

Elliptic versus Parabolic Regularization for the Equation of Prescribed Mean Curvature*

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1. INTRODUCTION

Consider the Dirichlet problem for the equation of prescribed mean curvature,

$$\operatorname{div}((1 + |Du|^2)^{-1/2} Du) + h(x) = 0 \quad \text{in } \Omega, \quad (1.1a)$$

$$u = \Phi(x) \quad \text{on } \partial\Omega, \quad (1.1b)$$

where Ω is a bounded open set in R^n with a C^2 boundary, $h(x)$ is a given Lipschitz continuous function in Ω , and $\Phi(x)$ is a given smooth function on $\partial\Omega$ (for many of our results a Lipschitz boundary would be sufficient). The expression

$$A(u) \equiv \operatorname{div}(g(Du)), \quad (1.2)$$

where

$$g(Du) \equiv (1 + |Du|^2)^{-1/2} Du, \quad (1.3)$$

is the curvature (sum of the principal curvatures) of the graph of the function u . However, for physical intuition it is worthwhile to think of $u(x)$ as the “temperature” in the region Ω and of $g(Du)$ as the “flux” (the negative of the “heat flux”) in the region. The $h(x)$ is then the given “heat source function”.

It is well known that if $h(x)$ is “too big” on Ω then there exists no solution on all Ω of the partial differential Eq. (1.1a). This is because the flux $g(Du)$ saturates with norm 1 as $|Du| \rightarrow \infty$. Integrating by parts, one has

$$\int_G h(x) dx = - \int_{\partial G} g(Du) \cdot \nu dH_{n-1} \quad (1.4)$$

for every subset G of Ω , where dH_{n-1} denotes the surface measure on $\partial\Omega$ and ν is the outward unit normal vector. Thus the flux g can remove at most $P(G)$ (the perimeter of G , i.e. the measure of ∂G) in heat from any subset, and, since $-1 < g \cdot \nu < 1$, a *necessary condition* for existence of a classical solution of (1.1a) on all Ω is that both

$$\int_G h(x) dx < P(G), \quad (1.5a)$$

and

$$-P(G) < \int_G h(x) dx, \quad (1.5b)$$

for all proper subsets G of Ω .

In fact, Giaquinta [5] proved that (1.5ab) (with the strict $<$ there holding also for the set Ω itself) is a *sufficient condition* for existence of a *pseudosolution* of (1.1). This pseudosolution is a solution of the Dirichlet problem (1.1) in the sense of $BV(\Omega)$, and it is a classical $C^{2,\alpha}(\Omega)$ solution of the PDE (1.1a) in the interior. However, it may “detach from the desired boundary values” on some portions of $\partial\Omega$.

Moreover, Giusti [7] showed that if Ω is an *extremal set* for $h(x)$ (i.e. (1.5) holds with strict $<$ for proper subsets, but with $=$ in (1.5a) for Ω itself) then there exists an *extremal solution* $U(x)$ on Ω for the PDE (1.1a). It has the following properties:

- (i) it is a classical $C^{2,\alpha}$ solution in Ω ,
- (ii) $g(DU(x)) \rightarrow -v(x_0)$ as $x \rightarrow x_0 \in \partial\Omega$,
- (iii) it is unique to within an additive constant.

Here Giusti requires that $h(x)$ be Lipschitz and that $\partial\Omega$ be C^2 .

But what happens when $h(x)$ is “too large” on Ω (that is, if either (1.5a) or (1.5b) fail for some subsets G)? *Why* don’t the usual parabolic (adding a time derivative u_t to the equation) or elliptic (adding an $\varepsilon \Delta u$ term) regularizations yield a solution for (1.1) in the limit as $t \rightarrow \infty$ or as $\varepsilon \rightarrow 0$?

Merely by way of illustration, let us consider the 1-D case with $h(x) \equiv$ a constant on the interval $\Omega = (-1, 1)$, with zero boundary values. Thus we *hope* for a solution $u(x)$ of the original mean curvature problem (1.1), i.e.

$$A(u) + h = 0 \quad \text{in } \Omega. \quad (1.6)$$

We have the solution $u(x, t)$ of the parabolic regularization

$$u_t = A(u) + h \quad \text{in } \Omega \times (0, \infty), \quad (1.7)$$

with zero initial and boundary values. We also have the solution $u^\varepsilon(x)$ of the elliptic regularization

$$A(u) + \varepsilon \Delta u + h = 0 \quad \text{in } \Omega, \quad (1.8)$$

again with zero boundary values.

If the constant h is sufficiently small for (1.5) to hold (that is, if $|h| \leq 1$) one can show that both $u(x, t)$ and $u^\varepsilon(x)$ converge to the solution $u(x)$ of (1.6) as $t \rightarrow \infty$ or $\varepsilon \rightarrow 0$. However, if $h > 1$ we can show that the asymptotic behavior of $u(x, t)$ as $t \rightarrow \infty$, or of $u^\varepsilon(x)$ as $\varepsilon \rightarrow 0$, are as illustrated (for the case $h = 2$) in the moving finite element computations of Figures 1a and 1b respectively.

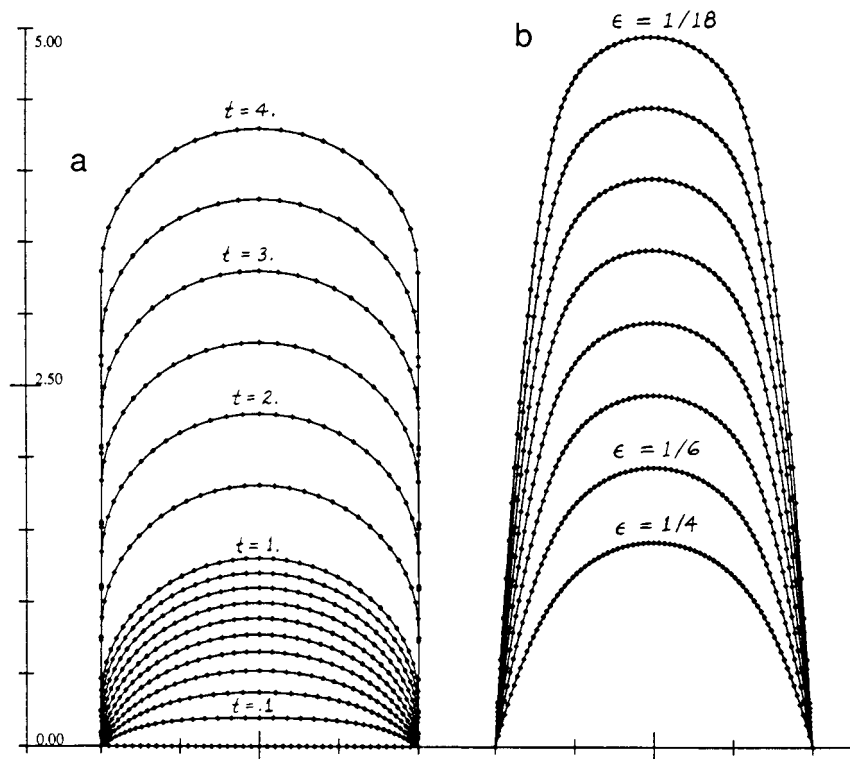


FIG. 1. (a) 80 node GWMFE solution $u(x, t)$ of $u_t = A(u) + 2$ on $\Omega = (-1, 1)$ with zero initial and boundary data, at times $t = 0, 1, \dots, 4$. Solution forms a rising elliptic cap on $\Omega^* = (-1, 1)$. (b) 80 node GWMFE solution $u^\epsilon(x)$ of $A(u) + \epsilon A u + 2 = 0$ on $\Omega = (-1, 1)$ with zero boundary data, with $\epsilon = \frac{1}{4}, \frac{1}{6}, \dots, \frac{1}{18}$. Solution forms a rising elliptic cap on $\tilde{\Omega} = (-\frac{1}{2}, \frac{1}{2})$.

Notice in Fig. 1a that $u(x, t)$ rises fastest as $t \rightarrow \infty$ on a certain subset Ω^* . (Here $\Omega^* =$ all of Ω , but that need not be the case in general.) On Ω^* , $u(x, t)$ is very quickly taking on the shape of a “rising elliptic cap” which in this case is a semicircle of curvature $A(u) = -1$, while on Ω^* the asymptotic speed with which this cap rises is $u_t = A(u) + 2 \approx 1$. Thus $u(x, t)$ grows without bound, but nevertheless with an asymptotic speed $v(x)$ (which is $\equiv 1$ on all Ω in this case).

Notice in Fig. 1b that $u^\epsilon(x)$ rises fastest as $\epsilon \rightarrow 0$ on a certain compactly contained subset $\tilde{\Omega}$, which in this case is the interval $(-\frac{1}{2}, \frac{1}{2})$. (Notice that, contrary to the parabolic case, $\tilde{\Omega}$ is compactly contained in Ω , which is a general fact for the elliptic case.) On $\tilde{\Omega}$, $u^\epsilon(x)$ is taking on the shape of a “rising elliptic cap” which in this case is a semicircle of curvature $A(u) = -h = -2$. We shall show that $u^\epsilon(x)$ grows without bound, but that nevertheless $\epsilon u^\epsilon(x)$ has a limit $w(x)$ which will be not identically zero.

We shall characterize these “parabolic and elliptic growth functions” v and w . In fact we will prove in Sections 4 and 6, in the general n -dimensional case, that $u(x, t)/t \rightarrow v(x)$ in $L^2(\Omega)$ and that $\varepsilon u^\varepsilon(x) \rightarrow w(x)$ in $W_0^{1,2}(\Omega)$. Here v is characterized as the unique minimizer in $BV(\bar{\Omega}) \cap L^2(\Omega)$ of the functional

$$F(u) = \int_{\bar{\Omega}} |Du| + \frac{1}{2} \int_{\Omega} u^2 dx - \int_{\Omega} hu dx. \quad (1.9)$$

Instead, w is characterized as the unique minimizer in $W_0^{1,2}(\Omega)$ of the functional

$$G(u) = \int_{\Omega} |Du| + \frac{1}{2} \int_{\Omega} |Du|^2 dx - \int_{\Omega} hu dx. \quad (1.10)$$

We point out that these expressions involving $|Du|$ on $\bar{\Omega}$ and Ω must be made precise in the BV sense, see Section 2.

The solutions of these two variational problems are quite different in nature because the change from u^2 in (1.9) to $|Du|^2$ in (1.10) yields minimizers of greater smoothness.

In Sections 2–6 we establish the existence and variational characterization of the parabolic and elliptic growth functions $v(x)$ and $w(x)$ described above. In Sections 7–9 we establish the properties of the parabolic and elliptic maximum sets Ω^* and $\bar{\Omega}$ (the sets on which $v(x)$ and $w(x)$ assume their maximums), based upon the variational formulations of (1.9) and (1.10). In Section 10, using the variational formulation, we establish the explicit formula for $v(x)$ in the case that $h(x)$ is a constant on a 2-D rectangle Ω . In Section 11 we show some numerical computations in 2-D by Carlson and Miller using their gradient-weighted Moving Finite Element codes (GWMFE) [2], [16]. In Section 12 we loosely consider the many possible phenomena arising when $h(x)$ is replaced by $h(x, u)$ in the parabolic and elliptic equations (1.6)–(1.8).

The parabolic phenomena established in this paper were first described in the authors' previous paper [14]. There we presented the general conjectures of Section 7 concerning the geometrically identifiable set Ω^* on which $u(x, t)$ grows fastest and the probable shape of the “rising elliptic cap” on that set. However, we were able to give proofs only in certain radially symmetric situations. The results of the present paper were presented without proof in the proceedings of the 1994 Levico conference on curvature flows [15]. Similar results to those of Section 10, the parabolic case on a 2-D square with constant h , have recently been established by Kawohl and Kutev [11] using completely different methods based on comparison functions and the maximum principle (somewhat as used in [14]).

2. NOTATION AND DEFINITIONS

In all cases when given an $L^1(\Omega)$ function u we consider it extended to be $\equiv 0$ in $R^n - \Omega$. We define *the total variation of u in Ω* by

$$\int_{\Omega} |Du| \equiv \sup \left\{ \int_{\Omega} u \operatorname{div} g \, dx: g \in C_0^1(\Omega, R^n), |g| \leq 1 \right\}. \quad (2.1)$$

Then, taking into account that u (extended to zero) may have jumps at the boundary, we define *the total variation of u on $\bar{\Omega}$* by

$$\int_{\bar{\Omega}} |Du| \equiv \sup \left\{ \int_{\Omega} u \operatorname{div} g \, dx: g \in C^1(\bar{\Omega}, R^n), |g| \leq 1 \right\}. \quad (2.2)$$

DEFINITION. $BV(\Omega)$ is the space of $L^1(\Omega)$ functions such that $\int_{\Omega} |Du|$ is finite. $BV(\bar{\Omega})$ is the space of $L^1(R^n)$ functions extended $\equiv 0$ outside Ω , such that $\int_{\bar{\Omega}} |Du|$ is finite.

It is easily seen (since $u \equiv 0$ outside Ω , $\partial\Omega$ is smooth, and our test fields g in (2.2) can be extended rather arbitrarily outside Ω) that

$$\int_{\bar{\Omega}} |Du| = \int_{R^n} |Du|. \quad (2.3)$$

Note that if $u \in W^{1,1}(\Omega) \cap BV(R^n)$ then u has an internal boundary trace u^- in $L^1(\partial\Omega)$, and that

$$\int_{\bar{\Omega}} |Du| = \int_{\Omega} |Du| \, dx + \int_{\partial\Omega} |u^- - 0| \, dH_{n-1}, \quad (2.4)$$

where dH_{n-1} denotes the $(n-1)$ -dimensional Hausdorff measure on $\partial\Omega$ (see Giusti [8], Remark 2.14).

We now define similar expressions for the area of the graph of a $BV(R^n)$ function which is $\equiv 0$ outside Ω . First we define *the area of the graph on Ω* by

$$\int_{\Omega} \sqrt{1 + |Du|^2} \equiv \sup \left\{ \int_{\Omega} g_0 + u \sum_{i=1}^n \frac{\partial g_i}{\partial x_i} \, dx: g \in C_0^1(\Omega, R^{n+1}), |g| \leq 1 \right\}. \quad (2.5)$$

Then we define *the area of the graph on $\bar{\Omega}$* , which includes the surface area of the possible vertical sides of the graph as it makes its transition to zero boundary values on $\partial\Omega$,

$$\int_{\bar{\Omega}} \sqrt{1 + |Du|^2} \equiv \sup \left\{ \int_{\Omega} g_0 + u \sum_{i=1}^n \frac{\partial g_i}{\partial x_i} \, dx: g \in C^1(\bar{\Omega}, R^{n+1}), |g| \leq 1 \right\}. \quad (2.6)$$

Note that the only difference between the two definitions is that the test vector fields g are in $C_0^1(\Omega, R^{n+1})$ in the first case and in $C^1(\bar{\Omega}, R^{n+1})$ in the second case. The analog of (2.4) then becomes, for $u \in W^{1,1}(\Omega) \cap BV(R^n)$, that

$$\int_{\bar{\Omega}} \sqrt{1 + |Du|^2} = \int_{\Omega} \sqrt{1 + |Du|^2} dx + \int_{\partial\Omega} |u^- - 0| dH_{n-1}. \quad (2.7)$$

We next recall the definition of the perimeter in the sense of De Giorgi of a measurable set in R^n . Let G be a measurable set in R^n ; then its *perimeter* is

$$P(G) \equiv \int_{R^n} |D\varphi_G| = \sup \left\{ \int_G \operatorname{div} g \, dx : g \in C_0^1(R^n, R^n), |g| \leq 1 \right\}, \quad (2.8)$$

where φ_G of course denotes the characteristic function of G . A *Caccioppoli* set is then defined to be a Borel set for which the above perimeter is finite.

We refer to the book of Giusti [8] for details and properties concerning the Definitions (2.1)–(2.8) above.

3. THE PARABOLIC PSEUDOSOLUTION

We want to consider the solution of the parabolic initial-boundary value problem

$$\begin{aligned} u_t &= \operatorname{div} \left(\frac{Du}{(1 + |Du|^2)^{1/2}} \right) + h(x) && \text{in } \Omega \times (0, \infty), \\ u(x, t) &= \Phi(x) && \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) &= u_0(x) && \text{on } \Omega, \end{aligned} \quad (3.1)$$

where u_0 and Φ are smooth functions on $\bar{\Omega}$ and $\partial\Omega$ giving the desired initial and *fixed* boundary values, and h is a given Lipschitz continuous function on the open bounded set Ω .

Classical solutions of (3.1), which assume the boundary values continuously, etc., may fail to exist. Instead we consider the pseudosolutions introduced by Lichnewsky and Temam [13]. These proceed by the regularized problem

$$\begin{aligned} u_t &= \operatorname{div} \left(\frac{Du}{(1 + |Du|^2)^{1/2}} + \varepsilon Du \right) + h(x) && \text{in } \Omega \times (0, \infty) \\ u(x, t) &= \Phi(x) && \text{on } \partial\Omega \times (0, \infty) \\ u(x, 0) &= u_0(x) && \text{on } \Omega. \end{aligned} \quad (3.2)$$

Because of the added εDu flux term in (3.2), this problem is uniformly elliptic and thus it has a unique classical ($C^{2,\alpha}$ on $\bar{\Omega} \times [0, \infty)$) solution, which we will denote $u^\varepsilon(x, t)$. Lichnewsky and Temam proved (among other things) that for each $t \geq 0$, $u^\varepsilon(x, t)$ converges weakly in $L^2(\Omega)$ to a limit function $u(x, t)$, which they call the *pseudosolution* for problem (3.1).

