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Regularity for Some Scalar Variational Problems Under General Growth Conditions

P. MARCELLINI[1]

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Abstract. We consider integrals of the calculus of variations over a set Ω of $\mathbb{R}^n$ and the related regularity result: are the minimizers smooth functions, say for example of class $C^\infty(\Omega)$? Classically, the so-called natural growth conditions on the integrand have been the main sufficient assumptions for regularity. In recent years, motivated also by application, the interest in the study of this problem has increased under more general growth assumptions. In this paper, we propose some general growth conditions that guarantee regularity for a class of scalar variational problems.

Key Words. Calculus of variations, direct methods, regularity, nonstandard growth conditions, general growth conditions.

1. Introduction

We are interested in the local (i.e., interior) Lipschitz continuity, and the local $C^{1,\alpha}$ regularity as a consequence, of minimizers of integral functionals of the calculus of variations of the type

$$F(v) = \int_\Omega f(Dv)\,dx, \tag{1}$$

where Ω is an open set of $\mathbb{R}^n$, $n \geq 2$, Du is the $N \times n$ matrix of the gradient of $u: \Omega \to \mathbb{R}^N$, $N \geq 1$, and f: $\mathbb{R}^{N\times n} \to \mathbb{R}$ is a given function.

Existence of minimizers in the Sobolev class $W^{1,p}(\Omega, \mathbb{R}^N)$ can be obtained by a coercivity assumption, for example of the type

$$f(\xi) \geq c_1 |\xi|^p, \qquad \forall \xi \in \mathbb{R}^{N\times n}, \tag{2}$$

for some constant $c_1 > 0$ and some exponent $p > 1$, and by a convexity condition on $f(\xi)$ that guarantees the lower semicontinuity of $F(v)$ in the weak

[1]Professor, Department of Mathematics U. Dini, University of Florence, Firenze, Italy.

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topology of $W^{1,p}(\Omega, \mathbb{R}^N)$, or more generally a quasiconvexity condition on $f(\xi)$ in the Morrey sense (Ref. 1), or a polyconvexity condition on $f(\xi)$ (see Ball in Ref. 2; see also the book of Dacorogna in Ref. 3) of the type

$$f(\xi)=(1+|\xi|^2)^{p/2}+g(\det \xi), \tag{3}$$

in the case $n=N$, where g is a convex function on $\mathbb{R}$ and $\det \xi$ is the determinant of the matrix $\xi \in \mathbb{R}^{n\times n}$. Functions $f(\xi)$ of the type (3), with $p<n$, are relevant as models of the energy density functionals in the theory of nonlinear elasticity; see Ball in Ref. 2 for more details and a more appropriate context.

In order to show that a weak solution in $W^{1,p}(\Omega, \mathbb{R}^N)$ belongs in fact to a class of smooth functions, i.e., to study the regularity problem, classically the so called natural growth conditions have been considered as main assumptions; see, for example, Refs. 1, 4, 5. These natural conditions can be stated in the form

$$f(\xi)\leq c_2(1+|\xi|^p), \qquad |Df(\xi)|\leq c_2(1+|\xi|^{p-1}), \tag{4a}$$

$$|D^2f(\xi)|\leq c_2(1+|\xi|^{p-2}), \qquad \forall \xi \in \mathbb{R}^{N\times n}, \tag{4b}$$

for some constant c_2, together with the ellipticity condition that, in the convex case, can be expressed for example in the form

$$\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(\xi)\lambda_i^\alpha \lambda_j^\beta \geq c_3[1+|\xi|^2]^{(p-2)/2}|\lambda|^2, \qquad \forall \lambda, \xi \in \mathbb{R}^{N\times n}, \tag{5}$$

for some constant $c_3>0$; here, $\xi=(\xi_i^\alpha)$, $i=1, 2, \ldots, n$, $\alpha=1, 2, \ldots, N$. This ellipticity condition rules out quasiconvex functions $f(\xi)$ of the type (3), because they are not convex and since they do not satisfy the natural conditions (4). In fact, if the function $g=g(t)$ goes to $+\infty$ as $t\to+\infty$, then by its convexity it grows at infinitely linearly or more than linearly; therefore, functions $f(\xi)$ in (3) are coercive with exponent p, but their growth can be controlled by a larger exponent q; i.e., we have the growth conditions (in the specific case $1<p<n\leq q$ and in general $p<q$)

$$1<p<q, \qquad c_1|\xi|^p\leq f(\xi)\leq c_4(1+|\xi|^q), \quad \forall \xi \in \mathbb{R}^{N\times n}, \tag{6}$$

with $c_4\geq c_1>0$. Therefore, it seems relevant, at least in the applications to nonlinear elasticity, to study regularity properties under growth conditions more general than the natural ones in (4). We are not yet able to treat the general quasiconvex case for $N>1$; however, we can propose an approach to regularity in the scalar case (i.e., $N=1$) under general growth conditions.

In recent years, interest has increased in the study of regularity of solutions of elliptic equations and systems, and of minimizers of integral functionals of the calculus of variations under nonstandard and general growth conditions. These studies started from the regularity results obtained by Marcellini (Refs. 6, 7), who considered integrals of the type (1), together with its natural generalizations, satisfying the so called (p, q) growth conditions; i.e., for functions $f\in C^2(\mathbb{R}^n)$,

$$m[1+|\xi|^2]^{(p-2)/2}|\lambda|^2\leq\sum_{i,j} f_{\xi_i\xi_j}(\xi)\lambda_i\lambda_j$$
$$\leq M[1+|\xi|^2]^{(q-2)/2}|\lambda|^2, \qquad \forall\lambda,\ \xi\in\mathbb{R}^n, \tag{7}$$

