$.

(ii) Let $L=\|v_2\|_{L^\infty}+\|Dv_2\|_{L^\infty}$. Then $|Dw| \leqslant |Dv_1|\cdot|v_2|+|v_1|\cdot|Dv_2| \leqslant L(|Dv_1|+|v_1|)$ and the conclusion can be obtained similarly to the previous part (i), by using Lemma 3.2.

(iii) The fact that the composite function $w=\Phi(v)$ belongs to $W^{1,1}(\Omega)$ and that the chain rule holds is well known; it is Stampacchia's result and it can be found in [14, Lemma 7.5]. The chain rule establishes that $D(\Phi(v))=\Phi'\cdot Dv$ a.e. in Ω; thus $|Dw|=|\Phi'|\cdot|Dv| \leqslant L\cdot|Dv|$ if $|\Phi'| \leqslant L$. The conclusion follows with $\tau=\sigma/L$.

LEMMA 3.4. *Let $g(t)$, $h(t)$ be two nonnegative, increasing functions on $[0,+\infty)$; then*:

(i) $$g(t_1)\cdot h(t_2) \leqslant g(t_1)\cdot h(t_1)+g(t_2)\cdot h(t_2), \qquad \forall t_1, t_2 \in [0,+\infty);$$

(ii) $$\sum_{i=1}^n g(t_i)\cdot\sum_{i=1}^n h(t_i) \leqslant n\sum_{i=1}^n g(t_i)\cdot h(t_i), \qquad \forall n \in \mathbb{N}, \forall t_1, t_2, ..., t_n \in [0,+\infty);$$

(iii) $$g(t_1+t_2) \leqslant g(\sigma t_1)+g((\sigma/(\sigma-1))\,t_2), \qquad \forall \sigma>1, \forall t_1, t_2 \in [0,+\infty);$$

(iv) $$g\left(\left(\sum_{i=1}^n t_i^p\right)^{1/p}\right) \leqslant \sum_{i=1}^n g(n^{1/p}t_i), \qquad \forall n \in \mathbb{N}, \forall p>0, \forall t_1, t_2, ..., t_n \in [0,+\infty);$$

(v) $$\int_a^b g(t)\,dt\cdot\int_a^b h(t)\,dt \leqslant (b-a)\int_a^b g(t)\cdot h(t)\,dt, \qquad \forall a, b \in [0,+\infty).$$

Proof. (i) If $t_2 \leqslant t_1$ then $g(t_1)\cdot h(t_2) \leqslant g(t_1)\cdot h(t_1) \leqslant g(t_1)\cdot h(t_1)+g(t_2)\cdot h(t_2)$; similarly if $t_2>t_1$.

(ii) By symmetry (with respect to i and j) we have

$$\sum_{i=1}^{n} g(t_i)\cdot\sum_{i=1}^{n} h(t_i)=\sum_{i=1}^{n}\sum_{j=1}^{n} g(t_i)\cdot h(t_j)$$

$$=\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}[g(t_i)\cdot h(t_j)+g(t_j)\cdot h(t_i)]; \tag{3.12}$$

$$n\sum_{i=1}^{n} g(t_i)\cdot h(t_i)=\sum_{i=1}^{n}\sum_{j=1}^{n} g(t_i)\cdot h(t_i)$$

$$=\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}[g(t_i)\cdot h(t_i)+g(t_j)\cdot h(t_j)]. \tag{3.13}$$

By subtracting side to side we obtain

$$n\sum_{i=1}^{n} g(t_i)\cdot h(t_i)-\sum_{i=1}^{n} g(t_i)\cdot\sum_{i=1}^{n} h(t_i)$$

$$=\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}[g(t_i)-g(t_j)]\cdot[h(t_i)-h(t_j)] \tag{3.14}$$

and the right hand side is positive, since $g(t)$, $h(t)$ are both increasing (we have not used here their positivity).

(iii) If $t_2\leqslant(\sigma-1)\,t_1$ then $g(t_1+t_2)\leqslant g(t_1+(\sigma-1)\,t_1)\leqslant g(\sigma t_1)+g((\sigma/(\sigma-1))\,t_2)$; while, if $(\sigma-1)\,t_1<t_2$, then $g(t_1+t_2)\leqslant g((1/(\sigma-1))\,t_2+t_2)\leqslant g(\sigma t_1)+g((\sigma/(\sigma-1))\,t_2)$.

(iv) $$g\left(\left(\sum_{i=1}^{n} t_i^p\right)^{1/p}\right)\leqslant g(n^{1/p}\cdot\max\{t_i\colon i=1,2,\ldots,n\})\leqslant\sum_{i=1}^{n} g(n^{1/p}t_i).$$

(v) This is well known and can be proved with the same method of (ii), by using double integrals.

LEMMA 3.5. *Under the assumptions* (2.4) *and* (2.5) *we have*

$$\left|\sum_{i,j} a^i_{\xi_j}(x,s,\xi)\,\lambda_j\eta_i\right|\leqslant c\left(\sum_{i,j} a^i_{\xi_j}(x,s,\xi)\,\lambda_i\lambda_j\right)^{1/2}[g_2(|\xi|)\,|\eta|^2]^{1/2} \tag{3.15}$$

for some constant c, for every $\xi,\lambda,\eta\in\mathbb{R}^n$, for $|s|\leqslant s_0$ and for a.e. $x\in\Omega$.

Proof. Let us define $b_{ij}=(a^i_{\xi_j}+a^j_{\xi_i})/2$, $c_{ij}=(a^i_{\xi_j}-a^j_{\xi_i})/2$. Since (b_{ij}) is a positive definite symmetric matrix, by the Cauchy–Schwarz inequality, by the fact that $\sum b_{ij}\lambda_i\lambda_j=\sum a^i_{\xi_j}\lambda_i\lambda_j$ and by (2.5) we obtain

$$\begin{aligned}\left|\sum_{i,j} b_{ij}\lambda_j\eta_i\right| &\leqslant \left(\sum_{i,j} b_{ij}\lambda_i\lambda_j\right)^{1/2}\left(\sum_{i,j} b_{ij}\eta_i\eta_j\right)^{1/2}\\ &\leqslant \left(\sum_{i,j} a^i_{\xi_j}\lambda_i\lambda_j\right)^{1/2}[Mg_2(|\xi|)\,|\eta|^2]^{1/2}. \end{aligned}\tag{3.16}$$

Moreover, by (2.4) and (2.5) we have also

$$\begin{aligned}\left|\sum_{i,j} c_{ij}\lambda_j\eta_i\right| &\leqslant \frac{n^2M}{2}(g_1g_2)^{1/2}\,|\lambda|\,|\eta|\\ &\leqslant \frac{n^2M}{2\sqrt{m}}\left(\sum_{i,j} a^i_{\xi_j}\lambda_i\lambda_j\right)^{1/2}[g_2(|\xi|)\,|\eta|^2]^{1/2}. \end{aligned}\tag{3.17}$$

Our result follows from (3.16), (3.17), since $a^i_{\xi_j}=b_{ij}+c_{ij}$.