This function u is characterized as the unique solution of a certain weak form of problem (3.1). Before giving the precise formulation, let us proceed heuristically to motivate that weak form. For a fixed $T > 0$, let Q denote the open cylinder $\Omega \times (0, T)$. Suppose that $u(x, t)$ were a $C^2(\bar{Q})$ solution of (3.1) and let $\varphi(x, t)$ be any test function in $C^2(\bar{Q})$ with the same boundary values on $\partial\Omega \times (0, T)$. Multiplying the equation (3.1) by $(\varphi - u)(x, \tau)$ and integrating by parts with respect to x , then integrating with respect to τ ($0 < \tau < t$), one has

$$\int_0^t (u_t - h, \varphi - u)_{L^2} d\tau + \int_0^t \int_\Omega \frac{Du \cdot (D\varphi - Du)}{\sqrt{1 + |Du|^2}} dx d\tau = 0. \quad (3.3)$$

By the convexity of the function $\xi \rightarrow (1 + |\xi|^2)^{1/2}$, we have for every ξ and η in R^n that

$$(1 + |\eta|^2)^{1/2} \geq (1 + |\xi|^2)^{1/2} + \frac{1}{\sqrt{1 + |\xi|^2}} \xi \cdot (\eta - \xi), \quad (3.4)$$

and thus

$$\int_0^t (u_t - h, \varphi - u)_{L^2} d\tau + \int_0^t \int_\Omega [(1 + |D\varphi|^2)^{1/2} - (1 + |Du|^2)^{1/2}] dx d\tau \geq 0. \quad (3.5)$$

Now we get rid of the u_t derivative by essentially integrating by parts, adding to (3.5) the equality

$$\begin{aligned} \int_0^t (\varphi_t - u_t, \varphi - u)_{L^2} d\tau &= \frac{1}{2} \int_0^t \frac{d}{d\tau} (\varphi - u, \varphi - u)_{L^2} d\tau \\ &= \frac{1}{2} \|\varphi(t) - u(t)\|_{L^2}^2 - \frac{1}{2} \|\varphi(0) - u(0)\|_{L^2}^2. \end{aligned} \quad (3.6)$$

Then the u_t is removed from (3.5) and replaced by a φ_t ,

$$\begin{aligned} \int_0^t (\varphi_t, \varphi - u_t)_{L^2} d\tau + \int_0^t \int_\Omega [(1 + |D\varphi|^2)^{1/2} - (1 + |Du|^2)^{1/2}] dx d\tau \\ \geq \frac{1}{2} \|\varphi(t) - u(t)\|_{L^2}^2 - \frac{1}{2} \|\varphi(0) - u(0)\|_{L^2}^2. \end{aligned} \quad (3.7)$$

The above derivation assumed that both u and φ smoothly assume the boundary values $\Phi(x)$ on $\partial\Omega \times (0, T)$. In fact the regularized solution $u^\varepsilon(x, t)$ may build up thin boundary layers on portions of $\partial\Omega \times (0, T)$ and the limiting pseudosolution will “detach from the desired boundary values $\Phi(x)$ ” at places. It is for that reason that we included the area of the vertical portions of the graph in our definition (2.6) which replaces the integrals involving Du and $D\varphi$ in (3.7).

In fact, to make our notation and definitions simpler, we have assumed a zero boundary function $\Phi(x) \equiv 0$ on $\partial\Omega$ in our definition of (2.6). Thus *henceforth we simplify the formulation of (3.1) by assuming the boundary function $\Phi(x)$ is $\equiv 0$* . The precise formulation then yields the following.

THEOREM 3.1 (Lichnewsky and Temam [13]). *Suppose $\Phi(x) \equiv 0$, $h(x) \in W^{1,2}(\Omega)$ and*

$$u_0 \in L^2(\Omega) \cap W_{\text{loc}}^{1,2}(\Omega) \cap W^{1,1}(\Omega). \quad (3.8)$$

Then there exists a unique function $u(x, t)$ having the following properties: For each positive number T ,

$$u \in L^1(0, T; W^{1,1}(\Omega)) \cap L^\infty(0, T; W_{\text{loc}}^{1,2}(\Omega)), \quad (3.9)$$

$$u \in C([0, T]; L^2(\Omega)), \quad (3.10)$$

$$u(0) = u_0, \quad (3.11)$$

and for every $t \in [0, T]$ and for every test function $\varphi \in L^2(Q)$ such that $\varphi_t \in L^2(Q)$ and $D\varphi \in L^1(Q, \mathbb{R}^n)$, we have

$$\begin{aligned} & \int_0^t (\varphi_t - h, \varphi - u)_{L^2} d\tau + \int_0^t \int_\Omega [(1 + |D\varphi|^2)^{1/2} - (1 + |Du|^2)^{1/2}] dx d\tau \\ & \geq \frac{1}{2} \|\varphi(t) - u(t)\|_{L^2}^2 - \frac{1}{2} \|\varphi(0) - u_0\|_{L^2}^2. \end{aligned} \quad (3.12)$$

Furthermore, if $h \in L^\infty(\Omega)$ and $u_0 \in L^\infty(\Omega)$ then $u \in L^\infty(Q)$.

In the proof of this theorem u was constructed as the limit as $\varepsilon \rightarrow 0$ of the regularized solution u^ε of (3.2). Among other things, it was established that for each t in $[0, T]$, $u^\varepsilon(\cdot, t)$ converges to $u(\cdot, t)$ weakly in $L^2(\Omega)$. In fact, although it was not explicitly stated, one easily establishes that “weakly” here can be replaced by “strongly”, i.e.

$$\begin{aligned} & \text{for each } t \text{ in } [0, T], u^\varepsilon(\cdot, t) \text{ converges to} \\ & u(\cdot, t) \text{ strongly in } L^2(\Omega). \end{aligned} \quad (3.13)$$

This is because it was established (Lemma 2.2 of [13]) that Du^ε remains bounded in $L^2(\Omega')$ as $\varepsilon \rightarrow 0$ for every $\Omega' \subset\subset \Omega$. Thus by Rellich and the weak convergence of u^ε to u one obtains that $u^\varepsilon \rightarrow u$ strongly in $L^2(\Omega')$, for every $\Omega' \subset\subset \Omega$. Moreover, it was established that the u^ε (and thus their limit u) are bounded in $L^\infty(\Omega)$ (Lemma 2.3). Hence we have

$$\|u^\varepsilon - u\|_{L^2(\Omega)}^2 \leq \|u^\varepsilon - u\|_{L^2(\Omega')}^2 + (\|u^\varepsilon\|_{L^\infty}^2 + \|u\|_{L^\infty}^2) \operatorname{mis}(\Omega - \Omega') \quad (3.14)$$

which can be made arbitrarily small.

Moreover, in a second theorem it was established that $u(x, t)$ is a solution of the original PDE in (3.1) in the sense of distributions, and that also the boundary condition of (3.1) is satisfied in the sense that

$$g(Du) \cdot \nu \in \operatorname{Sign}(u - 0) \quad \text{a.e. on } \partial\Omega \times (0, T). \quad (3.15)$$

In a more recent terminology, the function $u(x, t)$, the limit as $\varepsilon \rightarrow 0$ of the regularized solution $u^\varepsilon(x, t)$ of (3.2), is a “*viscosity solution*” of (3.1) (see the approach of Kawohl and Kutev in [11]).

4. THE PARABOLIC GROWTH FUNCTION v

Let $u^\varepsilon(x, t)$ and $u(x, t)$ denote the regularized solutions and the pseudosolution of the previous section. It is probable that $u_\varepsilon(x, t)$ tends to a limit as $t \rightarrow \infty$; instead we can show only that $u(x, t)/t$ tends to a limit $v(x)$, which we shall call *the parabolic rate function*. We shall then characterize v as the unique solution of a certain variational problem. From this variational formulation will then follow many of the properties of this v ; in particular we shall discover necessary and sufficient conditions for v to be not identically zero.

THEOREM 4.1. *As $t \rightarrow \infty$, $u(x, t)/t$ converges in $L^2(\Omega)$ to a certain function $v(x)$. This function is independent of the particular initial function $u_0(x)$.*

Proof. We will use a result on contraction semigroups due to Crandall, as reported by Brezis [1, p. 166] and by Pazy [17, p. 305, Theorem 3.9]. The result (slightly simplified for our purposes) is the following: Let $S(t)$ be a strongly continuous contraction semigroup on a Hilbert space \mathcal{H} . Then there exists an element v in \mathcal{H} such that for every element u_0 in \mathcal{H} , we have

$$\frac{S(t)u_0}{t} \rightarrow v \quad \text{in } \mathcal{H} \quad \text{as } t \rightarrow \infty. \quad (4.1)$$

Thus it suffices for us now to show that the evolution operator $S(t)$ for the L&T pseudosolutions can be extended to give a strongly continuous semigroup on all $L^2(\Omega)$.

For sufficiently smooth initial functions u_0 , let $S^\varepsilon(t)$ and $S(t)$ be the evolution operators defined by

$$S^\varepsilon(t) u_0 = u^\varepsilon(t), \quad (4.2)$$

$$S(t) u_0 = u(t), \quad (4.3)$$

where $u^\varepsilon(x, t)$ and $u(x, t)$ are the regularized solutions and limiting pseudosolutions corresponding to the initial function $u_0(x)$ on Ω and zero boundary values $\Phi(x) \equiv 0$ on $\partial\Omega \times (0, \infty)$. Then L&T (see Theorem 3.1 and also (3.8)) showed that for all u_0 in the dense subset

$$\mathcal{D} \equiv L^2(\Omega) \cap W_{\text{loc}}^{1,2}(\Omega) \cap W^{1,1}(\Omega), \quad (4.4)$$

we have, as seen in (3.13), that

$$S^\varepsilon(t) u_0 \rightarrow S(t) u_0 \quad \text{in } L^2, \quad \text{for all } t \geq 0 \quad (4.5)$$

and that moreover

$$S(t) u_0 \text{ stays in } \mathcal{D}, \text{ for almost every } t \geq 0. \quad (4.6)$$

Now the regularized parabolic evolution $S^\varepsilon(t)$ is clearly a contraction on \mathcal{D} . That is, for u_0 and $w_0 \in \mathcal{D}$ we have that the regularized solutions $u^\varepsilon(t) \equiv S^\varepsilon(t) u_0$ and $w^\varepsilon(t) \equiv S^\varepsilon(t) w_0$ satisfy

$$\|u^\varepsilon(t) - w^\varepsilon(t)\|_{L^2} \leq \|u^\varepsilon(s) - w^\varepsilon(s)\|_{L^2}, \quad \text{for } 0 \leq s \leq t. \quad (4.7)$$

This follows from integration by parts and convexity of the function $\xi \rightarrow (1 + \xi^2)^{-1/2} \xi + \varepsilon \xi$ on R^n . Then, from the strong convergence in (4.5) we get that this contraction property on \mathcal{D} holds also in the limit as $\varepsilon \rightarrow 0$; i.e. for u_0 and w_0 in \mathcal{D} ,

$$\|u(t) - w(t)\|_{L^2} \leq \|u(s) - w(s)\|_{L^2} \quad \text{for } 0 \leq s \leq t. \quad (4.8)$$

Moreover, the regularized evolution $S^\varepsilon(t)$ clearly has the semigroup property for u_0 in \mathcal{D} , i.e.

$$S^\varepsilon(t+s) u_0 = S^\varepsilon(t)(S^\varepsilon(s) u_0) \quad \text{for } t, s \geq 0 \quad \text{and} \quad u_0 \in \mathcal{D}. \quad (4.9)$$

Now let $\varepsilon \rightarrow 0$. The left-hand side in (4.9) tends to $S(t+s) u_0$ by (4.5) since $u_0 \in \mathcal{D}$. Consider the right-hand side in (4.9). Letting $\varepsilon \rightarrow 0$, since $u_0 \in \mathcal{D}$,

$$S^\varepsilon(s) u_0 \rightarrow S(s) u_0. \quad (4.10)$$

Thus, by the fact that the $S^\varepsilon(t)$ are contractions

$$\begin{aligned} & \|S^\varepsilon(t)(S^\varepsilon(s) u_0) - S(t)(S(s) u_0)\| \\ & \leq \|S^\varepsilon(t)(S^\varepsilon(s) u_0) - S^\varepsilon(t)(S(s) u_0)\| + \|S^\varepsilon(t)(S(s) u_0) - S(t)(S(s) u_0)\| \\ & \leq \|S^\varepsilon(s) u_0 - S(s) u_0\| + \|S^\varepsilon(t)(S(s) u_0) - S(t)(S(s) u_0)\|. \end{aligned} \quad (4.11)$$

The first term $\rightarrow 0$ by (4.10), the second term $\rightarrow 0$ since by (4.6) $S(s) u_0 \in \mathcal{D}$. Thus in the limit as $\varepsilon \rightarrow 0$, (4.9) yields

$$S(t+s) u_0 = S(t)(S(s) u_0) \quad \text{for } u_0 \in \mathcal{D}. \quad (4.12)$$

Thus both $S(t)$ and $S^\varepsilon(t)$ are contraction semigroups for u_0 in \mathcal{D} . Hence by continuity we can extend them uniquely to be contraction semigroups for all $u_0 \in L^2(\Omega)$.

Finally, we note that this extended $S(t)$ is strongly continuous with respect to t . It suffices to show that

$$S(t) u_0 \rightarrow u_0 \quad \text{as } t \rightarrow 0^+, \quad \text{for arbitrary } u_0 \in L^2(\Omega). \quad (4.13)$$

This was shown by [13], see (3.5), for u_0 in the dense set \mathcal{D} . Thus for arbitrary u_0 in L^2 , given $\delta > 0$ choose w_0 in \mathcal{D} such that $\|u_0 - w_0\| \leq \delta$, then we have

$$\begin{aligned} \|S(t) u_0 - u\| & \leq \|S(t) u_0 - S(t) w_0\| + \|S(t) w_0 - w_0\| \\ & \leq \|u_0 - w_0\| + \|S(t) w_0 - w_0\|. \end{aligned} \quad (4.14)$$

The first term is $\leq \delta$, the second is arbitrarily small for small t , as noted in (4.13), since $w_0 \in \mathcal{D}$.

This completes the proof that $S(t)$ is a strongly continuous contraction semigroup on $L^2(\Omega)$, and hence of Theorem 4.1.

THEOREM 4.2. *The parabolic growth function $v(x)$ discovered in Theorem 4.1 (i.e. the limit in $L^2(\Omega)$ of $u(x, t)/t$ as $t \rightarrow \infty$) is the unique minimizer in $BV(\overline{\Omega}) \cap L^2(\Omega)$ of the functional*

$$F(u) = \int_{\overline{\Omega}} |Du| + \frac{1}{2} \int_{\Omega} u^2 dx - \int_{\Omega} hu dx. \quad (4.15)$$

Note. The variational problem just stated in (4.15) can be shown to have a unique minimizer using standard methods of lower semicontinuity and compactness with respect to $L^1(\Omega)$ convergence. See the book of Giusti [8]. The boundedness of $\|u^k\|_{L^2}$ for a minimizing sequence $\{u^k\}$

comes from the dominance of the u^2 term over the hu term in the integral. The uniqueness of the minimizer comes from the strict convexity of the u^2 term.

Proof of Theorem 4.2. We need to show that

$$F(v) \leq F(w) \quad (4.16)$$

for every w in $BV(\bar{\Omega}) \cap L^2(\Omega)$. Note that such $F(w)$ can be approached arbitrarily closely by $F(w^k)$ where $\{w^k\}$ is a sequence of $C^1(\bar{\Omega})$ functions. See Giusti [8]. Thus it suffices to prove (4.16) where w is henceforth a given and fixed $C^1(\bar{\Omega})$ function.

Step 1. We use the test function

$$\varphi(x, t) \equiv tw(x) \quad (4.17)$$

in the variational inequality (3.12) for $u(x, t)$ of Lichnewsky and Temam. We obtain

$$\begin{aligned} & \int_0^t \int_{\Omega} (w(x) - h(x))(\tau w(x) - u(x, \tau)) \, dx \, d\tau \\ & + \int_0^t \left\{ \int_{\bar{\Omega}} (1 + |D(\tau w)|^2)^{1/2} - \int_{\bar{\Omega}} (1 + |Du|^2)^{1/2} \right\} d\tau \\ & \geq \frac{1}{2} \int_{\Omega} (tw(x) - u(x, t))^2 \, dx - \frac{1}{2} \int_{\Omega} (0 - u_0(x))^2 \, dx. \end{aligned} \quad (4.18)$$

Let us denote by A, B, C, D, E the five terms above, thus (4.18) can be written as

$$A + B - C \geq D - E. \quad (4.19)$$

Now we divide each side by t^2 and compute separately the limit (or lim inf) of each term as $t \rightarrow \infty$.

We use L'Hôpital's rule in the form

$$\lim_{t \rightarrow \infty} \frac{f(t)}{g(t)} = \lim_{t \rightarrow \infty} \frac{f'(t)}{g'(t)}, \quad (4.20)$$

provided that f and g are differentiable, $g'(t) \neq 0$, $g(t) \rightarrow +\infty$, and the second limit exists. Here $g(t) = t^2$ of course.

Step 2.

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{A}{t^2} &= \lim_{t \rightarrow \infty} \frac{1}{2} \int_{\Omega} (w(x) - h(x)) \left(w(x) - \frac{u(x, t)}{t} \right) dx \\ &= \frac{1}{2} \int_{\Omega} (w - h)(w - v) dx, \end{aligned} \quad (4.21)$$

since $u/t \rightarrow v$.

Step 3. Since w is $C^1(\bar{\Omega})$, the formal integral over $\bar{\Omega}$ in (4.18) can by (2.7) be written as an integral over Ω plus a boundary integral, i.e.,

$$B = \int_0^t \left\{ \int_{\Omega} (1 + |D\tau w|^2)^{1/2} dx + \int_{\partial\Omega} |\tau w| dH_{n-1} \right\} d\tau. \quad (4.22)$$

Thus

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{B}{t^2} &= \lim_{t \rightarrow \infty} \frac{1}{2} \int_{\Omega} (t^{-2} + |Dw|^2)^{1/2} dx + \frac{1}{2} \int |w| dH_{n-1} \\ &= \frac{1}{2} \int_{\Omega} |Dw| dx + \frac{1}{2} \int_{\partial\Omega} |w| dH_{n-1} = \frac{1}{2} \int_{\Omega} |Dw|. \end{aligned} \quad (4.23)$$

Here we have used the uniform convergence of the continuous integrand and also (2.7) again.

Step 4. For the C term we use the definitions (2.2) and (2.6) and note that the “integral” of (2.6) is certainly \geq the “integral” of (2.2). Thus

$$\begin{aligned} \frac{C}{t^2} &\geq \frac{1}{t^2} \int_0^t \left(\int_{\bar{\Omega}} |Du| \right) d\tau \\ &\geq \frac{1}{t^2} \int_0^t \left(\int_{\Omega} u(x, \tau) \operatorname{div} g(x) dx \right) d\tau \equiv \frac{1}{t^2} C_1, \end{aligned} \quad (4.24)$$

for every $g \in C^1(\bar{\Omega}; R^n)$ with $|g| \leq 1$.

