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Implicit Second Order Partial Differential Equations

BERNARD DACOROGNA – PAOLO MARCELLINI

This paper is dedicated to the memory of Ennio De Giorgi, who was a master for us, as well as a gentle and modest person, in spite of his great personality. Always ready to listen, to discuss and to advise. We miss him.

Abstract. We prove existence of $W^{2,\infty}$ solutions to Dirichlet-Neumann problems related to some second order nonlinear partial differential equations in implicit form.

1. – Introduction

Motivated by some studies of existence problems of the calculus of variations without convexity assumptions (see [9]), we considered the following Dirichlet problem for systems of first order P.D.E.

$$\begin{cases} F_1(Du) = \ldots = F_N(Du) = 0, & \text{a.e. in } \Omega \\ u = \varphi, & \text{on } \partial\Omega, \end{cases}$$

with Ω an open set of $\mathbb{R}^n$, $n \geq 1$, u vector-valued function (i.e. $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ for some $m \geq 1$), Du the jacobian matrix of the gradient of u, $F_i : \mathbb{R}^{m\times n} \to \mathbb{R}$, $i = 1, 2, \ldots, N$, and the boundary datum φ vector-valued function of class $C^1(\overline{\Omega}; \mathbb{R}^m)$ (or piecewise of class C^1). By the existence results obtained in [10], [11], [12], [13] it has been possible to treat, for example, the *singular values* case (see also below in this introduction for more details), a generalization of the *eikonal equation* and attainment for the problem of *potential wells* (introduced by J. M. Ball and R. D. James [2] in the context of nonlinear elasticity; for a similar attainment result see also S. Müller and V. Sverak [21]).

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In this paper we study *second order* equations and systems, more precisely some boundary value problems associated to them. If we start from the case of one equation, the equations that we study take the form

$$F(x, u(x), Du(x), D^2u(x)) = 0, \quad x \in \Omega; \tag{1}$$

here u is a scalar function (i.e. $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$) and $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_s \to \mathbb{R}$ is a continuous function; since the matrix $D^2u(x)$ of the second derivatives is symmetric, then for every fixed $x \in \Omega$ this matrix is an element of the subset

$$\mathbb{R}^{n\times n}_s = \{\xi \in \mathbb{R}^{n\times n} : \xi = \xi^t\}$$

of the $n \times n$ matrices $\mathbb{R}^{n\times n}$.

We say that (1) is a second order partial differential equation of *implicit type*, since our hypotheses exclude that it is a quasilinear equation, i.e. it is not possible to write it as an equivalent equation which is linear with respect to the matrix of the second derivatives $D^2u(x)$.

We can consider, for example, the P.D.E.

$$|\Delta u| = 1, \quad \text{a.e. in } \Omega, \tag{2}$$

together with a boundary datum $u = \varphi$ on $\partial\Omega$. Instead, we could simply solve the Dirichlet problem with the same boundary datum, for the linear equation $\Delta u = 1$. But, the interesting fact is that, if we remain with the original nonlinear equation, then we can solve even a Dirichlet-Neumann problem of the type

$$\begin{cases} |\Delta u| = 1, & \text{a.e. in } \Omega \\ u = \varphi, & \text{on } \partial\Omega \\ \partial u/\partial \nu = \psi, & \text{on } \partial\Omega. \end{cases}$$

Independently of the differential equation, if a smooth function u is given on a smooth boundary $\partial\Omega$, then it is known also its tangential derivative. Therefore to prescribe Dirichlet and Neumann conditions at the same time is equivalent to give u and Du together.

This means that the Dirichlet-Neumann problem that we consider will be written, in the specific context of (2), under the form

$$\begin{cases} |\Delta u| = 1, & \text{a.e. in } \Omega \\ u = \varphi, & \text{on } \partial\Omega \\ Du = D\varphi, & \text{on } \partial\Omega \end{cases} \tag{3}$$

(note the compatibility condition that we have imposed on the boundary gradient, to be equal to the gradient $D\varphi$ of the boundary datum φ; of course we assume that φ is defined all over $\overline{\Omega}$). We can formalize problem (3) as

$$\begin{cases} F(D^2u(x)) = 0, & \text{a.e. in } \Omega \\ u = \varphi, & \text{on } \partial\Omega \\ Du = D\varphi, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $F : \mathbb{R}^{n\times n}_s \to \mathbb{R}$ is the convex function given by

$$F(\xi) = |tr\ \xi| - 1 = \left|\sum_{i=1}^{n} \xi_{ii}\right| - 1, \quad \forall\, \xi \equiv (\xi_{ij}) \in \mathbb{R}^{n\times n}_s .$$

To compare with the statement of Theorem 2.1 below let us remark that F is not coercive in the usual sense, but it is only coercive, for example, with respect to the ξ_{11} real variable, in the sense that, if ξ vary on a bounded set of $\mathbb{R}^{n\times n}_s$, then there exists a constant q such that

$$F(\xi + t\, e_{11}) \geq |t| - q, \quad \forall\, t \in \mathbb{R}$$

and for every ξ that vary on the bounded set; here e_{11} is the matrix with all the entries equal to 0, but the entry with indices 1, 1, which is equal to 1. We will say that F is coercive in the rank-one direction e_{11} (see (9) for the definition of coercivity of a function F in a direction $\lambda \in \mathbb{R}^{n\times n}_s$ with rank $\{\lambda\} = 1$).

Returning to the equation (1), more generally we will consider in this paper Dirichlet-Neumann problems of the form (5)

$$\text{(5)} \qquad \begin{cases} F(x, u(x), Du(x), D^2u(x)) = 0, & \text{a.e. in } \Omega \\ u = \varphi,\ Du = D\varphi, & \text{on } \partial\Omega . \end{cases}$$

We look for solutions u in the class $W^{2,\infty}(\Omega)$ and in general we cannot expect that $u \in C^2(\Omega)$. An existence theorem for the problem (5) will be obtained in Section 2, while in Section 3 we will consider the case of Dirichlet-Neumann problems for systems of second order implicit equations. We will obtain a multiplicity result for solutions to problem (5); more precisely, we will prove that there exists a set of solutions dense in the $C^1(\overline{\Omega})$−norm on a given functional metric space contained in $W^{2,\infty}(\Omega)$. In particular we will prove the following result: *Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. Let $F(x, s, p, \xi)$ be a continuous function, convex and coercive with respect to $\xi \in \mathbb{R}^{n\times n}_s$ in at least one rank-one direction. Let φ be a function of class $C^2(\overline{\Omega})$ satisfying the compatibility condition*

$$\text{(6)} \qquad F\left(x, \varphi(x), D\varphi(x), D^2\varphi(x)\right) \leq 0, \quad \forall\, x \in \overline{\Omega} .$$

Then there exists (a dense set of) $u \in W^{2,\infty}(\Omega)$ that solves the Dirichlet-Neumann problem (5).

We need the compatibility condition (6) first to be sure that the function F is equal to zero somewhere (consequence of the compatibility condition and the coercivity assumption). More relevant, however is the implication by the convexity assumption through Jensen's inequality: in fact, for example, if the problem (4) with special boundary datum φ equal to a polynomial of degree at most two (i.e. $D^2\varphi\,(x) = \xi_0$ for some $\xi_0 \in \mathbb{R}^{n\times n}_s$ and for every $x \in \Omega$) has a

solution $u \in W^{2,\infty}(\Omega)$, then, since $F\left(D^2u\right) = 0$ a.e. in Ω and $Du = D\varphi$ on $\partial\Omega$, we obtain the necessary compatibility condition

$$\begin{aligned} 0 &= \frac{1}{|\Omega|}\int_\Omega F\left(D^2u\,(x)\right)\,dx \geq F\left(\frac{1}{|\Omega|}\int_\Omega D^2u\,(x)\,dx\right) \\ &= F\left(\frac{1}{|\Omega|}\int_\Omega D^2\varphi\,(x)\,dx) = F(\xi_0\right) = F(D^2\varphi)\,. \end{aligned}$$

A difficulty in the proof of the result stated above relies in the fact that limits of sequences of solutions (or of approximate solutions) of a given equation in implicit form in general are not solutions. For example, if $u_k \in W^{2,\infty}(\Omega)$ satisfies $|\Delta u_k(x)| = 1$, a.e. $x \in \Omega$, then functions u, limits in the weak* topology of $W^{2,\infty}(\Omega)$ of converging subsequences, in general only satisfy the condition $|\Delta u(x)| \leq 1$, a.e. $x \in \Omega$. Instead to go to the limit in a sequence of approximate solutions, we "go to the limit" in a sequence of *sets* of approximate solutions; more precisely, we consider the intersection of a sequence of sets of approximate solutions and, by mean of Baire theorem, we show that the intersection is not empty. This method has been introduced by A. Cellina [6] for ordinary differential inclusions. See also F. S. De Blasi-G. Pianigiani [14], [15] and A. Bressan-F. Flores [4].

Let us make a remark related to the important case of second order elliptic *fully nonlinear* partial differential equations: a continuous function $F(x, s, p, \xi)$, which is coercive with respect to $\xi \in \mathbb{R}^{n\times n}_s$ in a rank-one direction, is not elliptic in the sense of L. Caffarelli, L. Nirenberg, J. Spruck [5], M. G. Crandall-H. Ishii-P. L. Lions [7], L. C. Evans [16], N. S. Trudinger [22]. In fact the ellipticity condition $\sum_{ij} \partial F/\partial\xi_{ij}\lambda_i\lambda_j \geq 0$, for every $\lambda \equiv (\lambda_i) \in \mathbb{R}^n$, excludes the coercivity of F in any rank-one direction.

We point out some examples of problems that can be solved by the existence results of this paper. We already gave a first example in (3); there the compatibility condition on the boundary datum $\varphi \in C^2(\overline{\Omega})$ is $|\Delta\varphi(x)| \leq 1$, for every $x \in \Omega$. An other reference equation in this context is the implicit equation (see the example 1.4 below)

$$\left|\det D^2u\,(x)\right| = 1, \quad \text{a.e. } x \in \Omega,$$

that cannot be solved by the Monge-Ampère equation $\det D^2u\,(x) = 1$ if associated to the Dirichlet-Neumann conditions $u = \varphi$ and $Du = D\varphi$. Here the function $F(\xi) = |\det \xi| - 1$ is not convex in $\mathbb{R}^{n\times n}_s$, but is only *quasiconvex* in Morrey sense (see [20] and the definition (35) in Section 2 of this paper). In Sections 2 and 3 we will treat also the quasiconvex case. We can also prove existence in the cases stated below.