for some constants $M\geq m>0$ and exponents $q\geq p\geq 2$ such that the ratio q/p is sufficiently close to 1 (in dependence of the dimension n). Examples of singular (more precisely, unbounded) weak minimizers of integrals of (p, q) growth have been given in Refs. 7–9 with ratio q/p large. A sharp condition has been introduced by Boccardo, Marcellini, and Sbordone in Ref. 10 to study the special anisotropic case, i.e., for example a scalar case of the type

$$F(v)=\int_\Omega \{(1+|Dv|^2)^{p/2}+\sum_i |v_{x_i}|^{q_i}\}\, dx, \tag{8}$$

for some exponents p, $q_i>1$, for $i=1, 2, \ldots, n$. Further results for anisotropic functionals and equations, as well as for variational problems with nonstandard growth conditions can be found in the bibliography of Ref. 11. In Refs. 11, 12 an approach has been proposed to the regularity of a class of elliptic problems under general growth conditions, including the slow exponential growth, i.e., the case

$$f(\xi)\sim\exp(|\xi|^p), \qquad \text{as } |\xi|\to+\infty, \tag{9}$$

of growth $\exp|\xi|^p$, by assuming that p is a positive real number close to zero. We mention also the Lipschitz regularity and the $C^{1,\alpha}_{\mathrm{loc}}$ regularity obtained for systems and the regularity results specific for functions depending on the modulus of the gradient, i.e., $f(\xi)=g(|\xi|)$, quoted in the bibliography of Ref. 13. For completeness, we quote the regularity of weak solutions under natural growth conditions: main references are the books in Refs. 1, 4, 5; more recent results can be found in Refs. 12, 14–18. A priori gradient estimates for classical solutions of non-uniformly elliptic equations have been given in Refs. 19–23.

In this paper, we propose some regularity results for minimizers of the integral in (1) that can be applied not only to the case of arbitrarily large exponential growth, as in (9) with $p>0$ arbitrarily large, but also to functions whose behavior at infinity can be represented by any finite composition of the following type:

$$f(\xi) \sim (\exp(\ldots(\exp(\exp|\xi|^{p_0})^{p_1})^{p_2})\ldots)^{p_k}, \quad \text{as } |\xi| \to +\infty, \tag{10}$$

with $p_i > 0$, $\forall i = 0, 1, 2, \ldots, k$.

Theorem 2.1 that we propose in the next section is related to a general integrand $f(\xi)$; it is an extension, in the scalar case, of a similar result proved by the author (Ref. 13) in the vector-valued case with integrands $f(\xi) = g(|\xi|)$ depending on the modulus $|\xi|$ of the gradient variable ξ. It can also be considered a unified approach to the regularity of minimizers of the integral in (1), since it can be applied not only to the critical examples in (9), (10), but also to the case of (p, q) growth in (7).

2. Statements of Results

We assume that $f: \mathbb{R}^n \to \mathbb{R}$ is a nonnegative function of class $C^2(\mathbb{R}^n)$ satisfying the following general growth conditions: there exist two increasing functions $g_1, g_2: [0, +\infty) \to [0, +\infty)$, not identically zero, and a constant $C > 0$ such that

$$g_1(|\xi|)|\lambda|^2 \leq \sum_{i,j} f_{\xi_i \xi_j}(\xi)\lambda_i\lambda_j \leq g_2(|\xi|)|\lambda|^2, \qquad \forall \lambda, \xi \in \mathbb{R}^n, \tag{11}$$

$$g_2(t)t^2 \leq C\left\{1 + \int_0^t \sqrt{g_1(s)}\, ds\right\}^{\alpha}, \qquad \forall t \geq 0, \tag{12}$$

$$g_2(|\xi|)|\xi|^2 \leq C\{1 + f(\xi)\}^{\beta}, \qquad \forall \xi \in \mathbb{R}^n, \tag{13}$$

for some real exponents α, β related by the constraints

$$2 \leq \alpha < 2n/(n-2), \qquad 1 \leq \beta < (2/n)[\alpha/(\alpha - 2)]. \tag{14}$$

When $n = 2$, in (14) α can be any real number greater than or equal to 2; if $\alpha = 2$, then β can be any real number greater than or equal to 1.

Since our assumptions imply that f has at least quadratic growth at infinity (but not necessarily polynomial growth), by a minimizer of the integral functional (1) we mean a function $u \in W^{1,2}_{\text{loc}}(\Omega)$ such that $f(Du) \in L^1_{\text{loc}}(\Omega)$, with the property that

$$F(u) \leq F(u + \varphi), \qquad \text{for every } \varphi \in C^1_0(\Omega).$$

Let us denote by B_ρ, B_R balls of radii respectively ρ and R, $\rho < R$, whose closure is contained in Ω, with the same center.

Theorem 2.1. Under the general growth conditions (11)–(14), every minimizer of the integral functional (1) is locally Lipschitz continuous and, for every $R>\rho>0$, there exists a constant $c=c(n, \alpha, \beta, \rho, R)$ such that

$$\|Du\|^2_{L^\infty(B_\rho;\mathbb{R}^n)}\leq c\left\{\int_{B_R} [1+f(Du)]\, dx\right\}^\theta, \tag{15}$$

for some exponent $\theta=\theta(\alpha, \beta, n)\geq 1$.