Then for fixed $g(x)$ we go to the limit as $t \rightarrow \infty$, using L'Hôpital's rule

$$\lim_{t \rightarrow \infty} \frac{C_1}{t^2} = \lim_{t \rightarrow \infty} \frac{1}{2} \int_{\Omega} \frac{u(x, t)}{t} \operatorname{div} g(x) dx = \frac{1}{2} \int_{\Omega} v(x) \operatorname{div} g(x) dx. \quad (4.25)$$

This means that

$$\liminf_{t \rightarrow \infty} \frac{C}{t^2} \geq \frac{1}{2} \int_{\Omega} v(x) \operatorname{div} g(x) dx \quad (4.26)$$

for every $g \in C^1(\bar{\Omega}, R^n)$ with $|g| \leq 1$. Taking the supremum with respect to such g in (4.26) and using again the definition (2.2), we have

$$\liminf_{t \rightarrow \infty} \frac{C}{t^2} \geq \frac{1}{2} \int_{\bar{\Omega}} |Dv|. \quad (4.27)$$

Step 5. We now take the limits of the $D - E$ terms in (4.19). We have easily

$$\lim_{t \rightarrow \infty} \frac{D - E}{t^2} = \frac{1}{2} \int_{\bar{\Omega}} (w - v)^2 dx, \quad (4.28)$$

since $u/t \rightarrow v$ in $L^2(\Omega)$.

Step 6. We combine (4.21), (4.23), (4.27), (4.28) to obtain

$$\int_{\Omega} (w - h)(w - v) dx + \int_{\bar{\Omega}} |Dw| - \int_{\bar{\Omega}} |Dv| \geq \int_{\Omega} (w - v)^2 dx. \quad (4.29)$$

Note that

$$w(w - v) - (w - v)^2 \leq \frac{1}{2}(w^2 - v^2). \quad (4.30)$$

Thus (4.29) plus (4.30) yields (4.16) as desired.

5. THE RADially SYMMETRIC ELLIPTIC CASE

In the radially symmetric case, with $h = h(r)$ and zero boundary values on the ball of radius R , we can easily discover *everything* about the elliptically regularized solution $u^\varepsilon(r)$ and its limiting behavior. This is because the total flux issuing out of the ball of radius r merely equals the total heat being produced in the ball by the heat source function $h(r)$, and hence can be obtained by integrating h and is independent of ε .

Our regularized solution $u^\varepsilon(r)$ has “flux” (the negative of the “heat flux”) in the outward radial direction of magnitude $g_\varepsilon(u_r^\varepsilon(r))$ where g_ε is the function

$$g_\varepsilon(\xi) \equiv (1 + \xi^2)^{-1/2} \cdot \xi + \varepsilon \xi, \quad \text{for all real } \xi, \quad \text{and all } \varepsilon \geq 0. \quad (5.1)$$

Balancing the heat flux and the heat produced, we have for each ball $B(r)$ of radius r that

$$-\omega_n r^{n-1} g_\varepsilon(u_r^\varepsilon(r)) = \int_0^r \omega_n s^{n-1} h(s) ds, \quad (5.2)$$

where ω_n is the surface area of the unit ball in R^n , or that

$$-g_\varepsilon(u_r^\varepsilon(r)) = H(r) \equiv \frac{1}{r^{n-1}} \int_0^r s^{n-1} h(s) ds. \quad (5.3)$$

Thus the flux is determined by the integral $H(r)$ above, and is completely independent of ε . The derivative of u^ε is then given by the inverse function

$$u_r^\varepsilon(r) = g_\varepsilon^{-1}(-H(r)). \quad (5.4)$$

Thus it suffices to study the limiting behavior as $\varepsilon \rightarrow 0$ of the function g_ε^{-1} , or of the function $\varepsilon g_\varepsilon^{-1}$. One observes that

$$g_\varepsilon^{-1}(\xi) \rightarrow g_0^{-1}(\xi) \quad \text{as } \varepsilon \rightarrow 0, \quad \text{if } |\xi| < 1, \quad (5.5)$$

and that this convergence is uniform on compact subsets of $(-1, +1)$. On the other hand, $g_\varepsilon^{-1}(\xi)$ is unbounded as $\varepsilon \rightarrow 0$ if $|\xi| \geq 1$. For those values one has to consider $\varepsilon g_\varepsilon^{-1}(\xi)$ instead. One sees that

$$\varepsilon g_\varepsilon^{-1}(\xi) \rightarrow \psi(\xi) \quad \text{as } \varepsilon \rightarrow 0, \quad \text{uniformly for } \xi \in R^1, \quad (5.6)$$

where ψ is the function

$$\psi(\xi) = \begin{cases} 0 & \text{for } |\xi| \leq 1 \\ \xi - 1 & \text{for } \xi > 1 \\ \xi + 1 & \text{for } \xi < -1. \end{cases} \quad (5.7)$$

In fact, since the error function $\varepsilon g_\varepsilon^{-1}(y) - \psi(y)$ is increasing for $0 \leq y < 1$ and decreasing for $1 < y < \infty$, the error is largest at the value $y = 1$. At that point analysis shows that

$$g_\varepsilon^{-1}(1) \cdot \varepsilon^{2/3} \rightarrow 2^{1/3} \quad \text{as } \varepsilon \rightarrow 0, \quad (5.8)$$

and that hence $(\varepsilon g_\varepsilon^{-1} - 0)$ is $O(\varepsilon^{1/3})$ there.

We can thus consider the function

$$w^\varepsilon \equiv \varepsilon u^\varepsilon \quad (5.9)$$

instead. Its derivative $v_r^\varepsilon(r)$ converges uniformly, because of (5.6). We have

$$w_r^\varepsilon(r) \rightarrow "w_r"(r) \quad \text{uniformly on } 0 \leq r \leq R, \quad (5.10)$$

where $"w_r"$ is the following function:

- (a) $"w_r"(r) = 0$ at any r where $|H(r)| \leq 1$,
 - (b) $"w_r"(r) = -H(r) + 1$ at any r where $H(r) > 1$,
 - (c) $"w_r"(r) = -H(r) - 1$ at any r where $H(r) < -1$.
- (5.11)

Then, because of the zero boundary condition, the w^ε also converge uniformly to a limiting function w , where

$$w^\varepsilon(r) = \int_R^r w_r^\varepsilon(\tau) d\tau \rightarrow \int_R^r "w_r"(\tau) d\tau \equiv w(r). \quad (5.12)$$

Since h is Lipschitz, the integral $H(r)$ is $C^{1,1}$ on $[0, R]$. But then the " w_r " of (5.11) is $C^{1,1}$ in open intervals where $H(r) \neq \pm 1$, but is perhaps only Lipschitz near transition points where $H(r) = \pm 1$. Hence its integral w is $C^{2,1}$ except near the transition points, where it is perhaps only $C^{1,1}$.

Thus, from the integral $H(r)$ one can easily determine the limiting behavior of the u^ε and of the $w^\varepsilon \equiv \varepsilon u^\varepsilon$. Consider for example an $H(r)$ behaving as shown in Figure 2a. By (5.11) and (5.12) the derivative " w_r " of w is $\equiv 0$ in the intervals $[0, r_2]$, $[r_3, r_5]$, $[r_6, r_8]$, $= -H(r) + 1 (< 0)$ in (r_2, r_3) , $= -H(r) - 1 (> 0)$ in (r_5, r_6) and in $(r_8, R]$. Hence $w(r)$ looks somewhat like Fig. 2b.

Let's examine instead the behavior of the u^ε in those intervals $[0, r_1)$, (r_3, r_5) , (r_6, r_8) where $|H(r)|$ remains < 1 . On those intervals, according to (5.5),

$$u_r^\varepsilon(r) \rightarrow g_0^{-1}(-H(r)) \equiv "u_r"(r) \quad \text{as } \varepsilon \rightarrow 0, \quad (5.13)$$

the convergence being uniform on compact subsets. We normalize by subtracting off the value at some point r^* inside each interval, say $r^* = 0$, or r_4 , or r_7 as shown. Then on each of these open intervals let $u(r)$ denote that function which equals zero at r^* and has " u_r " as its derivative.

Thus

$$u^\varepsilon(r) - u^\varepsilon(r^*) = \int_{r^*}^r u_r^\varepsilon(s) ds \rightarrow \int_{r^*}^r "u_r"(s) ds \equiv u(r), \quad (5.14)$$

the convergence being uniform on compact subsets of these intervals.

Because of (5.2) and (5.3) with $\varepsilon = 0$, this u is a solution of the equation of the unregularized prescribed mean curvature equation (1.1) on the disc or annulus corresponding to those intervals. Moreover, u has outward normal derivative $= -\infty$ on the boundary of the disc $[0, r_1)$ and $= +\infty$ on the boundary of the annulus (r_3, r_5) . Thus u is the (unique to within an additive constant) Giusti extremal solution for $h(r)$ in each of those two regions. Hence our result (5.14) might be phrased more loosely as " $u^\varepsilon(x)$ takes on the shape of the Giusti extremal solution $u(x)$ (for $h(x)$)" on each of these two regions.

The solution u on the annulus (r_6, r_8) is a different type of extremal solution, however; it has instead outward derivative $= -\infty$ on the inner boundary and $= +\infty$ on the outer boundary.

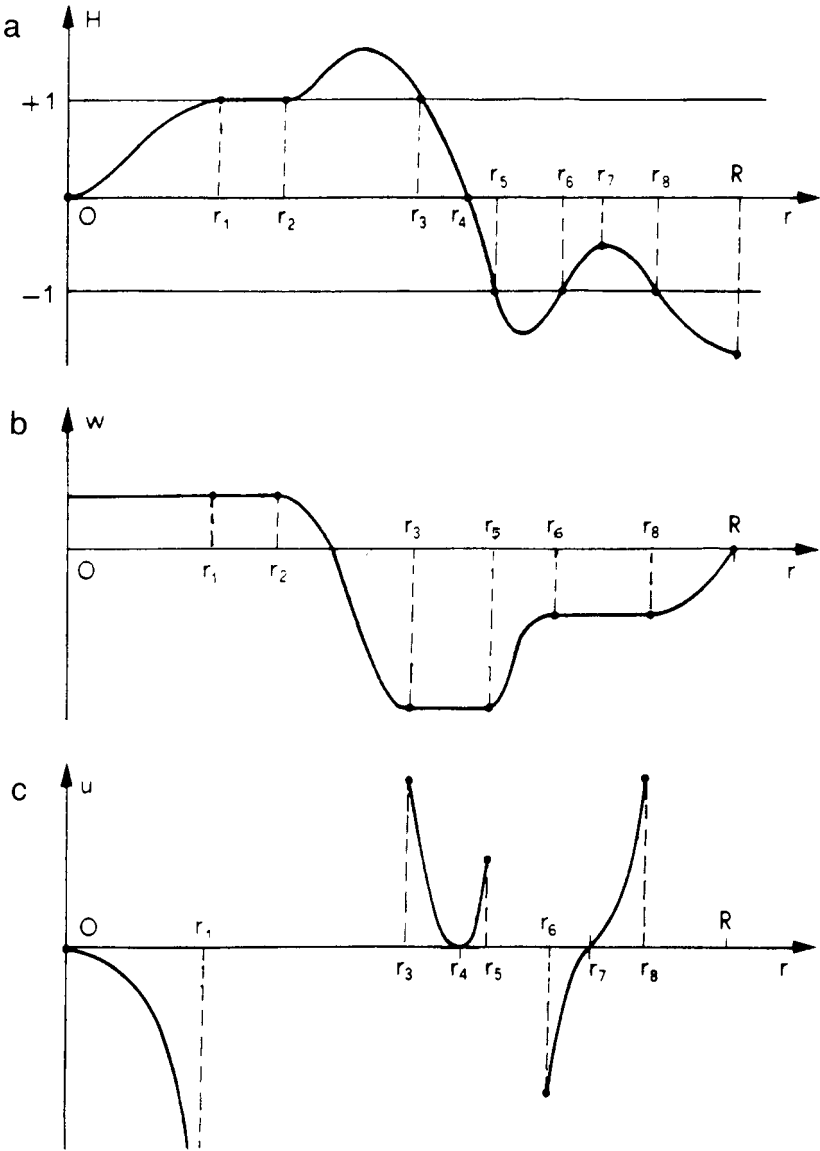


FIG. 2. (a) The integral function $H(r)$ of (5.3) giving the flux at radius r . (b) The corresponding elliptic growth function $w(r)$. (c) The limiting shapes of the solution $u^\epsilon(r)$ in the three plateau regions of the growth function $w(r)$.

We have indicated on Figure 2c that these extremal solutions u are bounded as $r \rightarrow r_3^+$, or r_5^- , or r_6^+ , or r_8^- , but unbounded as $r \rightarrow r_1^-$. This is because $H(r)$ approaches ± 1 with nonzero slope in the former cases, and with zero slope in the latter case. In fact, since h is Lipschitz and thus H is $C^{1,1}$, we have $1 - |H(r)| \geq \text{const. } |r - r_i|^2$ in the latter tangent case. Since the $u_r \equiv "u_r" = g_0^{-1}(-H(r))$ from (5.13) satisfies

$$|u_r| \approx [2(1 - |H(r)|)]^{-1/2} \quad \text{as } |H(r)| \rightarrow 1, \quad (5.15)$$

we have that the integral of u_r converges in the former cases as $r \rightarrow r_i$, and diverges in the latter case.

Incidentally, since

$$H'(r) = h(r) - (n-1) \frac{1}{r} \quad (5.16)$$

at any r where $H(r) = +1$, we see that at the point r_1 above, $H'(r_1)$ is > 0 if and only if " $h(r_1)$ is greater than $(n-1)$ times the mean curvature of the boundary at that point". Thus our condition on H' (as stated in the previous paragraph) for the boundedness or unbounded of the extremal solution u as one approaches the boundary of the extremal set agrees with that discovered by Giusti [7] for the general (nonradial) situation.

Finally, in the interval (r_1, r_2) we have what we shall call a "mush region" where $H(r) \equiv 1$. In that case the derivative u_r^ε has the constant value $g_\varepsilon^{-1}(-1)$ on this interval, which according to (5.8) is asymptotic to $-2^{1/3}\varepsilon^{-2/3}$ as $\varepsilon \rightarrow 0$.

6. THE ELLIPTIC GROWTH FUNCTION w .

Consider the regularized solutions $u^\varepsilon(x, t)$ of (3.2) (where we added both the u_t term and the $\varepsilon \Delta u$ term to the equation). For our *parabolic regularization* we let $\varepsilon \rightarrow 0$ first, to get $u(x, t)$; we then studied the behavior of $u(x, t)$ as $t \rightarrow \infty$. For our *elliptic regularization* we will instead let $t \rightarrow \infty$ first, to get $u^\varepsilon(x)$; we will then study the behavior of $u^\varepsilon(x)$ as $\varepsilon \rightarrow 0$.

Here u^ε is the solution of the regularized elliptic problem

$$\begin{aligned} \operatorname{div} \left(\frac{Du^\varepsilon}{(1 + |Du^\varepsilon|^2)^{1/2}} + \varepsilon Du^\varepsilon \right) + h(x) &= 0 & \text{in } \Omega, \\ u^\varepsilon &= \Phi(x) & \text{on } \partial\Omega. \end{aligned} \quad (6.1)$$

The solution u^ε to this uniformly elliptic problem exists and is a classical solution which is $C^{2,\alpha}$ in the interior and assumes the desired boundary values continuously, etc.

If $h(x)$ is “sufficiently small” then as $\varepsilon \rightarrow 0$, $u^\varepsilon(x)$ will converge to an elliptic pseudosolution $u(x)$ of the original Dirichlet problem (1.1). See Giaquinta [5]. In more recent terminology (see Kawohl and Kutev [11]), this pseudosolution $u(x)$, the limit as $\varepsilon \rightarrow 0$ of the regularized solution $u^\varepsilon(x)$ of (6.1), is called a “viscosity solution” of the Dirichlet problem (1.1). This pseudosolution $u(x)$ may detach from its desired boundary values. This pseudosolution is a classical $C^{2,\alpha}$ solution in the interior, thanks to the given Lipschitz continuity of $h(x)$. However, we point out that simple 1-D examples show that (even when $h(x)$ is “sufficiently small”) $u(x)$ may have internal discontinuities if $h(x)$ is discontinuous. Consider for example $\Omega = (-1, 1)$ with $h(x) = -b$ on $(-1, 0)$, $= +b$ on $(0, 1)$ with b slightly less than 2, and with the boundary values $u(\mp 1) = \mp 10$. These however, are not the pathologies which interest us.

We are interested instead in the behavior of u^ε when “ $h(x)$ is too large on Ω ”, i.e. when the inequalities (1.5) on subsets G fail to be satisfied. We study then the behavior of the function

$$w^\varepsilon \equiv \varepsilon u^\varepsilon. \quad (6.2)$$

Because the boundary values $\varepsilon \Phi(x)$ of w^ε tend to zero, the terms in the following considerations involving these boundary values would vanish in the limit as $\varepsilon \rightarrow 0$. Thus we simplify our arguments and our notation greatly by assuming *henceforth that* $\Phi(x) \equiv 0$. Then w^ε is the solution of

$$\begin{aligned} \operatorname{div} \left(\frac{Dw^\varepsilon}{(\varepsilon^2 + |Dw^\varepsilon|^2)^{1/2}} + Dw^\varepsilon \right) + h(x) &= 0 & \text{in } \Omega, \\ w^\varepsilon &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (6.2)$$

Thus w^ε is the unique minimizer in $W_0^{1,2}(\Omega)$ of the functional

$$G_\varepsilon(u) \equiv \int_\Omega (\varepsilon^2 + |Du|^2)^{1/2} dx + \frac{1}{2} \int_\Omega |Du|^2 dx - \int_\Omega hu dx. \quad (6.3)$$

Note that as $\varepsilon \rightarrow 0$ this functional $G_\varepsilon(u)$ tends to a limiting form $G(u)$ with a $|Du|$ in the first integrand. We show now that w^ε tends to the minimizer w of that limiting functional.