EXAMPLE 1.1. We consider the Dirichlet-Neumann problem in an open set $\Omega \subset \mathbb{R}^n$

$$\begin{cases} \left|\dfrac{\partial^2 u}{\partial x_1^2} - \displaystyle\sum_{i=2}^n \dfrac{\partial^2 u}{\partial x_i^2}\right| = 1, & \text{a.e. in } \Omega \\ u = \varphi,\ Du = D\varphi, & \text{on } \partial\Omega\,. \end{cases}$$

Existence Theorem 2.1 applies, with the compatibility condition on $\varphi \in C^2\left(\overline{\Omega}\right)$

$$\left|\frac{\partial^2\varphi}{\partial x_1^2} - \sum_{i=2}^{n} \frac{\partial^2\varphi}{\partial x_i^2}\right| \leq 1, \qquad \forall x \in \Omega.$$

EXAMPLE 1.2. Consider the generalization of the *eikonal equation* to second derivatives, related to a continuous function $a(x, s, \xi)$,

$$\begin{cases} \left|D^2u(x)\right| = a(x, u(x), Du(x)), & \text{a.e. in } \Omega \\ u = \varphi, \ Du = D\varphi, & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Then the compatibility condition on $\varphi \in C^2\left(\overline{\Omega}\right)$ for existence is that

$$\left|D^2\varphi(x)\right| \leq a(x, \varphi(x), D\varphi(x)), \qquad \forall x \in \Omega.$$

EXAMPLE 1.3. Let $0 \leq \lambda_1(\xi) \leq \ldots \leq \lambda_n(\xi)$ denote the singular values (i.e the absolute value of the eigenvalues) of $\xi \in \mathbf{R}_s^{n\times n}$. Then we can solve the Dirichlet-Neumann problem

$$\begin{cases} \lambda_1\left(D^2u\right) = \ldots = \lambda_n\left(D^2u\right) = 1, & \text{a.e. in } \Omega \\ u = \varphi, \ Du = D\varphi, & \text{on } \partial\Omega, \end{cases} \tag{8}$$

under the compatibility condition for the boundary datum $\varphi \in C^2\left(\overline{\Omega}\right)$

$$\lambda_n(D^2\varphi(x)) < 1, \qquad \forall x \in \Omega.$$

EXAMPLE 1.4. The problem

$$\begin{cases} \left|\det D^2u\,(x)\right| = 1, & \text{a.e. } x \in \Omega \\ u = \varphi, \ Du = D\varphi, & \text{on } \partial\Omega, \end{cases}$$

with the compatibility condition $\lambda_n(D^2\varphi(x)) < 1$, for every $x \in \Omega$ (for example $\varphi \equiv 0$), can be reduced to the case of the previous Example 1.3, since

$$\prod_{i=1}^{n} \lambda_i(\xi) = |\det \xi|.$$

2. – The case of one implicit equation

Let $\Omega \subset \mathbb{R}^n$ be an open set. We start by considering a Dirichlet-Neumann problem related to a single second order P.D.E. in implicit form, with a function $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_s \to \mathbb{R}$ continuous in the variables $(x, s, p, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_s$ and convex with respect to the last variable $\xi \in \mathbb{R}^{n\times n}_s$. We denote by $\lambda \in \mathbb{R}^{n\times n}_s$ a matrix of rank one, for example

$$\lambda = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

We say that $F(x, s, p, \xi)$ is *coercive* with respect to the last variable ξ *in the rank-one direction* λ, if $\lambda \in \mathbb{R}^{n\times n}_s$ with $\operatorname{rank}\{\lambda\} = 1$, and for every bounded set of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_s$ there exist constants $m, q \in \mathbb{R}$, $m > 0$, such that

$$F(x, s, p, \xi + t\lambda) \geq m|t| - q \tag{9}$$

for every $t \in \mathbb{R}$ and for every (x, s, p, ξ) that vary on the bounded set of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_s$.

THEOREM 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. Let $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_s \to \mathbb{R}$ be a continuous function, convex with respect to the last variable and coercive in a rank-one direction λ. Let φ be a function of class $C^2(\overline{\Omega})$ (or piecewise C^2) satisfying*

$$F\left(x, \varphi(x), D\varphi(x), D^2\varphi(x)\right) \leq 0, \quad \forall x \in \overline{\Omega}. \tag{10}$$

Then there exists (a dense set of) $u \in W^{2,\infty}(\Omega)$ such that

$$\begin{cases} F(x, u(x), Du(x), D^2u(x)) = 0, & \text{a.e. in } \Omega \\ u = \varphi, \ Du = D\varphi, & \text{on } \partial\Omega. \end{cases} \tag{11}$$

REMARK 2.1. (i) By the notation $u \in W^{2,\infty}(\Omega)$, $u = \varphi$, $Du = D\varphi$, on $\partial\Omega$, we mean that $u - \varphi \in W^{2,\infty}_0(\Omega)$.

(ii) We have assumed the Lipschitz continuity of the boundary to be able to define $D\varphi$ on $\partial\Omega$, but we could also consider a more general open set Ω provided there exists $\Omega_1 \supset \Omega$ with Lipschitz boundary and such that $\varphi \in C^2(\overline{\Omega}_1)$. This remark will be used at the end of Section 2 in order to conclude the proof of Theorem 2.1.

The proof of the theorem will be divided into several lemmas. We give at the same time the proof of Theorem 2.1 and of Theorem 2.5 below. The two proofs are almost identical; thus we emphasize the differences in parentheses, by quoting explicitly the quasiconvexity assumption of Theorem 2.5.

We also observe that the following proof can be proposed with similar arguments in the kth order case, $k > 2$, for implicit functions involving derivatives $D^i u(x)$ from $i = 0$ up to the order k.

LEMMA 2.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. Let $t \in [0,1]$ and $A, B \in \mathbb{R}^{n\times n}_s$ with rank $\{A - B\} = 1$. Let φ be a polynomial of degree at most 2 such that*

$$D^2\varphi(x) = tA + (1-t)B, \quad x \in \overline{\Omega}.$$

Then, for every $\varepsilon > 0$, there exists $u \in W^{2,\infty}(\Omega)$, piecewise polynomial of degree at most 2 up to a set of measure less than ε, (i.e. *$u \in P_\varepsilon$, see the definition* (21) *below*) *and there exist disjoint open sets $\Omega_A, \Omega_B \subset \Omega$, with Lipschitz boundary, such that*

$$\begin{cases} |\text{meas}\,\Omega_A - t\ \text{meas}\,\Omega|\,,\ |\text{meas}\,\Omega_B - (1-t)\,\text{meas}\,\Omega| \le \varepsilon \\ u(x) = \varphi(x), \quad Du(x) = D\varphi(x), & x \in \partial\Omega \\ |u(x) - \varphi(x)| \le \varepsilon, \quad |Du(x) - D\varphi(x)| \le \varepsilon, & x \in \Omega \\ D^2u(x) = \begin{cases} A & \text{in } \Omega_A \\ B & \text{in } \Omega_B \end{cases} \\ \text{dist}\left(D^2u(x), co\,\{A, B\}\right) \le \varepsilon, \quad \text{a.e. in } \Omega, \end{cases}$$

where $co\,\{A, B\} = [A, B]$ is the convex hull of $\{A, B\}$, that is the closed segment joining A and B.

PROOF. Step 1: Let us first assume that the matrix $A - B$ has the form

$$A - B = \begin{pmatrix} \alpha & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

We can express Ω as union of cubes with faces parallel to the coordinate axes and a set of small measure. Then, by posing $u = \varphi$ on the set of small measure, and by homotethies and translations, we can reduce ourselves to work with Ω equal to the unit cube.

Let Ω_ε be a set compactly contained in Ω and let $\eta \in C_0^2(\Omega)$ and $L > 0$ be such that

$$\begin{cases} \text{meas}\,(\Omega - \Omega_\varepsilon) \le \varepsilon \\ 0 \le \eta(x) \le 1, & \forall x \in \Omega \\ \eta(x) = 1, & \forall x \in \Omega_\varepsilon \\ |D\eta(x)| \le L/\varepsilon, & \forall x \in \Omega - \Omega_\varepsilon \\ \left|D^2\eta(x)\right| \le L/\varepsilon^2, & \forall x \in \Omega - \Omega_\varepsilon \end{cases} \tag{12}$$

Let us define a function v in the following way: given $\delta > 0$, $v = v(x_1)$ depends only on the real variable $x_1 \in (0,1)$ and the interval $(0,1)$ is divided into two finite unions I, J of open subintervals such that

$$\begin{cases} \overline{I} \cup \overline{J} = [0,1],\ I \cap J = \emptyset \\ \text{meas}\,I = t,\ \text{meas}\,J = 1 - t \\ v''(x_1) = \begin{cases} (1-t)\,\alpha & \text{if } x_1 \in I \\ -t\,\alpha & \text{if } x_1 \in J \end{cases} \\ \left|v'(x_1)\right| \le \delta, \quad \forall x_1 \in (0,1) \\ |v(x_1)| \le \delta, \quad \forall x_1 \in (0,1). \end{cases}$$

In particular $v''(x_1)$ can assume the two values $(1-t)\alpha$ and $-t\alpha$, and at the same time $v(x_1)$ and $v'(x_1)$ can be small, i.e. in absolute value less than or equal to δ, since 0 is a convex combination of the two values, with coefficients t and $1-t$. If we choose $\delta = \min\{\varepsilon^2/(4L), \varepsilon^3/(2L)\}$, then by the last two inequalities we obtain

$$|Dv(x)| \leq \varepsilon^2/(4L), \quad \forall x \in \Omega, \tag{13}$$

$$|v(x)| \leq \varepsilon^3/(2L), \quad \forall x \in \Omega. \tag{14}$$

We define u as a convex combination of $v+\varphi$ and φ in the following way

$$u = \eta(v+\varphi) + (1-\eta)\varphi = \eta v + \varphi.$$

Then u satisfies the conclusion, with

$$\Omega_A = \{x \in \Omega_\varepsilon : x_1 \in I\}, \quad \Omega_B = \{x \in \Omega_\varepsilon : x_1 \in J\}.$$

In fact $u(x) = \varphi(x)$ and $Du(x) = D\varphi(x)$ for every $x \in \partial\Omega$, and

$$|u(x) - \varphi(x)| = |\eta\, v| \leq \varepsilon^3/(2L) \leq \varepsilon, \quad \forall x \in \Omega,$$

for ε sufficiently small. Since

$$Du = D\eta \cdot v + \eta \cdot Dv + D\varphi,$$

then, again for ε sufficiently small

$$|Du(x) - D\varphi(x)| = |D\eta \cdot v + \eta \cdot Dv| \leq \varepsilon^2/2 + \varepsilon^2/(4L) \leq \varepsilon, \quad \forall x \in \Omega.$$

Finally

$$D^2u = D^2\eta \cdot v + 2\, D\eta \cdot Dv + \eta D^2 v + D^2\varphi, \tag{15}$$

and so

$$D^2u = D^2v + D^2\varphi = D^2v + tA + (1-t)B = \begin{cases} A & \text{in } \Omega_A \\ B & \text{in } \Omega_B. \end{cases}$$

Since both $D^2v + D^2\varphi$ ($= A$ or B) and $D^2\varphi = tA + (1-t)B$ belong to $co\{A, B\}$, then by (15) and by (12), (14), (13) for almost every $x \in \Omega$ we obtain

$$\begin{aligned} \operatorname{dist}\left(D^2u, co\{A, B\}\right) &\leq \left|D^2u - \left[\eta\left[D^2v + D^2\varphi\right] + (1-\eta)D^2\varphi\right]\right| \\ &\leq \left|D^2\eta\right| \cdot |v| + 2\,|D\eta| \cdot |Dv| \leq \varepsilon. \end{aligned}$$

Step 2: Since $A - B$ is symmetric and of rank one, by the usual reduction of a symmetric matrix to diagonal form, by the well known properties of eigenvalues (c.f. Bellman [3], Theorem 2, page 54) we can always find $R \in SO(n)$ so that

$$A - B = R \begin{pmatrix} \alpha & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} R^t .$$

Then we set

$$\begin{cases} \widetilde{\Omega} = R^t \Omega \Rightarrow \operatorname{meas} \widetilde{\Omega} = \operatorname{meas} \Omega \\ \widetilde{A} = R^t A R, \quad \widetilde{B} = R^t B R \end{cases}$$

and by Step 1 we find $\widetilde{\Omega}_{\widetilde{A}}$, $\widetilde{\Omega}_{\widetilde{B}}$ and $\widetilde{u} \in W^{2,\infty}(\widetilde{\Omega})$ with the claimed properties. By posing

$$\begin{cases} u(x) = \widetilde{u}(R^t x), \quad x \in \Omega \\ \Omega_A = R\widetilde{\Omega}_{\widetilde{A}}, \quad \Omega_B = R\widetilde{\Omega}_{\widetilde{B}} \end{cases}$$

(note that $Du(x) = RD\widetilde{u}(R^t x)$ and $D^2 u(x) = RD^2\widetilde{u}(R^t x)R^t$) we get the result. □

We now prove the theorem for functions F independent of (x, s, p) and for boundary data that are polynomial of degree at most two.