Remark 2.1. In the natural growth conditions case, assumptions (11)–(14) hold with $\alpha=2$ and $\beta=1$. We can verify that (11)–(14) also hold with arbitrary α, β satisfying the strict inequalities in (14) in the case of general exponential growth as in (9), (10). The (p, q) growth case in (7) corresponds to the assumptions (11)–(14), with

$$q/p=\alpha/2=\beta<1+2/n,$$

and we obtain the regularity result of Theorem 2.1 if

$$1\leq q/p<1+2/n.$$

More generally, we can consider functions g_1, g_2: $[0, +\infty)\rightarrow[0, +\infty)$ satisfying the conditions

$$(d/dt)g_1(t)\leq Cg_1(t)/t, \qquad \forall t>0, \tag{16}$$

$$g_2(|\xi|)|\xi|^2\leq C\{1+\min\{g_1(|\xi|)|\xi|^2, f(\xi)\}\}^\beta, \qquad \forall\xi\in\mathbb{R}^n, \tag{17}$$

for some exponent β such that

$$1\leq\beta<1+2/n. \tag{18}$$

As proved at the end of Section 3, conditions (16)–(18) imply (11)–(14) with $\alpha=2\beta$. Note that the uniformly elliptic case (uniformly at infinity) is realized in (16)–(18) when

$$g_2\leq\text{const}(1+g_1) \quad \text{and} \quad \beta=1.$$

Once we have the estimate for the L^∞-norm of the gradient, then the behavior as $t\rightarrow+\infty$ of $g_1(t)$, $g_2(t)$ becomes irrelevant to obtain further regularity of solutions. Thus, we can apply the results known under natural growth conditions (see Refs. 1, 4, 5) and we obtain, for example, the following regularity result in the classical sense.

Corollary 2.1. Let us assume that the general growth conditions (11)–(14) hold with strictly positive functions g_1, g_2: $[0, +\infty)\rightarrow(0, +\infty)$. Then, if $f \in C^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $k \geq 2$, every minimizer of the integral (1) is of class $C^{k,\alpha}_{\mathrm{loc}}(\Omega)$.

3. Proofs

Proof of Theorem 2.1. This proof includes five steps.

Step 1. Let u be a minimizer of (1). First, let us make the following supplementary assumptions (that will be removed in Step 5): f is uniformly strictly convex and u is locally Lipschitz continuous; more precisely, there exists a positive constant m, and for every R there exists a constant $L = L(R)$, such that

$$m|\lambda|^2 \leq \sum_{i,j} f_{\xi_i\xi_j}(\xi)\lambda_i\lambda_j, \qquad \forall \lambda, \xi \in \mathbb{R}^n, \tag{19a}$$

$$\|Du\|^2_{L^\infty(B_R;\mathbb{R}^n)} \leq L(R). \tag{19b}$$

In the first four steps, we will derive the estimate (15) with constant c independent of m and L; then in Step 5, by an approximation argument, we will obtain the conclusion (15) without the supplementary assumptions (19).

Since $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, it satisfies the Euler equation: for every open set Ω' compactly contained in Ω, we have

$$\int_\Omega \sum_i f_{\xi_i}(Du)\phi_{x_i}\, dx = 0, \qquad \forall \phi \in W^{1,2}_0(\Omega'). \tag{20}$$

By also using (19a), with the technique of the difference quotient, we obtain (see, for example Theorem 1.1 of Chapter II of Ref. 5; in the specific context of nonstandard growth conditions see also Refs. 6, 7, 11, 24) that $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ and it satisfies the condition

$$\int_\Omega \sum_{i,j} f_{\xi_i\xi_j}(Du)u_{x_jx_k}\phi_{x_i}\, dx = 0,$$
$$\forall k = 1, 2, \ldots, n,\ \forall \phi \in W^{1,2}_0(\Omega'). \tag{21}$$

For fixed $k \in \{1, 2, \ldots, n\}$, let $\eta \in C^1_0(\Omega')$ and $\phi = \eta^2 u_{x_k}\Phi(|Du|)$, where Φ is a positive, increasing, locally Lipschitz continuous function in $[0, +\infty)$. Then, a.e. in Ω,

$$\phi_{x_i} = 2\eta\eta_{x_i}u_{x_k}\Phi(|Du|) + \eta^2 u_{x_ix_k}\Phi(|Du|)$$
$$+ \eta^2 u_{x_k}\Phi'(|Du|)(|Du|)_{x_i},$$

and from Equation (21), we deduce that

$$\int_\Omega 2\eta\Phi(|Du|)\sum_{i,j} f_{\xi_i\xi_j}(Du)u_{x_jx_k}\eta_{x_i}u_{x_k}\,dx$$
$$+\int_\Omega \eta^2\Phi(|Du|)\sum_{i,j} f_{\xi_i\xi_j}(Du)u_{x_jx_k}u_{x_ix_k}\,dx$$
$$+\int_\Omega \eta^2\Phi'(|Du|)\sum_{i,j} f_{\xi_i\xi_j}(Du)u_{x_jx_k}u_{x_k}(|Du|)_{x_i}\,dx=0. \tag{22}$$

We can estimate the first integral in (22) by using the Cauchy–Schwarz inequality and the inequality

$$2ab\le(1/2)a^2+2b^2.$$

We have

$$\left|\int_\Omega 2\eta\Phi(|Du|)\sum_{i,j} f_{\xi_i\xi_j}(Du)u_{x_jx_k}\eta_{x_i}u_{x_k}\,dx\right|$$
$$\le\int_\Omega 2\Phi(|Du|)\left(\eta^2\sum_{i,j} f_{\xi_i\xi_j}(Du)u_{x_ix_k}u_{x_jx_k}\right)^{1/2}$$
$$\times\left(\sum_{i,j} f_{\xi_i\xi_j}(Du)\eta_{x_i}u_{x_k}\eta_{x_j}u_{x_k}\right)^{1/2}dx$$
$$\le(1/2)\int_\Omega \eta^2\Phi(|Du|)\sum_{i,j} f_{\xi_i\xi_j}(Du)u_{x_ix_k}u_{x_jx_k}\,dx$$
$$+2\int_\Omega \Phi(|Du|)\sum_{i,j} f_{\xi_i\xi_j}(Du)\eta_{x_i}u_{x_k}\eta_{x_j}u_{x_k}\,dx.$$

