Research Article

Michela Eleuteri, Paolo Marcellini* and Elvira Mascolo

Regularity for scalar integrals without structure conditions

https://doi.org/10.1515/acv-2017-0037
Received July 2, 2017; revised November 1, 2017; accepted February 6, 2018

Abstract: Integrals of the Calculus of Variations with p, q-growth may have not smooth minimizers, not even bounded, for general p, q exponents. In this paper we consider the scalar case, which contrary to the vector-valued one allows us not to impose structure conditions on the integrand $f(x, \xi)$ with dependence on the modulus of the gradient, i.e. $f(x, \xi) = g(x, |\xi|)$. Without imposing structure conditions, we prove that if $\frac{q}{p}$ is sufficiently close to 1, then every minimizer is locally Lipschitz-continuous.

Keywords: Elliptic equations, local minimizers, local Lipschitz continuity, p, q-growth, general growth conditions

MSC 2010: Primary 35J60, 35B65, 49N60; secondary 35J70, 35B45

Communicated by: Juha Kinnunen

1 Introduction

The fundamental classical problem of the Calculus of Variations in the scalar case usually is formulated as finding a function u assuming a given value u_0 at the boundary $\partial\Omega$ of an open bounded set $\Omega \subset \mathbb{R}^n$ which minimizes the integral

$$\int_\Omega f(x, Dv)\, dx \tag{1.1}$$

among all functions $v : \Omega \to \mathbb{R}$, assuming the same boundary value u_0 as u. The precise functional space where to look for solutions depends on the growth conditions of $f = f(x, \xi)$ as $\xi \in \mathbb{R}^n$ grows in modulus to $+\infty$. Usually this growth is stated in terms of an inequality of the type

$$f(x, \xi) \geq M_1 |\xi|^p \tag{1.2}$$

for a.e. $x \in \Omega$, $\xi \in \mathbb{R}^n$ and for some positive constant M_1. Here $p = 1$ is associated to the $\mathrm{BV}(\Omega)$ space of functions with bounded variation, while $p > 1$ is related to the Sobolev space $W^{1,p}(\Omega)$. Usually the condition $p > 1$ and the strict convexity of $f(x, \xi)$ with respect to ξ are sufficient conditions for the existence and uniqueness of minimizers.

A different problem is the regularity of minimizers. A large literature is known about regularity (see for instance [26, 28, 30]) partly based on the nowadays classical well-known Hölder continuity result by De Giorgi [16]. To this aim it seems necessary to impose also a growth condition from above, to be associated

Michela Eleuteri, Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università degli Studi di Modena e Reggio Emilia, via Campi 213/b, 41125 Modena, Italy, e-mail: michela.eleuteri@unimore.it. http://orcid.org/0000-0003-4043-1852
***Corresponding author: Paolo Marcellini,** Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Viale Morgagni 67/A, 50134 Firenze, Italy, e-mail: marcellini@math.unifi.it. http://orcid.org/0000-0002-9350-1351
Elvira Mascolo, Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Viale Morgagni 67/A, 50134 Firenze, Italy, e-mail: mascolo@math.unifi.it. http://orcid.org/0000-0001-7489-2011

to the growth condition from below in (1.2), of the type

$$f(x,\xi) \le M_2(1+|\xi|^q) \tag{1.3}$$

for a.e. $x \in \Omega$, $\xi \in \mathbb{R}^n$, for $q \ge p$ and for some positive constant M_2. The so-called "natural growth conditions" appear if $q = p$, while the more general assumption $q > p$ allows us to consider a much larger class of integrals of the Calculus of Variations, such as for example

$$f(\xi) = |\xi|^p \log(1+|\xi|) \tag{1.4}$$

or

$$f(x,\xi) = |\xi|^{p(x)} \quad \text{or} \quad f(x,\xi) = (1+|\xi|^2)^{\frac{p(x)}{2}}. \tag{1.5}$$

We recall also the integrands recently considered in [11, 12, 19, 20], see also [2–4],

$$f(x,\xi) = a(x)|\xi|^p + b(x)|\xi|^q, \tag{1.6}$$

where $a(x), b(x) \ge 0$ and possibly zero on some part of Ω, being at least one of the two coefficients positive at almost every $x \in \Omega$. The above examples (1.4), (1.5), (1.6) enter in the theory presented in this paper. However, here we study the more general case with $f = f(x,\xi)$ without a structure, i.e. not necessarily depending on the modulus of ξ of the type $f(x,\xi) = g(x,|\xi|)$.

