MR1948872 (2003m:49020) 49J45 (35J20)
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On some sharp conditions for lower semicontinuity in L^1. (English summary)
Differential Integral Equations **16** (2003), *no. 1,* 51–76.

A classical problem in the calculus of variations is studying the lower semicontinuity properties of multiple integrals $I(u)=\int_\Omega f(x,u,\nabla u)\,dx$ along sequences of functions in $W^{1,1}_{\rm loc}(\Omega)$ that are strongly convergent in $L^1_{\rm loc}(\Omega)$ to some function in $W^{1,1}_{\rm loc}(\Omega)$ so that no control on the derivatives is generally available. Here Ω is an open, bounded set in $\mathbb{R}^N$.

It is well known since Aronszajn's example [see C. Pauc, *La Méthode Métrique en Calcul des Variations*, Hermann et Cie., Paris, 1941; MR0012736 (7,67b); G. Dal Maso, Manuscripta Math. **30** (1979/80), no. 4, 387–416; MR0567216 (81i:28007)] that the natural hypotheses (i) f is real-valued and continuous on $\Omega\times\mathbb{R}\times\mathbb{R}^N$; (ii) $f\ge 0$; and (iii) $f(x,s,\xi)$ is convex as a function of ξ for every (x,s) are not sufficient for establishing this property. Indeed, a celebrated result by J. Serrin [Trans. Amer. Math. Soc. **101** (1961), 139–167; MR0138018 (25 #1466)] states that in order to have such a property one has to assume also one of the following additional hypotheses: (a) $f(x,s,\xi)\to+\infty$ as $|\xi|\to+\infty$ for every (x,s); or (b) $f(x,s,\xi)$ is strictly convex as a function of ξ for every (x,s); or also (c) the derivatives f_x, f_ξ and $f_{x\xi}$ exist and are continuous. It is clear that (a) and (b) go in the direction of strengthening the convexity properties of f whereas (c) calls for smooth enough dependence on the variables x and ξ. Note in particular that Serrin's hypothesis (c) does not allow for Caratheodory integrands.

Several extensions (in various directions) of Serrin's theorem have been given so far, and recently M. Gori and P. Marcellini have improved Serrin's result by replacing the condition (c) on the derivatives of f with a much weaker local, Lipschitz continuity condition with respect to x only, namely (d) $|f(x_2,s,\xi)-f(x_1,s,\xi)|\le L|x_2-x_1|$ for all points (x_i,s,ξ), $i=1,2$, in every compact set K with some constant $L=L(K)$. However, this hypothesis does not allow for Caratheodory integrands.

Moreover, in the same paper, the authors show by suitably modifying Aronszajn's example that this result is sharp: (d) cannot be replaced by local, Hölder continuity with respect to x with exponent α. But, for every $\alpha\in(0,1)$, a counterexample is obtained by an N-dimensional multiple integral with $N=N(\alpha)>4\alpha/(1-\alpha)$ and this leaves the question open whether an optimal bound on the space dimension for the lower semicontinuity of I in the class of local, Hölder continuous integrands exists.

In the paper under review, first the authors answer this question negatively by once more refining Aronszajn's example so as to get the same counterexample to the lower semicontinuity for every α in dimension $N=1$ and then improve Serrin's theorem again by requiring only, besides the usual assumptions of positivity and convexity with respect to ξ, that $f(x,s,\xi)$ is in $W^{1,1}_{\rm loc}(\Omega)$ as a function of x for every (s,ξ) and that, for every $\Omega'\subset\subset\Omega$, the integral $\int_{\Omega'}|f_x(x,s,\xi)|\,dx$ remains bounded as (s,ξ) ranges in bounded sets. It is clear that this latter hypothesis is somewhat intermediate between Lipschitz and Hölder continuity, and it allows for Caratheodory integrands. Finally, some extensions to the vector valued case are also considered.

Reviewed by *Pietro Celada*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.