4. Higher Integrability of the Gradient

We consider again Eq. (2.1) under the assumptions (2.2)–(2.7), with (2.2)(iii) replaced by the condition

$$g_2(t)\cdot(1+t^2)\leqslant c\left\{\int_0^t (g_1(s))^{1/2}\,ds\right\}^{2^*/\gamma},\qquad \forall t\geqslant 1,\tag{4.1}$$

where $\gamma>1$ and $c>0$ and where, as usual, $2^*=2n/(n-2)$ if $n>2$, while 2^* is any fixed real number greater than 2, if $n=2$. In the next theorem we prove the summability of $g_2(|Du|)(1+|Du|^2)$ to a power greater than 1 (where u is a weak solution) in terms of the summability of the same quantity to a power less than 1. An application of this higher summability result is given in Section 6.

Theorem 4.1. *Let the previous assumptions* (2.2)–(2.7) *hold, with* (2.2)(iii) *replaced by* (4.1). *Let $u\in W^{1,G}_{\mathrm{loc}}(\Omega)$ be a weak locally bounded solution to Eq.* (2.1). *Let α be a positive real number smaller than $2/2^*$. Then, for every ρ, R $(0<\rho<R\leqslant\rho+1)$, there is a constant c such that*

$$\begin{aligned}&\int_{B_\rho}[g_2(|Du|)(1+|Du|^2)]^\gamma\,dx\\ &\quad\leqslant c\left\{\int_{B_R}[g_2(|Du|)(1+|Du|^2)]^{(1-\gamma\alpha)/(1-\alpha)}\,dx\right\}^{(1-\alpha)/(2/2^*-\alpha)}.\end{aligned}\tag{4.2}$$

Remark 4.2. In the estimate (4.2) we are not allowed to consider $\alpha = 1/\gamma$. This depends on the limitations $\alpha < 2/2^*$ and

$$2/2^* \leqslant 1/\gamma. \tag{4.3}$$

To prove (4.3) we start from (4.1) and we observe that since g_1 is increasing

$$\begin{aligned}\frac{1}{c} g_2(t) \cdot (1+t^2) &\leqslant \left\{ \int_0^t (g_1(s))^{1/2} \, ds \right\}^{2^*/\gamma} \\ &\leqslant \{(g_1(t))^{1/2} t\}^{2^*/\gamma} \leqslant \{g_1(t) \cdot (1+t^2)\}^{2^*/(2\gamma)}. \end{aligned} \tag{4.4}$$

Moreover, since $g_1 \leqslant g_2$, we obtain

$$\frac{1}{c} \leqslant \{g_1(t) \cdot (1+t^2)\}^{2^*/(2\gamma)-1}, \qquad \forall t \geqslant 1, \tag{4.5}$$

that forces the exponent $2^*/(2\gamma) - 1$ to be greater than or equal to zero.

Proof of Theorem 4.1. We consider again the estimate (2.29) of Section 2, with $\Phi(t) = t$ (i.e., $\Phi'(t) = 1$, which disappear in the estimate):

$$\begin{aligned}&\int_\Omega \int_0^1 \eta^2 g_1 \cdot |\Delta_h Du|^2 \, dx \, dt \\ &\quad \leqslant c_2 \int_\Omega \int_0^1 (\eta^2 + |Dn|^2) \, g_2 \cdot (1 + |Du + th\Delta_h Du|^2 + |\Delta_h u|^2) \, dx \, dt. \end{aligned} \tag{4.6}$$

Let us denote by $G(t)$ the primitive of $(g_1(t))^{1/2}$ in $[0, +\infty)$ such that $G(0) = 1$ and let us compute

$$\begin{aligned}\Delta_h G(|Du|) &= \frac{1}{h} \int_0^1 \frac{d}{dt} G(|Du + th\Delta_h Du|) \, dt \\ &= \int_0^1 G' \sum_{j=1}^n \frac{u_{x_j} + th\, \Delta_h u_{x_j}}{|Du + th\Delta_h Du|} \Delta_h u_{x_j} \, dt; \end{aligned} \tag{4.7}$$

therefore, since $G'(t) = (g_1(t))^{1/2}$, by Jensen's (or, equivalently, by Hölder's) inequality, we obtain

$$|\Delta_h G(|Du|)|^2 \leqslant \left(\int_0^1 (g_1)^{1/2} \, |\Delta_h Du| \, dt \right)^2 \leqslant \int_0^1 g_1 \cdot |\Delta_h Du|^2 \, dt; \tag{4.8}$$

about the derivability of $G(|Du + th\Delta_h Du|)$ with respect to t, we note that either $\Delta_h Du$ is equal to zero or not (note that x is fixed in Ω); if $\Delta_h Du = 0$, then $Du(x+h)=Du(x)$ and $\Delta_h G(|Du|)=0$, thus (4.8) holds. If $\Delta_h Du \neq 0$, then (4.7) and its consequence (4.8) hold, since the derivative with respect to t of $G(|Du+th\Delta_h Du|)$ is defined and bounded for a.e. $t \in [0, 1]$.

Then, (4.8) together with (4.6) gives

$$\int_\Omega \eta^2 |\Delta_h G(|Du|)|^2 \, dx \leqslant c_2 \int_\Omega \int_0^1 (\eta^2 + |D\eta|^2) \, g_2 \cdot (1 + |Du + th\Delta_h Du|^2 + |\Delta_h u|^2) \, dx \, dt. \tag{4.9}$$

Now in (4.9) we take $\eta = 1$ on an open subset $\Omega' \subset\subset \Omega$, with $\operatorname{supp}(\eta) \subset\subset \Omega'' \subset\subset \Omega$. With the same method used to obtain (2.33) we deduce the estimate

$$\int_{\Omega'} |\Delta_h G(|Du|)|^2 \, dx \leqslant 3c_4 \int_{\Omega''} g_2(|Du|) \cdot (1 + |Du|^2) \, dx. \tag{4.10}$$

Since the right hand side is bounded by a constant independent of h, then $G(|Du|)$ has first partial derivative $\partial G(|Du|)/\partial x_k$ in $L^2(\Omega')$. In (4.9) we pass to the limit as $h \to 0$, we sum up for $k = 1, 2, \ldots, n$, and we obtain

$$\int_\Omega \eta^2 |DG(|Du|)|^2 \, dx \leqslant 2nc_2 \int_\Omega (\eta^2 + |D\eta|^2) \, g_2(|Du|) \cdot (1 + |Du|^2) \, dx. \tag{4.11}$$

Then, similarly to (2.40), for some constant c_7 we have

$$\left\{ \int_\Omega [\eta G(|Du|)]^{2^*} \, dx \right\}^{2/2^*} \leqslant c_7 \int_\Omega (\eta^2 + |D\eta|^2) \, g_2(|Du|)(1 + |Du|^2) \, dx. \tag{4.12}$$

By recalling the definition of G and assumption (4.1), for some constant c_8 we have

$$[G(t)]^{2^*} = \left[1 + \int_0^t (g_1(s))^{1/2} \, ds \right]^{2^*} \geqslant c_8 [g_2(t) \cdot (1 + t^2)]^{\gamma} \tag{4.13}$$

for every $t \geqslant 0$, since it holds by assumption if $t \geqslant 1$ and it is satisfied for $0 \leqslant t \leqslant 1$ too, because the right hand side is bounded, while the left hand side is greater than or equal to 1.