From (22), we obtain

$$(1/2)\int_\Omega \eta^2\Phi(|Du|)\sum_{i,j} f_{\xi_i\xi_j}(Du)u_{x_jx_k}u_{x_ix_k}\,dx$$
$$+\int_\Omega \eta^2\Phi'(|Du|)\sum_{i,j} f_{\xi_i\xi_j}(Du)u_{x_jx_k}u_{x_k}(|Du|)_{x_i}\,dx$$
$$\le 2\int_\Omega \Phi(|Du|)\sum_{i,j} f_{\xi_i\xi_j}(Du)\eta_{x_i}u_{x_k}\eta_{x_j}u_{x_k}\,dx. \tag{23}$$

Since a.e. in Ω

$$(|Du|)_{x_i} = [1/|Du|] \sum_k u_{x_i x_k} u_{x_k}, \tag{24}$$

then it is natural in (23) to sum up with respect to $k = 1, 2, \ldots, n$; we obtain

$$\sum_k \sum_{i,j} f_{\xi_i \xi_j}(Du) u_{x_j x_k} u_{x_k} (|Du|) x_i$$

$$= |Du| \sum_{i,j} f_{\xi_i \xi_j}(Du) (|Du|)_{x_j} (|Du|)_{x_i}.$$

Therefore from (23), we deduce the estimate

$$\int_\Omega \eta^2 \Phi(|Du|) \sum_{k,i,j} f_{\xi_i \xi_j}(Du) u_{x_j x_k} u_{x_i x_k}\, dx$$

$$+ \int_\Omega \eta^2 |Du| \Phi'(|Du|) \sum_{i,j} f_{\xi_i \xi_j}(Du) (|Du|)_{x_j} (|Du|)_{x_i}\, dx$$

$$\leq 4 \int_\Omega \Phi(|Du|) \sum_{k,i,j} f_{\xi_i \xi_j}(Du) \eta_{x_i} u_{x_k} \eta_{x_j} u_{x_k}\, dx.$$

By applying the Cauchy–Schwarz inequality to (24), we have

$$|(|Du|)_{x_i}|^2 \leq \sum_k (u_{x_i x_k})^2, \qquad \forall i = 1, 2, \ldots, n.$$

Thus,

$$|D(|Du|)|^2 \leq |D^2 u|^2,$$

and by the ellipticity condition in (11), we finally obtain

$$\int_\Omega \eta^2 [\Phi(|Du|) + |Du| \Phi'(|Du|)] g_1(|Du|) |D(|Du|)|^2\, dx$$

$$\leq 4 \int_\Omega |D\eta|^2 \Phi(|Du|) g_2(|Du|) |Du|^2\, dx. \tag{25}$$

Step 2. Let us define

$$G(t)=1+\int_0^t \sqrt{\Phi(s)g_1(s)}\,ds, \qquad \forall t\geq 0.$$

Since g_1 and Φ are increasing, then we have

$$\begin{aligned}[G(t)]^2 &\leq [1+t\sqrt{\Phi(t)g_1(t)}]^2\\ &\leq 2(1+\Phi(t)g_1(t)t^2)\\ &\leq 2(1+\Phi(t)g_2(t)t^2),\end{aligned}$$

and

$$\begin{aligned}&|D(\eta G(|Du|)|^2\\ &\leq 2|D\eta|^2[G(|Du|)]^2+2\eta^2[G'(|Du|)]^2|D(|Du|)|^2\\ &\leq 4|D\eta|^2(1+\Phi(|Du|)g_2(|Du|)|Du|^2)\\ &+2\eta^2\Phi(|Du|)g_1(|Du|)|D(|Du|)|^2. \end{aligned}\tag{26}$$

From (25), (26), we deduce that

$$\begin{aligned}&\int_\Omega |D(\eta G(|Du|)|^2\,dx\\ &\leq 4\int_\Omega |D\eta|^2(1+3\Phi(|Du|)g_2(|Du|)|Du|^2)\,dx.\end{aligned}\tag{27}$$

Let us denote by 2^* the Sobolev exponent related to 2; i.e.,

$$2^*=2n/(n-2), \qquad \text{if } n\geq 3,$$

while 2^* is any fixed real number greater than α, if $n=2$. By the Sobolev inequality, there exists a constant c_1 such that

$$\left\{\int_\Omega [\eta G(|Du|)]^{2^*}\,dx\right\}^{2/2^*}\leq c_1\int_\Omega |D(\eta G(|Du|)|^2\,dx.\tag{28}$$

Let us define

$$\Phi(t)=t^{2\gamma}, \qquad \text{with } \gamma\geq 0,$$

so that Φ is increasing. Since the functions $t \to t^\gamma$ and $t \to g_1(t)$ are increasing, then we have [see for example Lemma 3.4(v) of Ref. 11]

$$\begin{aligned} G(t) &= 1 + \int_0^t \sqrt{\Phi(s) g_1(s)}\, ds \\ &= 1 + \int_0^t s^\gamma \sqrt{g_1(s)}\, ds \\ &\geq 1 + (1/t) \int_0^t s^\gamma\, ds \int_0^t \sqrt{g_1(s)}\, ds \\ &= 1 + [t^\gamma/(\gamma+1)] \int_0^t \sqrt{g_1(s)}\, ds. \end{aligned} \tag{29}$$