THEOREM 6.1. *As $\varepsilon \rightarrow 0$, $w^\varepsilon \equiv \varepsilon u^\varepsilon \rightarrow w$ in $W_0^{1,2}(\Omega)$, where w is the unique minimizer in $W_0^{1,2}(\Omega)$ of the functional*

$$G(u) = \int_\Omega |Du| dx + \frac{1}{2} \int_\Omega |Du|^2 dx - \int_\Omega hu dx. \quad (6.4)$$

Proof. Note first that existence and uniqueness of the minimizer w is standard.

Step 1. We show that $G_\varepsilon(w^\varepsilon) \rightarrow G(w)$ as $\varepsilon \rightarrow 0$. Since $\sqrt{\varepsilon^2 + |\xi|^2} \leq \varepsilon + |\xi|$, one has, by integrating with $\xi = Dw$,

$$G_\varepsilon(w) \leq \varepsilon |\Omega| + G(w). \quad (6.5)$$

Moreover, since w is a minimizer of G , G_ε is monotone with respect to ε and w^ε is a minimizer of G_ε , we have

$$G(w) \leq G(w^\varepsilon) \leq G_\varepsilon(w^\varepsilon) \leq G_\varepsilon(w). \quad (6.6)$$

From (6.5) and (6.6) we then obtain

$$G(w) \leq G_\varepsilon(w^\varepsilon) \leq \varepsilon |\Omega| + G(w). \quad (6.7)$$

Step 2. We show that w^ε converges weakly to w in $W_0^{1,2}(\Omega)$. First note that G_ε is coercive on $W_0^{1,2}(\Omega)$. That is, there exist positive constants c_1 and c_2 such that

$$G_\varepsilon(u) \geq c_1 \int_\Omega |Du|^2 dx - c_2 \quad \text{for all } u \in W_0^{1,2}(\Omega). \quad (6.8)$$

This is because, using the Poincaré inequality,

$$\begin{aligned} G_\varepsilon(u) &\geq \frac{1}{2} \int_\Omega |Du|^2 dx - \int_\Omega hu dx \\ &\geq \frac{1}{2} \|Du\|_{L^2}^2 - \|h\|_{L^2} \|u\|_{L^2} \\ &\geq \frac{1}{2} \|Du\|_{L^2}^2 - c_3 \|h\|_{L^2} \|Du\|_{L^2} \geq \frac{1}{4} \|Du\|_{L^2}^2 - c_2 \end{aligned} \quad (6.9)$$

This combined with (6.7), which establishes that $G_\varepsilon(w^\varepsilon)$ is bounded with respect to ε , yields that w^ε is bounded in $W_0^{1,2}(\Omega)$. Thus, by weak compactness, there exists a subsequence $w^{\varepsilon'}$ that converges weakly to a function v in $W_0^{1,2}(\Omega)$. By the lower semicontinuity of G , by the fact that $G \leq G_{\varepsilon'}$, and by Step 1 we obtain

$$\begin{aligned} G(v) &\leq \liminf_{\varepsilon' \rightarrow 0} G(w^{\varepsilon'}) \\ &\leq \liminf_{\varepsilon' \rightarrow 0} G_{\varepsilon'}(w^{\varepsilon'}) = \lim_{\varepsilon \rightarrow 0} G_\varepsilon(w^\varepsilon) = G(w) = \min G. \end{aligned} \quad (6.10)$$

Thus v is a minimizer of G and, by uniqueness, $v = w$. This also implies by a standard argument that the whole sequence w^ε weakly converges to w .

Step 3. We claim that, with the notation

$$f_\varepsilon(\xi) \equiv \sqrt{\varepsilon^2 + |\xi|^2} + \frac{1}{2} |\xi|^2, \quad \xi \in R^n, \quad (6.11)$$

we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} (Df_\varepsilon(Dw), Dw^\varepsilon - Dw) dx = 0. \quad (6.12)$$

To this end let us define the vector valued function

$$g_3(\xi) \equiv \begin{cases} \xi/|\xi| & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0. \end{cases} \quad (6.13)$$

Then, for almost every x in Ω , $Dw(x)$ exists and hence we have

$$\lim_{\varepsilon \rightarrow 0} Df_\varepsilon(Dw(x)) = \lim_{\varepsilon \rightarrow 0} \left\{ \frac{Dw(x)}{\sqrt{\varepsilon^2 + |Dw(x)|^2}} + Dw(x) \right\} = g_3(Dw(x)), \quad (6.14)$$

Then, since $|Df_\varepsilon(Dw)| \leq 1 + |Dw|$, by the Lebesgue dominated convergence theorem we obtain that

$$\lim_{\varepsilon \rightarrow 0} \|Df_\varepsilon(Dw) - g_3(Dw)\|_{L^2} = 0. \quad (6.15)$$

Therefore,

$$\begin{aligned} \left| \int_{\Omega} (Df_\varepsilon(Dw), Dw^\varepsilon - Dw) dx \right| &\leq \|Df_\varepsilon(Dw) - g_3(Dw)\|_{L^2} \cdot \|Dw^\varepsilon - Dw\|_{L^2} \\ &\quad + \left| \int_{\Omega} (g_3(Dw), Dw^\varepsilon - Dw) dx \right|. \end{aligned} \quad (6.16)$$

Now by the weak convergence of w^ε to w in $W_0^{1,2}(\Omega)$, by the fact that $|g_3(Dw)| \leq 1 + |Dw|$ is in $L^2(\Omega)$, by (6.15) and the boundedness of w^ε in $W_0^{1,2}$, we see that the right-hand side of (6.16) tends to zero as $\varepsilon \rightarrow 0$, thereby yielding (6.12) as claimed.

Step 4. We show that w^ε converges strongly to w in $W_0^{1,2}(\Omega)$. By the convexity of the function $\sqrt{\varepsilon^2 + |\xi|^2}$ and the uniform convexity of $\frac{1}{2} |\xi|^2$ on R^n we obtain

$$f_\varepsilon(\xi) \geq f_\varepsilon(\xi_0) + (Df_\varepsilon(\xi_0), \xi - \xi_0) + \frac{1}{2} |\xi - \xi_0|^2. \quad (6.17)$$

Using this with $\xi = Dw^\varepsilon(x)$ and $\xi_0 = Dw(x)$ we have

$$\begin{aligned} G_\varepsilon(w^\varepsilon) - G_\varepsilon(w) &= \int_{\Omega} \{f_\varepsilon(Dw^\varepsilon) - f_\varepsilon(Dw)\} dx - \int_{\Omega} h(w^\varepsilon - w) dx \\ &\geq \int_{\Omega} (Df_\varepsilon(Dw), Dw^\varepsilon - Dw) dx \\ &\quad + \frac{1}{2} \int_{\Omega} |Dw^\varepsilon - Dw|^2 dx - \int_{\Omega} h(w^\varepsilon - w) dx. \end{aligned} \quad (6.18)$$

By the monotone convergence theorem $G_\varepsilon(w)$ tends to $G(w)$ as $\varepsilon \rightarrow 0$. By Step 1, $G_\varepsilon(w^\varepsilon) \rightarrow G(w)$. Thus the left-hand side in (6.18) tends to zero.

By Step 3 the first integral on the right-hand side of (6.18) tends to zero. The final term on the right side also tends to zero, by Step 2. Thus we can conclude that the second integral on the right-hand side converges to zero, as desired.

7. CONJECTURE ABOUT THE PARABOLIC MAXIMUM SET

We repeat from [14] our longstanding conjecture about the behavior of $u(x, t)$ on the set where it is growing fastest. Integrating (2.1a) by parts on any subset $G \subset \Omega$ we have

$$\int_G u_t dx = \int_{\partial G} g \cdot \nu dH_{n-1} + \int_G h(x) dx \geq -P(G) + \int_G h(x) dx, \quad (7.1)$$

with equality if and only if the outward normal derivative $\partial u / \partial \nu$ is equal $-\infty$ on all ∂G . Thus, dividing by the measure of G ,

$$\int_G u_t dx / |G| \geq -\frac{P(G)}{|G|} + \frac{1}{|G|} \int_G h(x) dx \equiv MR(G). \quad (7.2)$$

Here $MR(G)$ is the *minimum rate* at which this mean value of u on G could be increasing, with equality if and only if $\partial u / \partial \nu = -\infty$ on all ∂G .

We *believe* that there exists a subset Ω^* on which u asymptotically grows fastest, all at the same asymptotic rate λ^* . Hence, $\partial u / \partial \nu$ should be tending to $-\infty$ on all $\partial \Omega^*$. Thus λ^* should equal the minimum rate function $MR(\Omega^*)$. But for any other subset G , on which the mean value of u is growing at a slower rate, we would have

$$MR(G) \leq \text{mean value } u_t \text{ on } G < \lambda^* = MR(\Omega^*). \quad (7.3)$$

Hence Ω^* should be a set which maximizes $MR(G)$ over all subsets G , and λ^* should be the value c^* of this maximum; this result we have proved, for the maximum value λ^* and maximum set Ω^* of the parabolic growth function $v(x)$, as described in Section 8. Note that when $h(x) = \text{constant}$, this means that Ω^* is a set which minimizes the ratio $P(G)/|G|$ over all subsets $G \subset \Omega$.

A further part of our longstanding conjecture concerns the detailed asymptotic shape of $u(x, t)$ in Ω^* . We believe that Ω^* contains an open subset Ω^{**} such that $\partial u / \partial \nu$ is tending to $-\infty$ on $\partial \Omega^{**}$ but Du is staying bounded on compact subsets of Ω^{**} . (This subset might be the whole interior of Ω^* , but radial examples involving a “mush” phenomenon in a band surrounding Ω^{**} show that it may at times be a proper subset of the interior of the maximum set Ω^* .) Assume also that $u_t \rightarrow \lambda^*$ in Ω^{**} ; then $u(x, t)$ in Ω^{**} should asymptotically satisfy the conditions

$$\begin{aligned} A(u) + (h(x) - \lambda^*) &\approx 0 && \text{in } \Omega^{**}, \\ Du &\text{ stays bounded on compact subsets of } \Omega^{**}, \\ \frac{\partial u}{\partial \nu} &\approx -\infty && \text{on } \partial \Omega^{**}. \end{aligned} \tag{7.4}$$

Thus Ω^{**} should be an extremal set for the function $h(x) - \lambda^*$, and in the limit $u(x, t)$ should take on the shape of the unique (to within an additive constant) Giusti extremal solution $V(x)$ (for $h(x) - \lambda^*$). This part of the conjecture we have *not* been able to establish in the general (nonradial) situation.

8. PROPERTIES OF THE PARABOLIC MAXIMUM SET

Now we use the variational formulation of Theorem 4.2 to study the properties of the parabolic growth function $v(x) = \lim u(x, t)/t$ as $t \rightarrow \infty$. There it was shown that v is the unique minimizer in $BV(\bar{\Omega}) \cap L^2(\Omega)$ of the functional

$$F(u) = \int_{\bar{\Omega}} |Du| + \frac{1}{2} \int_{\Omega} u^2 dx - \int_{\Omega} hu dx. \tag{8.1}$$

Let λ^* and Ω^* denote the essential supremum and “maximum set” for the function v on Ω , that is

$$\lambda^* = \text{ess sup } v, \quad \Omega^* = \bigcap_{\lambda < \lambda^*} \Omega_{\lambda} \tag{8.2}$$

where Ω_λ is the level set

$$\Omega_\lambda \equiv \{x \in \Omega : v(x) > \lambda\}. \quad (8.3)$$

Notice that it is easy to see that

$$\|v\|_\infty \leq \|h\|_\infty \quad (8.4)$$

because of the maximum principle applied to the regularized parabolic solution $u^\varepsilon(x, t)$. That is, $(t \|h\|_\infty + \text{const.})$ and $-(t \|h\|_\infty + \text{const.})$ are supersolutions and subsolutions for the parabolic problem (3.2). As a consequence, λ^* is finite; however this will also follow from the following Lemma 8.2.

We first introduce some machinery. Let φ_λ denote the “ λ th cap function of v ”; that is

$$\varphi_\lambda(x) = \max\{v(x) - \lambda, 0\}. \quad (8.5)$$

LEMMA 8.1. *For every real number α with $-1 \leq \alpha < \infty$ we have, since $v \in BV(\bar{\Omega})$,*

$$\int_{\bar{\Omega}} |D(v + \alpha \varphi_\lambda)| = \int_{\bar{\Omega}} |Dv| + \alpha \int_{\bar{\Omega}} |D\varphi_\lambda|. \quad (8.6)$$

Proof. This follows from the coarea formula (see [8, p. 20]). We have that

$$v + \alpha \varphi_\lambda = \psi \circ v \quad (8.7)$$

where ψ is the increasing function

$$\psi(t) = \begin{cases} t & \text{for } t \leq \lambda \\ t + (1 + \alpha)(t - \lambda) & \text{for } t > \lambda. \end{cases} \quad (8.8)$$

The coarea formula states that for any $u \in BV(\bar{\Omega}) \equiv \{v \in BV(R^n) \text{ with } v(x) \equiv 0 \text{ outside } \bar{\Omega}\}$, we have

$$\int_{\bar{\Omega}} |Du| = \int_{-\infty}^{\infty} P(\Omega_t) dt. \quad (8.9)$$

Thus, since the functions v and $\psi \circ v$ (where ψ is any piecewise smooth increasing function with bounded derivative) have “shared level sets”, one can see by a change of the t variable in the integral above that

$$\int_{\bar{\Omega}} |D(\psi \circ v)| = \int_{-\infty}^{\infty} \psi'(t) P(\Omega_t) dt. \quad (8.10)$$

This, in our case, gives the desired formula (8.6), because the above becomes

$$\begin{aligned} \int_{-\infty}^{\infty} 1 \cdot P(\Omega_t) dt + \int_{\lambda}^{\infty} (1 + \alpha) P(\Omega_{\lambda}) dt &= \int_{-\infty}^{\infty} 1 \cdot P(\Omega_t) dt + \alpha \int_{\lambda}^{\infty} P(\Omega_{\lambda}) dt \\ &= \int |Dv| + \alpha \int_{\bar{\Omega}} |D\varphi_{\lambda}|. \end{aligned} \quad (8.11)$$

We now prove that the maximum set Ω^* has positive measure. In fact

LEMMA 8.2. *Suppose $\lambda^* > 0$. There exists an absolute constant C , depending only on the dimension n , such that*

$$|\Omega^*| \geq C(\|h\|_{\infty})^{-n}. \quad (8.12)$$

Remark. Before proceeding with the formal proof, we explain more intuitively our reasoning. The derivative u_t on Ω^* should be asymptotically approaching λ^* and thus the total flux flowing out of Ω^* should be approximately equal to the integral there of $h(x) - \lambda^*$. However because u_t is smaller outside Ω^* , the outward normal derivative $\partial u / \partial v$ should be approaching $-\infty$, and hence this total flux should be approaching the perimeter $P(\Omega^*)$. Thus we should have

$$P(\Omega^*) = \int_{\Omega^*} (h(x) - \lambda^*) dx, \quad (8.13)$$

see (8.19) for the formal proof. Hence, since $\lambda^* \geq 0$ and by the isoperimetric inequality (see Corollary 1.29 in [8]),

$$|\Omega^*|^{1-1/n} \leq c_1 P(\Omega^*) \leq c_1 \|h\|_{\infty} |\Omega^*|. \quad (8.14)$$

Assuming that $|\Omega^*| > 0$ and cancelling, we would get the desired estimate (8.12).

Proof. We will derive a uniform lower bound for the measure of each of the level sets Ω_{λ} with $0 < \lambda < \lambda^*$, and hence for Ω^* . We now perturb our minimizer v by the test function $-\varphi_{\lambda}$ of (8.5). Because of (8.6) with $\alpha = -1$, one has

$$\begin{aligned} 0 &\geq F(v) - F(v - \varphi_{\lambda}) = \int_{\bar{\Omega}} |D\varphi_{\lambda}| + \frac{1}{2} \int_{\Omega} (v^2 - (v - \varphi_{\lambda})^2) dx - \int_{\Omega} h\varphi_{\lambda} dx \\ &\geq \int_{\bar{\Omega}} |D\varphi_{\lambda}| - \|h\|_{L^{\infty}} \int_{\Omega_{\lambda}} |\varphi_{\lambda}| dx. \end{aligned} \quad (8.15)$$

Now we use the Hölder inequality, followed by Sobolev inequality for $BV(\mathbb{R}^n)$ functions with compact support (see Theorem 1.28 in [8]) (which is equivalent to an isoperimetric inequality)

$$\begin{aligned} \int_{\Omega_\lambda} |\varphi_\lambda| \, dx &\leq |\Omega_\lambda|^{1/n} \left(\int_{\Omega_\lambda} |\varphi_\lambda|^{n/(n-1)} \, dx \right)^{(n-1)/n} \\ &\leq |\Omega_\lambda|^{1/n} C \int_{\mathbb{R}^n} |D\varphi_\lambda| = |\Omega_\lambda|^{1/n} C \int_{\Omega} |D\varphi_\lambda|. \end{aligned} \quad (8.16)$$

Combining (8.15), (8.16) we get

$$\int_{\Omega} |D\varphi_\lambda| \cdot (1 - \|h\|_{L^\infty} |\Omega_\lambda|^{1/n} C) \leq 0. \quad (8.17)$$

Since $0 < \lambda < \lambda^*$, the integral in (8.17) is positive. Cancelling, we get (8.12) as desired.

THEOREM 8.3. (a) *For every measurable subset G of Ω we have*

$$\int_G (h(x) - \lambda^*) \, dx \leq P(G). \quad (8.18)$$

(b) *However, if $\lambda^* > 0$, then for the set Ω^* itself we have*

$$\int_{\Omega^*} (h(x) - \lambda^*) \, dx = P(\Omega^*). \quad (8.19)$$

Proof of (a). We perturb the minimizer v by the test function $\alpha\varphi$ where φ is the characteristic function of G and where α is a tiny positive constant. Then, by the subadditivity of the total variation on $\bar{\Omega}$,

$$\int_{\bar{\Omega}} |D(v + \alpha\varphi)| \leq \int_{\bar{\Omega}} |Dv| + \alpha \int_{\bar{\Omega}} |D\varphi|. \quad (8.20)$$

Hence, since v is the minimizer,

$$F(v) \leq F(v + \alpha\varphi)$$

$$\leq \int_{\bar{\Omega}} |Dv| + \alpha \int_{\bar{\Omega}} |D\varphi| + \frac{1}{2} \int_{\Omega} (v + \alpha\varphi)^2 \, dx - \int_{\Omega} h(v + \alpha\varphi) \, dx. \quad (8.21)$$

Cancelling, dividing by the positive α , and letting $\alpha \rightarrow 0$ we obtain

$$0 \leq \int_{\bar{\Omega}} |D\varphi| - \int_{\Omega} (h-v) \varphi \, dx. \quad (8.22)$$

Hence, since $v(x) \leq \lambda^*$, one obtains (8.18).

