On the $n$-dimensional Dirichlet problem for isometric maps

B. Dacorogna$^a$, P. Marcellini$^b$, E. Paolini$^b$

$^a$ Section de Mathématiques, EPFL, 1015 Lausanne, Switzerland
$^b$ Dipartimento di Matematica, Università di Firenze, Viale Morgagni 67/A, 50134 Firenze, Italy

Received 24 January 2008; accepted 2 October 2008
Available online 28 October 2008
Communicated by H. Brezis

Abstract

We exhibit explicit Lipschitz maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ which have almost everywhere orthogonal gradient and are equal to zero on the boundary of a cube. We solve the problem by induction on the dimension $n$.

© 2008 Elsevier Inc. All rights reserved.

Keywords: Dirichlet problem; Orthogonal gradients

1. Introduction

We consider in the general $n$-dimensional case ($n > 1$) the nonlinear system of PDE’s

$$Du^t Du = I,$$

where $Du^t$ denotes the transpose matrix of the gradient $Du$ of a map $u: \mathbb{R}^n \to \mathbb{R}^n$, while $I$ is the identity matrix. A map $u$ satisfying (1) is said to be an isometric map or rigid map and its gradient is an orthogonal matrix; briefly as usual we write $Du \in O(n)$.

To the system (1) we associate the homogeneous boundary condition $u = 0$ on the boundary of a bounded open set of $\mathbb{R}^n$. The Dirichlet problem that we obtain is critical; i.e., it is incompatible with classical solutions. In fact any isometric map $u: \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ of class $C^1$ on an open
connected set \( \Omega \) of \( \mathbb{R}^n \) is affine by the classical Liouville theorem, and it therefore cannot be equal to zero on its boundary \( \partial \Omega \). We can then consider Lipschitz continuous maps \( u : \mathbb{R}^n \to \mathbb{R}^n \), satisfying the system (1) almost everywhere; then, if \( u \) is equal to zero on the boundary \( \partial \Omega \) it must be not differentiable on any neighbourhood of any boundary point, thus presenting a fractal behaviour at the boundary.

In this paper we find an explicit Lipschitz solution to the differential problem

\[
\begin{cases}
Du(x) \in O(n) & \text{a.e. } x \in Q, \\
u(x) = 0 & x \in \partial Q,
\end{cases}
\]

where \( Q = (0, 1)^n \) is the unit cube and \( O(n) \) stands, as said above, for the set of orthogonal matrices in \( \mathbb{R}^{n \times n} \).

The study of differential inclusions of the form

\[
\begin{cases}
Du(x) \in E & \text{a.e. } x \in \Omega, \\
u(x) = u_0(x) & x \in \partial \Omega,
\end{cases}
\]

where \( E \subset \mathbb{R}^{N \times n} \), \( u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N \) and \( u_0 \) is a given map, has received considerable attention. In the vectorial case \( n, N \geq 2 \), general theories of existence have been developed either via the Baire category method (see Dacorogna–Marcellini [3–5]) or via the convex integration method by Gromov (see Müller–Sverak [9]). These methods are purely existential and do not give a way of constructing explicit solutions. In parallel, for some special problems mostly related to the case when \( E \) is the set of orthogonal matrices, some solutions were provided in a constructive way. This started with the work of Cellina–Perrotta [1] when \( n = N = 3 \) and \( u_0 = 0 \), Dacorogna–Marcellini–Paolini [6,7] when \( n = N = 2 \) or \( n = N = 3 \) and Iwaniec–Verchota–Vogel [8] for \( n = N = 2 \), for related results see [2]. In this context there are also some related unpublished arguments by R.D. James for \( n = N = 2 \). In [7] the connection between this problem with isometric immersions and origami has been made. Moreover in [7] we also dealt with inhomogeneous linear boundary data.

In the present article we give a self contained and purely analytical construction in any dimension. Despite its generality our proof is shorter than the existing ones which were, however, restricted to the cases \( n = 2, 3 \). We first solve the problem by induction on the dimension in the half-space \( (0, \infty) \times \mathbb{R}^{n-1} \). We then get the solution to our problem by composing the solution in the half-space with a map that sends the whole boundary of the unit cube in \( \mathbb{R}^n \) to one of its faces. We should point out that our construction in fact solves the problem in a more precise way: instead of considering matrices in the whole of \( O(n) \), we use only a finite number of them, namely permutation matrices whose non zero entries are \( \pm 1 \).