LEMMA 2.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. Let $F : \mathbb{R}^{n\times n}_s \to \mathbb{R}$ be a convex function, coercive in a rank-one direction λ,* i.e. *there exists a matrix $\lambda \in \mathbb{R}^{n\times n}_s$ with rank $\{\lambda\} = 1$, and constants $m, q \in \mathbb{R}$, $m > 0$, such that*

$$F(\xi + t\lambda) \geq m|t| - q \tag{16}$$

for every $t \in \mathbb{R}$ and for every ξ that vary on a bounded set of $\mathbb{R}^{n\times n}_s$. Let φ be a polynomial of degree at most two (i.e. $D^2\varphi(x) = \xi_0$, *for every $x \in \Omega$) satisfying*

$$F\left(D^2\varphi\right) = F(\xi_0) \leq 0 . \tag{17}$$

Then there exists (a dense set of) $u \in W^{2,\infty}(\Omega)$ such that

$$\begin{cases} F\left(D^2 u(x)\right) = 0, & \text{a.e. } x \in \Omega \\ u = \varphi, \ Du = D\varphi, & \text{on } \partial\Omega . \end{cases} \tag{18}$$

PROOF. We assume without loss of generality that Ω is bounded (since we can cover Ω with a countable family of bounded sets with Lipschitz boundary and prove the lemma on each of these sets). We can assume also that

$$F(\xi_0) < 0$$

otherwise φ is a solution of our problem. For $r > 0$ given we define the subset of $\mathbb{R}^{n\times n}_s$

$$K = \left\{\eta \in \mathbb{R}^{n\times n}_s : \eta = \xi + t\lambda, \ \xi \in \overline{B_r(\xi_0)}, \ t \in \mathbb{R} \text{ and } m|t| - q \leq 0\right\} \tag{19}$$

where m, q are the constants that appear in the coercivity assumption (16), when ξ vary in $\overline{B_r(\xi_0)}$, the closed ball centered at ξ_0 with radius r. The set K is compact and convex. Furthermore its interior is given by

$$\text{(20)} \quad \operatorname{int} K = \left\{\eta \in \mathbb{R}_s^{n\times n} : \eta = \xi + t\lambda, \xi \in B_r(\xi_0), t \in \mathbb{R} \text{ and } m|t| - q < 0\right\}.$$

Moreover by (16) the following inclusion holds

$$\left\{\eta \in \mathbb{R}_s^{n\times n} : \eta = \xi + t\lambda,\ \xi \in \overline{B_r(\xi_0)},\ t \in \mathbb{R} \text{ and } F(\eta) \le 0\right\} \subset K.$$

We next define for every $\varepsilon \ge 0$, P_ε to be the set of piecewise polynomial of degree at most two, up to a set of measure less than ε. More precisely

$$\text{(21)} \quad \begin{cases} P_\varepsilon \text{ is the set of functions } u \in W^{2,\infty}(\Omega) \text{ such that there exist} \\ \text{an open set } \Omega_0, \text{ with Lipschitz boundary and with meas}\,\Omega_0 \le \varepsilon, \\ \text{and a countable family of disjoint open sets } \{\Omega_j\}_{j=1}^{\infty} \\ \text{with Lipschitz boundary satisfying } \bigcup_{j=1}^{\infty} \overline{\Omega_j} = \overline{\Omega \backslash \Omega_0}, \text{ such that} \\ u|_{\Omega_j}, j = 1, 2, \dots, \text{ is a polynomial of degree at most two.} \end{cases}$$

Note that (for $\varepsilon = 0$) P_0 is the set of C^1 functions which are piecewise polynomial of degree at most two. Moreover $P_0 \subset P_\varepsilon$ for every $\varepsilon \ge 0$. In particular $\varphi \in P_\varepsilon$ for every $\varepsilon \ge 0$.

We then let V be the set of functions $u \in C^1(\overline{\Omega})$ such that there exist sequences $\varepsilon_k \to 0$ and $u_k \in P_{\varepsilon_k}$ satisfying

$$\begin{cases} u_k \to u \text{ in } C^1(\overline{\Omega}) & \\ D^2 u_k(x) \in \operatorname{int} K, & \text{a.e. } x \in \Omega \\ F\left(D^2 u_k(x)\right) < 0, & \text{a.e. } x \in \Omega \\ u_k = \varphi \text{ and } Du_k = D\varphi & \text{on } \partial\Omega. \end{cases}$$

The set V is non empty since $\varphi \in V$. We endow V with the C^1–norm and thus V is a metric space. By classical diagonal process we get that V is closed in $C^1(\overline{\Omega})$; thus V is a complete metric space.

Furthermore since K is bounded, V is bounded in $W^{2,\infty}$; then any sequence in V contains a subsequence which converges in the weak* topology of $W^{2,\infty}$. Since F is convex (quasiconvex in Theorem 2.5, where we use the semicontinuity in the weak* topology of $W^{2,\infty}$ of integrals depending on second derivatives; see Meyers [19]; see also [1], [17], [18]) we obtain

$$\text{(22)} \qquad F\left(D^2 u(x)\right) \le 0, \quad \text{a.e. } x \in \Omega, \quad \forall u \in V.$$

In fact, if $\eta \in C^0(\overline{\Omega})$, $\eta \ge 0$ and u_k weak* converges to u in $W^{2,\infty}$, with $F(D^2 u_k) \le 0$, then

$$\int_\Omega \eta(x) \cdot F\left(D^2 u(x)\right) dx \le \liminf_{k\to+\infty} \int_\Omega \eta(x) \cdot F\left(D^2 u_k(x)\right) dx,$$

and thus

$$\int_\Omega \eta(x) \cdot F\left(D^2 u(x)\right) dx \leq 0, \quad \forall \eta \in C^0(\overline{\Omega}), \quad \eta \geq 0,$$

which implies (22). Therefore

$$V \subset \left\{u \in \varphi + W_0^{2,\infty}(\Omega) : F\left(D^2 u(x)\right) \leq 0, \text{ a.e. } x \in \Omega\right\}. \tag{23}$$

For $k \in \mathbb{N}$ we define

$$V^k = \{u \in V : \int_\Omega F\left(D^2 u(x)\right) dx > -1/k\}.$$

The set V^k is open in V. Indeed by the boundedness in $W^{2,\infty}$ of V and by the convexity (quasiconvexity in Theorem 2.5) of F we deduce that $V \backslash V^k$ is closed in V.

Now we show that V^k is dense in V. So let $v \in V$; we can assume, by construction of V, that there exist $\varepsilon > 0$ and v_ε such that

$$\begin{cases} v_\varepsilon \in P_\varepsilon & \\ D^2 v_\varepsilon \in \operatorname{int} K & \text{a.e. } x \in \Omega \\ F\left(D^2 v_\varepsilon(x)\right) < 0 & \text{a.e. } x \in \Omega \\ v_\varepsilon = \varphi, \quad D v_\varepsilon = D\varphi & \text{on } \partial\Omega \\ \|v_\varepsilon - v\|_{C^1(\overline{\Omega})} \leq \varepsilon/2. & \end{cases}$$

Therefore there exist disjoint open sets Ω_j, $j = 0, 1, 2, \ldots$, so that $\operatorname{meas} \Omega_0 \leq \varepsilon$ and $v_\varepsilon|_{\Omega_j}$ is a polynomial of degree at most two, i.e. $D^2 v_\varepsilon = \xi_j \in \operatorname{int} K$ in Ω_j, with $F\left(\xi_j\right) < 0$, for every $j = 1, 2, \ldots$.

Next we consider the function which at every $t \in \mathbb{R}$ associates $\xi_j(t) = \xi_j + t\lambda \in \mathbb{R}_s^{n\times n}$. Note that $\xi_j(t) \in \operatorname{int} K$ for $t \in \mathbb{R}$ as long as $F\left(\xi_j(t)\right) < 0$. Since $F\left(\xi_j\right) < 0$, by (16) we can find $t_1 < 0 < t_2$ such that $F\left(\xi_j(t_1)\right) = F\left(\xi_j(t_2)\right) = 0$. By the continuity of F for ε sufficiently small we can find δ_1, $\delta_2 > 0$ such that

$$F\left(\xi_j(t_1 + \delta_1)\right) = F\left(\xi_j(t_2 - \delta_2)\right) = -\varepsilon.$$

We then apply the previous lemma with $A = \xi_j(t_1 + \delta_1)$, $B = \xi_j(t_2 - \delta_2)$ (observe that $A - B = (t_1 - t_2 + \delta_1 + \delta_2)\lambda$ is a rank one matrix) and $t = (t_2 - \delta_2) / (t_2 - \delta_2 - t_1 - \delta_1)$, with φ replaced by v_ε and ε replaced by $\min\left\{\delta, \varepsilon/2^j\right\}$ with δ to be chosen below. Therefore we get $v_{\varepsilon,j} \in P_\epsilon \subset W^{2,\infty}(\Omega)$ and $\widetilde{\Omega}_j\ (= \Omega_A \cup \Omega_B) \subset \Omega_j$ such that

$$\begin{cases} \operatorname{meas}(\Omega_j - \widetilde{\Omega}_j) \leq \varepsilon/2^j & \\ v_{\varepsilon,j}(x) = v_\varepsilon(x), \quad D v_{\varepsilon,j}(x) = D v_\varepsilon(x), & x \in \partial\Omega_j \\ \left\|v_{\varepsilon,j} - v_\varepsilon\right\|_{C^1(\overline{\Omega}_j)} \leq \varepsilon/2^j \leq \varepsilon/2 & \\ F\left(D^2 v_{\varepsilon,j}(x)\right) = -\varepsilon & \text{a.e. } x \in \Omega_A \cup \Omega_B = \widetilde{\Omega}_j \\ D^2 v_{\varepsilon,j}(x) \in \operatorname{int} K \text{ and } F\left(D^2 v_{\varepsilon,j}(x)\right) < 0 & \text{a.e. } x \in \Omega_j. \end{cases}$$

The facts that $D^2 v_{\varepsilon,j}(x) \in \operatorname{int} K$ and $F\left(D^2 v_{\varepsilon,j}(x)\right) < 0$ are consequences of

$$\begin{cases} co\ \{A, B\} \subset \operatorname{int} K \\ F\,|_{co\{A,B\}} \leq -\varepsilon < 0 \\ \operatorname{dist}\left(D^2 v_{\varepsilon,j}, co\ \{A, B\}\right) < \delta\,, \end{cases}$$

and the possibility to choose δ arbitrarily small; note above that $F(A) = F(B) = -\varepsilon < 0$ and, by convexity, $F(\xi) < 0$ for every $\xi \in co\ \{A, B\}$ (in Theorem 2.5 the quasiconvex function F is also convex on the segment $co\ \{A, B\}$, since $A - B \in \mathbb{R}^{n\times n}_s$ and $\operatorname{rank}\{A - B\} = 1$). We define the function u_ε by

$$u_\varepsilon(x) = \begin{cases} v_\varepsilon(x) & \text{if } x \in \overline{\Omega}_0 \\ v_{\varepsilon,j}(x) & \text{if } x \in \overline{\Omega}_j, \quad j = 1, 2, \ldots \end{cases}$$

Then $u_\varepsilon \in P_{2\varepsilon} \cap V$ and $\|u_\varepsilon - v\|_{C^1(\overline{\Omega})} \leq \varepsilon$. It remains to show that $u_\varepsilon \in V^k$. To this aim we compute

$$\int_\Omega F\left(D^2 u_\varepsilon(x)\right) dx = \int_{\Omega_0} F\left(D^2 v_\varepsilon(x)\right) dx + \sum_{j=1}^{\infty} \int_{\Omega_j} F\left(D^2 v_{\varepsilon,j}(x)\right) dx$$

$$= \int_{\Omega_0} F\left(D^2 v_\varepsilon(x)\right) dx + \sum_{j=1}^{\infty} \int_{\Omega_j - \widetilde{\Omega}_j} F\left(D^2 v_{\varepsilon,j}(x)\right) dx - \varepsilon \sum_{j=1}^{\infty} \operatorname{meas} \widetilde{\Omega}_j\,.$$

We use the inequalities

$$\operatorname{meas} \Omega_0 < \varepsilon, \qquad \sum_{j=1}^{\infty} \operatorname{meas}\left(\Omega_j - \widetilde{\Omega}_j\right) < \varepsilon\,,$$

and the fact that $D^2 v_\varepsilon(x)$, $D^2 v_{\varepsilon,j}(x)$ belong to the compact set K a.e. $x \in \Omega$, to deduce, since F is continuous on K, that

$$\int_\Omega F\left(D^2 u_\varepsilon(x)\right) dx > -1/k$$

for ε sufficiently small. Therefore $u_\varepsilon \in V^k$ and the density of V^k in V has been established.