Like in Section 2, we denote by B_R and B_ρ balls compactly contained in Ω, of radii respectively R, ρ $(R - \rho \leqslant 1)$, with the same center and we consider a test function η equal to 1 in B_ρ, with compact support contained in B_R, such that $|D\eta| \leqslant 2/(R-\rho)$. Then, by (4.12) and (4.13), there exists a constant c_9 such that

$$\left\{\int_{B_\rho} [v(x)]^\gamma \, dx\right\}^{2/2^*} \leqslant \frac{c_9}{(R-\rho)^2} \int_{B_R} v(x)\, dx,$$
$$\text{where} \quad v(x) = g_2(|Du|) \cdot (1+|Du|^2). \tag{4.14}$$

Like in the statement of the theorem (see also Remark 4.2), let α be a real number such that $0<\alpha<\min\{1/\gamma; 2/2^*\}$; by Hölder's inequality with exponents $1/\alpha$ and $1/(1-\alpha)$ we have

$$\left\{\int_{B_\rho} v^\gamma \, dx\right\}^{2/2^*} \leqslant \frac{c_9}{(R-\rho)^2} \int_{B_R} v^{\gamma\alpha} \cdot v^{1-\gamma\alpha}\, dx$$
$$\leqslant \frac{c_9}{(R-\rho)^2} \left\{\int_{B_R} v^\gamma \, dx\right\}^{\alpha} \cdot \left\{\int_{B_R} v^{(1-\gamma\alpha)/(1-\alpha)}\, dx\right\}^{1-\alpha}. \tag{4.15}$$

For $R_0 > \rho_0$ we define the sequence $\rho_i = R_0 - 2^{-i}(R_0-\rho_0)$; then in the previous estimate we consider $\rho = R_{i-1}$ and $R = R_i$. Since $R-\rho = 2^{-i}(R_0-\rho_0)$, we obtain

$$\left\{\int_{B_{\rho_{i-1}}} v^\gamma \, dx\right\}^{2/2^*} \leqslant \frac{c^9 \cdot 4^i}{(R_0-\rho_0)^2} \left\{\int_{B_{\rho_i}} v^\gamma \, dx\right\}^{\alpha} \cdot I,$$
$$\text{where} \quad I = \left\{\int_{B_{R_0}} v^{(1-\gamma\alpha)/(1-\alpha)}\, dx\right\}^{1-\alpha}. \tag{4.16}$$

By iterating in (4.16) we have

$$\int_{B_{\rho_0}} v^\gamma \, dx \leqslant \left\{\int_{B_{\rho_i}} v^\gamma \, dx\right\}^{[\alpha 2^*/2]^i} \cdot \prod_{i=1}^{\infty} \left\{\frac{c_9 \cdot 4^i}{(R_0-\rho_0)^2} \cdot I\right\}^{\alpha^{i-1} \cdot [2^*/2]^i}$$
$$\leqslant c_{10} \left\{\int_{B_{R_0}} v^\gamma \, dx\right\}^{[\alpha 2^*/2]^i} \cdot I^{[2^*/2] \cdot \sum_{i=0}^{\infty} [\alpha 2^*/2]^i}$$
$$= c_{10} \left\{\int_{B_{R_0}} v^\gamma \, dx\right\}^{[\alpha 2^*/2]^i} \cdot I^{1/(2/2^*-\alpha)}, \tag{4.17}$$

where, since $\alpha 2^*/2 < 1$,

$$c_{10} = \prod_{i=1}^{\infty} \left\{\frac{c_9 \cdot 4^i}{(R_0-\rho_0)^2}\right\}^{\alpha^{i-1} \cdot [2^*/2]^i}$$
$$= \exp\left(\frac{2^*}{2} \sum_{i=0}^{\infty} \left(\alpha \frac{2^*}{2}\right)^i \log \frac{c_9 \cdot 4^i}{(R_0-\rho_0)^2}\right) < +\infty. \tag{4.18}$$

We go to the limit as $i \to +\infty$ in (4.17); again, since $\alpha 2^*/2 < 1$, we obtain the conclusion

$$\int_{B_{\rho_0}} v^{\gamma}\, dx \leqslant c_{10} I^{1/(2/2^* - \alpha)} = c_{10} \left\{ \int_{B_{R_0}} v^{(1-\gamma\alpha)/(1-\alpha)}\, dx \right\}^{(1-\alpha)/(2/2^* - \alpha)}. \quad (4.19)$$

Remark 4.3. The exponents of formula (4.2) in Theorem 4.1 have been obtained by an iteration procedure; we can have a quick formal control of which exponents should appear in the formula, by posing formally $B_\rho = B_R$ in (4.15) (of course, without the term $1/(R-\rho)^2$),

$$\left\{ \int_{B_R} v^{\gamma}\, dx \right\}^{2/2^* - \alpha} \leqslant \text{const} \left\{ \int_{B_R} v^{(1-\gamma\alpha)/(1-\alpha)}\, dx \right\}^{1-\alpha}, \quad (4.20)$$

that corresponds (by replacing B_R in the left hand side by B_ρ) to the correct conclusion (4.2).

5. On the Definition of the Functional Set and of the Notion of Weak Solution: Euler's Equation

Similarly to Section 3, we introduce the following functional sets (we call them functional sets, instead of functional spaces, since, in this paper, we do not introduce on them any structure of topological type; about this point see [19] and the forthcoming paper [8]), related to the function $G(t) = g_2(t) \cdot (1 + t^2)$ and to a constant $\sigma > 0$:

$$W^{1,G,\sigma}(\Omega) = \left\{ v \in W^{1,2}(\Omega) : \int_\Omega G(\sigma\, |Dv|)\, dx < +\infty \right\}; \quad (5.1)$$

$$W^{1,G,\sigma}_{\text{loc}}(\Omega) = \left\{ v \in W^{1,2}_{\text{loc}}(\Omega) : \int_{\Omega'} G(\sigma\, |Dv|)\, dx < +\infty, \forall \Omega' \subset\subset \Omega \right\}. \quad (5.2)$$

Finally we define $W^{1,G,\sigma}_0(\Omega) = W^{1,G,\sigma}(\Omega) \cap W^{1,2}_0(\Omega)$; when $\sigma = 1$, like in Section 2 we use the simpler notations $W^{1,G,1}(\Omega) = W^{1,G}(\Omega)$, $W^{1,G,1}_{\text{loc}}(\Omega) = W^{1,G}_{\text{loc}}(\Omega)$, $W^{1,G,1}_0(\Omega) = W^{1,G}_0(\Omega)$.

Remark 5.1. If the function g_2 (and then G) satisfies the Δ_2 condition ($g_2(2t) \leqslant \text{const} \cdot g_2(t)$, $\forall t \geqslant 0$), then the sets defined previously are independent of $\sigma > 0$; in this case these sets are linear vector spaces. On the contrary, if G does not satisfy the Δ_2 condition, in general the previous sets are not linear vector spaces, but (by the convexity of G) are only convex sets.

Remark 5.2. The functional sets in (5.1), (5.2) can be defined equivalently through the function $t \to (1 + g_2(t)) \cdot (1 + t^2)$, or through the function $t \to g_2(t) \cdot t^2$, instead of $G(t) = g_2(t) \cdot (1 + t^2)$. This depends on the fact that, like in (2.3), $g_2(t) \geq \text{const} \cdot (1 + g_2(t))$. For the same reason we can define equivalently the previous functional sets as in (3.10), with the condition $v \in W^{1,1}_{\text{loc}}(\Omega)$ instead of $v \in W^{1,2}_{\text{loc}}(\Omega)$.