2. The fundamental brick

Define \( f : \mathbb{R} \to \mathbb{R} \) by

\[
f(t) = \min\{t, 1 - t\}.
\]

Then define \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) by

\[
h(x, y) = \begin{cases} (h^1(x, y), h^2(x, y)) & \text{if } x \leq y, \\
(y, f(x)) & \text{if } x \geq y.
\end{cases}
\]
Finally we define a map \( \phi_n : \mathbb{R}^n \to \mathbb{R}^n \), for \( n = 2, 3, \ldots \), by induction on \( n \):

\[
\begin{align*}
\phi_2(x_1, x_2) &= h(x_1, x_2), \\
\phi_{n+1}(x_1, x_2, \ldots, x_{n+1}) &= (\phi_n(h^1(x_1 - n + 1, x_{n+1}) + n - 1, x_2, \ldots, x_n), h^2(x_1 - n + 1, x_{n+1})).
\end{align*}
\]

More in details, \( \phi_{n+1} \) can be written as a composition of the following maps:

\[
\begin{align*}
(x_1, \ldots, x_{n+1}) &\mapsto (x_1 - n + 1, x_2, \ldots, x_{n+1}), \\
(y_1, \ldots, y_{n+1}) &\mapsto (h^1(y_1, y_{n+1}), y_2, \ldots, y_n, h^2(y_1, y_{n+1})), \\
(z_1, \ldots, z_{n+1}) &\mapsto (z_1 + n - 1, z_2, \ldots, z_{n+1}), \\
(w_1, \ldots, w_{n+1}) &\mapsto (\phi_n(w_1, \ldots, w_n), w_{n+1}).
\end{align*}
\]

We recall that \( u : \mathbb{R}^n \to \mathbb{R}^n \) is called a rigid map, or an isometric map, if it is Lipschitz continuous and \( Du(x) \in O(n) \) for almost every \( x \in \mathbb{R}^n \); i.e., if \( u \) satisfies (1) for almost every \( x \in \mathbb{R}^n \).

**Theorem 1 (Properties of \( \phi_n \)).** The map \( \phi_n : \mathbb{R}^n \to \mathbb{R}^n \), for every \( n = 2, 3, \ldots \), satisfies the following properties:

(i) \( \phi_n \) is a piecewise affine rigid map;

(ii) if \( x_2, \ldots, x_{n+1} \in [0, 1] \) then

\[
\phi_n(0, x_2, \ldots, x_n) = (0, f(x_2), \ldots, f(x_n));
\]

(iii) on the cube \([n - 1, n] \times [0, 1]^{n-1}\) the map \( \phi_n \) is affine.

**Proof.** We will use the following properties of the map \( h \) defined in (2):

(1) \( h \) is a piecewise affine rigid map;

(2) if \( y \geq 0 \) and \( x \leq 0 \) then \( h(x, y) = (x, f(y)) \);

(3) if \( x \geq 1 \) and \( y \in [0, 1] \) then \( h(x, y) = (y, 1 - x) \).

We prove the theorem by induction on \( n \). In the case \( n = 2 \) the claims are direct consequences of the properties of \( h \). We assume now that the theorem holds true for \( n \), and we prove it for \( n + 1 \).

Claim (i) is a consequence of the fact that the composition of piecewise affine rigid maps is again a piecewise affine rigid map.

To prove (ii) we compute, for any \( x_2, \ldots, x_{n+1} \in [0, 1] \),

\[
\phi_{n+1}(0, x_2, \ldots, x_{n+1}) = (\phi_n(h^1(1 - n, x_{n+1}) + n - 1, x_2, \ldots, x_n), h^2(1 - n, x_{n+1}));
\]

since \( 1 - n \leq 0 \leq x_{n+1} \), by property (2) of the function \( h \) we have \( h(1 - n, x_{n+1}) = (1 - n, f(x_{n+1})) \), hence we continue

\[
= (\phi_n(0, x_2, \ldots, x_n), f(x_{n+1})).
\]
and by the induction hypothesis
\[ = (0, f(x_2), \ldots, f(x_n), f(x_{n+1})) \]
which is the claim.

Let us conclude by proving (iii). Let \( x_1 \in [n, n+1] \) and \( x_2, \ldots, x_{n+1} \in [0, 1] \). We have
\[
\phi_{n+1}(x_1, x_2, \ldots, x_{n+1}) = (\phi_n(h^1(x_1 - n + 1, x_{n+1}) + n - 1, x_2, \ldots, x_n), h^2(x_1 - n + 1, x_{n+1}));
\]
since \( x_1 - n + 1 \geq 1 \) and \( x_{n+1} \in [0, 1] \), by the property (3) of \( h \) we find that
\[
h(x_1 - n + 1, x_{n+1}) = (x_{n+1} + n - x_1, n - x_1);
\]
since now \( x_{n+1} + n - 1 \in [n - 1, n] \), by the induction hypothesis we conclude that \( \phi_{n+1} \) is affine on this region. \( \square \)