Proof of (b). Here one needs the usual inequality in (8.20) to hold also for tiny *negative* α in order to reverse the inequality in (8.18). Let us choose our φ to be φ_λ , the “ λ th top cap of v ” as defined in (8.5) with $\lambda > 0$ chosen slightly smaller than λ^* . For *this* test function, thanks to (8.6), we have equality in (8.20) for all $\alpha \geq -1$.

Thus (8.21) becomes

$$\begin{aligned} F(v) &\leq F(v + \alpha \varphi_\lambda) \\ &= \int_{\bar{\Omega}} |Dv| + \alpha \int_{\bar{\Omega}} |D\varphi_\lambda| + \frac{1}{2} \int_{\Omega} (v + \alpha \varphi_\lambda)^2 \, dx - \int_{\Omega} h(v + \alpha \varphi_\lambda) \, dx. \end{aligned} \quad (8.23)$$

Cancelling, dividing by negative α (which reverses the usual inequality) or by a positive α (which leaves the inequality alone), and taking the limit as $\alpha \rightarrow 0^-$ or 0^+ , one obtains the equality

$$0 = \int_{\bar{\Omega}} |D\varphi_\lambda| - \int_{\Omega} (h-v) \varphi_\lambda \, dx. \quad (8.24)$$

Now one normalizes φ_λ so that its maximum value is 1, i.e. (8.24) holds with φ_λ replaced by

$$\psi_\lambda = \frac{\varphi_\lambda}{\lambda^* - \lambda}. \quad (8.25)$$

As $\lambda \rightarrow \lambda^*$, ψ_λ converges pointwise by (8.2), and hence, by the dominated convergence theorem, in L^1 to φ_{Ω^*} , the characteristic function of Ω^* . Thus by the lower semicontinuity of total variation with respect to L^1 convergence, plus (8.24),

$$\begin{aligned} P(\Omega_*) &\equiv \int_{\bar{\Omega}} |D\varphi_{\Omega^*}| \leq \liminf_{\lambda \rightarrow \lambda^*} \int_{\bar{\Omega}} |D\psi_\lambda| \\ &= \lim_{\lambda \rightarrow \lambda^*} \int_{\Omega} (h-v) \psi_\lambda \, dx = \int_{\Omega^*} (h-v) \, dx. \end{aligned} \quad (8.26)$$

This inequality, together with the opposite inequality of (8.18) for $G = \Omega^*$, gives the desired equality of (8.19).

Note that if $\lambda^* \leq 0$ then (8.18) gives that

$$\int_G h(x) \leq P(G), \quad (8.27)$$

for all measurable subsets $G \subset \Omega$. However, if $\lambda^* > 0$ then (8.19) reveals that the subset $G = \Omega^*$ (of positive measure by Lemma 8.2) satisfies the opposite strict inequality

$$\int_{\Omega^*} h(x) = P(\Omega^*) + \lambda^* |\Omega^*| > P(\Omega^*). \quad (8.28)$$

Thus we have part (a) of the following Lemma. Part (b) of course follows by considering instead $\lambda^{**} = \text{ess inf } v$, Ω^{**} = the “minimum set” of v , bottom caps of v , etc.

COROLLARY 8.4. (a) $\lambda^* \equiv \text{ess sup } v$ is ≤ 0 if and only if

$$\int_G h \, dx \leq P(G) \quad \text{for all } G \subset \bar{\Omega}, \quad (8.29)$$

(b) $\lambda^{**} \equiv \text{ess inf } v$ is ≥ 0 if and only if

$$-P(G) \leq \int_G h \, dx \quad \text{for all } G \subset \bar{\Omega}. \quad (8.30)$$

(c) Thus v is $\equiv 0$ on Ω if and only if

$$-P(G) \leq \int_G h \leq P(G) \quad \text{for all } G \subset \bar{\Omega}. \quad (8.31)$$

Note that if $\lambda^* > 0$ then (8.18) combined with (8.19) reveal that λ^* is the *smallest* constant c such that

$$\int_G (h(x) - c) \, dx \leq P(G), \quad (8.32)$$

for all subsets $G \subset \bar{\Omega}$ of positive measure. Thus dividing by $|G|$ one obtains the geometrically identifiable value for λ^* which was conjectured in Section 7.

COROLLARY 8.5. *If $\lambda^* > 0$, then $\lambda^* = c^*$, where*

$$c^* = \sup\{MR(G): G \text{ is a subset of } \Omega \text{ of positive measure}\}, \quad (8.33)$$

and where MR is the “minimum rate” of (7.2), i.e.

$$MR(G) \equiv -\frac{P(G)}{|G|} + \frac{1}{|G|} \int_G h(x) dx. \quad (8.34)$$

In fact, the set Ω^* assumes this supremum.

Remark. In particular, when $h(x)$ is a constant, Ω^* minimizes the ratio $P(G)/|G|$ over all subsets G of Ω of positive measure. This is a variational problem which has been studied previously, for example, see Keller [12] and Gonzales, Massari and Tamanini [10].

Remark. Note that (8.18) and (8.19) establish that, if $\lambda^* > 0$, then the maximum set Ω^* is *almost* a Giusti extremal set for the function $h(x) - \lambda^*$. We have the desired equality in (8.19) for the set Ω^* itself; and all that is lacking is a *strict* inequality in (8.18) for all proper subsets G of Ω^* .

We point out however that strict inequality in (8.18) is not necessarily true, even for proper subsets of Ω^* . See for example the “mush” region $r_1 < r < r_2$ surrounding the Giusti extremal region $0 \leq r < r_1$ discussed in the final paragraph of Section 5.

9. PROPERTIES OF THE ELLIPTIC MAXIMUM SET

We use the variational formulation of Theorem 6.1 to study the properties of the elliptic growth function $w(x) \equiv \lim_{\varepsilon \rightarrow 0} \varepsilon u^\varepsilon(x)$ as $\varepsilon \rightarrow 0$. It was shown that w is the unique minimizer in $W_0^{1,2}(\Omega)$ of the functional

$$G(u) = \int_\Omega |Du| dx + \frac{1}{2} \int_\Omega |Du|^2 dx - \int_\Omega hu dx. \quad (9.1)$$

Let $\tilde{\lambda}$, $\tilde{\Omega}$, and Ω_λ denote the essential supremum, “maximum set”, and λ th level set of w , in analogy with (8.2) and (8.3). We will also use the “cap function” φ_λ for w analogous to (8.5) and its properties given in Lemma 8.1. That is, $\varphi_\lambda(x) = \max\{w(x) - \lambda, 0\}$ and, for every real number α with $-1 \leq \alpha < \infty$, we have

$$\int_\Omega |D(w + \alpha\varphi_\lambda)| dx = \int_\Omega |Dv| dx + \alpha \int_\Omega |D\varphi_\lambda| dx. \quad (9.2)$$

Notice that it is easy to get bounds on the L^∞ norm of w in terms of the L^∞ norm of h because of maximum principle arguments applied to the regularized solutions $u^\varepsilon(x)$ of (6.1). Let Ω be contained in the ball B of radius R about the origin. Then let $z(x)$ be the solution of the simple radially symmetric Dirichlet problem

$$\begin{aligned} \varepsilon \Delta z + \|h\|_\infty &= 0 & \text{in } B \\ z &= 0 & \text{on } \partial B. \end{aligned} \quad (9.3)$$

The solution is $z(x) = (R^2 - |x|^2) \|h\|_\infty / (2n\varepsilon)$. By the maximum principle applied in Ω one has $u^\varepsilon(x) \leq z(x)$ in Ω . Thus $\varepsilon u^\varepsilon$ and its L^2 limit W are L^∞ bounded as follows

$$\|\varepsilon u^\varepsilon\|_\infty \quad \text{and} \quad \|w\|_\infty \quad \text{are} \quad \leq (R^2/2n) \|h\|_\infty. \quad (9.4)$$

However an L^∞ bound on $w(x)$ will also follow directly from the variational formulation (9.1) as a consequence of the following lemma.

We point out here that there are regularity theorems for variational problems of the form (9.1). Let w be a minimizer in $W_0^{1,p}(\Omega)$ of a functional of the form

$$\mathcal{F}(u) = \int_{\Omega} F(x, u, Du) \, dx \quad (9.5)$$

where there exist positive constants c_1, c_2, c_3, c_4 , and $p_1 > 1$ such that

$$c_1 |\xi|^p - c_2 \leq F(x, s, \xi) \leq c_3 |\xi|^p + c_4 \quad (9.6)$$

for every x in Ω , every s in R with $|s| \leq \|w\|_\infty$, and every ξ in R^n . (For our purposes $p = 2$.) Then it is established in Theorem 7.6 and Theorem 7.8 of Giusti [9] that there exists a Holder coefficient, $0 < \alpha < 1$, such that w is in $C^{0,\alpha}(\bar{\Omega})$. Here F is continuous with respect to (s, ξ) but it is not required that F be differentiable with respect to its arguments (or even that F be convex with respect to ξ !).

This result applies to our case (9.1) and hence our w is $C^{0,\alpha}(\bar{\Omega})$ and assumes its zero boundary values continuously.

LEMMA 9.1. *Suppose $\tilde{\lambda} > 0$. There exists an absolute constant C , depending only on the dimension n , such that*

$$|\tilde{\Omega}| \geq C(\|h\|_{L^\infty})^{-n}. \quad (9.7)$$

Proof. Our proof is similar to that of the parabolic case in Lemma 8.2. We will actually establish (9.7) for each of the Ω_λ , with $0 < \lambda < \tilde{\lambda}$, and

hence for $\tilde{\Omega}$ which is the intersection of these level sets. Because of the continuity of w on $\bar{\Omega}$, these level sets Ω_λ are open, $\Omega_\lambda \subset \subset \Omega$ for each $\lambda > 0$, and $\Omega_{\lambda_2} \subset \subset \Omega_{\lambda_1}$ for $0 < \lambda_2 < \lambda_1 < \tilde{\lambda}$.

We now perturb our w by the test function $-\varphi_\lambda$. Because of (9.2) with $\alpha = -1$, one has

$$\begin{aligned} 0 &\geq F(w) - F(w - \varphi_\lambda) = \int_{\Omega_\lambda} |D\varphi_\lambda| \, dx + \frac{1}{2} \int_{\Omega_\lambda} |D\varphi_\lambda|^2 \, dx - \int_{\Omega_\lambda} h\varphi_\lambda \, dx \\ &\geq \int_{\Omega_\lambda} |D\varphi_\lambda| \, dx - \|h\|_\infty \int_{\Omega_\lambda} |\varphi_\lambda| \, dx. \end{aligned} \quad (9.8)$$

Note that $\varphi_\lambda \in W_0^{1,2}(\Omega_\lambda)$. Hence by the Hölder inequality, followed by a standard Sobolev inequality in $W_0^{1,1}(\Omega_\lambda)$, see [6], one has

$$\int_{\Omega_\lambda} |\varphi_\lambda| \, dx \leq |\Omega_\lambda|^{1/n} \left(\int_{\Omega_\lambda} \varphi_\lambda^{n/(n-1)} \, dx \right)^{(n-1)/n} \leq |\Omega_\lambda|^{1/n} C \int_{\Omega_\lambda} |D\varphi_\lambda| \, dx, \quad (9.9)$$

where C is an absolute constant. Combining (9.8) and (9.9) and using the important fact that $D\varphi_\lambda \not\equiv 0$, one has the inequality (9.7) for each $|\Omega_\lambda|$, as claimed.

We now prove a theorem for the elliptic case similar to the Theorem 8.3 of the parabolic case. However we have to qualify our hypotheses for part (a) a bit by *assuming* that $\tilde{\Omega}$ has a nonempty interior (which seems very probable because of Lemma 9.1, but which we are as yet unable to prove).

THEOREM 9.2. (a) *If $\tilde{\lambda} > 0$, then for the set $\tilde{\Omega}$ we have*

$$\int_{\tilde{\Omega}} h(x) \, dx \geq P(\tilde{\Omega}). \quad (9.10)$$

(b) *Assume that $\tilde{\Omega}$ has a nonempty interior $\tilde{\Omega}^{\text{int}}$. If G is any compactly contained measurable subset of $\tilde{\Omega}^{\text{int}}$, then*

$$\int_G h(x) \, dx \leq P(G). \quad (9.11)$$

Proof of (a). As in the proof of Theorem 8.4b we perturb w by the top cap φ_λ of w , with $\lambda > 0$ chosen slightly smaller than $\tilde{\lambda}$. By (9.2) with tiny positive or negative α we have

$$\begin{aligned} 0 &\geq G(w) - G(w + \alpha\varphi) \\ &= \alpha \int_{\Omega} |D\varphi_\lambda| \, dx + \frac{1}{2} \int_{\Omega} (|Dw|^2 - |Dw + \alpha D\varphi|^2) \, dx - \alpha \int_{\Omega} h\varphi_\lambda \, dx. \end{aligned} \quad (9.12)$$

Cancelling, dividing by a negative or positive α , and taking the limit as $\alpha \rightarrow 0$, one obtains

$$0 = \int_{\Omega} |D\varphi_{\lambda}| dx + \int_{\Omega} |D\varphi_{\lambda}|^2 dx - \int_{\Omega} h\varphi_{\lambda} dx. \quad (9.13)$$

Thus, since the second term is positive,

$$\int_{\Omega} |D\varphi_{\lambda}| dx < \int_{\Omega} h\varphi_{\lambda} dx. \quad (9.14)$$

Once again, one normalizes φ_{λ} to ψ_{λ} with maximum value 1 as in (8.25). Then as $\lambda \rightarrow \tilde{\lambda}$, ψ_{λ} tends to the characteristic function of $\tilde{\Omega}$ in L^1 , hence by lower semicontinuity, one has (9.10) as in (8.26).

Proof of (b). Let φ be any test function in $C_0^{\infty}(\tilde{\Omega}^{\text{int}})$. Then, for positive α ,

$$\begin{aligned} G(w) &\leq G(w + \alpha\varphi) \\ &\leq G(u) + \alpha \left(\int_{\Omega} |D\varphi| dx + \int_{\Omega} Dw \cdot d\varphi dx - \int_{\Omega} h\varphi dx \right) + \alpha^2 \int_{\Omega} |D\varphi|^2 dx. \end{aligned} \quad (9.15)$$

Dividing by α and letting $\alpha \rightarrow 0^+$ one has

$$0 \leq \int_{\Omega} |D\varphi| dx + \int_{\Omega} Dw \cdot D\varphi dx - \int_{\Omega} h\varphi dx. \quad (9.16)$$

However, since w is constant a.e. in $\tilde{\Omega}$, Dw is $=0$ a.e. in $\tilde{\Omega}$, a standard result about functions in Sobolev space. Thus

$$\int_{\Omega} h\varphi \leq \int_{\Omega} |D\varphi| \quad \text{for every } \varphi \in C_0^{\infty}(\tilde{\Omega}^{\text{int}}). \quad (9.17)$$

However, for any Caccioppoli subset G in R^n one has that

$$P(G) = \lim_{\varepsilon \rightarrow 0} \int_{R^n} |D(\psi_G)_{\varepsilon}| dx \quad (9.18)$$

where $(\psi_G)_{\varepsilon}$ denotes the standard ε -mollification of the characteristic function ψ_G of G . See Giusti [8, Remark 1.16]. Thus for G compactly contained in $\tilde{\Omega}^{\text{int}}$, this $(\psi_G)_{\varepsilon}$ is a valid test function in (9.16). As $\varepsilon \rightarrow 0$, we find (9.11). Furthermore if G is measurable but not Caccioppoli (i.e. $P(G) = +\infty$) then (9.11) is trivially satisfied. This completes the proof.

The flux in the elliptically regularized equation of (6.1) is

$$\frac{D(\varepsilon u^\varepsilon)}{(\varepsilon^2 + |D\varepsilon u^\varepsilon|^2)^{1/2}} + D(\varepsilon u^\varepsilon). \quad (9.19)$$

We proved in Theorem 6.1 that as $\varepsilon \rightarrow 0$, $D(\varepsilon u^\varepsilon) \rightarrow Dw$ in $L^2(\Omega, \mathbb{R}^n)$. Thus at almost every point where $Dw \neq 0$ we have the limiting flux

$$\left(\frac{Dw}{|Dw|} + Dw \right), \quad (9.20)$$

We suspect that for most λ , $0 < \lambda < \tilde{\lambda}$, we should have $Dw \neq 0$ everywhere on $\partial\Omega_\lambda$, moreover this should be in the direction of $-\nu$, where ν is the unit outward normal vector. Thus for these λ we would have

$$\begin{aligned} \int_{\Omega_\lambda} h \, dx &= \int_{\partial\Omega_\lambda} \left(\frac{Dw}{|Dw|} + Dw \right) \cdot \nu \, dH_{n-1} \\ &= \int_{\partial\Omega_\lambda} 1 + |Dw| \, dH_{n-1} = P(\Omega_\lambda) + \int_{\partial\Omega_\lambda} |Dw| \, dH_{n-1}. \end{aligned} \quad (9.21)$$

Thus we suspect that for many of these sets Ω_λ we have a strict inequality

$$\int_{\Omega_\lambda} h \, dx > P(\Omega_\lambda). \quad (9.22)$$

However, because it is difficult to say a priori for which λ one has $Dv \neq 0$ everywhere on $\partial\Omega_\lambda$, we instead will find an integral version of (9.22).