Recall that the function $g_1: [0, +\infty) \to [0, +\infty)$ is increasing and not identically zero. Then, there exists $t_0 > 0$ such that

$$g_1(t) > 0, \qquad \text{for every } t \geq t_0.$$

Up to a rescaling, we can assume that $t_0 \leq 1$, so that

$$g_1(1) = c_2 > 0, \qquad \int_0^1 \sqrt{g_1(s)}\, ds = c_3 > 0. \tag{30}$$

Then, for every $t \geq 1$, we have

$$\begin{aligned} 2 \int_0^t \sqrt{g_1(s)}\, ds &\geq c_3 + \int_0^t \sqrt{g_1(s)}\, ds \\ &\geq \min\{c_3, 1\} \left\{ 1 + \int_0^t \sqrt{g_1(s)}\, ds \right\}. \end{aligned} \tag{31}$$

Now, we use assumptions (12) and (14). By (12) and by (31), there exists a constant c_4 such that

$$\left\{ \int_0^t \sqrt{g_1(s)}\, ds \right\}^\alpha \geq c_4 g_2(t) t^2, \qquad \forall t \geq 1. \tag{32}$$

Recall that, by (14), $\alpha \in [2, 2^*)$. Then, by (30), for every $t \geq 1$ we obtain

$$\begin{aligned} \left\{ \int_0^t \sqrt{g_1(s)}\, ds \right\}^{2^*} &\geq \left\{ \int_0^t \sqrt{g_1(s)}\, ds \right\}^\alpha \{c_2(t-1) + c_3\}^{2^*-\alpha} \\ &\geq (\min\{c_2, c_3\})^{2^*-\alpha} c_4 g_2(t) t^{2^*-\alpha+2}, \qquad \forall t \geq 1. \end{aligned} \tag{33}$$

By (33) and by (29), we deduce that

$$[G(t)]^{2^*} \geq (1/2)\left[1 + \left\{[t^\gamma/(\gamma+1)] \int_0^t \sqrt{g_1(s)}\, ds\right\}^{2^*}\right]$$
$$\geq 1/2 + \{[\min\{c_2, c_3\}]^{2^*-\alpha} c_4/[2(\gamma+1)^2]\}$$
$$\times t^{(\gamma+1)2^*-\alpha+2} g_2(t), \qquad \forall t \geq 1.$$

That is, there exists a constant c_5 such that

$$c_5(\gamma+1)^{2^*}[G(t)]^{2^*} \geq 1 + t^{(\gamma+1)2^*-\alpha+2} g_2(t), \tag{34}$$

for every $t \geq 1$, and also for $t \in [0, 1)$, since the left-hand side is bounded for $t \in [0, 1)$ by the constant (independent of γ) $1 + g_2(1)$, while the right-hand side is bounded from below away from zero.

From (27), (28), (34), we obtain

$$\left\{\int_\Omega \eta^{2^*}(1 + |Du|^{(\gamma+1)2^*-\alpha+2} g_2(|Du|))\, dx\right\}^{2/2^*}$$
$$\leq c_6(\gamma+1)^2 \int_\Omega |D\eta|^2 (1 + |Du|^{2(\gamma+1)} g_2(|Du|))\, dx.$$

Let us denote by B_R and B_ρ balls compactly contained in Ω, of radii respectively R, ρ, with the same center. Let η be a test function equal to 1 in B_ρ, whose support is contained in B_R, such that

$$|D\eta| \leq 2/(R-\rho).$$

We finally obtain

$$\left\{\int_{B_\rho} (1 + |Du|^{(\gamma+1)2^*-\alpha+2} g_2(|Du|))\, dx\right\}^{2/2^*}$$
$$\leq c_6[2(\gamma+1)/(R-\rho)]^2 \int_{B_R} (1 + |Du|^{2(\gamma+1)} g_2(|Du|))\, dx. \tag{35}$$

Step 3. We define by induction a sequence γ_k in the following way:

$$\gamma_1 = 0, \qquad \gamma_{k+1} = (2^*/2)\gamma_k + (2^*-\alpha)/2, \quad \forall k \in \mathbb{N}; \tag{36}$$

we note in particular that γ_k satisfies the property

$$(\gamma_k+1)2^* - \alpha + 2 = 2(\gamma_{k+1}+1), \qquad \forall k \in \mathbb{N}. \tag{37}$$

It is easy to prove by induction the following representation formula for γ_k:

$$\gamma_k = [(2^* - \alpha)/(2^* - 2)][(2^*/2)^{k-1} - 1], \qquad \forall k \in \mathbb{N}. \tag{38}$$

Since $\alpha \geq 2$, from (37), (38), we deduce the inequality

$$(\gamma_k + 1)2^* - \alpha + 2 \geq 2[(2^* - \alpha)/(2^* - 2)](2^*/2)^k. \tag{39}$$

For fixed R_0 and ρ_0, for all $k \in \mathbb{N}$, we rewrite (35) with $R = \rho_{k-1}$ and $\rho = \rho_k$, where

$$\rho_k = \rho_0 + (R_0 - \rho_0)/2^k;$$

moreover, for $k = 1, 2, 3, \ldots, i$, with i fixed in $\mathbb{N}$, we put γ equal to γ_k. By iterating (35), by (37) we obtain

$$\left\{\int_{B_{\rho_i}} (1 + |Du|^{(\gamma_i+1)2^* - \alpha + 2} g_2(|Du|))\, dx\right\}^{(2/2^*)^i}$$

$$\leq c_7 \int_{B_{R_0}} (1 + |Du|^2 g_2(|Du|))\, dx. \tag{40}$$

Since

$$R - \rho = \rho_{k-1} - \rho_k = (R_0 - \rho_0)2^{-k}, \qquad \text{for every } k \in \mathbb{N},$$

we have

$$\begin{aligned} c_7 &= \prod_{k=1}^{\infty} [c_3 2^8 (2^*)^{k-1}/(R_0 - \rho_0)^2]^{(2/2^*)^{k-1}} \\ &= (2^*)^{\sum_{k=1}^{\infty}(k-1)(2/2^*)^{k-1}} [c_3 2^8/(R_0 - \rho_0)^2]^{\sum_{k=1}^{\infty}(2/2^*)^{k-1}} \\ &= c_8 (R_0 - \rho_0)^{-2/(1-2/2^*)} \\ &= c_8 (R_0 - \rho_0)^{-n} \end{aligned} \tag{41}$$

if $n \geq 3$; otherwise, if $n = 2$, then for every $\epsilon > 0$ we can choose 2^* so that

$$c_7 = c_8 (R_0 - \rho_0)^{-2-\epsilon}, \qquad \text{for some constant } c_8.$$