We assume that $f : \Omega \times \mathbb{R}^n \to [0,+\infty)$ is a convex function with respect to the gradient variable and it is strictly convex only at infinity. More precisely, there exists $M_0 > 0$ such that $f_{\xi\xi}, f_{\xi x}$ are Carathéodory functions satisfying

$$\begin{cases} M_1|\xi|^{p-2}|\lambda|^2 \le \sum_{i,j} f_{\xi_i\xi_j}(x,\xi)\lambda_i\lambda_j, \\ |f_{\xi\xi}(x,\xi)| \le M_2|\xi|^{q-2}, \\ |f_{\xi x}(x,\xi)| \le h(x)|\xi|^{\frac{p+q-2}{2}} \end{cases} \tag{1.7}$$

for a.e. $x \in \Omega$ and for all $\lambda, \xi \in \mathbb{R}^n$, with $|\xi| \ge M_0$ and for positive constants M_1, M_2. Here $1 < p \le q$ and $h \in L^r(\Omega)$ for some $r > n$.

Model integrands satisfying condition (1.7) are, for instance, the function $f(x,\xi)$ in (1.6) and also

$$f(x,\xi) = |\xi|^p + c(x)|\xi|^s + |\xi_n|^q, \tag{1.8}$$

ξ_n being the last component (or any other component) of the vector $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, when

$$s \le \frac{p+q}{2}.$$

For instance, when $s = p$ and $q \ge p$, we are considering energy integrals with integrand of the type (we denote here $a(x) = 1 + c(x)$ a generic positive coefficient)

$$f(x,\xi) = a(x)|\xi|^p + |\xi_n|^q. \tag{1.9}$$

Note that the cases (1.8) and (1.9) can be handled with Theorem 1.1. On the other hand example (1.10) below enters in Theorem 1.2:

$$f(x,\xi) = |\xi|^p + b(x)|\xi|^q. \tag{1.10}$$

The main regularity result that we prove here is the following a-priori estimate.

Theorem 1.1 (A-priori estimate). *Let $u \in W^{1,p}(\Omega)$ be a smooth local minimizer of the integral functional* (1.1) *with exponents p, q fulfilling*

$$\frac{q}{p} < 1 + 2\Big(\frac{1}{n} - \frac{1}{r}\Big). \tag{1.11}$$

Under the growth assumption (1.7), *there exist positive constants C, β, γ depending on $n, r, p, q, M_0, M_1, M_2$ such that, for every $0 < \rho < R \le \rho + 1$,*

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^n)} \le C\Big(\frac{\|1+h\|_{L^r(\Omega)}}{R-\rho}\Big)^{\beta\gamma}\Big(\int_{B_R}\{1+|Du|^p\}\,dx\Big)^{\frac{\gamma}{p}}. \tag{1.12}$$

Note that to get regularity of solutions it is natural, and also necessary, to assume that the gap $q - p$ is small or that $\frac{q}{p}$ is close to 1, because of the known counterexamples [27, 31, 33].

The L^∞-bound of the gradient is obtained through several steps. The first step of the a-priori estimate is Lemma 2.3 below, where on the right-hand side of the a-priori estimate there is the norm of the minimizer u in $W^{1,qm}(\Omega)$ ($m = \frac{r}{r-2}$), and the exponents p, q are related by the condition

$$\frac{q}{p} < 1 + \frac{2n}{n-2}\left(\frac{1}{n} - \frac{1}{r}\right) \tag{1.13}$$

with $n \geq 3$. Note that if $r = +\infty$ in (1.11) and (1.13), we recover the bounds in [33, Theorems 2.1 and 3.1]. An interpolation method allows us to obtain (1.12).