To motivate our definition of the functional set where we look for solutions, we use an heuristic approach. We consider as the model problem the variational problem

$$\text{minimize, on a functional set } V \text{, the functional } v \to \int_\Omega f(x, Dv)\, dx, \tag{5.3}$$

where $f = f(x, \xi)$ is a real function defined for $x \in \Omega$ and $\xi = (\xi_i) \in \mathbb{R}^n$. We assume that u is a minimizer on V and we study Euler's first variation

$$\left[\frac{d}{d\varepsilon} \int_\Omega f(x, Du + \varepsilon\, D\phi)\, dx\right]_{\varepsilon = 0} = 0, \qquad \forall \phi \in V, \tag{5.4}$$

with the condition that $u + \varepsilon\phi \in V$ for every ε (positive and negative) sufficiently small. By the mean value theorem for functions of several real variables we have the well known computation

$$\begin{aligned} &\left[\frac{d}{d\varepsilon} \int_\Omega f(x, Du + \varepsilon\, D\phi)\, dx\right]_{\varepsilon = 0} \\ &\qquad = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_\Omega \left[f(x, Du + \varepsilon\, D\phi) - f(x, Du)\right] dx \\ &\qquad = \lim_{\varepsilon \to 0} \int_\Omega \sum_{i=1}^n f_{\xi_i}(x, Du + \theta\varepsilon\, D\phi) \cdot \phi_{x_i}\, dx, \end{aligned} \tag{5.5}$$

where $\theta = \theta(x) \in [0, 1]$ for almost every $x \in \Omega$.

We estimate the integrand with the aim to go to the limit as $\varepsilon \to 0$. We start with some of the assumptions in (2.5) and (2.6), precisely,

$$|f_{\xi_i \xi_j}(x, \xi)| \leq M g_2(|\xi|), \qquad |f_{\xi_i x_k}(x, 0)| \leq M, \forall i, j, k \tag{5.6}$$

(we recall that, under the notations of Section 2, in the variational problem of this section we have $a^i = f_{\xi_i}$) and we obtain

$$\begin{aligned} f_{\xi_i}(x, \xi) &= f_{\xi_i}(x, 0) + \int_0^1 \frac{d}{dt} f_{\xi_i}(x, t\xi)\, dt \\ &= f_{\xi_i}(x, 0) + \int_0^1 \sum_{j=1}^n f_{\xi_i \xi_j}(x, t\xi)\, \xi_j\, dt; \end{aligned} \tag{5.7}$$

thus, by (5.6), (5.7), and since g_2 is increasing and strictly positive, there is a constant c_1 such that

$$|f_{\xi_i}(x, \xi)| \leqslant c_1\, g_2(|\xi|) \cdot (1 + |\xi|), \qquad \forall i = 1, 2, ..., n. \tag{5.8}$$

Again since g_2 is increasing, by Lemma 3.4(i) we obtain

$$\begin{aligned} \left| \sum_{i=1}^{n} f_{\xi_i}(x, \xi) \cdot \eta_i \right| &\leqslant \sqrt{n}\, c_1\, g_2(|\xi|)(1 + |\xi|) \cdot |\eta| \\ &\leqslant \sqrt{n}\, c_1 \{ g_2(|\xi|)(1 + |\xi|^2) + g_2(|\eta|)(1 + |\eta|^2) \}. \end{aligned} \tag{5.9}$$

Thus, if $|\varepsilon| \leqslant 1$, since $|\theta\varepsilon| \leqslant 1$, from Lemma 3.4(iii) with $g(t) = g_2(t) \cdot (1 + t^2)$ we deduce that, for every $\sigma > 1$

$$\begin{aligned} &\left| \sum_{i=1}^{n} f_{\xi_i}(x, \xi + \theta\varepsilon\eta) \cdot \eta_i \right| \\ &\quad \leqslant \sqrt{n}\, c_1 \{ g_2(|\xi + \theta\varepsilon\eta|) \cdot (1 + |\xi + \theta\varepsilon\eta|^2) + g_2(|\eta|) \cdot (1 + |\eta|^2) \} \\ &\quad \leqslant \sqrt{n}\, c_1 \left\{ g_2(\sigma\, |\xi|)(1 + \sigma^2\, |\xi|^2) + g_2\left(\frac{\sigma}{\sigma - 1}\, |\eta| \right)\left(1 + \left(\frac{\sigma}{\sigma - 1} \right)^2 |\eta|^2 \right) \right. \\ &\qquad \left. + g_2(|\eta|)(1 + |\eta|^2) \right\}. \end{aligned} \tag{5.10}$$

The right hand side is independent of ε. Thus, if we assume that there exist $\sigma > 1$ such that $u \in W^{1,G,\sigma}_{\mathrm{loc}}(\Omega)$ and $\phi \in W^{1,G,\sigma/(\sigma-1)}_0(\Omega')$ for some $\Omega' \subset\subset \Omega$, then by (5.10) we can go to the limit as $\varepsilon \to 0$ in the integral in the right hand side of (5.5) and we obtain the usual Euler's equation

$$\int_\Omega \sum_{i=1}^{n} f_{\xi_i}(x, Du) \cdot \phi_{x_i}\, dx = 0, \qquad \forall \phi \in W^{1,G,\sigma/(\sigma-1)}_0(\Omega') \colon \Omega' \subset\subset \Omega. \tag{5.11}$$

Let us observe that, again by Lemma 3.4(iii), if $u \in W^{1,G,\sigma}_{\mathrm{loc}}(\Omega)$ and $\phi \in W^{1,G,\sigma/(\sigma-1)}_0(\Omega')$, then $u + \varepsilon\phi \in W^{1,G}(\Omega)$ for $|\varepsilon| \leqslant 1$. Finally, we can multiply both sides of the previous equation by the constant $(\sigma - 1)/\sigma$ and change the test function $x \to \phi(x)$ by $x \to (\sigma - 1)/\sigma \cdot \phi(x)$ that is a generic function of $W^{1,G,1}_0(\Omega') = W^{1,G}_0(\Omega')$. Thus we have proved the following result:

THEOREM 5.3. *Let $f = f(x, \xi)$ be a measurable function with respect to x and locally Lipschitz continuous function with respect to $\xi \in \mathbb{R}^n$, satisfying* (5.6) *with g_2 as in* (2.2)(i), (2.2)(ii). *Let u be a minimizer of the integral* (5.3) *in the functional set $W^{1,G}(\Omega)$, with some fixed boundary conditions. If $u \in W^{1,G,\sigma}_{\mathrm{loc}}(\Omega)$ for some $\sigma > 1$, then it satisfies the weak form of Euler's equation*

$$\int_\Omega \sum_{i=1}^{n} f_{\xi_i}(x, Du) \cdot \phi_{x_i}\, dx = 0, \qquad \forall \phi \in W^{1,G}_0(\Omega') \colon \Omega' \subset\subset \Omega. \tag{5.12}$$

More generally we can consider integrals of the calculus of variations of the type

$$v \in W^{1,G}(\Omega) \to \int_\Omega f(x, v, Dv)\, dx, \tag{5.13}$$

where the function $f = f(x, s, \xi)$ depends explicitly on x and s too and it satisfies the growth conditions

$$\begin{aligned} |f_{\xi_i}(x, s, \xi)| &= Mg_2(|\xi|) \cdot (1 + |\xi|), \quad \forall i = 1, 2, ..., n, \\ |f_s(x, s, \xi)| &= Mg_2(|\xi|) \cdot (1 + |\xi|^2), \end{aligned} \tag{5.14}$$

for every $\xi, \lambda \in \mathbb{R}^n$, for a.e. $x \in \Omega$, for every $s \in [-s_0, s_0]$, and for some constant $M = M(s_0)$, with s_0 generic in $\mathbb{R}^+$.