3. The pyramid construction

Let us start with some notations. Let \( f \) be as in the previous section. For every \( x = (x_1, \ldots, x_n) \) we order the real numbers \( f(x_1), \ldots, f(x_n) \) so that
\[
f(x_{i_1}) \leq f(x_{i_2}) \leq \cdots \leq f(x_{i_n}).
\]
We then define \( v : [0, 1]^n \to \mathbb{R}^n \) as
\[
v(x) = (f(x_{i_1}), f(x_{i_2}), \ldots, f(x_{i_n})).
\]
Note that for \( v(x) = (v^1(x), \ldots, v^n(x)) \) we have
\[
v^1(x) = \min_{i=1,\ldots,n} \{ f(x_i) \}, \quad v^n(x) = \max_{i=1,\ldots,n} \{ f(x_i) \},
\]
\[
v^k(x) = \max_{i_1,\ldots,i_{k-1}} \left[ \min_{i \neq i_1,\ldots,i_{k-1}} \{ f(x_i) \} \right], \quad k = 2, \ldots, n - 1,
\]
in particular, when \( n = 3 \),
\[
v^2(x) = \max\{\min\{f(x_1), f(x_2)\}, \min\{f(x_1), f(x_3)\}, \min\{f(x_2), f(x_3)\}\}.
\]

**Theorem 2** (Pyramid construction). Let \( Q = (0, 1)^n \subset \mathbb{R}^n \). The map \( v : \overline{Q} \to \mathbb{R}^n \) defined above, has the following properties:

(i) \( v \) is a piecewise affine rigid map;
(ii) \( v(Q) \subset (0, 1/2]^n \subset \{ x \in \mathbb{R}^n : x_1 > 0 \} \);
(iii) \( v(\partial Q) \subset \{ x \in \mathbb{R}^n : x_1 = 0 \} \), meaning that \( v^1 = 0 \) on \( \partial Q \).
Proof. The map \( v \) is constructed as the composition of piecewise affine rigid maps, so it is piecewise affine rigid. The second property is a consequence of the fact that if \( x_1, \ldots, x_n \in (0, 1) \) then \( f(x_1), \ldots, f(x_n) \in (0, 1/2] \). If we take \( x \in \partial Q \) we know that at least one component \( x_k \) of \( x \) is equal to either 0 or 1. So \( f(x_k) = 0 \). Since \( f(x_j) \geq 0 \) for every \( x_j \in [0, 1] \) we conclude that \( f(x_k) = 0 \) is the first component of \( v(x) \).

4. The solutions to the Dirichlet problem

Now we are going to construct a locally piecewise rigid map \( w : [0, +\infty) \times \mathbb{R}^{n-1} \to \mathbb{R}^n \) with zero boundary condition.

First we consider the zigzag function \( F : \mathbb{R} \to \mathbb{R} \) which is defined by the conditions

\[
\begin{cases}
F(t) = 2 f(t/2) = \min\{t, 2 - t\}, & \text{when } t \in [0, 2], \\
F(t) = F(t + 2), & \text{for every } t \in \mathbb{R}.
\end{cases}
\]

We also consider the affine map \( x \mapsto Jx + a \), with \( J \in O(n) \), \( a \in \mathbb{R}^n \), such that (as mentioned in Theorem 1)

\[
\phi_n(x) = Jx + a \quad \text{when } x_1 \in [n - 1, n] \text{ and } x_2, \ldots, x_n \in [0, 1].
\]

Define, for \( k \in \mathbb{Z} \), the vector \( b_k \in \mathbb{R}^n \) as

\[
b_k = \sum_{j=k}^{+\infty} \frac{J^{-j}}{2^{j+1}} a',
\]

where \( a' = (n - 1, 0, \ldots, 0) + J^{-1}a \).

Let \( H = (0, +\infty) \times \mathbb{R}^{n-1} \). Given \( x \in H \) there exists \( k \in \mathbb{Z} \) such that

\[
(n - 1)2^{-k} \leq x_1 < (n - 1)2^{1-k}.
\]

Then, for such a point \( x \), we define

\[
w(x_1, \ldots, x_n) = 2^{-k} J^{-k} \phi_n(2^k x_1 - n + 1, F(2^k x_2), \ldots, F(2^k x_n)) + b_k,
\]

where \( \phi_n \) is the map considered in Theorem 1, while for \( x_1 = 0 \) we define

\[
w(0, x_2, \ldots, x_n) = 0 \quad \text{for all } x_2, \ldots, x_n \in \mathbb{R}.
\]

Theorem 3 (Solution in the half-space). Let \( H = (0, +\infty) \times \mathbb{R}^{n-1} \). The map \( w : \overline{H} \to \mathbb{R}^n \) is locally piecewise affine in \( H \) and it is rigid on \( \overline{H} \). Moreover \( w(\partial H) = 0 \).