By Baire category theorem we have that

$$\bigcap_{k\in\mathbb{N}} V^k = \left\{u \in V : \int_\Omega F\left(D^2 u(x)\right) dx \geq 0\right\}$$

is dense in V, in particular it is not empty. Since every $u \in V$ by (23) satisfies the condition $F\left(D^2 u(x)\right) \leq 0$ a.e. $x \in \Omega$, we obtain that every element u of this intersection solves the equation $F\left(D^2 u(x)\right) = 0$ a.e. $x \in \Omega$, and thus the thesis (18) holds. □

LEMMA 2.4. *Under the hypotheses of Theorem* 2.1 *and under the further assumption that the boundary datum* $\varphi \in C^2(\overline{\Omega})$ *satisfies the condition*

$$F\left(x, \varphi(x), D\varphi(x), D^2\varphi(x)\right) < 0, \quad \forall x \in \overline{\Omega}, \tag{24}$$

then the conclusion (11) *of Theorem* 2.1 *holds.*

PROOF. As usual we can assume that Ω is bounded. Letting $w = u - \varphi$ we obtain the equivalent formulation: find $w \in W^{2,\infty}(\Omega)$ satisfying

$$\begin{cases} G(x, w(x), Dw(x), D^2w(x)) = 0, & \text{a.e. in } \Omega \\ w = 0, \ Dw = 0, & \text{on } \partial\Omega, \end{cases}$$

where

$$G(x, s, p, \xi) = F\left(x, s + \varphi(x), p + D\varphi(x), \xi + D^2\varphi(x)\right).$$

Since $\varphi \in C^2(\overline{\Omega})$, then G is continuous with respect to (x, s, p, ξ), convex (quasiconvex in Theorem 2.5) and coercive in the λ direction with respect to the ξ variable. By the compatibility assumption (24), if we denote by $\psi = 0$ the function identically equal to zero on $\overline{\Omega}$, then

$$G\left(x, \psi(x), D\psi(x), D^2\psi(x)\right) = G(x, 0, 0, 0) < 0, \quad \forall x \in \overline{\Omega}. \tag{25}$$

Let $r > 0$. If $v \in W_0^{2,\infty}(\Omega)$ and $\left|D^2v(x)\right| \le r$ a.e. $x \in \Omega$, then $|v(x)| + |Dv(x)| \le L \cdot r$, for every $x \in \overline{\Omega}$ and for some positive constant L depending on the diameter of Ω. Therefore we can consider the coercivity assumption (9) with $x \in \overline{\Omega}$, $|s| + |p| \le L \cdot r$, $|\xi| \le r$ and thus we can find constants q, $m > 0$, such that

$$G(x, s, p, \xi + t\lambda) \ge m|t| - q, \quad \forall t \in \mathbb{R}. \tag{26}$$

Then we consider the (compact and convex) set $K \subset \mathbb{R}_s^{n\times n}$ as in (19)

$$K = \left\{\eta \in \mathbb{R}_s^{n\times n} : \eta = \xi + t\lambda, \ \xi \in \overline{B_r(0)}, \ t \in \mathbb{R} \text{ and } m|t| - q \le 0\right\}.$$

By the coercivity assumption (26) for every $x \in \overline{\Omega}$, $|s| + |p| \le L \cdot r$, the following inclusion holds

$$\left\{\eta \in \mathbb{R}_s^{n\times n} : \eta = \xi + t\lambda, \ \xi \in \overline{B_r(0)}, \ t \in \mathbb{R} \text{ and } G(x, s, p, \eta) \le 0\right\} \subset K.$$

By considering again the set $P_\varepsilon \subset W^{2,\infty}(\Omega)$ of piecewise polynomial of degree at most two up to a set of measure less than ε, as in Lemma 2.3, we can define

W as the set of functions $u \in C^1(\overline{\Omega})$ such that there exist sequences $\varepsilon_k \to 0$ and $u_k \in P_{\varepsilon_k}$ satisfying

$$\begin{cases} u_k \to u \text{ in } C^1(\overline{\Omega}) & \\ D^2 u_k(x) \in \operatorname{int} K, & \text{a.e. } x \in \Omega \\ G\left(x, u_k(x), Du_k(x), D^2 u_k(x)\right) < 0 & \text{a.e. } x \in \Omega \\ u_k = 0 \text{ and } Du_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Then W is non empty, since $0 \in W$. As in Lemma 2.3 we endow W with the C^1–norm and W becomes a complete metric space. Again, similarly to (23) of Lemma 2.3, we can see that

$$W \subset \left\{u \in W_0^{2,\infty}(\Omega) : G\left(x, u(x), Du(x), D^2 u(x)\right) \leq 0, \text{ a.e. } x \in \Omega\right\}. \tag{27}$$

For every $k \in \mathbb{N}$ we define

$$W^k = \{u \in W : \int_\Omega G\left(x, u(x), Du(x), D^2 u(x)\right) dx > -1/k\}.$$

By the lower semicontinuity of the integral in the weak* topology of $W^{2,\infty}(\Omega)$ the set W^k is open in W.

We also show that W^k is dense in W. So let $v \in W$; we can assume, by definition of W, that there exist $\varepsilon > 0$ and v_ε such that

$$\begin{cases} v_\varepsilon \in P_\varepsilon & \\ D^2 v_\varepsilon \in \operatorname{int} K, & \text{a.e. } x \in \Omega \\ G\left(x, v_\varepsilon(x), Dv_\varepsilon(x), D^2 v_\varepsilon(x)\right) < 0, & \text{a.e. } x \in \Omega \\ v_\varepsilon = 0, \quad Dv_\varepsilon = 0 & \text{on } \partial\Omega \\ \|v_\varepsilon - v\|_{C^1(\overline{\Omega})} \leq \varepsilon/2. & \end{cases}$$

Therefore there exist disjoint open sets Ω_j, $j = 0, 1, 2, \ldots$, so that $\operatorname{meas} \Omega_0 \leq \varepsilon$ and, for $j = 1, 2, \ldots$, $v_\varepsilon|_{\Omega_j}$ is a polynomial of degree at most two, i.e. $D^2 v_\varepsilon = \xi_j \in \operatorname{int} K$ in Ω_j, with

$$G\left(x, v_\varepsilon(x), Dv_\varepsilon(x), \xi_j\right) < 0, \quad \forall x \in \overline{\Omega}_j, \quad \forall j = 1, 2, \ldots$$

Given $\delta > 0$, for every $j = 1, 2, \ldots$. we consider $\delta_j \leq \delta$ satisfying

$$0 < \delta_j < -\max\left\{G\left(x, v_\varepsilon(x), Dv_\varepsilon(x), \xi_j\right) : \ x \in \overline{\Omega}_j\right\}. \tag{28}$$

By the uniform continuity of G and the equicontinuity in $\overline{\Omega}_j$ of the first derivatives of elements of W, there exists a finite number H_j of disjoint open sets $\{\Omega_{jh}\}_{h=1,\ldots,H_j}$ with Lipschitz boundary, with the property that the closure of their union is equal to $\overline{\Omega}_j$ and such that, if $x_h \in \Omega_{jh}$ for $h = 1, \ldots, H_j$, then

$$|G(x, v_1(x), Dv_1(x), \xi) - G(x_h, v_2(x_h), Dv_2(x_h), \xi)| < \delta_j \tag{29}$$

for every $x \in \overline{\Omega}_{jh}$, for every $\xi \in K$ and for every $v_1, v_2 \in W^{2,\infty}(\Omega_{jh})$, $v_1 - v_2 \in W_0^{2,\infty}(\Omega_{jh})$ such that $D^2 v_1(x), D^2 v_2(x) \in K$ (which is a compact set) a.e. $x \in \Omega_{jh}$. From (28) we deduce

$$G\left(x_h, v_\varepsilon(x_h), Dv_\varepsilon(x_h), \xi_j\right) < -\delta_j \tag{30}$$

for every $h = 1, \dots, H_j$. Recalling that $D^2 v_\varepsilon \equiv \xi_j$ in $\overline{\Omega}_j = \cup \overline{\Omega}_{jh}$, by the compatibility condition (30) we can apply Lemma 2.3 to solve the problem

$$\begin{cases} G\left(x_h, v_\varepsilon(x_h), Dv_\varepsilon(x_h), D^2 w(x)\right) = -\delta_j, & \text{a.e. } x \in \Omega_{jh} \\ w(x) = v_\varepsilon(x), \ Dw(x) = Dv_\varepsilon(x), & x \in \partial\Omega_{jh} \end{cases} \tag{31}$$

and find $w = w_{jh} \in V \subset W^{2,\infty}(\Omega_{jh})$ (see the definition of V in the proof of Lemma 2.3). Recall that, by the definition of V, for j, h fixed there exist sequences $\varepsilon_k \to 0$ and $u_{jhk} \in P_{\varepsilon_k}$ satisfying

$$\begin{cases} u_{jhk} \to w_{jh} \text{ in } C^1\left(\overline{\Omega}\right) & \text{as } k \to +\infty \\ D^2 u_{jhk} \in \operatorname{int} K & \text{a.e. } x \in \Omega \\ G\left(x_h, v_\varepsilon(x_h), Dv_\varepsilon(x_h), D^2 u_{jhk}(x)\right) < -\delta_j, & \text{a.e. } x \in \Omega_{jh} \\ u_{jhk}(x) = v_\varepsilon(x), \ Du_{jhk}(x) = Dv_\varepsilon(x), & x \in \partial\Omega_{jh}. \end{cases} \tag{32}$$

By (29), with $v_1 = u_{jhk}$ and $v_2 = v_\varepsilon$ we get

$$G\left(x, u_{jhk}(x), Du_{jhk}(x), D^2 u_{jhk}(x)\right) < 0, \quad \text{a.e. } x \in \Omega_{jh}. \tag{33}$$

We then define the function $w \in W$ in Ω by

$$w(x) = \begin{cases} v_\varepsilon(x) & \text{if } x \in \overline{\Omega}_0 \\ w_{jh}(x) & \text{if } x \in \overline{\Omega}_{jh}, \quad j = 1, 2, \dots, \quad h = 1, \dots, H_j. \end{cases}$$

The fact that $w \in W$ is a consequence of the definition of v_ε, (31), (32) and (33). We then go back to $\delta_j \le \delta$ in (28); as $\delta \to 0$ the function w converges to v_ε pointwise in Ω, and thus in $C^1(\overline{\Omega})$, since $D^2 w$ is uniformly bounded. To show that $w \in W^k$ we compute

$$\begin{aligned} &\int_\Omega G\left(x, w(x), Dw(x), D^2 w(x)\right) dx \\ &= \int_{\Omega_0} G\left(x, v_\varepsilon(x), Dv_\varepsilon(x), D^2 v_\varepsilon(x)\right) dx \\ &\quad + \sum_{jh} \int_{\Omega_{jh}} G\left(x, w_{jh}(x), Dw_{jh}(x), D^2 w_{jh}(x)\right) dx \\ &= \int_{\Omega_0} G\left(x, v_\varepsilon(x), Dv_\varepsilon(x), D^2 v_\varepsilon(x)\right) dx + \sum_{j=1}^{\infty} \int_{\Omega_j} (-\delta_j)\, dx \\ &\quad + \sum_{jh} \int_{\Omega_{jh}} \Big\{ G\left(x, w_{jh}(x), Dw_{jh}(x), D^2 w_{jh}(x)\right) \\ &\quad - G\left(x_h, v_\varepsilon(x_h), Dv_\varepsilon(x_h), D^2 w_{jh}(x)\right) \Big\} dx \\ &\ge -M\varepsilon - 2|\Omega|\delta, \end{aligned}$$

since $\operatorname{meas}\Omega_0 < \varepsilon$ and $\left|G\left(x, v_\varepsilon(x), Dv_\varepsilon(x), D^2v_\varepsilon(x)\right)\right| \leq M$, for some constant M (in fact $D^2v_\varepsilon(x)$ belongs to the compact set K a.e. $x \in \Omega$) and where we have used (29) with $v_1 = w_{jh}$ and $v_2 = v_\varepsilon$. Therefore $w \in W^k$ for ε and δ sufficiently small and the density of W^k in W has been proved. We obtain the conclusion by using Baire category theorem, as in the proof of Lemma 2.3. In fact we have that

$$\bigcap_{k\in\mathbb{N}} W^k = \left\{u \in W : \int_\Omega G\left(x, u(x), Du(x), D^2u(x)\right) dx \geq 0\right\}$$

is dense in W, in particular it is not empty. Then every $u \in W$ by (27) satisfies the condition $G\left(x, u(x), Du(x), D^2u(x)\right) \leq 0$ a.e. $x \in \Omega$, and every element u of this intersection solves the equation $G\left(x, u(x), Du(x), D^2u(x)\right) = 0$ a.e. $x \in \Omega$. Thus the conclusion (11) holds. □

We are ready to conclude the proof of Theorem 2.1; it only remains to reduce ourselves to the case of compatibility condition (24).