THEOREM 5.4. *Let $f = f(x, s, \xi)$ be a measurable function with respect to x and locally Lipschitz continuous function with respect to $(s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, satisfying* (5.14) *with g_2 as in* (2.2)(i), (2.2)(ii). *Let u be a minimizer of the integral* (5.13) *in the functional set $W^{1,G}(\Omega)$, with some fixed boundary conditions. If $u \in W^{1,G,\sigma}_{\mathrm{loc}}(\Omega)$ for some $\sigma > 1$ and it is locally bounded in Ω, then it satisfies the weak form of Euler's equation*

$$\int_\Omega \left\{ \sum_{i=1}^n f_{\xi_i}(x, u, Du) \cdot \phi_{x_i} + f_s(x, u, Du) \cdot \phi \right\} dx = 0,$$

$$\forall \phi \in W_0^{1,G}(\Omega') \cap L^\infty(\Omega') : \Omega' \subset\subset \Omega. \tag{5.15}$$

Proof. We can proceed as in the proof of Theorem 5.3. First we can replace the assumption (5.6) by the weaker condition (5.8) (more precisely, by the first condition in (5.14)). Then, we use the growth condition on f_s in (5.14), together with Lemma 3.4(iii) and the fact that ϕ is locally bounded, to obtain

$$\begin{aligned} &|f_s(x, u + \theta\varepsilon\phi, Du + \theta\varepsilon\, D\phi) \cdot \phi| \\ &\quad \leqslant c_2 \cdot g_2(|Du + \theta\varepsilon\, D\phi|) \cdot (1 + |Du + \theta\varepsilon\, D\phi|^2) \\ &\quad \leqslant c_2 \left\{ g_2(\sigma\, |Du|) \cdot (1 + \sigma^2\, |Du|^2) + g_2\left(\frac{\sigma}{\sigma - 1}\, |Du|\right) \right. \\ &\qquad \left. \cdot \left(1 + \left(\frac{\sigma}{\sigma - 1}\right)^2 |D\phi|^2\right) \right\}, \end{aligned} \tag{5.16}$$

for $|\varepsilon| \leqslant 1$. Then we can go to the limit as $\varepsilon \to 0$ like in the previous proof.

6. Integrals of the Calculus of Variations, Possibly with Exponential Growth

In this section we prove two regularity results for minimizers of some integrals-functionals of the calculus of variations with non-standard growth conditions, in particular with exponential growth. One of the peculiarities of functionals with exponential growth is that they do not give rise to uniformly elliptic problems. Let us explain this fact by an example; let us consider integrals of the type

$$\int_\Omega f(Dv)\,dx, \qquad \text{with } \; f(\xi)=\Phi(|\xi|), \tag{6.1}$$

where Φ: $[0,+\infty)\to\mathbb{R}$ is a convex function of class $C^2([0,+\infty))$, with $\Phi'(0)=0$. By a computation we can see that

$$\sum_{i,j} f_{\xi_i\xi_j}(\xi)\,\lambda_i\lambda_j=\left(\frac{\Phi''(|\xi|)}{|\xi|^2}-\frac{\Phi'(|\xi|)}{|\xi|^3}\right)(\xi,\lambda)^2+\frac{\Phi'(|\xi|)}{|\xi|}\,|\lambda|^2, \tag{6.2}$$

and thus

$$\begin{aligned}\min\left\{\Phi''(|\xi|);\frac{\Phi'(|\xi|)}{|\xi|}\right\}|\lambda|^2&\leqslant\sum_{i,j} f_{\xi_i\xi_j}(\xi)\,\lambda_i\lambda_j\\ &\leqslant\max\left\{\Phi''(|\xi|);\frac{\Phi'(|\xi|)}{|\xi|}\right\}|\lambda|^2.\end{aligned} \tag{6.3}$$

Therefore the problem is uniformly elliptic if and only if there are positive constants c_1, c_2 such that

$$c_1\Phi''(t)\leqslant\frac{\Phi'(t)}{t}\leqslant c_2\Phi''(t). \tag{6.4}$$

It is easy to be convinced that the behavior in (6.4) is typical of functions $\Phi(t)$ of power growth, or, for example, of the type $\Phi(t)=t^p\cdot\log(1+t)$, while it does not hold for functions $\Phi(t)$ of exponential growth. In fact, if

$$f(\xi)=\Phi(|\xi|), \qquad \text{with } \; \Phi(t)=\exp(t^\alpha),\ \forall t\geqslant t_0, \tag{6.5}$$

for some $\alpha>0$ (and for t_0 sufficiently large in dependence of α when $\alpha\in(0,1)$), then f satisfies the ellipticity (but not uniformly ellipticity) conditions

$$\begin{aligned}\alpha\exp(|\xi|^\alpha)\cdot|\xi|^{\alpha-2}\,|\lambda|^2&\leqslant\sum_{i,j} f_{\xi_i\xi_j}(\xi)\,\lambda_i\lambda_j\\ &\leqslant M_\alpha\exp(|\xi|^\alpha)\cdot|\xi|^{2\alpha-2}\,|\lambda|^2, \qquad \forall\,|\xi|\geqslant t_0.\end{aligned} \tag{6.6}$$

Thus the ratio between the maximum $g_2(|\xi|)$ and the minimum $g_1(|\xi|)$ eigenvalue of the matrix D^2f, of the second derivatives of f, is not bounded. On the contrary, due to the exponential growth, for our function f in (6.5) we have that

$$\forall \varepsilon > 0 \ \exists c_\varepsilon > 0, \qquad (g_2(|\xi|) \cdot (1 + |\xi|^2))^{1-\varepsilon} \leqslant c_\varepsilon \cdot f; \tag{6.7}$$

we adopt (6.7) as an assumption in the regularity results of this section. We note that (6.7) is a coercivity condition.

We consider integrals of the calculus of variations of the type

$$\int_\Omega f(x, v, Dv)\, dx, \tag{6.8}$$

where $f = f(x, s, \xi)$ is a function of class $C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ whose partial derivatives satisfy the following *general growth conditions*: for every $s_0 > 0$, there are positive constants m, M such that, for every ξ, $\lambda \in \mathbb{R}^n$, for a.e. $x \in \Omega$ and for every $s \in [-s_0, s_0]$,

$$\begin{gathered} m g_1(|\xi|)\, |\lambda|^2 \leqslant \sum_{i,j} f_{\xi_i \xi_j}(x, s, \xi)\, \lambda_i \lambda_j \leqslant M g_2(|\xi|)\, |\lambda|^2; \\ |f_{\xi_i s}(x, s, \xi)| \leqslant M[g_1(|\xi|)\, g_2(|\xi|)]^{1/2}; \\ |f_{\xi_i x_k}(x, s, \xi)| \leqslant M(1 + |\xi|)[g_1(|\xi|)\, g_2(|\xi|)]^{1/2}, \quad \forall i, k; \\ |f_{ss}(x, s, \xi)| \leqslant M g_2(|\xi|); \qquad |f_{s x_k}(x, s, \xi)| \leqslant M(1 + |\xi|)\, g_2(|\xi|), \quad \forall k; \end{gathered} \tag{6.9}$$

the functions g_1, g_2 that appear in (6.9) are increasing and strictly positive in $[0, +\infty)$, the product function $G(t) = g_2(t) \cdot (1 + t^2)$ is convex, and they are related by the condition

$$g_2(\sqrt{n}\, t) \cdot (1 + t^2) \leqslant c \left\{ \int_0^t (g_1(s))^{1/2}\, ds \right\}^{2^*/\gamma}, \qquad \forall t \geqslant 1, \tag{6.10}$$

for some $\gamma > 1$ and $c > 0$, where $2^* = 2n/(n-2)$ if $n > 2$ (while 2^* can be any fixed number greater than 2 if $n = 2$).