THEOREM 9.3. *Suppose $\tilde{\lambda} > 0$. Then for every pair λ_1, λ_2 with $0 < \lambda_1 < \lambda_2 < \tilde{\lambda}$, let φ be the “horizontal slice” of w , i.e.*

$$\varphi(x) = \varphi_{\lambda_1}(x) - \varphi_{\lambda_2}(x). \quad (9.23)$$

Then one has the equality

$$\int_{\lambda_1}^{\lambda_2} \left\{ P(\Omega_\lambda) - \int_{\Omega_\lambda} h(x) \, dx \right\} d\lambda = - \int_{\Omega} |D\varphi|^2 \, dx. \quad (9.24)$$

Since the right-hand side of (9.24) is strictly negative, one must have that the strict inequality (9.22) is satisfied on a dense subset of λ 's of positive measure.

Proof. Once again we have, analogous to (8.6) and (9.2), that

$$\int_{\Omega} |D(w + \alpha\varphi)| = \int_{\Omega} |Dw| + \alpha \int_{\Omega} |D\varphi| \quad \text{for } -1 \leq \alpha < \infty. \quad (9.25)$$

The proof is as in Lemma 8.1, except that now we use the Lipschitz composite function $\psi \circ w$ where the piecewise linear $\psi(\lambda)$ has $\psi'(\lambda) = 1$ for $\lambda < \lambda_1$, $= 1 + \alpha$ for $\lambda_1 < \lambda < \lambda_2$, $= 1$ for $\lambda > \lambda_2$. Rather than the coarea formula (8.10), we use that $D(\psi \circ w) = \psi'(w)Dw$ in Sobolev space.

Thus we perturb w by $\alpha\varphi$ with a tiny positive or negative α . We have

$$\begin{aligned} G(w) &\leq G(w + \alpha\varphi) \\ &= G(w) + \alpha \left(\int_{\Omega} |D\varphi| + \int_{\Omega} Dw \cdot D\varphi \, dx - \int_{\Omega} h\varphi \, dx \right) + \alpha^2 \int_{\Omega} |D\varphi|^2 \, dx. \end{aligned} \quad (9.26)$$

Dividing by the positive or negative α , letting $\alpha \rightarrow 0$, one finds

$$0 = \int_{\Omega} |D\varphi| - \int_{\Omega} h\varphi \, dx + \int_{\Omega} |D\varphi|^2 \, dx. \quad (9.27)$$

Now as in (8.9) one has the coarea formula for φ (= the “slice” of the function $w \in W_0^{1,2}(\Omega)$)

$$\int_{\Omega} |D\varphi| = \int_{\lambda_2}^{\lambda_1} P(\Omega_{\lambda}) \, d\lambda. \quad (9.28)$$

Moreover, one easily gets a similar formula in terms of level sets for the second integral in (9.27). One finds that

$$\int_{\Omega} h\varphi \, dx = \int_{\lambda_2}^{\lambda_1} \left(\int_{\Omega_{\lambda}} h(x) \, dx \right) d\lambda. \quad (9.29)$$

The proof involves working with the subgraph of the function $\varphi(x)$ in R^{n+1} . That is, for $x \in \Omega$ and $\lambda \in R^1$ let

$$\chi(x, \lambda) = \begin{cases} 1 & \text{if } 0 < \lambda - \lambda_1 < \varphi(x) \\ 0 & \text{otherwise.} \end{cases} \quad (9.30)$$

Then, using Fubini,

$$\begin{aligned} \int_{\Omega} h(x) \varphi(x) \, dx &= \int_{\Omega} \left(h(x) \left(\int_{\lambda_2}^{\lambda_1} \chi(x, \lambda) \, d\lambda \right) \right) dx \\ &= \int_{\lambda_2}^{\lambda_1} \left(\int_{\Omega} h(x) \chi(x, \lambda) \, dx \right) d\lambda \\ &= \int_{\lambda_2}^{\lambda_1} \left(\int_{\Omega_{\lambda}} h(x) \, dx \right) d\lambda. \end{aligned} \quad (9.31)$$

From (9.27)–(9.29) one obtains (9.24) as desired.

10. THE PARABOLIC CASE ON A 2-D RECTANGLE WITH CONSTANT h

In this case, $\Omega =$ a rectangle in 2-D with a positive constant h , one can actually *guess* the formula for the parabolic growth function and then prove that our guessed function $v(x)$ is the unique minimizer of the variational problem given in Theorem 4.2.

Recall that the problem was to find the unique minimizer in $BV(\Omega) \cap L^2(\Omega)$ of the functional

$$F(u) = \int_{\Omega} |Du| + \frac{1}{2} \int_{\Omega} u^2 dx - \int_{\Omega} hu dx. \quad (10.1)$$

Our guessed function v will be piecewise $C^2(\bar{\Omega})$ and it will be sufficient to show that

$$F(v) \leq F(v + \varphi) \quad \text{for all } C^2(\bar{\Omega}) \text{ test functions } \varphi. \quad (10.2)$$

Also, for such functions our total variation on $\bar{\Omega}$ becomes, by (2.4),

$$\int_{\bar{\Omega}} |D(v + \varphi)| = \int_{\Omega} |D(v + \varphi)| dx + \int_{\partial\Omega} |v + \varphi| dH_{n-1}. \quad (10.3)$$

Note that by convexity one has

$$|D(v + \varepsilon\varphi)| \geq |Dv| + \varepsilon g(Dv) \cdot D\varphi, \quad (10.4)$$

for all real ε , where $g(\xi)$ denotes any “subderivative” of the function $|\xi|$ on R^n , i.e.

$$g(\xi) = \begin{cases} \xi/|\xi| & \text{if } \xi \neq 0 \\ \text{any vector } g \text{ with } |g| \leq 1 & \text{if } \xi = 0. \end{cases} \quad (10.5)$$

Likewise,

$$|v + \varepsilon\varphi| \geq |v| + \varepsilon \text{“sign } v\text{” } \varphi, \quad (10.6)$$

for all real ε , where “sign η ” denotes any subderivative of the function $|\eta|$ on R^1 , i.e.

$$\text{“sign } \eta\text{”} = \begin{cases} \text{sign } \eta & \text{if } \eta \neq 0 \\ \text{any number } \beta \text{ with } |\beta| \leq 1 & \text{if } \eta = 0. \end{cases} \quad (10.7)$$

Thus, integrating (10.4) and (10.6) over Ω , one sees that for (10.2) to occur it suffices that v satisfy the variational condition

$$\begin{aligned} 1 \text{ st var} \equiv & \int_{\Omega} g(Dv) \cdot D\varphi \, dx + \int_{\partial\Omega} \text{“sign } v\text{” } \varphi \, dH_{n-1} \\ & + \int_{\Omega} (v-h) \varphi \, dx = 0 \quad \text{for all } \varphi \in C^2(\bar{\Omega}), \end{aligned} \quad (10.8)$$

where $g(\xi)$ and “sign v ” are as in (10.5) and (10.7).

Therefore, in any open subset of Ω where $Dv \neq 0$ and where v is C^2 , after integration by parts, one sees that v must satisfy the equation

$$\operatorname{div} \left(\frac{Dv}{|Dv|} \right) + (h-v) = 0. \quad (10.9)$$

Likewise in any open subset of Ω where v is \equiv a constant, and hence $Dv \equiv 0$, it would be sufficient to find a C^1 vector field $g(x)$, with $|g| \leq 1$, satisfying the equation

$$\operatorname{div}(g(x)) + (h-v) = 0, \quad (10.10)$$

where $g(x)$ also satisfies certain “matching conditions” at the boundary of this open set.

From (10.9), since the $\operatorname{div}(Dv/|Dv|)$ term is known to be the negative of the curvature of the level set Ω_λ of v through each point (see (10.16) later), one sees that the “free part” of the boundary of each level set Ω_λ must have constant curvature $= h - \lambda$, and thus in 2-D must be arcs of circles of this curvature. Further considerations involving the integral over $\partial\Omega$ term in (10.8) will show that these free arcs should also meet the boundary $\partial\Omega$ tangentially.

Thus we guess that our desired $v(x)$ will have level curves as shown in Figure 5b; here because of symmetry we show only the lower left quarter of the rectangle Ω . In region A we have $v \equiv 0$. In region B we have $Dv \neq 0$ and the λ -th level curve is the tangent circle of curvature $c(\lambda) \equiv h - \lambda$. Finally, at a certain value λ^* , \equiv the maximum of v , these level curves cease and we have a plateau with $v \equiv \lambda^*$ in the region C . Clearly this function v is piecewise $C^2(\bar{\Omega})$ and patches together continuously (but not in C^1 fashion) at the interfaces between the open sets A , B , and C .

Finally, we need to discover what is the curvature γ of the free boundary of the maximum set $C(\equiv \Omega^*)$. According to Section 7 and Corollary 8.5, Ω^* should be a set which minimizes the ratio $P(G)/|G|$ over all subsets of Ω of positive measure. This is a classical problem, studied for example by [10] and [12]. The solution exists and must be a set of the form C shown in Figure 5b, that is, its free boundary is a tangential circle of a certain curvature γ . Thus, writing $P(C)/|C|$ in terms of γ , one has an elementary

minimum problem for a function of the single variable γ . Solving this minimum problem, see formula (10.17) below for the solution, one has our desired curvature γ for the free boundary of the region $C \equiv \Omega^*$. Notice that γ , and hence the region C , was determined geometrically, completely independently of the particular constant h .

Thus, if h is sufficiently large that $h - \gamma$ is strictly positive, our guessed $v(x)$ will have the positive maximum value $\lambda^* = h - \gamma$, and the region B will be nonempty. Otherwise our guessed v will be $\equiv 0$ on all Ω .

In the region B we already have the flux $g(Dv) = Dv/|Dv|$ constructed so as to satisfy (10.9). Now we need to find a suitable "flux" vector field $g(x)$ satisfying (10.10) in the regions A and C .

In the region C we get the desired flux vector field from the Giusti extremal solution $V(x)$ for $h - \lambda^*$ in this region. Notice that because C is the *unique* subset which minimizes $P(G)/|G|$ over all subsets, we have

$$\int_G (h - \lambda^*) dx < P(G) \quad (10.11)$$

for all proper subsets $G \subset C$, with equality for $G = C$ itself. Thus C is an extremal set for the function $h - \lambda^*$ and hence there exists the (unique to within an additive constant) Giusti extremal solution $U(x)$ for the (constant) function $h - \lambda^*$. Thus in the region C let $g(x)$ be the flux from that extremal solution, i.e.

$$g(x) \equiv (1 + DU(x))^{-1/2} DU(x). \quad (10.12)$$

Thus, as desired in (10.10), one has

$$\operatorname{div} g + (h - \lambda^*) = 0 \quad \text{in } C. \quad (10.13)$$

Moreover

$$g \cdot \nu = -1 \quad \text{on } \partial C. \quad (10.14)$$

This will be seen to be necessary since the "sign v " in (10.7) is $= 1$ on $\partial C \cap \partial \Omega$.

Now consider the region A where $v(x) \equiv 0$. Here we have greater freedom in choosing a subderivative $g(x)$. We need $|g(x)| \leq 1$ and by (10.10) we need the equation

$$\operatorname{div} g + (h - 0) = 0. \quad (10.15)$$

However, because $v \equiv 0$ on ∂A we will not require a condition such as (10.14) on $\partial A \cap \partial \Omega$. Here we apply the useful fact that for a smooth *unit* vector field $g(x)$ its divergence is given by

$$-\operatorname{div} g(x) = k(x), \quad (10.16)$$

where $k(x)$ denotes the curvature of the orthogonal curve (in 2-D, or surface in n-D) to the vector field through the point x . Thus here let us draw the family of circular arcs obtained by translating the interface arc between regions A and B (of curvature $h - 0$) in the -135° direction. Then let $g(x)$ be the field of unit normals to these circular arcs. Since each of these arcs is orthogonal to $g(x)$ and is of curvature $k(x) = h$, we have from (10.16) that (10.15) is satisfied in the region A .

We conclude by multiplying the equations (10.9) in B , (10.13) in C , and (10.15) in A by our $C^2(\bar{\Omega})$ test function φ , integrating by parts on A , B , and C separately and adding. The result is that the first variation of (10.8) equals an integral of zero on Ω , plus an integral involving the fluxes $g(Dv)$ on the interfaces between A and B and between B and C , plus an integral on $\partial\Omega$ of $(g \cdot v + \text{"sign } v")\varphi$. Now, the interface integrals are zero because the fluxes $g(Dv)$ patch together continuously there. The integral on $\partial\Omega$ is zero because on $\partial C \cap \partial\Omega$ and $\partial B \cap \partial\Omega$ we have $v > 0$, hence $\text{sign } v = +1$ but $g \cdot v = -1$, and because on $\partial A \cap \partial\Omega$ our $v \equiv 0$ and hence our "sign v " is allowed to be any number ≤ 1 in magnitude, which is true for our $g \cdot v$ since g is a unit vector field there. Thus the first variation in (10.8) is zero, which was seen to be sufficient for v to be the desired (unique) minimizer.

The above proof was for the case that $h - \gamma$ is strictly positive, and then $\lambda^* = h - \gamma$. If $h - \gamma = 0$ then we guess $v \equiv 0$ and a similar proof holds, but with the set B collapsed to the empty set. If $h - \gamma < 0$ however, then the strict inequalities (1.5a) and (1.5b) hold, hence there exists by the results of Giaquinta [5], as mentioned following (1.5), a pseudosolution u to the stationary problem (1.1). The corresponding flux $g(x) \equiv (1 + |Du|^2)^{-1/2} Du$ can then be used to show that the function $v \equiv 0$ satisfies the variational equation (10.8). Thus we can conclude that λ^* is positive if and only if $h - \gamma$ is positive.

Alternatively, the result of the previous paragraph, that $v \equiv 0$ if $h - \gamma \leq 0$, follows directly from Corollary 8.4(c), since γ is the minimum of $P(G)/|G|$.

A straightforward 1-D minimization shows that the desired value of γ (that curvature for the circular arcs of C for which the minimum value of $P(G)/|G|$ occurs) is given by

$$\gamma = \frac{a + b + \sqrt{(a - b)^2 + \pi ab}}{2ab} \quad (10.17)$$

where $2a$ and $2b$ are the side lengths of our rectangle.

Recall that we showed that λ^* , the maximum value of our parabolic growth function $v(x)$ for this case of constant h on the rectangle, is given by

$$\lambda^* = \max\{h - \gamma, 0\} \quad (10.18)$$

where γ is given by (10.17).

Note that the radius of curvature $1/\gamma$ in (10.17) is strictly less than both a and b , and thus the minimizing set C for the ratio $P(G)/|G|$ always has nontrivial flat boundary portions on each end of the rectangle. Moreover, let $a = 1$ and let b vary from 1 to ∞ ; the resulting γ then satisfies

$$\begin{aligned} \gamma &= \frac{2 + \sqrt{\pi}}{2} \approx 1.8862269, & \text{when } a = 1 \text{ and } b = 1, \\ \gamma &\rightarrow 1 & \text{when } a = 1 \text{ and } b \rightarrow \infty. \end{aligned} \quad (10.19)$$

This critical value γ for h thus tends to the 1-D critical value (i.e. 1) as $b \rightarrow \infty$.

We refer to the next section for numerical computations and graphs of this $v(x)$ function on the rectangle.

The results above, establishing the specific formula for the asymptotic behavior of $u(x, t)/t$ in the case of a constant h on the square Ω , were recently established by Kawohl and Kutev [11] by using the maximum principle with lower and upper comparison functions.

11. NUMERICAL EXAMPLES IN 2-D

We show the results of some numerical computations in 2-D by Carlson and Miller using a slight modification of their general purpose gradient-weighted moving finite element code GWMFE2DS, see [2], [16].

For our parabolic results we solve the regularized problem (3.2), i.e.

$$\begin{aligned} u_t &= A(u) + h(x) + \varepsilon \Delta u & \text{in } \Omega & \text{ for } t > 0 \\ u(x, t) &= 0 & \text{on } \partial\Omega & \text{ for } t > 0 \\ u(x, 0) &= 0 & \text{on } \Omega, \end{aligned} \quad (11.1)$$

with an extremely tiny ε , where Ω is a 2-D rectangle. For our elliptic results we solve the same problem with a considerably larger (but still quite small) ε , but we solve out to steady state and for numerical reasons we use a quite large “internodal viscosity” in our GWMFE computations.

Figures 3–6 show the solution $u(x, t)$, at the time $t = 30$, of the parabolic problem (11.1) with the exceedingly tiny $\varepsilon = 10^{-7}$ on the rectangle $\Omega = (-2, 2) \times (-1, 1)$. By symmetry only the quarter of the solution on $(-2, 0) \times (-1, 0)$ is computed and displayed. Here h is constant with the value $h = \gamma + 1.0$, where

$$\gamma = \frac{3 + \sqrt{1 + 2\pi}}{4} = 1.42468443 \quad (11.2)$$

is that critical value of h above which there exists no steady state, as given in (10.17).

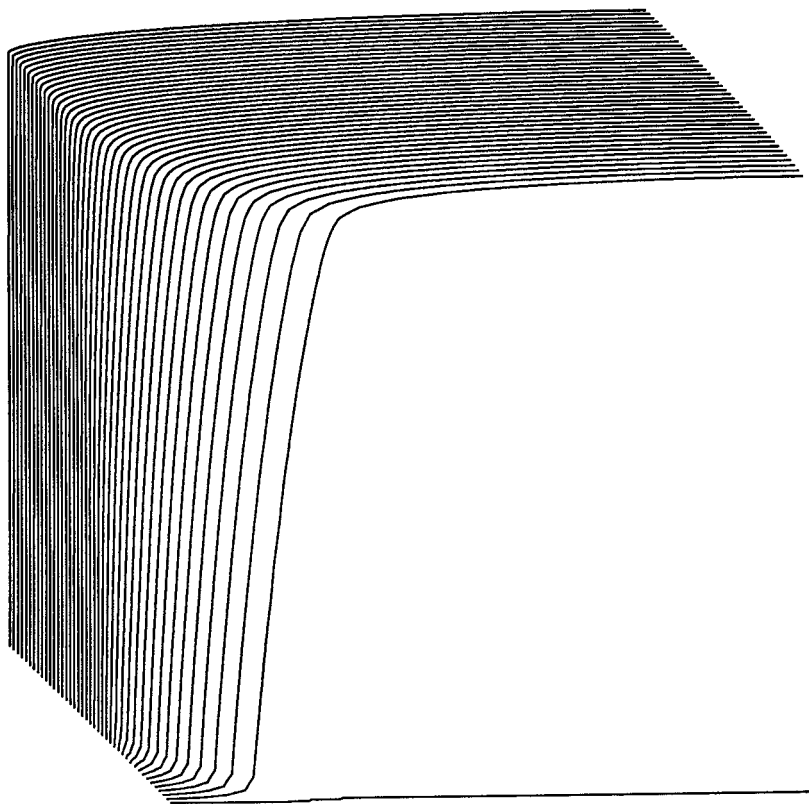


FIG. 3. 40 slices of the GWMFE solution at $t = 30$ of $u_t = A(u) + 2.42468$ on the rectangle $(-2, 2) \times (-1, 1)$ with zero initial and boundary data. Solution greatly shortened vertically by the graphics; height at center is 30.76. This graph essentially shows the shape of the parabolic growth function $v(x) = \lim_{t \rightarrow \infty} u(x, t)/t$. Evident are the cap region $C = \Omega^*$, the corner region A , the transition region B between them, and the vertical sides of the graph over much of $\partial\Omega$.