Since g_2 is increasing, then

$$g_2(t) \geq g_1(t) \geq g_1(1) > 0, \qquad \text{for every } t \geq 1.$$

Therefore for $r \geq s \geq 0$, we have

$$g_2(t)t^r + 1 \geq g_1(1)t^s, \qquad \text{if } t \geq 1,$$

$$g_2(t)t^r + 1 \geq 1 \geq t^s, \qquad \text{if } 0 \leq t \leq 1.$$

Thus, by posing

$$c_9 = \min\{g_1(1), 1\},$$

we obtain

$$g_2(t)t^r + 1 \geq c_9 t^s, \qquad \forall t \geq 0, \qquad \forall r \geq s \geq 0. \tag{42}$$

Now, we go to the limit in (40) as $i \to +\infty$. We use the inequalities (39), (40), (42); we obtain

$$\sup\{|Du(x)|^{2[(2^* - \alpha)/(2^* - 2)]}: x \in B_{\rho_0}\}$$

$$= \lim_{i \to +\infty} \left\{ \int_{B_{\rho_0}} |Du|^{2[(2^* - \alpha)/(2^* - 2)](2^*/2)^i} \, dx \right\}^{(2/2^*)^i}$$

$$\leq \limsup_{i \to +\infty} \left\{ (1/c_9) \int_{B_{\rho_i}} (1 + |Du|^{(\gamma_i + 1)2^* - \alpha + 2} g_2(|Du|)) \, dx \right\}^{(2/2^*)^i}$$

$$\leq (c_7/c_9) \int_{B_{R_0}} (1 + |Du|^2 g_2(|Du|)) \, dx,$$

and by the representation of c_7 in (41), we finally obtain

$$\|Du\|^{2[(2^* - \alpha)/(2^* - 2)]}_{L^\infty(B_{\rho_0}; \mathbb{R}^n)}$$

$$\leq [c_{10}/(R_0 - \rho_0)^n] \int_{B_{R_0}} (1 + |Du|^2 g_2(|Du|)) \, dx. \tag{43}$$

Step 4. Let us consider again the estimates (27), (28), with Φ identically equal to 1,

$$\left\{ \int_\Omega [\eta G(|Du|)]^{2^*} \, dx \right\}^{2/2^*}$$

$$\leq 4c_1 \int_\Omega |D\eta|^2 (1 + 3g_2(|Du|)|Du|^2) \, dx. \tag{44}$$

We use assumption (12) with $a \in [2, 2^*)$, and we represent α under the form $\alpha = 2^*/\delta$; thus, $1 < \delta \leq 2^*/2$. Then, like in the previous step, with $\Phi \equiv 1$ and $\gamma = 0$, we pose

$$G(t) = 1 + \int_0^t \sqrt{g_1(s)} \, ds, \qquad \forall t \geq 0.$$

By assumption (12) (we denote by c_2 the constant there appearing), we obtain

$$2[G(t)]^{2^*/\delta} \geq 1 + (1/c_2) g_2(t) t^2, \qquad \forall t \geq 0. \tag{45}$$

From (44) and (45), we deduce that

$$\left\{ \int_\Omega \eta^{2^*} (1 + |Du|^2 g_2(|Du|))^\delta \, dx \right\}^{2/2^*}$$
$$\leq \int_\Omega c_3 \, |D\eta|^2 (1 + |Du|^2 g_2(|Du|)) \, dx.$$

Like in the previous Step 3, we consider a test function η equal to 1 in B_ρ, with support contained in B_R and such that

$$|D\eta| \leq 2/(R - \rho).$$

We obtain

$$\left\{ \int_{B_\rho} V^\delta \, dx \right\}^{2/2^*} \leq [4c_3/(R-\rho)^2] \int_{B_R} V \, dx,$$

where

$$V = V(x) = 1 + |Du|^2 g_2(|Du|).$$

Let $\gamma > 2^*/2$ to be chosen later. By the Hölder inequality, then we have

$$\begin{aligned} \left\{ \int_{B_\rho} V^\delta \, dx \right\}^{2/2^*} &\leq [4c_3/(R-\rho)^2] \int_{B_R} V^{\delta/\gamma} V^{1-\delta/\gamma} \, dx \\ &\leq [4c_3/(R-\rho)^2] \left\{ \int_{B_R} V^\delta \, dx \right\}^{1/\gamma} \\ &\quad \times \left\{ \int_{B_R} V^{(\gamma-\delta)/(\gamma-1)} \, dx \right\}^{(\gamma-1)/\gamma}. \end{aligned} \tag{46}$$