Remark 6.1. The vector field $a^i(x, s, \xi) = f_{\xi_i}(x, s, \xi)$ satisfies all the assumptions of Section 2; in particular (2.7) holds with $\alpha(x, s, \xi) = 0$ and $\beta(x, s, \xi) = f_s(x, s, \xi)$.

Remark 6.2. For the function f in (6.5), (6.6) we have $g_1(t)=\exp(t^\alpha)\cdot t^{\alpha-2}$ and $g_2(t)=\exp(t^\alpha)\cdot t^{2\alpha-2}$. Then, for every $\varepsilon>0$ there exists a constant c_ε such that

$$\begin{aligned}(g_1(t))^{1/2} &\geqslant c_\varepsilon \exp((1-\varepsilon)\,\tfrac{1}{2}\,t^\alpha);\\ g_2(\sqrt{n}\,t)\cdot(1+t^2) &\leqslant c_\varepsilon \exp((1+\varepsilon)\,n^{\alpha/2}t^\alpha);\end{aligned} \tag{6.11}$$

now it is easy to see that, in the present case, (6.10) holds for some $\gamma>1$ if f has *slow exponential growth*, i.e., if α is small; more precisely if

$$n^{\alpha/2}<\frac{2^*}{2}, \qquad \text{that is if} \quad \alpha<2\,\frac{\log(2^*/2)}{\log n}. \tag{6.12}$$

In particular (6.10) is satisfied by the function f in (6.5), (6.6) with any positive α if $n=2$, with $\alpha<2$ if $n=3$, and with $\alpha<1$ if $n=4$.

THEOREM 6.3. *Let $f=f(x,s,\xi)$ be a function of class $C^2(\Omega\times\mathbb{R}\times\mathbb{R}^n)$ satisfying* (6.7), (6.9), (6.10). *Let u be a minimizer of the integral* (6.8) *in the functional set $W^{1,G}(\Omega)$, with some fixed boundary conditions. If $u\in W^{1,G,\sigma}_{\mathrm{loc}}(\Omega)$ for some $\sigma>1$ and it is locally bounded in Ω, then it is of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and there exists a positive exponent δ with the property that for every ρ, R $(0<\rho<R\leqslant\rho+1)$ there is a constant c such that*

$$\sup\{|Du(x)|: x\in B_\rho\}\leqslant c\left\{\int_{B_R} f(x,u,Du)\,dx\right\}^\delta. \tag{6.13}$$

Moreover, if $f\in C^{k,\alpha}_{\mathrm{loc}}(\Omega\times\mathbb{R}\times\mathbb{R}^n)$ for some $k\geqslant 2$, then u is of class $C^{k,\alpha}_{\mathrm{loc}}(\Omega)$.

Remark 6.4. If $f=f(x,\xi)$ is independent of s, then in the previous theorem it is not necessary to assume a priori the local boundedness of the minimizer u (see Remark 2.6).

Proof of Theorem 6.3. Since the minimizer u belongs to $W^{1,G,\sigma}_{\mathrm{loc}}(\Omega)$, then, by the results of Section 5, it is a weak solution to Euler's equation (5.15). Let us now make use of Theorem 4.1: since the (positive) exponent $(1-\gamma\alpha)/(1-\alpha)$, that appears in the right hand side of (4.2), is smaller than 1, then by (4.2) and (6.7) we have

$$\int_{B_\rho}[g_2(|Du|)(1+|Du|^2)]^\gamma\,dx\leqslant c_1\left\{\int_{B_R} f(x,u,Du)\,dx\right\}^{(1-\alpha)/(2/2^*-\alpha)}, \tag{6.14}$$

for $0<\alpha<2/2^*$, for every ρ, R $(0<\rho<R\leqslant\rho+1)$ and for some constant c_1. Then, by Theorem 2.3 with $1+\beta/2=\gamma$, for every $\rho'<\rho$ there exists a constant c_2 such that

$$\sup\{|Du(x)|: x\in B_{\rho'}\} \leqslant c_2\left\{\int_{B_R} f(x,u,Du)\,dx\right\}^{(1-\alpha)/2(\gamma-1)(2/2^*-\alpha)}. \tag{6.15}$$

We conclude this section with a consequence of the a priori regularity result of Theorem 6.3; it applies to integrals of the type

$$F(\Omega,v)=\int_\Omega f(Dv)\,dx, \tag{6.16}$$

with f of class $C^2(\mathbb{R}^n)$ such that

$$mg_1(|\xi|)\,|\lambda|^2\leqslant\sum_{i,ju} f_{\xi_i\xi_j}(\xi)\,\lambda_i\lambda_j\leqslant Mg_2(|\xi|)\,|\lambda|^2, \tag{6.17}$$

with g_1 and g_2 satisfying (6.7), (6.10).

THEOREM 6.5. *Let $f=f(\xi)$ be a nonnegative function of class $C^2(\mathbb{R}^n)$, satisfying* (6.17), (6.7), (6.10). *Let $u_0\in W^{1,1}(\Omega)$ be a boundary datum with the property that $f(Du_0(x))\in L^1(\Omega)$. Then there exists a minimizer of the integral* (6.16) *in the class $u_0+W_0^{1,1}(\Omega)$. The minimizer is of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$, it is unique, and, if $f\in C^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $k\geqslant 2$, then it is of class $C^{k,\alpha}_{\mathrm{loc}}(\Omega)$.*

Proof. The existence and uniqueness of a minimizer $u\in u_0+W_0^{1,1}(\Omega)$, or equivalently in the class $u_0+W_0^{1,2}(\Omega)$, follows from the strict convexity and coercivity of the integral F (in fact, by the ellipticity condition in the left hand side of (6.17), with the same method of Section 5, we can prove the existence of two constants c_1, c_2 such that $f(\xi)\geqslant c_1\,|\xi|^2-c_2$).

Let α_ε be a smooth mollifier with compact support and let us define in $\Omega'\subset\subset\Omega$

$$v_\varepsilon(x)=\int_{R^n}\alpha_\varepsilon(y)\cdot v(x-y)\,dy,\qquad \forall x\in\Omega', \tag{6.18}$$

where $v\in W^{1,2}(\Omega)$ is any function such that $f(Dv(x))\in L^1(B_R)$ on a ball $B_R\subset\subset\Omega$ and such that $v=u$ on the boundary ∂B_R of B_R. Then v_ε is a net of smooth functions that converges, as $\varepsilon\to 0$, to v in $W^{1,2}(\Omega)$. Moreover

$$\lim_{\varepsilon\to0}\int_{B_R} f(Dv_\varepsilon)\,dx=\int_{B_R} f(Dv)\,dx; \tag{6.19}$$

in fact, we can use the argument of [31, Corollary 4.2]: by Jensen's inequality we have

$$f(Dv_\varepsilon(x)) = f\left(\int_{R^n} \alpha_\varepsilon(y)\cdot Dv(x-y)\,dy\right)$$

$$\leqslant \int_{R^n} \alpha_\varepsilon(y)\cdot f(Dv(x-y))\,dy, \qquad \forall x\in B_R; \tag{6.20}$$

by integrating over B_R and by considering another ball $B_{R'}$ with radius $R'>R$ and with the same center as B_R, for ε sufficiently small we obtain

$$\int_{B_R} f(Dv_\varepsilon(x))\,dx \leqslant \int_{R^n} \alpha_\varepsilon(y)\,dy \int_{B_R} f(Dv(x-y))\,dx \leqslant \int_{B_{R'}} f(Dv)\,dx; \tag{6.21}$$

first we take the maximum limit as $\varepsilon\to 0$ of both sides, then the limit as $R'\to R$ in the right hand side and we obtain an inequality that, together with the lower semicontinuity of the integral, concludes the proof of (6.19).