The mathematical literature on the regularity under p, q-growth is now very large; we refer to [32–35] and to [36] for a complete survey on the subject. A new impulse to the subject has been given by the recent articles already cited [11, 12, 15, 20] for the case of elliptic equations and by [6–8] for the case of parabolic equations and systems under p, q-growth. We observe that here the ellipticity and growth assumptions hold only for large values of the gradient variable, i.e. we consider functionals which are *uniformly convex only at infinity*. In this context see [10, 14, 25] and recently [13, 19, 20]. The Sobolev dependence on x has recently been considered in [1, 37] and for obstacle problems in [21].

The previous a-priori estimate, more precisely Theorem 2.6, under assumptions (1.7) where the last condition is replaced by

$$|f_{\xi x}(x, \xi)| \leq h(x)|\xi|^{q-1} \tag{1.14}$$

for a.e. $x \in \Omega$, with $|\xi| \geq M_0$, allows us to obtain the following existence and regularity result.

Theorem 1.2 (Existence and regularity). *Assume that f satisfies* (1.7) *and* (1.14) *with $1 < p \leq q$ and*

$$\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}.$$

The Dirichlet problem $\min\{F(u) : u \in W_0^{1,p}(\Omega) + u_0\}$, *with F defined in* (1.15) *below and $u_0 \in W^{1,q}(\Omega)$, has at least one locally Lipschitz continuous solution.*

Here we emphasize the definition (i.e. the precise meaning) of the integral $F(u)$ to be minimized; in fact, the integral in (1.1) is well defined if $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$, due to the growth assumption in (1.3), but a-priori it is not uniquely defined if $u \in W^{1,p}(\Omega) \setminus W^{1,q}_{\mathrm{loc}}(\Omega)$. In this context of x-dependence, we cannot a-priori exclude the Lavrentiev phenomenon; however, note that in Section 5 we assume a special form of f to rule out this possibility.

For the gap in the Lavrentiev phenomenon we refer to [9, 39] and recently [22–24] for related results.

For the functional F we adopt the classical definition which refers to the pioneering research by Serrin [38] (see also [29]), which is related to the Γ-convergence theory by De Giorgi [17]. Precisely, for all $u \in W^{1,p}(\Omega)$,

$$F(u) = \inf\left\{\liminf_k \int_\Omega f(x, Du_k)\,dx : u_k \in W_0^{1,q}(\Omega) + u_0,\ u_k \xrightarrow{w} u \text{ in } W^{1,p}(\Omega)\right\}. \tag{1.15}$$

We discuss more in details in Section 3 the definition of F in (1.15), while in Section 2 we give the proof of the a-priori estimate. Finally, in Section 4 we give the proof of Theorem 1.2.

2 A-priori estimates

Let us start with two technical lemmas.

Lemma 2.1. *The inequality*

$$(1+t)^\beta \leq c_\beta\left(1 + \int_0^t (1+s)^{\beta-2} s\,ds\right) \tag{2.1}$$

holds for every $t \in [0, +\infty)$ and every $\beta \in (0, +\infty)$, where

$$c_\beta = \frac{\beta}{1 - (1 + \beta(\beta-1))^{\frac{1}{1-\beta}}} \tag{2.2}$$

if $\beta \neq 1$, while (by continuity)

$$c_1 = \lim_{\beta \to 1} c_\beta = \frac{e}{e-1}. \tag{2.3}$$

Proof. In order to prove inequality (2.1) we first consider the case $\beta = 1$.

Step 1 ($\beta = 1$). We compute the integral on the right-hand side of (2.1):

$$\int_0^t (1+s)^{-1} s \, ds = \int_0^t (1 - (1+s)^{-1}) \, ds = t - \log(1+t)$$

and inequality (2.1) becomes

$$1 + t \leq c_1(1 + t - \log(1+t)),$$

which is equivalent to

$$\frac{\log(1+t)}{1+t} \leq \frac{c_1 - 1}{c_1}.$$

A computation shows that $g(t) =: \frac{\log(1+t)}{1+t}$ is positive for $t \in (0, +\infty)$ and has a maximum at $t = e - 1$, thus

$$g(t) =: \frac{\log(1+t)}{1+t} \leq g(e-1) = \frac{1}{e};$$

with the position $\frac{c_1-1}{c_1} =: \frac{1}{e}$ we find (2.3).