For fixed R_0 and ρ_0, for all $i \in \mathbb{N}$, we consider (46) with $R = \rho_i$ and $\rho = \rho_{i-1}$, with

$$\rho_i = R_0 - (R_0 - \rho_0)/2^i.$$

By iterating (46), since

$$R - \rho = (R_0 - \rho_0) 2^{-i},$$

similarly to the computation in (41) we obtain

$$\begin{aligned}
&\int_{B_{\rho_0}} V^\delta\,dx \\
&\leq \left\{\int_{B_{\rho_i}} V^\delta\,dx\right\}^{(2^*/2\gamma)^i} \prod_{i=1}^{\infty} \{c_3 4^{i+1}/(R_0-\rho_0)^2\}^{\gamma(2^*/2\gamma)^i} \\
&\quad\times\left\{\int_{B_{R_0}} V^{(\gamma-\delta)/(\gamma-1)}\,dx\right\}^{(\gamma-1)(2^*/2\gamma)^i} \\
&\leq \left\{\int_{B_{\rho_i}} V^\delta\,dx\right\}^{(2^*/2\gamma)^i} c_4\{1/(R_0-\rho_0)^2\}^{2^*\gamma/(2\gamma-2^*)} \\
&\quad\times\left\{\int_{B_{R_0}} V^{(\gamma-\delta)/(\gamma-1)}\,dx\right\}^{2^*(\gamma-1)/(2\gamma-2^*)}.
\end{aligned} \tag{47}$$

Now, we use assumption (13) in the form

$$\begin{aligned}
V&=1+|Du|^2 g_2(|Du|) \\
&\leq 2\max\{1, C\}\{1+f(Du)\}^\beta,
\end{aligned} \tag{48}$$

where

$$\beta=\beta(\gamma)=(\gamma-1)/(\gamma-\delta)=(\gamma-1)/(\gamma-2^*/\alpha)\geq 1. \tag{49}$$

Since

$$\lim_{\gamma\to 2^*/2} \beta(\gamma)=[\alpha/(\alpha-2)][(2^*-2)/2^*]=(2/n)[\alpha/(\alpha-2)]$$

[if $n=2$ then, by choosing 2^* sufficiently large, the limit of $\beta(\gamma)$ can be any number less than $\alpha/(\alpha-2)$], then by assumption (14), it is possible to choose $\gamma\in(2^*/2, +\infty)$ so that the representation (49) of β in terms of γ holds. We go to the limit in (47) as $i\to+\infty$, and we use (48), (49); we obtain

$$\begin{aligned}
&\int_{B_{\rho_0}} V^\delta\,dx \\
&\leq c_5\{1/(R_0-\rho_0)^2\}^{2^*\gamma/(2\gamma-2^*)}\left\{\int_{B_{R_0}} [1+f(Du)]\,dx\right\}^{2^*(\gamma-1)/(2\gamma-2^*)}.
\end{aligned}$$

Then, we get

$$\int_{B_{\rho_0}} V\,dx \leq \text{meas}\{B_{\rho_0}\}^{1-1/\delta}\left\{\int_{B_{\rho_0}} V^\delta\,dx\right\}^{1/\delta}$$
$$\leq c_6\{1/(R_0-\rho_0)^2\}^{2^*\gamma/[(2\gamma-2^*)\delta]}$$
$$\times\left\{\int_{B_{R_0}} [1+f(Du)]\,dx\right\}^{2^*(\gamma-1)/[(2\gamma-2^*)\delta]} \tag{50}$$

That is, for every ρ_0, R_0 with $0<\rho_0<R_0$, there exist a constant $c_7 = c_7(n, \alpha, \beta, \rho_0, R_0)$ such that

$$\int_{B_{\rho_0}} (1+|Du|^2 g_2(|Du|))\,dx$$
$$\leq c_7\left\{\int_{B_{R_0}} [1+f(Du)]\,dx\right\}^{2^*(\gamma-1)/[(2\gamma-2^*)\delta]} \tag{51}$$

By the previous Step 3 [see (43) with different ρ_0, R_0] and by posing

$$\theta=[2^*(\gamma-1)/(2\gamma-2^*)\delta][(2^*-2)/(2^*-\alpha)]\geq 1, \tag{52}$$

finally we deduce that, for every ρ_0, R_0 with $0<\rho_0<R_0$, a constant $c_8 = c_8(n, \alpha, \beta, \rho_0, R_0)$ exists such that

$$\|Du\|^2_{L^\infty(B_{\rho_0};\mathbb{R}^n)} \leq c_8\left\{\int_{B_{R_0}} [1+f(Du)]\,dx\right\}^\theta. \tag{53}$$

We emphasize that the constant c_8 does not depend on the parameters m, $L(R)$ in (19).

Step 5. Let u be a minimizer of the integral F in (1). Let α_ϵ be a smooth mollifier with compact support in Ω. For every $v\in W^{1,2}_{\text{loc}}(\Omega)$ such that $f(Dv)\in L^1_{\text{loc}}(\Omega)$, and for every open set Ω' compactly contained in Ω, we define

$$v_\epsilon(x)=\int_{\mathbb{R}^n} \alpha_\epsilon(y)v(x-y)\,dy, \qquad \forall x\in\Omega'.$$

Then, v_ϵ is a sequence of smooth functions that, as $\epsilon\to 0$, converges to v in $W^{1,2}(\Omega')$.

Let B_ρ be a ball compactly contained in Ω, and let us assume that $f(Dv) \in L^1(B_\rho)$. Then,

$$\lim_{\epsilon\to 0} \int_{B_\rho} f(Dv_\epsilon)\,dx = \int_{B_\rho} f(Dv)\,dx, \qquad \forall \rho : f(Dv) \in L^1(B_\rho). \tag{54}$$

To prove (54), we use the Jensen inequality

$$\begin{aligned} f(Dv_\epsilon(x)) &= f\left(\int_{\mathbb{R}^n} \alpha_\epsilon(y) Dv(x-y)\,dy\right) \\ &\leq \int_{\mathbb{R}^n} \alpha_\epsilon(y) f(Dv(x-y))\,dy, \qquad \forall x \in B_\rho. \end{aligned} \tag{55}$$

By integrating over B_ρ and by considering another ball B_R with radius $R > \rho$ and with the same center as B_ρ, for ϵ sufficiently small we obtain

$$\begin{aligned} \int_{B_\rho} f(Dv_\epsilon(x))\,dx &\leq \int_{\mathbb{R}^n} \alpha_\epsilon(y)\,dy \int_{B_\rho} f(Dv(x-y))\,dx \\ &\leq \int_{B_R} f(Dv)\,dx. \end{aligned} \tag{56}$$

First, we take the maximum limit as $\epsilon \to 0$ of both sides, then the limit as $R \to \rho$ in the right-hand side, and we obtain an inequality that, together with the lower semicontinuity of the integral, conclude the proof of (54).