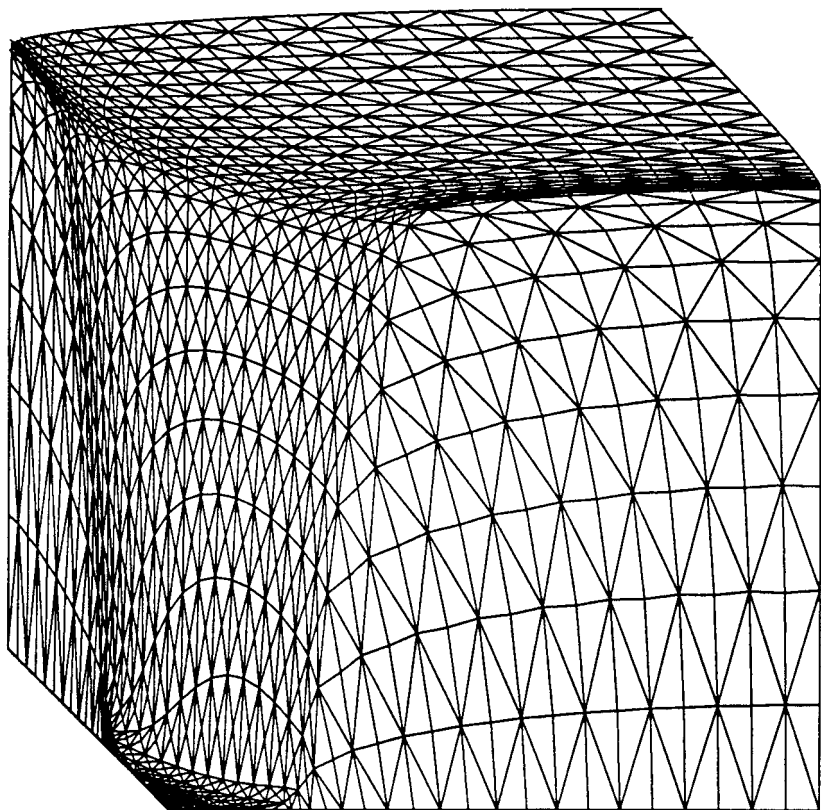


FIG. 4. The 32×32 GWMFE grid corresponding to Fig. 3.

Figure 3 shows 40 slices, sheared by $(-50\%, +50\%)$ in (x_1, x_2) , of the solution $u(x, t)$, which has a height of 30.76 at the center $(0, 0)$. Clearly visible are the cap region $C = \Omega^*$ on which the solution rises fastest, the corner region A on which the solution reaches a bounded steady-state, the transition region B joining the two, and the vertical sides on much of $\partial\Omega$ where the solution has “detached” from its zero boundary values. Figure 4 shows the corresponding 32×32 GWMFE grid for this solution. Notice that the grid points have concentrated at the edge of the region C where the “elliptic cap” of the solution goes vertical. Figures 5a and 5b show the contours of the solution with contour intervals of 0.2 and 2 respectively, adjusted vertically so that one contour passes through the maximum value at the center. Figure 5a thus shows the shape of the “elliptic cap” in region $C = \Omega^*$. Comparison of the solution contours at $t = 10, 20, 30$ shows that the shape of this cap remains unchanged. Figure 6 shows the grid of the

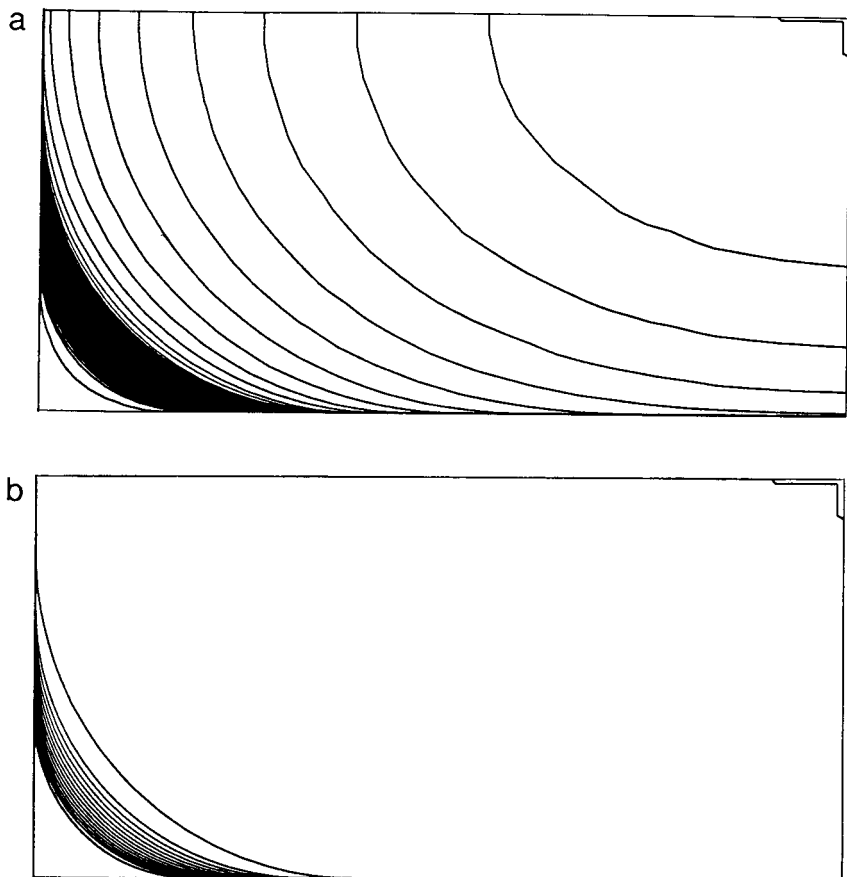


FIG. 5. (a) The contours corresponding to Fig. 3, with contour interval 2. These contours, adjusted vertically to hit the center value at $(0, 0)$, show the shape of the “elliptic cap” which quickly stabilizes on Ω^* . Evident are the cap region $C = \Omega^*$. The corner region A and the transition region B between them. (b) The same with contour interval 2.

solution at the much earlier time $t = 2$. The height of solution at the center is 2.713. Clearly visible are the “cap” (in fact the shape of this cap had nearly stabilized by this early time) and the vertical sides on much of $\partial\Omega$.

These computations confirm that the asymptotic speed at which this cap is rising is given by

$$c^* = h - \gamma = 1, \quad (11.3)$$

as proved in Section 10. Here the height at the center at times $t = 2, 10, 20, 30$ is 2.713, 10.80, 20.78, 30.76. Hence the average speeds on these time intervals are 1.011, 0.998, 0.998, in close agreement with (11.3).

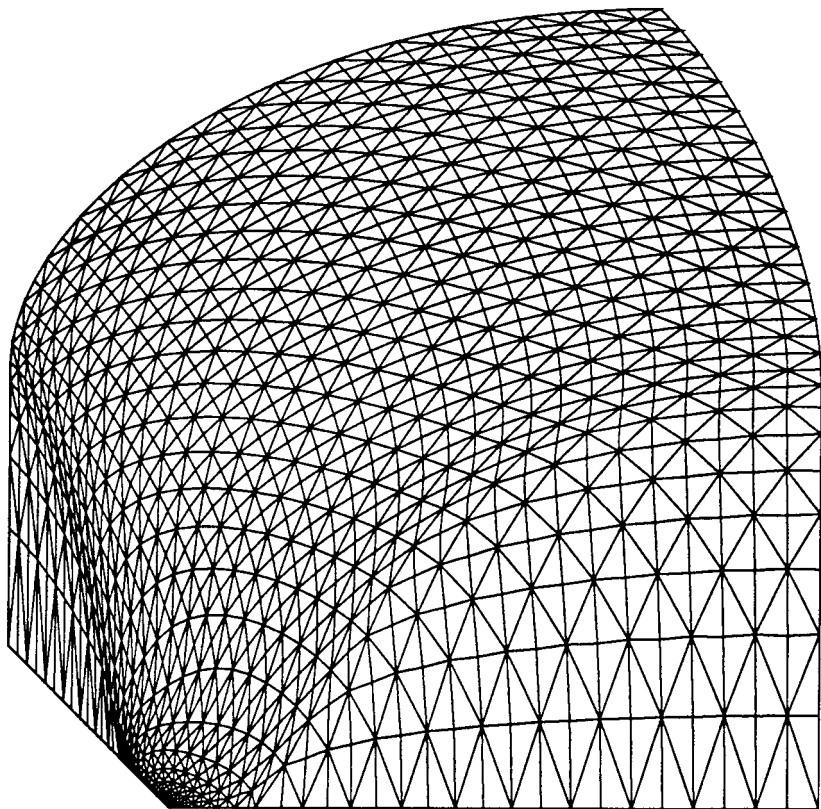


FIG. 6. Grid of the solution at $t=2$. Height at the center is 2.713. This shows the shape of the “elliptic cap” which has already nearly stabilized on Ω^* at this early time.

Since the vertical scale in Figures 3–4 is normalized such that the graph in the cap and corner regions are essentially “flat”, these figures show the shape of the parabolic growth function $v(x) = \lim_{t \rightarrow \infty} u(x, t)/t$.

Figure 7 instead shows the elliptic solution $u^\varepsilon(x)$ (at steady-state at $t=5000$) on the same rectangle $\Omega = (-2, 2) \times (-1, 1)$ with $h=2$ and $\varepsilon=0.0005$. Again, because of the normalization of the vertical scale, this figure essentially shows the shape of the elliptic growth function $w(x) = \lim_{\varepsilon \rightarrow 0} \varepsilon u^\varepsilon(x)$. The height of the solution at the center is 162.1. The height of the solution with $\varepsilon=0.001$ was 81.60; thus we see that the value of $w(x)$ at the center is approximately 0.081. Clearly visible is the cap region $\tilde{\Omega}$ on which $w(x)$ assumes its maximum. Note that $\tilde{\Omega}$ (contrary to Ω^*) is compactly contained in Ω and that the solution never detaches from its zero

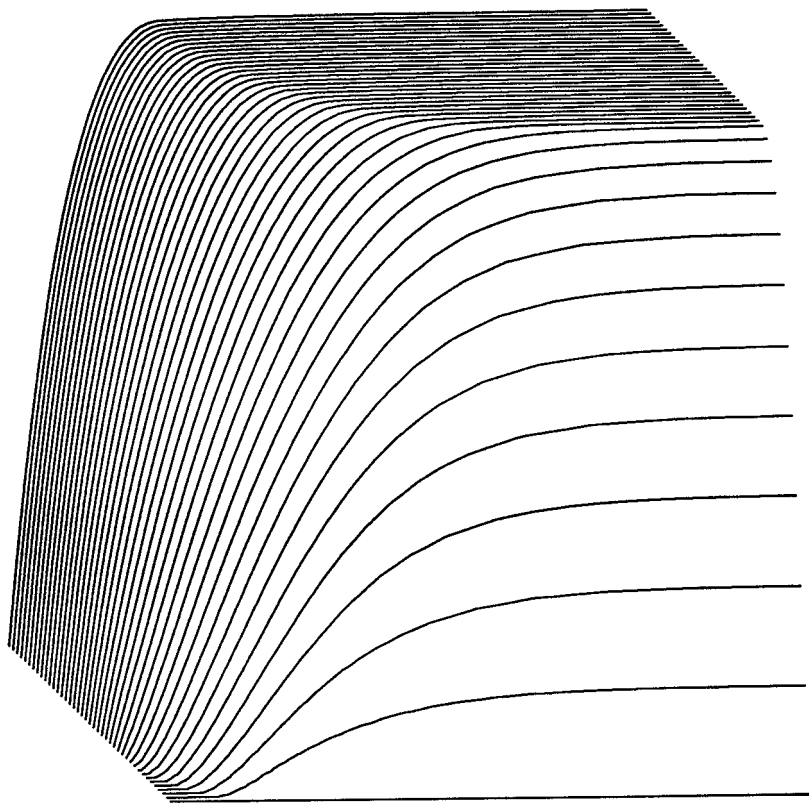


FIG. 7. 40 slices of the elliptic solution of $A(u) + \varepsilon \Delta u + 2 = 0$ with $\varepsilon = 0.0005$ on the rectangle $(-2, 2) \times (-1, 1)$ with zero boundary values. Height at center is 162.1. This graph essentially shows the shape of the elliptic growth function $w(x) = \lim_{\varepsilon \rightarrow 0} \varepsilon u^\varepsilon(x)$. Evident are the cap region $\tilde{\Omega}$ (compactly contained in Ω), the fact that $w(x)$ does not detach from its boundary values, and the fact that $w(x)$ (unlike $v(x)$) seems to merge into its maximum plateau in C^1 fashion.

boundary values on $\partial\Omega$. Note also that $w(x)$ seems to be identically zero in a small region near the corner $(-1, 1)$.

Finally in Figures 8 and 9 we consider the parabolic problem on the square $\Omega = (-1, 1) \times (-1, 1)$ with a nonconstant $h(x)$ and with $\varepsilon = 10^{-7}$. We choose $h(x)$ of the radial form

$$h(x) = h_0(|x|^4 - \tfrac{1}{2}|x|^6). \quad (11.4)$$

Note that this function is nonnegative on Ω . With $h_0 = 10$ or 20 the structure of $u(x, t)/t$ as $t \rightarrow \infty$ is largely uninteresting. It seems to have only

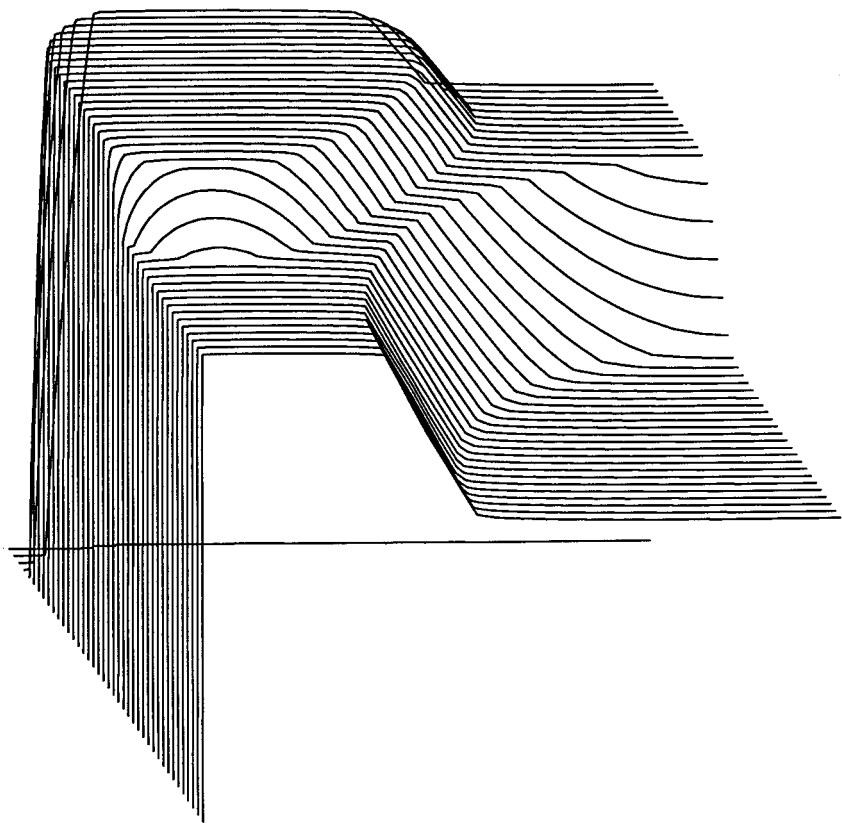


FIG. 8. 40 slices of the solution at $t = 10$ of $u_t = A(u) + h(x)$, with $h(x) = 30(|x|^4 - 0.5|x|^6)$, with zero initial and boundary data. Maximum height is 80.49. This essentially shows the shape of the parabolic growth function $v(x)$. Evident are four plateaus.

a single central plateau corresponding to the maximum set Ω^* , plus a zero valued plateau near the corner $(-1, -1)$. For $h_0 = 30$ however the solution develops far more interesting structure.

Shown in Figures 8 and 9 is the solution at time $t = 10$ with $h_0 = 30$. The “slices” of the solution in Figure 8, sheared by $(30\%, -50\%)$ in (x_1, x_2) , clearly show that the $u(x, t)/t$ has developed four plateau regions. The maximum plateau on Ω^* has the maximum height of $u = 80.49$ at this time, the minimum plateau has the height 43.8 at the center, the intermediate plateau has a height of ≈ 69 , and there is also a ≈ 0 value plateau near the corner $(-1, 1)$. The contours of Figure 9, with contour interval $= 1$, also clearly reveal these four plateaus plus the transition regions between them.