Proof. (Conclusion of the proof of Theorem 2.1). First we observe that if φ is piecewise C^2 we can do the following construction on each set where φ is of class C^2 and hence obtain the result on the whole of Ω. We will therefore assume that $\varphi \in C^2(\overline{\Omega})$. We also assume that Ω is bounded, otherwise we cover Ω with a countable family of bounded sets with Lipschitz boundary and prove the theorem on each of these sets. We define

$$\Omega_0 = \left\{x \in \Omega : F\left(x, \varphi(x), D\varphi(x), D^2\varphi(x)\right) = 0\right\}.$$

Since φ is C^2, the set $\Omega - \Omega_0$ is open. We define $u = \varphi$ in Ω_0 and we solve in $W^{2,\infty}(\Omega - \Omega_0)$ the Dirichlet-Neumann problem

$$\begin{cases} F(x, u(x), Du(x), D^2u(x)) = 0, & \text{a.e. in } \Omega - \Omega_0 \\ u = \varphi, \ Du = D\varphi, & \text{on } \partial(\Omega - \Omega_0) \end{cases}$$

(note that, by Remark 2.1 (ii), we do not care of the Lipschitz continuity of the boundary of $\Omega - \Omega_0$) with the compatibility condition

$$F\left(x, \varphi(x), D\varphi(x), D^2\varphi(x)\right) < 0, \quad \forall x \in \Omega - \Omega_0.$$

For every $t < 0$ let us define the subset of $\Omega - \Omega_0 \subset \mathbb{R}^n$

$$\Omega^t = \left\{x \in \Omega - \Omega_0 : F\left(x, \varphi(x), D\varphi(x), D^2\varphi(x)\right) = t\right\}.$$

Then we can find an increasing sequence $t_k < 0$, converging to zero as $k \to +\infty$, such that

$$\operatorname{meas}\Omega^{t_k} = 0, \quad \forall k \in \mathbb{N}. \tag{34}$$

To prove (34) we define for every $k \in \mathbb{N}$

$$T_k = \left\{ t < 0 : \ \frac{\text{meas}\,(\Omega - \Omega_0)}{k+1} < \text{meas}\,\Omega^t \leq \frac{\text{meas}\,(\Omega - \Omega_0)}{k} \right\} .$$

This set T_k, for k fixed, contains at most a finite number of real numbers. In fact, if T_k would contain infinitely many numbers, then, from the fact that

$$\Omega - \Omega_0 = \bigcup_{t<0} \Omega^t \supset \bigcup_{t \in T_k} \Omega^t ,$$

we would obtain

$$\begin{aligned} \text{meas}\,(\Omega - \Omega_0) &\geq \sum_{t \in T_k} \text{meas}\,\Omega^t \\ \geq \sum_{t \in T_k} \frac{\text{meas}\,(\Omega - \Omega_0)}{k+1} &= \frac{\text{meas}\,(\Omega - \Omega_0)}{k+1} \sum_{t \in T_k} 1 = +\infty , \end{aligned}$$

which contradicts the fact that Ω is bounded. Therefore the set

$$\{t < 0 : \ \text{meas}\,\Omega^t > 0\} \subset \bigcup_{k=1}^{\infty} T_k$$

is countable and the set $\{t < 0 : \ \text{meas}\,\Omega^t = 0\}$ is dense in the interval $(-\infty, 0)$. This proves (34). Then we define

$$\Omega_k = \left\{ x \in \Omega - \Omega_0 : \ t_k < F\left(x, \varphi(x), D\varphi(x), D^2\varphi(x)\right) < t_{k+1} \right\} .$$

Then for every $k \in \mathbb{N}$, since Ω_k is an open set with boundary

$$\partial\Omega_k \subset \partial(\Omega - \Omega_0) \cup \Omega^{t_k} \cup \Omega^{t_{k+1}} ,$$

and since meas $\Omega^{t_k} = 0$, meas $\Omega^{t_{k+1}} = 0$ by (34), we can define in Ω a function u by

$$u(x) = \begin{cases} u_k(x) & \text{if } x \in \overline{\Omega}_k \\ \varphi(x) & \text{if } x \in \Omega_0 , \end{cases}$$

where $u_k \in W^{2,\infty}(\Omega_k)$ is a solution given by Lemma 2.4 (c.f. Remark 2.1) to the problem

$$\begin{cases} F(x, u(x), Du(x), D^2u(x)) = 0, & \text{a.e. in } \Omega_k \\ u = \varphi, \ Du = D\varphi, & \text{on } \partial\Omega_k . \end{cases}$$

Then u is a $W^{2,\infty}(\Omega)$ solution to the initial problem (11). □

Following C.B. Morrey [20], N.G. Meyers [19] (see also [1], [17], [18]), we recall that a continuous function $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_s \to \mathbb{R}$ is quasiconvex with respect to the last variable if

$$\int_\Omega F\left(x, s, p, \xi + D^2\varphi(x)\right) dy \geq |\Omega| \cdot F(x, s, p, \xi), \tag{35}$$

for every $(x, s, p, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_s$ and for all $\varphi \in C^2_0(\Omega)$. A function of this type has the property that the function of one real variable $t \to F(x, s, p, \xi + t\lambda)$ is convex with respect to $t \in \mathbb{R}$ for every $\lambda \in \mathbb{R}^{n\times n}_s$, with $\operatorname{rank}\lambda = 1$. Every convex function is also quasiconvex; an example of quasiconvex function which is not convex is given by

$$F(\xi) = |\det \xi|, \quad \xi \in \mathbb{R}^{n\times n}_s.$$

With the proof given above in this section we have the following

THEOREM 2.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. Let $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_s \to \mathbb{R}$ be a continuous function, quasiconvex with respect to the last variable and coercive in a rank-one direction λ (see* (9)). *Let φ be a function of class $C^2(\overline{\Omega})$ (or piecewise C^2) satisfying*

$$F\left(x, \varphi(x), D\varphi(x), D^2\varphi(x)\right) \leq 0, \quad \forall x \in \overline{\Omega}.$$

Then there exists (a dense set of) $u \in W^{2,\infty}(\Omega)$ such that

$$\begin{cases} F(x, u(x), Du(x), D^2u(x)) = 0, & \text{a.e. in } \Omega \\ u = \varphi, \ Du = D\varphi, & \text{on } \partial\Omega. \end{cases}$$

3. – Systems of implicit second order equations

We say that $f : \mathbb{R}^{n\times n}_s \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is rank one convex if

$$f(tA + (1-t)B) \leq tf(A) + (1-t)f(B)$$

for every $A, B \in \mathbb{R}^{n\times n}_s$ with $\operatorname{rank}\{A - B\} = 1$ and for every $t \in [0, 1]$. As well known a quasiconvex function is necessarily rank one convex.

For $E \subset \mathbb{R}^{n\times n}_s$ and

$$F_E = \left\{f : \mathbb{R}^{n\times n}_s \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}, f|_E = 0\right\}$$

we let

$$RcoE = \left\{\xi \in \mathbb{R}^{n\times n}_s : f(\xi) \leq 0, \quad \forall f \in F_E, \ f \text{ rank one convex}\right\},$$

called the *rank one convex hull* of E.

The proof of the following proposition can be found in [13] (Proposition 2.3).

PROPOSITION 3.1. *Let $E \subset \mathbb{R}^{n\times n}_s$ and define by induction*

$$R_o coE = E,$$

$$R_{i+1}coE = \{\xi \in \mathbb{R}^{n\times n}_s : \xi = tA + (1-t)B,\ t \in [0,1],\\ A, B \in R_i coE,\ \operatorname{rank}\{A - B\} = 1\}\ .$$

Then $RcoE = \cup_{i\in\mathbb{N}} R_i coE$.

We now give the main theorem of this section.

THEOREM 3.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. Let $F_i^\delta(x, s, p, \xi)$, $i = 1, \dots, N$, be quasiconvex functions with respect to $\xi \in \mathbb{R}^{n\times n}_s$ and continuous with respect to $(x, s, p) \in \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n$ and with respect to $\delta \in [0, \delta_0)$, for some $\delta_0 > 0$. Assume that, for every $(x, s, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$,*

$$\text{(i) } Rco\left\{\xi \in \mathbb{R}^{n\times n}_s : F_i^\delta(x, s, p, \xi) = 0,\ i = 1, \dots, N\right\}\\ = \left\{\xi \in \mathbb{R}^{n\times n}_s : F_i^\delta(x, s, p, \xi) \le 0,\ i = 1, \dots, N\right\},\ \forall\delta \in [0, \delta_0)$$

and it is bounded;

$$\text{(ii)}\quad \left\{\xi \in \mathbb{R}^{n\times n}_s : F_i^\delta(x, s, p, \xi) = 0,\ i = 1, \dots, N\right\}\\ \subset \left\{\xi \in \mathbb{R}^{n\times n}_s : F_i^{\delta'}(x, s, p, \xi) < 0,\ i = 1, \dots, N\right\},\ \forall 0 \le \delta' < \delta < \delta_0.$$

If $\varphi \in C^2(\overline{\Omega})$ (or piecewise C^2) satisfies

$$F_i^0(x, \varphi(x), D\varphi(x), D^2\varphi(x)) < 0,\ \forall x \in \Omega,\ i = 1, \dots, N, \tag{36}$$

then there exists (a dense set of) $u \in W^{2,\infty}(\Omega)$ such that

$$\begin{cases} F_i^0(x, u(x), Du(x), D^2u(x)) = 0, & \text{a.e. } x \in \Omega,\ i = 1, \dots, N\\ u(x) = \varphi(x),\ Du(x) = D\varphi(x), & x \in \partial\Omega\,. \end{cases}$$

In order to establish this theorem we proceed as in the previous section and we start with the case independent of $(x, u(x), Du(x))$.

LEMMA 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. Let $F_i^\delta : \mathbb{R}^{n\times n}_s \to \mathbb{R}$, $i = 1, \dots, N$, be quasiconvex and continuous with respect to $\delta \in [0, \delta_0)$, for some $\delta_0 > 0$. Let us assume that*

$$\text{(i) } Rco\left\{\xi \in \mathbb{R}^{n\times n}_s : F_i^\delta(\xi) = 0,\ i = 1, \dots, N\right\}\\ = \left\{\xi \in \mathbb{R}^{n\times n}_s : F_i^\delta(\xi) \le 0,\ i = 1, \dots, N\right\},\ \ \forall\delta \in [0, \delta_0)$$

and it is bounded;

$$\text{(ii)} \qquad \left\{\xi \in \mathbb{R}^{n\times n}_s : F_i^\delta(\xi) = 0,\ i = 1, \dots, N\right\} \subset \left\{\xi \in \mathbb{R}^{n\times n}_s : F_i^0(\xi) < 0,\ i = 1, \dots, N\right\}, \quad \forall \delta \in (0, \delta_0).$$

If φ is a polynomial of degree at most two (i.e. $D^2\varphi(x) = \xi_0$, *for every* $x \in \Omega$) *satisfying*

$$F_i^0(D^2\varphi(x)) = F_i^0(\xi_0) < 0,\ \forall x \in \Omega,\ i = 1, \dots, N,$$

then there exists (a dense set of) $u \in W^{2,\infty}(\Omega)$ such that

$$\begin{cases} F_i^0(D^2u(x)) = 0, & \text{a.e. } x \in \Omega,\ i = 1, \dots, N \\ u(x) = \varphi(x),\ Du(x) = D\varphi(x), & x \in \partial\Omega. \end{cases}$$

The proof of Lemma 3.3 will be consequence of the following lemma, which is an iteration of Lemma 2.2.