Let B_R be a ball compactly contained in Ω. For every $\epsilon > 0$, we consider also the integral

$$F^\epsilon(v) = \int_\Omega \{\epsilon|Dv|^2 + f(Dv)\}\,dx, \tag{57}$$

and we note that, of course, the integrand f^ϵ of F^ϵ is uniformly strictly convex in the sense (19a); i.e.,

$$2\epsilon|\lambda|^2 \leq \sum_{i,j} f^\epsilon_{\xi_i\xi_j}(\xi)\lambda_i\lambda_j, \qquad \forall \lambda, \xi \in \mathbb{R}^n. \tag{58}$$

Let us denote by u_ϵ the minimizer of F^ϵ in the class of function with prescribed boundary value equal to v_ϵ on the boundary ∂B_R. Then, by the bounded slope condition theorem (see, for example, Chapter 1 of the book of Giusti in Ref. 25), u_ϵ is Lipschitz continuous in B_R and

$$\|Du_\epsilon\|_{L^\infty(B_R;\mathbb{R}^n)} \leq \|Dv_\epsilon\|_{L^\infty(B_R;\mathbb{R}^n)}. \tag{59}$$

To go on, we need a bound independent of ϵ for the L^∞-norm of Du_ϵ. This bound comes from the estimate (53) of Step 4 applied to u_ϵ; this estimate can be used, since the supplementary assumptions (19) are satisfied; see (58), (59). Then, for every $\rho < R$, a constant $c_8 = c_8(n, \alpha, \beta, \rho, R)$ exists such that

$$\begin{aligned}\|Du_\epsilon\|^2_{L^\infty(B_\rho;\mathbb{R}^n)} &\leq c_8 \left\{ \int_{B_R} \{1 + f^\epsilon(Du_\epsilon)\}\, dx \right\}^\theta \\ &\leq c_8 \left\{ \int_{B_R} \{1 + \epsilon |Dv_\epsilon|^2 + f(Dv_\epsilon)\}\, dx \right\}^\theta \\ &\leq c_9, \end{aligned} \tag{60}$$

the second inequality being satisfied, since u_ϵ is a minimizer of F^ϵ. Then, u_ϵ converges, up to a subsequence, in the weak* topology of $W^{1,\infty}(B_\rho)$, for every $\rho < R$, to a function $w \in W^{1,\infty}_{\mathrm{loc}}(B_R)$. By the lower semicontinuity of the integral and by (54), we deduce that

$$\begin{aligned}\int_{B_R} f(Dw)\, dx &\leq \liminf_{\epsilon \to 0} \int_{B_R} f(Du_\epsilon)\, dx \\ &\leq \lim_{\epsilon \to 0} \int_{B_R} \{\epsilon |Dv_\epsilon|^2 + f(Dv_\epsilon)\}\, dx \\ &= \int_{B_R} f(Dv)\, dx. \end{aligned} \tag{61}$$

Since v is a generic function in $W^{1,2}_{\mathrm{loc}}(\Omega)$ such that $f(Dv) \in L^1_{\mathrm{loc}}$ we obtain that w is a minimizer of the integral of $f(Dv)$ over the ball B_R. Moreover, by (60),

$$\|Dw\|^2_{L^\infty(B_\rho;\mathbb{R}^n)} \leq c_8 \left\{ \int_{B_R} \{1 + f(Dv)\}\, dx \right\}^\theta. \tag{62}$$

Our assumptions on f do not guarantee uniqueness of the minimizer for the Dirichlet problem. However, $g_1(t)$ is positive for $t \geq 1$; thus, $f(\xi)$ is locally strictly convex for $|\xi| > 1$. This implies that

$$|Dw(x)| = |Du(x)|, \qquad \text{for almost every } x \in B_R \text{ such that } |Du(x)| > 1,$$

and thus also Du satisfies (62) for every v, in particular for $v = u$. This completes the proof of Theorem 2.1. □

Proof of Statement in Remark 2.1. We conclude this section by proving the statement made in Remark 2.1 that conditions (16)–(18) imply the validity of (11)–(14). In fact, by (16) we obtain

$$(d/dt)\sqrt{g_1(t)} = (1/2)g_1^{-1/2}(d/dt)g_1 \leq (C/2)g_1^{1/2}/t, \qquad \forall t>0,$$

and then

$$\begin{aligned}\int_0^t \sqrt{g_1(s)}\, ds &\geq (2/C)\int_0^t s(d/ds)g_1^{1/2}(s)\, ds \\ &= (2/C)\left\{tg_1^{1/2} - \int_0^t \sqrt{g_1(s)}\, ds\right\}.\end{aligned} \tag{63}$$

Therefore,

$$t\sqrt{g_1(t)} \leq (C/2+1)\int_0^t \sqrt{g_1(s)}\, ds,$$

which, together with (17), gives

$$g_2(|\xi|)|\xi|^2 \leq c_2\left\{1 + \int_0^{|\xi|} \sqrt{g_1(s)}\, ds\right\}^{2\beta}, \qquad \forall \xi \in \mathbb{R}^n. \tag{64}$$

We obtain the conclusion, with the position $\alpha = 2\beta$, that guarantees the validity of (14) if $1 \leq \beta < 1 + 2/n$. □

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