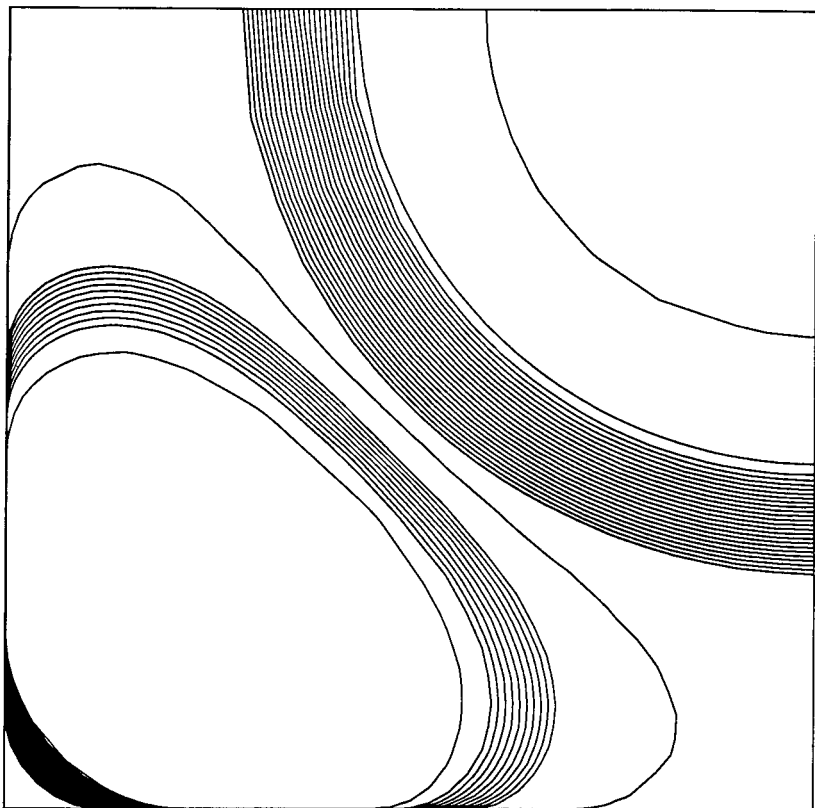


FIG. 9. The contours corresponding to Fig. 8, with contour interval 1.

12. STUDIES WITH $h(x, u)$

In all our results of the previous sections the function h has depended on x only. In this section we present some numerical examples, using a slightly modified version of the 1D GWMFE code of Carlson and Miller [2], [16], which indicate that some interesting new phenomenae occur when h is allowed to depend also on u . Some results on this case have been given by Chen [3]. See also Ural'tseva [18] for h depending on Du .

We consider $h(u)$ in the three forms λu , $\lambda \tanh u$, and $\lambda u(1 - (u/\beta)^2)$, where λ and β are positive constants. In all cases we consider the parabolic problem (3.1) on a 1-D interval with zero boundary values but nonzero initial values. Note that $u(x) \equiv 0$ is a steady state solution in all three cases since $h(0) = 0$.

One general observation for the first two cases is that for sufficiently large λ and u_0 the solution does not collapse to zero as $t \rightarrow \infty$ but instead develops plateaus and caps similar to those seen in previous sections. Moreover the asymptotic width of these caps seems to depend heavily on the initial data u_0 . This is in contrast to the previous sections where the maximum set Ω^* depends only on the function $h(x)$.

Case 1. Let $h(u) = \lambda u$, with $\lambda > 0$, on the interval $\Omega = (-1, 1)$. Note that if h were a constant then the critical value of h would be $h^* = 1$. For smaller h the solution would collapse to zero as $t \rightarrow \infty$; for larger h the solution would continue to grow. Here, with $h = \lambda u$, we seem to find that no matter how small the λ , we can get solutions which continue to grow by making the initial values sufficiently large.

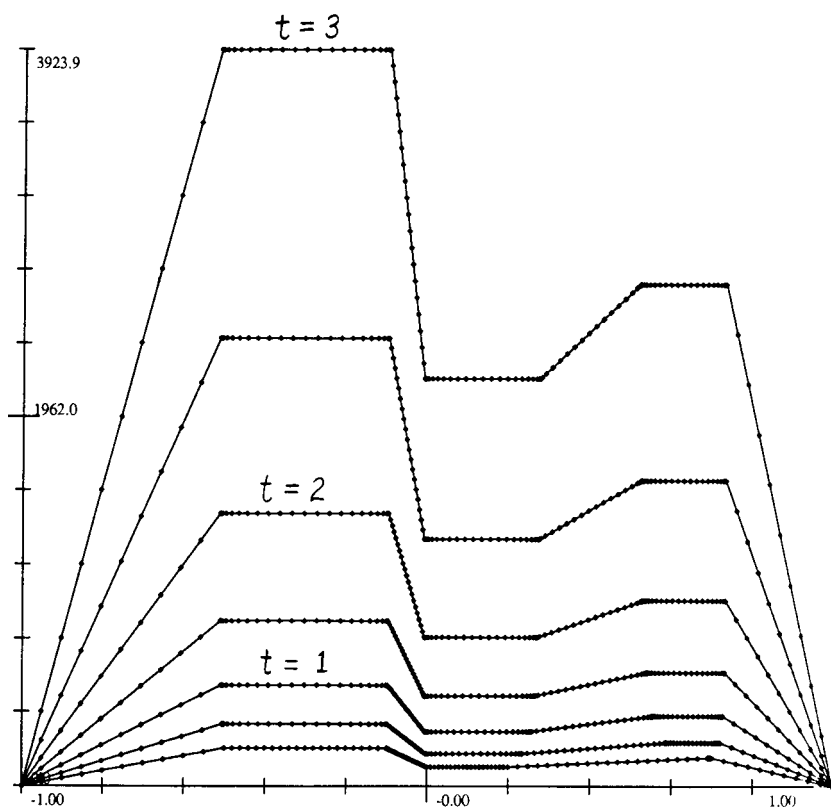


FIG. 10. The parabolic solution at $t = 0, .5, 1, \dots, 3$ of $u_t = A(u) + u$ on $(-1, 1)$. Evident are the three caps joined by steep transition zones.

In Figure 10 we show the solution $u(x, t)$ with $\lambda = 1$ and with piecewise linear $u_0(x)$ which has plateaus of height 200, 100, 150 on the intervals $(-0.5, -0.1)$, $(0, 0.2)$, $(0.69, 0.7)$. Because these initial values are sufficiently large this solution continues to grow. The solution forms three “caps”, whose asymptotic shape is shown in Figure 11. These are joined by extremely steep transition zones on which the curvature $A(u)$ is nearly zero and which therefore evolve essentially according to the ODE $u_t = u$. Once these steep transitions have been established they seem to limit the outward expansion of the caps; the first two caps expand hardly at all; the third cap (being initially so narrow) expands rapidly at first but soon also approaches an asymptotic width. Were our initial plateaus even higher (say ten times higher) then the third cap would expand even less. Thus the final “shape” of the growing solution $u(x, t)$, and the asymptotic widths of its “caps”, depends heavily upon the initial function $u_0(x)$. It might seem that

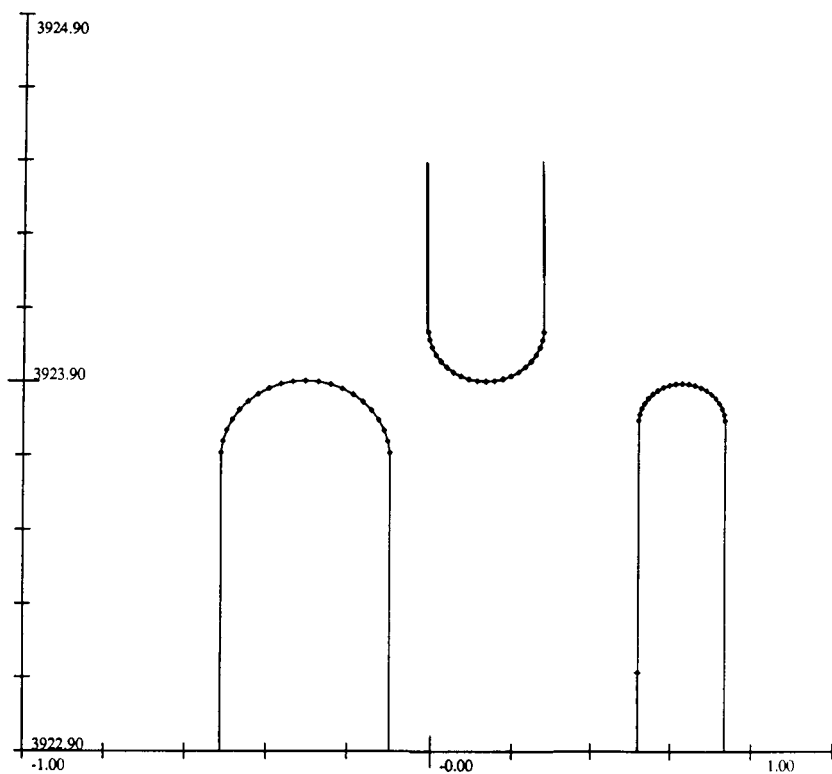


FIG. 11. The shape of the three caps of Fig. 10 at the final time $t = 3$. The cap on the left has maximum value 3923.9. The other two caps, of heights 2166.2 and 2669.1 have been adjusted vertically in the figure.

the second cap, since the $A(u)$ term is positive, might eventually overtake the first and third caps, but this is not the case at all since the transition zones are stretching vertically at such an exponential rate.

Note that the caps in Figure 11 are nearly circular since λu (with $\lambda = 1$) does not vary greatly across each cap. With much larger λ , however, (say $\lambda = 100$) the caps forming from these initial values $u_0(x)$ develop decidedly noncircular shapes.

Case 2. Let $h(u) = \lambda \tanh u$, with $\lambda > 0$ on the interval $(0,1)$. For small $|u|$ this resembles the previous case, but for large $|u|$ this $h(u)$ saturates at the two constant values $\pm \lambda$.

Shown in Figure 12 is the solution $u(x, t)$ with $\lambda = 10$ and with piecewise linear $u_0(x)$ which has two plateaus of heights $-2, 4$ on the intervals $(0.1, 0.3)$, $(0.5, 0.9)$. In this case, for $|u| \gg 1$ the $h(u)$ saturates with values

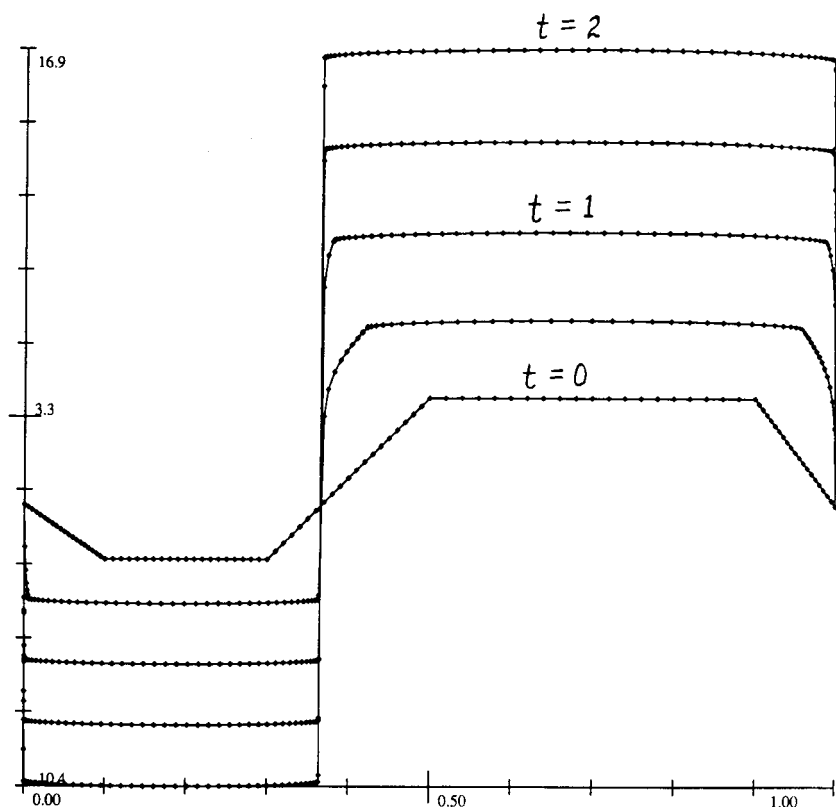


FIG. 12. The parabolic solution at $t = 0, 0.5, 1, 1.5, 2$ of $u_t = A(u) + 10 \tanh u$ on $(0, 1)$. Evident are the two caps which expand to fill the whole interval $(0, 1)$.

$\approx \pm \lambda$ and hence the transition zones between the developing caps do not stretch vertically at a huge rate. Thus the caps are able to expand in width and, asymptotically as $t \rightarrow \infty$, they fill the entire interval $(0,1)$. Note that the respective widths of these two caps depend strongly on the initial values $u_0(x)$. Because $h(u)$ is nearly $\mp \lambda$ in these two caps, each cap asymptotically evolves with a circular shape, separately in each subinterval as in the case of constant h .

Case 3. Let $h(u) = \lambda \cdot u(1 - (u/\beta)^2)$ with $\lambda, \beta > 0$, on the interval $(-1, 1)$. Here $h(u)$ changes sign for $|u| > \beta$; hence the solution will not continue growing without bound. Note that the linearization of this equation about the trivial solution $u(x, t) = 0$ is the equation $u_t = u_{xx} + \lambda u$. Hence the trivial solution can be expected to be unstable if $\lambda > \lambda_1$, where $\lambda_1 = (\pi/2)^2 \approx 2.47$ is the first eigenvalue of the Laplacian on this interval.

Figure 13 shows the evolving solution with $\beta = 1$, $\lambda = 6$, and piecewise linear $u_0(x)$ with plateaus of height ∓ 1 on the slightly nonsymmetric intervals $(-0.6, -0.3)$ and $(0.35, 0.65)$. Shown is the output at times $t = 0, 0.1661, 2.569, 1936.9, 2049.8, 2050.73, 2051.05, 2051.47, 2 \times 10^5$. The solution very quickly, certainly by $t = 2.569$, forms two caps of slightly differing widths. These caps are extremely near to steady state, with a nearly vertical interface between the two. We believe that in this 1-D situation the *exact* solution would evolve to a steady state consisting of two caps with vertical sides, much as shown at $t = 2.569$.

However, in these calculations the interface between the caps very slowly migrates to the right as shown at the times $t = 1936, 2049, 2050$. Finally at the time $t \approx 2051$ the positive cap becomes too narrow to sustain itself and collapses suddenly, as shown at times $t = 2051.05$ and 2051.47 . The solution then very quickly develops a stable steady state negative cap spanning the whole $(-1, 1)$ interval, as shown at $t = 2 \times 10^5$. It is our belief that this slow migration of the interface is an artifact of the nonzero diffusion coefficient which we are forced to use in our computations. We actually compute the solution $u^\varepsilon(x, t)$ of the slightly regularized equation (3.2) with a very tiny ε ($\varepsilon = 10^{-8}$ here and in Case 2, but $\varepsilon = 0$ in Case 1). We believe that the actual pseudosolution of (3.1), the limit of $u^\varepsilon(x, t)$ as $\varepsilon \rightarrow 0$, would exhibit no migration of the interface. This is because the computed migration gets slower and slower as ε is decreased. For example the collapse times for the positive cap occur at $t \approx 699$ with $\varepsilon = 10^{-5}$, at $t \approx 2051$ with $\varepsilon = 10^{-8}$, and at $t \approx 6488$ with $\varepsilon = 10^{-9}$.

Note that a larger λ , say $\lambda = 10$, leads to positive and negative caps which show an even slower migration.

If one changes to antisymmetric initial values $u_0(x)$ (i.e. change the $(0.35, 0.65)$ to $(0.3, 0.6)$) then with $\lambda = 6$ the solution stays antisymmetric and forms two steady-state caps of equal widths. However, with $\lambda = 5$ two

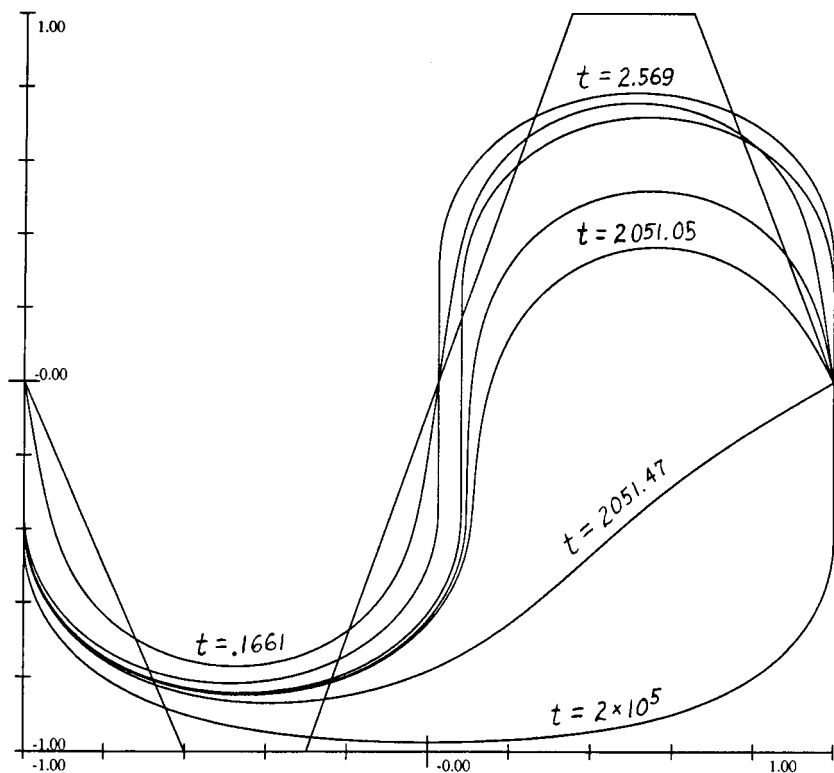


FIG. 13. The parabolic solution of $u_t = A(u) + \epsilon u_{xx} + 6u(1 - u^2)$ with a very tiny $\epsilon = 10^{-8}$. Output at times $t = 0, 0.1661, 2.569, 1936.9, 2049.8, 2050.73, 2051.05, 2051.47, 2 \times 10^5$. Solution quickly forms a near steady state with two caps and nearly vertical sides, as at $t = 2.569$. Gradual migration of the interface, leading to collapse of the positive cap and formation of a single negative gap, is believed to be due to the nonzero ϵ .

equal-width caps quickly form but then collapse to approximately zero amplitude. This solution then, over a long time, becomes unstable and grows into a single stable cap (of positive or negative amplitude depending on details of the numerics) spanning the whole $(-1, 1)$ interval.

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