LEMMA 3.4. *Let F_i^δ be as in the previous lemma and define E_δ to be*

$$E_\delta = \left\{\xi \in \mathbb{R}^{n\times n} :\ F_i^\delta(\xi) = 0, i = 1, \dots, N\right\}.$$

Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. Let $\xi \in RcoE_\delta$ and let φ be such that $D^2\varphi(x) = \xi$. Then, for every $\varepsilon > 0$, there exists $u \in W^{2,\infty}(\Omega)$, piecewise polynomial of degree at most 2 up to a set of measure less than ε (i.e. $u \in P_\varepsilon$, *see the definition* (21)) *and there exists an open set $\widetilde{\Omega} \subset \Omega$, with Lipschitz boundary, so that*

$$\begin{cases} \operatorname{meas}(\Omega - \widetilde{\Omega}) \le \varepsilon & \\ u(x) = \varphi(x),\ Du(x) = D\varphi(x), & x \in \partial\Omega \\ |u(x) - \varphi(x)| \le \varepsilon,\ |Du(x) - D\varphi(x)| \le \varepsilon, & x \in \Omega \\ D^2u(x) \in E_\delta, & \text{a.e. } x \in \widetilde{\Omega} \\ \operatorname{dist}\left(D^2u(x); RcoE_\delta\right) \le \varepsilon, & \text{a.e. } x \in \Omega. \end{cases}$$

PROOF. (Lemma 3.4). By Proposition 3.1, we have

$$RcoE_\delta = \bigcup_{i\in\mathbb{N}} R_icoE_\delta.$$

Since $\xi \in RcoE_\delta$, we deduce that $\xi \in R_icoE_\delta$ for some $i \in \mathbb{N}$. We proceed by induction on i.

STEP 1. We start with $i = 1$. We can hence write

$$\xi = \lambda A + (1 - \lambda)B,\ \operatorname{rank}\{A - B\} = 1, \quad A, B \in E_\delta.$$

We then use Lemma 2.2 and get the claimed result by setting $\widetilde{\Omega} = \Omega_A \cup \Omega_B$ and since $co\{A, B\} = [A; B] \subset RcoE_\delta$ and hence

$$\operatorname{dist}\left(D^2u(x); RcoE_\delta\right) \leq \varepsilon, \quad \text{a.e. in } \Omega.$$

STEP 2. We now let for $i \geq 1$

$$\xi \in R_{i+1}coE_\delta.$$

Therefore there exist $A, B \in \mathbb{R}^{n\times n}_s$ such that

$$\begin{cases} \xi = \lambda A + (1-\lambda)B, \\ \operatorname{rank}\{A - B\} = 1, \\ A, B \in R_i coE_\delta\,. \end{cases}$$

We then apply Lemma 2.2 and find that there exist a function $v \in P_\varepsilon$ and Ω_A, Ω_B disjoint open sets such that

$$\begin{cases} \operatorname{meas}(\Omega - (\Omega_A \cup \Omega_B)) \leq \varepsilon/2 & \\ v(x) = \varphi(x),\ Dv(x) = D\varphi(x)\,, & x \in \partial\Omega \\ |v(x) - \varphi(x)| \leq \varepsilon/2,\ \ |Dv(x) - D\varphi(x)| \leq \varepsilon/2\,, & x \in \Omega \\ D^2v(x) = \begin{cases} A \text{ in } \Omega_A \\ B \text{ in } \Omega_B \end{cases} & \\ \operatorname{dist}\left(D^2v(x); RcoE_\delta\right) \leq \varepsilon\,, & \text{a.e. in } \Omega\,. \end{cases}$$

We now use the hypothesis of induction on Ω_A, Ω_B and A, B. We then can find $\widetilde{\Omega}_A, \widetilde{\Omega}_B$, $v_A \in P_\varepsilon$, $v_B \in P_\varepsilon$ satisfying

$$\begin{cases} \operatorname{meas}(\Omega_A - \widetilde{\Omega}_A),\ \operatorname{meas}(\Omega_B - \widetilde{\Omega}_B) \leq \varepsilon/4 & \\ v_A(x) = v(x),\ \ Dv_A(x) = Dv(x) & \text{on } \partial\Omega_A \\ v_B(x) = v(x),\ \ Dv_B(x) = Dv(x) & \text{on } \partial\Omega_B \\ |v_A(x) - v(x)| \leq \varepsilon/2,\ \ |Dv_A(x) - Dv(x)| \leq \varepsilon/2 & \text{in } \Omega_A \\ |v_B(x) - v(x)| \leq \varepsilon/2,\ \ |Dv_B(x) - Dv(x)| \leq \varepsilon/2 & \text{in } \Omega_B \\ D^2v_A(x) \in E_\delta, \quad \text{a.e. in } \widetilde{\Omega}_A, \quad D^2v_B(x) \in E_\delta, & \text{a.e. in } \widetilde{\Omega}_B \\ \operatorname{dist}(D^2v_A, RcoE_\delta) \leq \varepsilon, & \text{a.e. in } \Omega_A \\ \operatorname{dist}(D^2v_B, RcoE_\delta) \leq \varepsilon, & \text{a.e. in } \Omega_B. \end{cases}$$

Letting $\widetilde{\Omega} = \widetilde{\Omega}_A \cup \widetilde{\Omega}_B$ and

$$u(x) = \begin{cases} v(x) & \text{in } \Omega - (\Omega_A \cup \Omega_B) \\ v_A(x) & \text{in } \Omega_A \\ v_B(x) & \text{in } \Omega_B \end{cases}$$

we have indeed obtained the result. □

PROOF. (Lemma 3.3). As in Theorem 2.1 we assume without loss of generality that Ω is bounded. We next define V to be the set of functions $u \in C^1(\overline{\Omega})$ such that there exist sequences $\varepsilon_k \to 0$ and $u_k \in P_{\varepsilon_k}$ satisfying

$$\begin{cases} u_k \to u & \text{in } C^1(\overline{\Omega}) \\ F_i^0\left(D^2 u_k(x)\right) < 0, & \text{a.e. } x \in \Omega,\ i = 1, \dots, N \\ u_k = \varphi \text{ and } Du_k = D\varphi & \text{on } \partial\Omega. \end{cases}$$

The set V is non empty since $\varphi \in V$. We endow V with the C^1–norm and thus V is a metric space. By classical diagonal process we get that V is closed in $C^1(\overline{\Omega})$; thus V is a complete metric space. Let

$$E = \left\{\xi \in \mathbb{R}_s^{n\times n} :\ F_i^0(\xi) = 0,\ i = 1, \dots, N\right\}.$$

Since E, and thus $RcoE$, are bounded, we get that V is bounded in $W^{2,\infty}$; then any sequence in V contains a subsequence which converges in the weak* topology of $W^{2,\infty}$. Since F_i^0 are quasiconvex we deduce, as in the proof of Lemma 2.3, that

$$V \subset \left\{u \in \varphi + W_0^{2,\infty}(\Omega) : F_i^0\left(D^2 u(x)\right) \le 0,\ \text{a.e. } x \in \Omega,\ i = 1, \dots, N\right\}.$$

For $k \in \mathbb{N}$ we define

$$V^k = \left\{u \in V : \sum_{i=1}^{N} \int_\Omega F_i^0\left(D^2 u(x)\right) dx > -1/k\right\}.$$

The set V^k is open in V. Indeed by the boundedness in $W^{2,\infty}$ of V and by the quasiconvexity of F_i^0 we deduce that $V \backslash V^k$ is closed in V.

Now we show that V^k is dense in V. So let $v \in V$; we can assume, by construction of V, that there exist $\varepsilon > 0$ and u_ε such that

$$\begin{cases} u_\varepsilon \in P_\varepsilon \\ F_i^0\left(D^2 u_\varepsilon(x)\right) < 0, & \text{a.e. } x \in \Omega,\ i = 1, \dots, N \\ u_\varepsilon = \varphi,\ \ Du_\varepsilon = D\varphi & \text{on } \partial\Omega \\ \|u_\varepsilon - v\|_{C^1(\overline{\Omega})} \le \varepsilon/2. \end{cases}$$

Therefore there exist disjoint open sets Ω_j, $j = 0, 1, 2, \dots$, so that meas $\Omega_0 \le \varepsilon$ and $u_\varepsilon|_{\Omega_j}$ is a polynomial of degree at most two, i.e. $D^2 u_\varepsilon = \xi_j$ in Ω_j, with $F_i^0(\xi_j) < 0$, for every $i = 1, \dots, N$ and for every $j = 1, 2, \dots$ By continuity of F_i^δ with respect to $\delta \ge 0$ and by assumption (i) we have that $\xi_j \in RcoE_{\delta_j}$ for a certain $\delta_j \in [0, \delta_0)$, where

$$E_{\delta_j} = \left\{\xi \in \mathbb{R}_s^{n\times n} :\ F_i^{\delta_j}(\xi) = 0,\ i = 1, \dots, N\right\};$$

we can also assume that $\delta_j \in [0, \delta_1]$ for every $j \in \mathbb{N}$. By the assumption (ii) and by the rank-one convexity and the openess of the set $\{\xi \in \mathbb{R}^{n\times n}_s : F_i^0(\xi) < 0\}$ we also have

$$RcoE_{\delta_j} \subset\subset \left\{\xi \in \mathbb{R}^{n\times n}_s : F_i^0(\xi) < 0\right\}. \tag{37}$$

By Lemma 3.4 we find a function $u_{\varepsilon,j} \in P_\varepsilon$ and an open set $\widetilde{\Omega}_j \subset \Omega_j$, with Lipschitz boundary, so that, for every $\varepsilon_j \in (0, \varepsilon)$, $j = 1, 2 \dots$

$$\begin{cases} \operatorname{meas}(\Omega_j - \widetilde{\Omega}_j) \le \varepsilon/2^j & \\ u_{\varepsilon,j}(x) = u_\varepsilon(x),\ Du_{\varepsilon,j}(x) = Du_\varepsilon(x),\ x \in \partial\Omega_j & \\ \left|u_{\varepsilon,j}(x) - u_\varepsilon(x)\right| \le \varepsilon/2, & \text{for every } x \in \Omega_j \\ \left|Du_{\varepsilon,j}(x) - Du_\varepsilon(x)\right| \le \varepsilon/2, & \text{for every } x \in \Omega_j \\ D^2u_{\varepsilon,j}(x) \in E_{\delta_j}, & \text{a.e. } x \in \widetilde{\Omega}_j \\ \operatorname{dist}\left(D^2u_{\varepsilon,j}(x); RcoE_{\delta_j}\right) \le \varepsilon_j, & \text{a.e. } x \in \Omega_j\,. \end{cases}$$

By (37), the last inequality implies that $D^2u_{\varepsilon,j}(x)$ is compactly contained in $\{\xi \in \mathbb{R}^{n\times n}_s : F_i^0(\xi) < 0\}$, provided that ε_j is sufficiently small.

Then the function v_ε defined as $v_\varepsilon(x) = u_\varepsilon(x)$, $x \in \overline{\Omega}_0$ and $v_\varepsilon(x) = u_{\varepsilon,j}(x)$, $x \in \overline{\Omega}_j$, $j = 1, 2 \dots$, belongs to $V \cap P_{2\varepsilon}$, since $u_\varepsilon, u_{\varepsilon,j} \in P_\varepsilon$, $v_\varepsilon = u_\varepsilon = \varphi$ and $Dv_\varepsilon = Du_\varepsilon = D\varphi$ on $\partial\Omega$ and $F_i^0\left(D^2u_\varepsilon\right)$, $F_i^0\left(D^2u_{\varepsilon,j}\right) < 0$, a.e. in Ω, $i = 1, \dots, N$.