Let us denote by u_ε the minimizer of $F(B_R,\cdot)$ in the class of function with prescribed boundary value equal to v_ε on ∂B_R. Then, by the "Bounded Slope Condition Theorem," u_ε is Lipschitz continuous in B_R, and

$$\|Du_\varepsilon\|_{L^\infty(B_R)} \leqslant \|Dv_\varepsilon\|_{L^\infty(\partial B_R)}. \tag{6.22}$$

To go on we need, for the L^∞-norm of Du_ε, a bound independent of ε. To this aim we observe that, by (6.22), $u_\varepsilon \in W^{1,G,\sigma}(B_R)$ for every σ; therefore, by Theorem 6.3, for $\rho<R'<R$ there exists a constant (independent of ε) such that

$$\|Du_\varepsilon\|_{L^\infty(B_\rho)} \leqslant c\left\{\int_{B_{R'}} f(Du)\,dx\right\}^\delta. \tag{6.23}$$

Then the u_ε converge (up to a subsequence) to a function $\bar u\in W^{1,\infty}_{\mathrm{loc}}(B_R)$ and, by the lower semicontinuity of the integral F, we have

$$\int_{B_R} f(D\bar u(x))\,dx \leqslant \liminf_{\varepsilon\to 0}\int_{B_R} f(Du_\varepsilon)\,dx$$

$$\leqslant \lim_{\varepsilon\to 0}\int_{B_R} f(Dv_\varepsilon)\,dx = \int_{B_R} f(Dv)\,dx. \tag{6.24}$$

Since v is a generic function, by the uniqueness of the minimizer, $\bar u = u$ in B_R and thus u is locally Lipschitz continuous (and satisfies (6.23) with u_ε replaced by u).

Remark 6.6. It is clear from the argument used in the previous proof that no Lavrentiev phenomenon can occur for the integral (6.16), under the convexity of $f(\xi)$ (even in the vectorial case). Thus the integral in (6.16) can be defined equivalently by the "naive" method of composition of f with the measurable vector Dv, or by extension by "lower semicontinuity," starting from the definition of the integral for smooth functions v.

7. A Generalization of the Uniformly Elliptic Case

With the notations of Section 2 let us consider again Eq. (2.1) and let $(a^i(x, s, \xi))$ be a locally Lipschitz continuous vector field in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ satisfying the ellipticity conditions

$$m g_1(|\xi|)\, |\lambda|^2 \leqslant \sum_{i,j} a^i_{\xi_j}(x, s, \xi)\, \lambda_i \lambda_j, \qquad |a^i_{\xi_j}(x, s, \xi)| \leqslant M g_2(|\xi|), \ \forall i, j, \tag{7.1}$$

and the other growth conditions (2.5), (2.6), (2.7), with g_1, g_2 related by (2.2)(i), (2.2)(ii), and by

$$\begin{gathered} g_1'(t) \leqslant c_1 \frac{g_1(t)}{t}, \ \forall t > 0, \\ g_2(\sqrt{n}\, t) \cdot t^2 \leqslant c_1 (g_1(t) \cdot t^2)^{2^*/(2\gamma)}, \ \forall t \geqslant 1, \end{gathered} \tag{7.2}$$

for some constants $c_1 > 0$ and $\gamma \geqslant 1$ (since g_1 is increasing, then its derivative exists a.e. in $[0, +\infty)$). In particular (7.2) becomes a *uniformly ellipticity condition* if

$$g_1(t) = g_2(t) \overset{\text{def}}{=} g(t) \qquad \text{and} \qquad g'(t) \leqslant c_1 \frac{g(t)}{t}, \ \forall t > 0; \tag{7.3}$$

note that if a function $g(t)$ satisfies (7.3) then it satisfies also the Δ_2-condition and thus the factor $\sqrt{n}$ in (7.2) becomes unnecessary.

Examples of variational problems which give rise to the uniformly ellipticity condition (7.3) are, for example, integral functionals with natural growth conditions and the integral functionals F_3 in (1.3) and F_5 in (1.9). Examples of integrals of the calculus of variations which can be treated with the more general condition (7.2) (for α smooth and $1 < \delta < 2^*/2$) and which do not satisfy (7.3) are

$$F_6(v) = \int_\Omega [\Phi(|Dv|)]^{\alpha(x)}\, dx \tag{7.4}$$

or

$$F_7(v) = \int_\Omega \{\Phi(|Dv|) + [\Phi(|v_{x_n}|)]^\delta\}\, dx,$$

where, for example, the function Φ is defined like in (1.9).

A relevant class to refer is the integral-functional class of the type (6.1), for which the condition of uniformly ellipticity is expressed by (6.4). We can easily verify that the conditions (6.4) and (7.3) are related to each other by the position $g(t) = \Phi'(t)/t$ and that

$$g'(t) \gtrless c\,\frac{g(t)}{t}, \quad \forall t > 0 \Leftrightarrow \Phi''(t) \gtrless (1+c)\,\frac{\Phi'(t)}{t}, \quad \forall t > 0. \tag{7.5}$$

Simon (see the second part of Section 4 of [34]) and more recently Lieberman [25] proved the local Lipschitz continuity of the gradient Du of weak solutions to Eq. (2.1) under the assumption (7.3) (in the notations of [34, 35], $g(t)$ is replaced by $g(t)/t$; but, similarly to the equivalence (7.5), we can easily see that the two cases are related to each other). Therefore, the next theorem is an extension of the quoted Simon and Lieberman results to the more general ellipticity condition (7.2).

THEOREM 7.1. *Let us assume that* (2.2)–(2.7) *hold, with* (2.2)(iii) *replaced either by the uniformly ellipticity condition* (7.3), *or by its generalization* (7.2) *with* $\gamma > 1$. *Then every weak locally bounded solution* $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$ *to Eq.* (2.1) *is of class* $W^{1,\infty}_{\mathrm{loc}}(\Omega)$. *More regularity, like in Corollary* 2.5, *and higher summability of the gradient, like in Theorem* 4.1, *hold.*

Proof. By the first condition in (7.2) we have

$$(g_1^{1/2})' = \frac{1}{2}\, g_1^{-1/2} \cdot g_1' \leqslant \frac{c_1}{2}\,\frac{g_1^{1/2}}{t}, \qquad \forall t > 0; \tag{7.6}$$

from which we obtain

$$\begin{aligned}\int_0^t (g_1(s))^{1/2}\, ds &\geqslant \frac{2}{c_1} \int_0^t s((g_1(s))^{1/2})'\, ds \\ &= \frac{2}{c_1}\left\{t(g_1(t))^{1/2} - \int_0^t (g_1(s))^{1/2}\, ds\right\};\end{aligned} \tag{7.7}$$

therefore

$$t(g_1(t))^{1/2} \leqslant \left(\frac{c_1}{2} + 1\right) \int_0^t (g_1(s))^{1/2}\, ds. \tag{7.8}$$