Proof. We first want to check the continuity of \( w \) on the planes \( x_1 = (n - 1)2^{-k} \), for every \( k \in \mathbb{Z} \). So let \( x \) be a point on such a plane and let us check that

\[
w(x_1, x_2, \ldots, x_n) = 2^{-k-1} J^{-k-1} \phi_n(2^{k+1} x_1 - n + 1, F(2^{k+1} x_2), \ldots, F(2^{k+1} x_n)) + b_{k+1}.
\]
With the substitution $x_1 = (n - 1)2^{-k}$ in the definition of $w$, the left-hand side of (3) becomes

$$2^{-k} J^{-k} \phi_n(0, F(2^k x_2), \ldots, F(2^k x_n)) + b_k;$$

by Theorem 1, since $F(t) \in [0, 1]$ for all $t$,

$$= 2^{-k} J^{-k} (0, f(F(2^k x_2)), \ldots, f(F(2^k x_n))) + b_k$$

and by the identity $f(F(t)) = F(2t)/2$,

$$= 2^{-k-1} J^{-k} (0, F(2^{k+1} x_2), \ldots, F(2^{k+1} x_n)) + b_k.$$

While the right-hand side of (3) is, for $x_1 = (n - 1)2^{-k}$, equal to

$$2^{-k-1} J^{-k-1} \phi_n(n - 1, F(2^{k+1} x_2), \ldots, F(2^{k+1} x_n)) + b_{k+1};$$

since $F(t) \in [0, 1]$ for every $t \in \mathbb{R}$, by Theorem 1 we can replace $\phi_n$ with the affine map $Jx + a$, and get

$$= 2^{-k-1} J^{-k-1} \left[ J(n - 1, F(2^{k+1} x_2), \ldots, F(2^{k+1} x_n)) + a \right] + b_{k+1}$$

$$= 2^{-k-1} J^{-k} (n - 1, F(2^{k+1} x_2), \ldots, F(2^{k+1} x_n)) + 2^{-k-1} J^{-k-1} a + b_{k+1}$$

$$= 2^{-k-1} J^{-k} (0, F(2^{k+1} x_2), \ldots, F(2^{k+1} x_n))$$

$$+ 2^{-k-1} J^{-k} (n - 1, 0, \ldots, 0) + 2^{-k-1} J^{-k-1} a + b_{k+1}$$

$$= 2^{-k-1} J^{-k} (0, F(2^{k+1} x_2), \ldots, F(2^{k+1} x_n)) + 2^{-k-1} J^{-k} a' + b_{k+1};$$

by recalling the definition of $b_k$, we obtain, as desired

$$= 2^{-k-1} J^{-k} (0, F(2^{k+1} x_2), \ldots, F(2^{k+1} x_n)) + b_k.$$

So the map $w$ on $H = (0, +\infty) \times \mathbb{R}^{n-1}$ is locally piecewise affine and rigid. We now inspect the boundary values of $w$. Take any $k \in \mathbb{Z}$, and $i_2, \ldots, i_n \in \mathbb{Z}$. We have

$$w(2^{-k}(n - 1), 2^{-k} i_1, \ldots, 2^{-k} i_n) = 2^{-k} J^{-k} \phi_n(0, F(i_1), \ldots, F(i_n)) + b_k;$$

by Theorem 1 we get

$$= 2^{-k} J^{-k} (0, f(F(i_1)), \ldots, f(F(i_n))) + b_k;$$

now notice that $F(i_k)$ is either 0 or 1 hence $f(F(i_k)) = 0$, so we find

$$= b_k.$$

Now since $b_k \to 0$ as $k \to +\infty$ and $w$ is Lipschitz continuous, we conclude that $w \to 0$ at every point of $\partial H$ and hence is continuous on the whole set $\overline{H}$. □
Theorem 4 (Solution in the cube). Let $Q = (0, 1)^n$, $w$ be as above and $v$ as in Section 3. The map $u = w \circ v : \overline{Q} \to \mathbb{R}^n$ is locally piecewise affine in $Q$ and it is rigid on $\overline{Q}$. Moreover $u(\partial Q) = 0$.

Proof. The map $w$ of Theorem 3 is a Lipschitz solution to the Dirichlet problem

$$\begin{cases}
    Dw \in O(n) & \text{a.e. in } H, \\
    w = 0 & \text{on } \partial H
\end{cases}$$

where $H$ is the half-space of $\mathbb{R}^n$. Since $u = w \circ v$, we clearly have that $u$ is rigid and so $Du \in O(n)$ a.e. Moreover since $v(\partial Q) \subset \partial H$ and $w(\partial H) = 0$, we get the condition $u(\partial Q) = 0$. \qed

Notice that we have solved a more precise problem, namely

$$Du(x) \in \Pi(n) \subset O(n)$$

where $\Pi(n)$ is the set of permutation matrices whose non-zero entries are $\pm 1$. In particular we have used at most $n!2^n$ different matrices in the construction of $w$ and $v$.

References