We let

$$M = \max\left\{\sum_{i=1}^N \left|F_i^0(\xi)\right| : F_i^0(\xi) \le 0,\ i = 1, \dots, N\right\}.$$

We then compute

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega F_i^0\left(D^2v_\varepsilon(x)\right) dx \\ &= \sum_{i=1}^N \int_{\Omega_0} F_i^0\left(D^2v_\varepsilon(x)\right) dx \\ &\quad + \sum_{i=1}^N \sum_{j=1}^\infty \int_{\Omega_j - \widetilde{\Omega}_j} F_i^0(D^2u_{\varepsilon,j})\, dx + \sum_{i=1}^N \sum_{j=1}^\infty \int_{\widetilde{\Omega}_j} F_i^0(D^2u_{\varepsilon,j})\, dx \\ &\ge -\left(\operatorname{meas}\Omega_0 + \sum_{j=1}^\infty \operatorname{meas}(\Omega_j - \widetilde{\Omega}_j)\right) \cdot M \\ &\quad + \sum_{i=1}^N \sum_{j=1}^\infty \int_{\widetilde{\Omega}_j} F_i^{\delta_j}(D^2u_{\varepsilon,j})\, dx + \sum_{i=1}^N \sum_{j=1}^\infty \int_{\widetilde{\Omega}_j} \left\{F_i^0(D^2u_{\varepsilon,j}) - F_i^{\delta_j}(D^2u_{\varepsilon,j})\right\} dx. \end{aligned}$$

In the right hand side the first addendum is small since

$$\operatorname{meas}\Omega_0 \le \varepsilon, \qquad \sum_{j=1}^{\infty} \operatorname{meas}(\Omega_j - \widetilde{\Omega}_j) \le \varepsilon;$$

the second addendum is zero since $D^2 u_{\varepsilon,j}(x) \in E_{\delta_j}$, a.e. $x \in \widetilde{\Omega}_j$, and $F_i^{\delta_j}(E_{\delta_j}) = 0$. Finally the third addendum in the right hand side is small because $D^2 u_{\varepsilon,j}(x)$ a.e. in $\widetilde{\Omega}_j$ belongs to the bounded set $\{\xi \in \mathbb{R}^{n\times n}_s : F_i^0(\xi) < 0\}$ and $F_i^\delta(\xi)$ is uniformly continuous as $\delta \to 0^+$ and since $\delta_j \le \delta_1$ for $j \in \mathbb{N}$. Therefore, for ε and δ_1 sufficiently small we have indeed that $v_\varepsilon \in V^k$.

Since V^k is a sequence of open and dense sets in V, by Baire category theorem we have that the intersection of the V^k, $k \in \mathbb{N}$, is still dense in V, in particular it is not empty. Any element of this intersection is a $W^{2,\infty}(\Omega)$ solution of the given Dirichlet-Neumann problem. □

PROOF. (Theorem 3.2). As before we can assume that Ω is bounded. If we let $w = u - \varphi$, we obtain the equivalent differential problem for $w \in W^{2,\infty}(\Omega)$

$$\begin{cases} G_i^0(x, w(x), Dw(x), D^2 w(x)) = 0, & \text{a.e. in } \Omega,\ i = 1, 2, \dots, N \\ w(x) = 0,\ Dw(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{38}$$

where, for every $i = 1, 2, \dots, N$ and $\delta \in [0, \delta_0)$

$$G_i^\delta(x, s, p, \xi) = F_i^\delta\left(x, s + \varphi(x), p + D\varphi(x), \xi + D^2\varphi(x)\right).$$

Since $\varphi \in C^2(\overline{\Omega})$, then G_i^δ are continuous with respect to (x, s, p, ξ) and with respect to $\delta \in [0, \delta_0)$, quasiconvex with respect to ξ; moreover, for every $(x, s, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ they satisfy

$$\begin{aligned} &Rco\left\{\xi \in \mathbb{R}^{n\times n}_s : G_i^\delta(x, s, p, \xi) = 0, i = 1, \dots, N\right\} \\ &\quad = \left\{\xi \in \mathbb{R}^{n\times n}_s : G_i^\delta(x, s, p, \xi) \le 0, i = 1, \dots, N\right\}, \end{aligned} \tag{39}$$

for every $\delta \in [0, \delta_0)$ and the sets in (39) are compact in $\mathbb{R}^{n\times n}_s$. While, for $0 \le \delta' < \delta < \delta_0$,

$$\begin{aligned} &\left\{\xi \in \mathbb{R}^{n\times n}_s : G_i^\delta(x, s, p, \xi) = 0,\ i = 1, \dots, N\right\} \\ &\quad \subset \left\{\xi \in \mathbb{R}^{n\times n}_s : G_i^{\delta'}(x, s, p, \xi) < 0,\ i = 1, \dots, N\right\}. \end{aligned} \tag{40}$$

By the compatibility assumption (36) we obtain

$$G_i^0(x, 0, 0, 0) < 0, \quad \forall x \in \overline{\Omega}, \quad \forall i = 1, \dots, N. \tag{41}$$

By considering again the set $P_\varepsilon \subset W^{2,\infty}(\Omega)$ of piecewise polynomial of degree at most two up to a set of measure less than ε, we can define W as the set of functions $u \in C^1(\overline{\Omega})$ such that there exist sequences $\varepsilon_k \to 0$ and $u_k \in P_{\varepsilon_k}$ satisfying

$$\begin{cases} u_k \to u \text{ in } C^1(\overline{\Omega}),\ u_k = 0 \text{ and } Du_k = 0 & \text{on } \partial\Omega \\ G_i^0\left(x, u_k(x), Du_k(x), D^2u_k(x)\right) < 0, & \text{a.e. } x \in \Omega,\ i = 1, \dots, N. \end{cases}$$

Then W is non empty, since $0 \in W$ by (41). As in Lemma 2.3 we endow W with the C^1–norm and W becomes a complete metric space. Again similarly to (23) of Lemma 2.3, we can see that

$$W \subset \left\{u \in W_0^{2,\infty}(\Omega) : G_i^0(x, u(x), Du(x), D^2u(x)) \le 0, \text{ a.e. } x \in \Omega,\ i = 1, \dots, N\right\}. \tag{42}$$

For every $k \in \mathbb{N}$ we define

$$W^k = \left\{u \in W : \sum_{i=1}^N \int_\Omega G_i^0\left(x, u(x), Du(x), D^2u(x)\right) dx > -1/k\right\}.$$

By the lower semicontinuity of the integral in the weak* topology of $W^{2,\infty}(\Omega)$ the set W^k is open in W.

We also show that W^k is dense in W. So let $v \in W$; we can assume, by definition of W, that for every $\varepsilon > 0$ we can find v_ε such that

$$\begin{cases} v_\varepsilon \in P_\varepsilon,\ v_\varepsilon = 0,\ Dv_\varepsilon = 0 & \text{on } \partial\Omega,\ \|v_\varepsilon - v\|_{C^1(\overline{\Omega})} \le \varepsilon/2 \\ G_i^0\left(x, v_\varepsilon(x), Dv_\varepsilon(x), D^2v_\varepsilon(x)\right) < 0, & \text{a.e. } x \in \Omega,\ i = 1, \dots, N. \end{cases}$$

Therefore there exist disjoint open sets Ω_j, $j = 0, 1, 2, \dots$, so that meas $\Omega_0 \le \varepsilon$ and, for $j = 1, 2, \dots$, $v_\varepsilon|_{\Omega_j}$ is a polynomial of degree at most two, i.e. $D^2v_\varepsilon = \xi_j$ in Ω_j, with

$$G_i^0\left(x, v_\varepsilon(x), Dv_\varepsilon(x), \xi_j\right) < 0,\ \forall x \in \overline{\Omega}_j,\ \forall i = 1, \dots, N,\ \forall j = 1, 2, \dots$$

Since $(x, v_\varepsilon(x), Dv_\varepsilon(x))$ vary on a compact set, we can find $\delta_j \in [0, \delta_0)$ such that

$$G_i^{\delta_j}\left(x, v_\varepsilon(x), Dv_\varepsilon(x), \xi_j\right) < 0,\ \forall x \in \overline{\Omega}_j,\ \forall i = 1, \dots, N,\ \forall j = 1, 2, \dots \tag{44}$$

By the fact that the set in (39) is bounded for every $\delta \in [0, \delta_0)$, there exists $r > 0$ such that $\left|D^2v(x)\right| \le r$ a.e. $x \in \Omega$ when $v \in W_0^{2,\infty}(\Omega)$ vary on this bounded set (39). Then $|v(x)| + |Dv(x)| \le L \cdot r$, for every $x \in \overline{\Omega}$ and for some positive constant L. Now we use the condition (40): by the uniform continuity of G_i^0 and the equicontinuity in $\overline{\Omega}_j$ of the first derivatives of elements of W,

there exists a finite number H_j of disjoint open sets $\{\Omega_{jh}\}_{h=1,\dots,H_j}$ with Lipschitz boundary, with the property that the closure of their union is equal to $\overline{\Omega}_j$ and such that, if $x_h \in \Omega_{jh}$ for $h = 1, \dots, H_j$, then

$$(45)\quad G_i^{\delta_j}(x_h, v_1(x_h), Dv_1(x_h), \xi) < 0 \implies G_i^0(x, v_2(x), Dv_2(x), \xi) < 0,$$

provided $x \in \Omega_{jh}$, $v_1, v_2 \in W^{2,\infty}(\Omega_{jh})$, $v_1 - v_2 \in W_0^{2,\infty}(\Omega_{jh})$, $|D^2 v_1|$, $|D^2 v_2|$, $|\xi| \le r$.

By the compatibility condition (44) with $x = x_h$ we can apply Lemma 3.3 to find a solution $w_{jh} \in V \subset W^{2,\infty}(\Omega_{jh})$ (see the definition of V in the proof of Lemma 3.3) to the problem

$$(46)\quad \begin{cases} G_i^{\delta_j}\left(x_h, v_\varepsilon(x_h), Dv_\varepsilon(x_h), D^2 w_{jh}(x)\right) = 0, \text{ a.e. } x \in \Omega_{jh},\ i = 1, \dots, N \\ w_{jh}(x) = v_\varepsilon(x), \quad Dw_{jh}(x) = Dv_\varepsilon(x), \quad \forall x \in \partial\Omega_{jh}. \end{cases}$$

Recall that, by the definition of V, for j, h fixed there exist sequences $\varepsilon_k \to 0$ and $u_{jhk} \in P_{\varepsilon_k}$ satisfying

$$(47)\quad \begin{cases} u_{jhk} \to w_{jh} \text{ in } C^1(\overline{\Omega}) \text{ as } k \to +\infty \\ G_i^{\delta_j}\left(x_h, v_\varepsilon(x_h), Dv_\varepsilon(x_h), D^2 u_{jhk}(x)\right) < 0, \text{ a.e. } x \in \Omega_{jh},\ i = 1, \dots, N \\ u_{jhk}(x) = v_\varepsilon(x), \quad Du_{jhk}(x) = Dv_\varepsilon(x), \quad \forall x \in \partial\Omega_{jh}. \end{cases}$$

By (45) with $v_1 = v_\varepsilon$ and $v_2 = u_{jhk}$ we also get

$$(48)\quad G_i^0\left(x, u_{jhk}(x), Du_{jhk}(x), D^2 u_{jhk}(x)\right) < 0, \text{ a.e. in } x \in \Omega_{jh},\ i = 1, \dots, N.$$

We then define the function $w \in W$ in Ω by

$$w(x) = \begin{cases} v_\varepsilon(x) & \text{if } x \in \overline{\Omega}_0 \\ w_{jh}(x) & \text{if } x \in \overline{\Omega}_{jh},\ j = 1, 2, \dots,\ h = 1, \dots, H_j. \end{cases}$$

By the definition of v_ε in (43) and by (46), (47) and (48) we get that $w \in W$. As $\delta_j \to 0$ the function w converges to v_ε in $C^1(\overline{\Omega})$, since $D^2 w$ is uniformly bounded. To show that $w \in W^k$ we compute