This estimate, together with the second condition in (7.2), gives

$$g_2(\sqrt{n}\,t)\cdot t^2 \leqslant c_2\left\{\int_0^t (g_1(s))^{1/2}\,ds\right\}^{2^*/\gamma}. \tag{7.9}$$

Then, if $\gamma=1$, the assumption (2.2)(iii) of Theorem 2.3 is satisfied; while, if $\gamma>1$, then the condition (2.12) of Theorem 2.4 with $\gamma=1+\beta/2$ (for local Lipschitz continuity) and assumption (4.1) (for higher summability of the gradient) are satisfied. This proves our theorem under assumption (7.2). It remains to observe that (7.3) is a particular case of (7.2), with $\gamma=2^*/2$ and with $g=g_1=g_2$ satisfying the Δ_2 property $g(2t)\leqslant 2^{c_1}g(t)$, as a consequence of the fact that the function $t\to g(t)/t^{c_1}$ is decreasing.

References

1. E. Acerbi and N. Fusco, Partial regularity under anisotropic (p, q) growth conditions, *J. Differential Equations*, in press.
2. T. Bhattacharya and F. Leonetti, A new Poincaré inequality and its applications to the regularity of minimizers of integral functionals with nonstandard growth, *Nonlinear Anal. Theory Methods Appl.* **17** (1991), 833–839.
3. T. Bhattacharya and F. Leonetti, $W^{2,2}$ regularity for weak solutions of elliptic systems with nonstandard growth, *J. Math. Anal. Appl.*, in press.
4. L. Boccardo, P. Marcellini, and C. Sbordone, L^∞-regularity for variational problems with sharp non standard growth conditions, *Boll. Un. Mat. Ital. A* **4** (1990), 219–225.
5. H. Brezis, "Analyse fonctionelle, Théorie et applications," Masson, Paris, 1983.
6. M. Chipot and L. C. Evans, Linearization at infinity and Lipschitz estimates for certain problems in the calculus of variations, *Proc. Roy. Soc. Edinburgh, Sect. A* **102** (1986), 291–303.
7. H. J. Choe, Interior behaviour of minimizers for certain functionals with nonstandard growth, *Nonlinear Anal. Theory Methods Appl.* **19** (1992), 933–945.
8. A. Dall'Aglio, E. Mascolo, and G. Papi, to appear.
9. E. Di Benedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal. Theory Methods Appl.* **7** (1983), 827–850.
10. L. C. Evans, A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e., *J. Differential Equations* **45** (1982), 356–373.
11. N. Fusco and C. Sbordone, Higher integrability of the gradient of minimizers of functionals with nonstandard growth conditions, *Comm. Pure Appl. Math.* **43** (1990), 673–683.
12. N. Fusco and C. Sbordone, Local boundedness of minimizers in a limit case, *Manuscripta Math.* **69** (1990), 19–25.
13. M. Giaquinta, Growth conditions and regularity, a counterexample, *Manuscripta Math.* **59** (1987), 245–248.
14. D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, *in* "Grundlehren Math. Wiss.," Vol. 224, Springer-Verlag, Berlin/Heidelberg/New York, 1977.
15. E. Giusti, Equazioni ellittiche del secondo ordine, *in* "Quaderni Unione Mat. Italiana," Vol. 6, Pitagora, Bologna, 1978.

16. HONG MIN-CHUNG, Some remarks on the minimizers of variational integrals with non standard growth conditions, *Boll. Un. Mat. Ital. A* **6** (1992), 91–101.
17. A. V. IVANOV, Quasilinear degenerate and nonuniformly elliptic and parabolic equations of second order, *in* "Proc. Steklov Inst. Math.," Vol. 160, Amer. Math. Soc., Providence, RI, 1984. [In Russian]
18. A. G. KOROLEV, On the boundedness of generalized solutions of elliptic differential equations with nonpower nonlinearities, *Mat. Sb.* **180** (1989), 78–100; *Math. USSR-Sb.* **66** (1990), 83–106.
19. M. A. KRASNOSEL'SKII AND Y. B. RUTICKI, "Convex Functions and Orlicz Spaces," Noordhoff, New York, 1961.
20. O. LADYZHENSKAYA AND N. URAL'TSEVA, Linear and quasilinear elliptic equations, *in* "Math. in Science and Engineering," Vol. 46, Academic Press, San Diego, 1968.
21. O. LADYZHENSKAYA AND N. URAL'TSEVA, Local estimates for gradients of solutions of non-uniformly elliptic and parabolic equations, *Comm. Pure Appl. Math.* **23** (1970), 677–703.
22. F. LEONETTI, Two-dimensional regularity of minima of variational functionals without standard growth conditions, *Ricerche Mat.* **38** (1989), 41–50.
23. F. LEONETTI, Weak differentiability for solutions to nonlinear elliptic systems with p, q-growth conditions, *Ann. Mat. Pura Appl.* **162** (1992), 349–366.
24. F. LEONETTI, Higher differentiability for weak solutions of elliptic systems with non-standard growth, *Ricerche Mat.*, in press.
25. G. M. LIEBERMAN, The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations, *Commun. Partial Differential Equations* **16** (1991), 311–361.
26. G. M. LIEBERMAN, On the regularity of the minimizer of a functional with exponential growth, preprint.
27. J. J. MANFREDI, Regularity for minima of functionals with p-growth, *J. Differential Equations* **76** (1988), 203–212.
28. P. MARCELLINI, Un exemple de solution discontinue d'un problème variationnel dans le cas scalaire, Istituto Matematico "U. Dini," No. 11, Università di Firenze, 1987.
29. P. MARCELLINI, Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions, *Arch. Rational Mech. Anal.* **105** (1989), 267–284.
30. P. MARCELLINI, Regularity and existence of solutions of elliptic equations with p, q-growth conditions, *J. Differential Equations* **90** (1991), 1–30.
31. P. MARCELLINI AND C. SBORDONE, Homogenization of non-uniformly elliptic operators, *Appl. Anal.* **8** (1978), 101–113.
32. G. MOSCARIELLO AND L. NANIA, Hölder continuity of minimizers of functionals with non standard growth conditions, *Ricerche Mat.* **40** (1991), 259–273.
33. J. SERRIN, Gradient estimates for solutions of nonlinear elliptic and parabolic equations, *in* "Contributions to Nonlinear Functional Analysis" (E. H. Zarantonello, Ed.), pp. 565–601, Academic Press, San Diego, 1971.
34. L. SIMON, Interior gradients bounds for non-uniformly elliptic equations, *Indiana Univ. Math. J.* **25** (1976), 821–855.
35. B. STROFFOLINI, Global boundedness of solutions of anisotropic variational problems, *Boll. Un. Mat. Ital. A* **5** (1991), 345–352.
36. G. TALENTI, Boundedness of minimizers, *Hokkaido Math. J.* **19** (1990), 259–279.
37. TANG QI, Regularity of minimizers of nonisotropic integrals in the calculus of variations, *Ann. Mat. Pura Appl.*, in press.
38. K. UHLENBECK, Regularity for a class of non-linear elliptic systems, *Acta Math.* **138** (1977), 219–240.