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega G_i^0\left(x, w(x), Dw(x), D^2 w(x)\right) dx \\ &= \sum_{i=1}^N \int_{\Omega_0} G_i^0\left(x, v_\varepsilon(x), Dv_\varepsilon(x), D^2 v_\varepsilon(x)\right) dx \\ &\quad + \sum_{i=1}^N \sum_{jh} \int_{\Omega_{jh}} G_i^{\delta_j}\left(x_h, v_\varepsilon(x_h), Dv_\varepsilon(x_h), D^2 w_{jh}(x)\right) dx \\ &\quad + \sum_{i=1}^N \sum_{jh} \int_{\Omega_{jh}} \Big\{ G_i^0\left(x, w_{jh}(x), Dw_{jh}(x), D^2 w_{jh}(x)\right) \\ &\quad - G_i^{\delta_j}\left(x_h, v_\varepsilon(x_h), Dv_\varepsilon(x_h), D^2 w_{jh}(x)\right) \Big\} dx. \end{aligned}$$

The second addendum in the right hand side is equal to zero by (46), while the third addendum is small as δ_j is small and the quantity

$$\left|G_i^0\left(x, v_2(x), Dv_2(x), \xi\right) - G_i^{\delta_j}\left(x_h, v_1(x_h), Dv_1(x_h), \xi\right)\right|$$

similarly to (29) or (45) can be considered arbitrarily small. Choosing ε and δ_j small enough we deduce that the first addendum is also small. Therefore $w \in W^k$ and the density of W^k is established. By Baire category theorem $\cap W^k$ is dense in W and (38) holds. □

We now turn to applications of Theorem 3.2 to Example 1.3 involving the singular values. Recall that for a given matrix $\xi \in \mathbb{R}_s^{n\times n}$ we denote by $0 \le \lambda_1(\xi) \le \ldots \le \lambda_n(\xi)$ the singular values of ξ (i.e. the absolute value of the eigenvalues of ξ), in particular we have (c.f. for example [8])

$$\begin{cases} |\xi|^2 = \sum_{i=1}^{n} (\lambda_i(\xi))^2 \\ |\det \xi| = \prod_{i=1}^{n} \lambda_i(\xi) \,. \end{cases}$$

THEOREM 3.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. Let $\varphi \in C^2(\overline{\Omega})$ (or piecewise C^2) satisfies*

$$\lambda_n(D^2\varphi(x)) < 1, \quad \forall x \in \Omega. \tag{49}$$

Then there exists (a dense set of) $u \in W^{2,\infty}(\Omega)$ such that

$$\begin{cases} \lambda_1\left(D^2u\right) = \ldots = \lambda_n\left(D^2u\right) = 1, & \text{a.e. } in\ \Omega \\ u = \varphi,\ Du = D\varphi, & on\ \partial\Omega\,. \end{cases}$$

REMARK 3.1. i) The above result can be stated equivalently as

$$\begin{cases} D^2u \in O(n), & \text{a.e. in } \Omega \\ u = \varphi,\ Du = D\varphi, & \text{on } \partial\Omega\,. \end{cases}$$

ii) The theorem implies in particular that if (49) holds then there exists $u \in W^{2,\infty}(\Omega)$ such that

$$\begin{cases} \left|\det D^2u\right| = \prod_{i=1}^{n} \lambda_i\left(D^2u\right) = 1, & \text{a.e. in } \Omega \\ u = \varphi,\ Du = D\varphi, & \text{on } \partial\Omega\,. \end{cases}$$

PROOF. We need to find the appropriate functions F_i^δ so as to apply Theorem 3.2. The first guess is to choose $F_i^\delta(\xi) = \lambda_i(\xi) - 1 + \delta$, however this is not the right one since, except for $i = n$, these functions are not quasiconvex

and furthermore do not satisfy hypothesis (i) of Theorem 3.2. To explain one of the possible choices we first observe that

$$\begin{aligned} \lambda_i(\xi) = 1,\ i = 1, \dots, n &\Leftrightarrow \sum_{\nu=i}^{n} (\lambda_\nu(\xi) - 1) = 0\ ,\ i = 1, \dots, n \\ &\Leftrightarrow \prod_{\nu=i}^{n} \lambda_\nu(\xi) = 1,\ i = 1, \dots, n. \end{aligned} \tag{50}$$

Therefore we set, for $0 \le \delta \le 1$,

$$F_i^\delta(\xi) = \sum_{\nu=i}^{n} (\lambda_\nu(\xi) - 1 + \delta) = F_i^0(\xi) + (n - i + 1)\delta = 0,\ i = 1, \dots, n$$

(we could also have chosen $F_i^\delta(\xi) = \prod_{\nu=i}^{n} \lambda_\nu(\xi) - 1 + \delta$). We claim that F_i^δ satisfies all the hypotheses of Theorem 3.2. Indeed all the F_i^δ are convex (see [8], Proposition 1.2 page 254) and thus quasiconvex in ξ, furthermore they are trivially continuous with respect to $0 \le \delta \le 1$.

Assume that we have shown (c.f. below) that, under the notation

$$\begin{aligned} E_\delta &= \left\{\xi \in \mathbb{R}_s^{n\times n} : \lambda_i(\xi) = 1 - \delta,\ i = 1, \dots, n\right\} \\ &= \left\{\xi \in \mathbb{R}_s^{n\times n} : F_i^\delta(\xi) = 0,\ i = 1, \dots, n\right\}, \end{aligned} \tag{51}$$

then

$$\begin{aligned} coE_\delta = RcoE_\delta &= \left\{\xi \in \mathbb{R}_s^{n\times n} : \lambda_n(\xi) \le 1 - \delta\right\} \\ &= \left\{\xi \in \mathbb{R}_s^{n\times n} : F_i^\delta(\xi) \le 0,\ i = 1, \dots, n\right\}. \end{aligned} \tag{52}$$

It is then clear that all the hypotheses of Theorem 3.2 are satisfied and thus the theorem applies and we get the result since (50) holds.

Therefore it only remains to show that (52) holds. To do this it is sufficient, in view of the definition of F_i^δ, to prove it for $\delta = 0$. Using (50) we have only to show that, if $E = \left\{\xi \in \mathbb{R}_s^{n\times n} : \lambda_i(\xi) = 1,\ i = 1, \dots, n\right\}$, then

$$coE = RcoE = \left\{\xi \in \mathbb{R}_s^{n\times n} : \lambda_n(\xi) \le 1\right\}.$$

We proceed as in Theorem 5.1 of [12]. We denote by

$$X = \left\{\xi \in \mathbb{R}_s^{n\times n} : \lambda_n(\xi) \le 1\right\}$$

and we observe that since the function $\xi \to \lambda_n(\xi)$ is convex and $E \subset X$, then $RcoE \subset coE \subset X$. Therefore we only need to show that any $\xi \in X$ belongs also to $RcoE$. Note that, since singular values remain unchanged by orthogonal transformations, there is no loss of generality in assuming that ξ is a diagonal matrix, i.e.

$$\xi = \begin{pmatrix} a_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & a_n \end{pmatrix}$$

with $0 \le a_1 \le \ldots \le a_n \le 1$. We then interpolate a_1 between ± 1 and get

$$\xi = \frac{1+a_1}{2}\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_n \end{pmatrix} + \frac{1-a_1}{2}\begin{pmatrix} -1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_n \end{pmatrix}.$$

Similarly, interpolating a_2 between ± 1 we obtain

$$\begin{pmatrix} \pm 1 & 0 & 0 & \ldots & 0 \\ 0 & a_2 & 0 & \ldots & 0 \\ 0 & 0 & a_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & a_n \end{pmatrix}$$

$$= \frac{1+a_2}{2}\begin{pmatrix} \pm 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & a_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & a_n \end{pmatrix} + \frac{1-a_2}{2}\begin{pmatrix} \pm 1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 0 & \ldots & 0 \\ 0 & 0 & a_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & a_n \end{pmatrix}.$$

Iterating the procedure with every $i = 3, \ldots n$, we get, in particular for $i = n$, that

$$\begin{pmatrix} \pm 1 & 0 & \ldots & 0 & 0 \\ 0 & \pm 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \pm 1 & 0 \\ 0 & 0 & \ldots & 0 & a_n \end{pmatrix}$$

$$= \frac{1+a_n}{2}\begin{pmatrix} \pm 1 & 0 & \ldots & 0 & 0 \\ 0 & \pm 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \pm 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} + \frac{1-a_n}{2}\begin{pmatrix} \pm 1 & 0 & \ldots & 0 & 0 \\ 0 & \pm 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \pm 1 & 0 \\ 0 & 0 & \ldots & 0 & -1 \end{pmatrix}$$

Hence the matrix in the left hand side is in $R_1 coE$. Going reversely we thus obtain that $\xi \in R_n coE \subset RcoE$, which is the claimed result. □

REFERENCES

[1] J. M. Ball – J. C. Curie – P. J. Olver, *Null Lagrangians, weak continuity and variational problems of arbitrary order*, J. Funct. Anal. **41** (1981), 135-174.

[2] J. M. Ball – R. D. James, *Fine phase mixtures as minimizers of energy*, Arch. Rational Mech. Anal. **100** (1987), 15-52.

[3] R. Bellman, "Introduction to matrix analysis", McGraw-Hill, New York, 1960.

[4] A. Bressan – F. Flores, *On total differential inclusions*, Rend. Sem. Mat. Univ. Padova **92** (1994), 9-16.

[5] L. CAFFARELLI – L. NIRENBERG – J. SPRUCK, *The Dirichlet problem for nonlinear second order elliptic equations, I: Monge-Ampère equations*, Comm. Pure Appl. Math. **37** (1984), 369-402.

[6] A. CELLINA, *On the differential inclusion*, $x' \in [-1, 1]$, Atti. Accad. Naz. Lincei, Rend. Sci. Fis. Mat. Nat. **69** (1980), 1-6.

[7] M. G. CRANDALL – H. ISHII – P. L. LIONS, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. **27** (1992), 1-67.

[8] B. DACOROGNA, "Direct methods in the calculus of variations", Applied Math. Sciences, 78, Springer, Berlin, 1989.

[9] B. DACOROGNA – P. MARCELLINI, *Existence of minimizers for non quasiconvex integrals*, Arch. Rational Mech. Anal. **131** (1995), 359-399.

[10] B. DACOROGNA – P. MARCELLINI, *Théorème d'existence dans le cas scalaire et vectoriel pour les équations de Hamilton-Jacobi*, C.R. Acad. Sci. Paris **322** (1996), 237-240.

[11] B. DACOROGNA – P. MARCELLINI, *Sur le problème de Cauchy-Dirichlet pour les systèmes d'équations non linéaires du premier ordre*, C. R. Acad. Sci. Paris **323** (1996), 599-602.

[12] B. DACOROGNA – P. MARCELLINI, *General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases*, Acta Mathematica, **178** (1997), 1-37.

[13] B. DACOROGNA – P. MARCELLINI, *Cauchy-Dirichlet problem for first order nonlinear systems*, J. Functional Analysis, to appear.

[14] F. S. DE BLASI – G. PIANIGIANI, *A Baire category approach to the existence of solutions of multivalued differential equations in Banch spaces*, Funkcialaj Ekvacioj **25** (1982), 153-162.

[15] F. S. DE BLASI – G. PIANIGIANI, *Non convex valued differential inclusions in Banach spaces*, J. Math. Anal. Appl. **157** (1991), 469-494.

[16] L. C. EVANS, *Classical solutions of fully nonlinear, convex, second-order elliptic equations*, Comm. Pure Appl. Math. **25** (1982), 333-363.

[17] N. FUSCO, *Quasi-convessità e semicontinuità per integrali multipli di ordine superiore*, Ricerche di Matematica **29** (1980), 307-323.

[18] M. GUIDORZI – L. POGGIOLINI, *Lower semicontinuity for quasiconvex integrals of higher order*, preprint 1997.

[19] N. G. MEYERS, *Quasiconvexity and the semicontinuity of multiple integrals*, Trans. Amer. Math. Soc. **119** (1965), 125-149.

[20] C. B. MORREY, "Multiple integrals in the calculus of variations", Springer, Berlin, 1966.

[21] S. MÜLLER – V. SVERAK, "Attainment results for the two-well problem by convex integration", edited by J. Jost, International Press 1996, pp. 239-251.

[22] N. S. TRUDINGER, *Fully nonlinear, uniformly elliptic equations under natural structure conditions*, Trans. Amer. Math. Soc. **278** (1983), 751-769.

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