Step 2 ($\beta \neq 1$). We compute the integral on the right-hand side of (2.1) under the condition $\beta \neq 1$ and with the notation $r =: 1 + s$:

$$\begin{aligned}
\int_0^t (1+s)^{\beta-2} s \, ds &= \int_1^{t+1} r^{\beta-2}(r-1) \, dr \\
&= \int_1^{t+1} r^{\beta-1} \, dr - \int_1^{t+1} r^{\beta-2} \, dr \\
&= \left[\frac{r^\beta}{\beta}\right]_{r=1}^{r=t+1} - \left[\frac{r^{\beta-1}}{\beta-1}\right]_{r=1}^{r=t+1} \\
&= \frac{(t+1)^\beta}{\beta} - \frac{(t+1)^{\beta-1}}{\beta-1} + \frac{1}{\beta(\beta-1)}.
\end{aligned}$$

Inequality (2.1) takes then the form

$$\frac{1}{c_\beta}(1+t)^\beta \leq 1 + \frac{(t+1)^\beta}{\beta} - \frac{(t+1)^{\beta-1}}{\beta-1} + \frac{1}{\beta(\beta-1)}.$$

We can write it equivalently

$$g(t) \leq 1 + \frac{1}{\beta(\beta-1)}, \tag{2.4}$$

where

$$g(t) =: \frac{(t+1)^{\beta-1}}{\beta-1} - \left(\frac{1}{\beta} - \frac{1}{c_\beta}\right)(t+1)^\beta.$$

We can compute the maximum of $g(t)$ when $t \in [0, +\infty)$. We find that the derivative $g'(t)$ is equal to zero if $t = \frac{\beta}{c_\beta - \beta}$ and, since $c_\beta > \beta$, we obtain

$$\max\{g(t) : t \in [0, +\infty)\} = g\left(\frac{\beta}{c_\beta - \beta}\right) = \left(\frac{c_\beta}{c_\beta - \beta}\right)^{\beta-1} \frac{1}{\beta(\beta-1)}.$$

Therefore inequality (2.4) holds if we choose c_β to satisfy the condition

$$\left(\frac{c_\beta}{c_\beta - \beta}\right)^{\beta-1} \frac{1}{\beta(\beta-1)} = 1 + \frac{1}{\beta(\beta-1)}.$$

A further computation gives for c_β the explicit expression in (2.2). Note that $c_\beta \to c_1$ as $\beta \to 1$. □

In the sequel we apply the previous lemma to get the a-priori estimates in particular to deal with the left-hand side of (2.26), with $\beta = \frac{\gamma}{2}+\frac{p}{2}$, for $\gamma \geq 0$; thus $\beta \geq \frac{p}{2}$. In the next result in fact we consider $\beta \in [\beta_0, +\infty)$ for some fixed $\beta_0 > 0$.

Lemma 2.2. *Let $\beta_0 > 0$. There exist constants c' and c'', depending on β_0 but independent of $\beta \geq \beta_0$ and of $t \geq 0$, such that*

$$(1+t)^\beta \leq c' \frac{\beta^2}{\log(1+\beta)}\left(1 + \int_0^t (1+s)^{\beta-2} s\, ds\right), \tag{2.5}$$

$$(1+t)^\beta \leq c'' \beta^2 \left(1 + \int_0^t (1+s)^{\beta-2} s\, ds\right) \tag{2.6}$$

for every $\beta \in [\beta_0, +\infty)$ and every $t \in [0, +\infty)$.

Proof. First we show that the constant c_β is bounded independently of $\beta \leq 1$ if $\beta \in [\beta_0, 1]$ (here we assume that $\beta_0 \in (0,1)$, otherwise nothing to be proved at this step). Precisely, we show that

$$c_\beta \leq \frac{\beta}{1-e^{-\beta_0}} \quad \text{for all } \beta \in [\beta_0, 1]. \tag{2.7}$$

In fact, by the inequality $\log t \leq t-1$, valid for all $t > 0$, by posing $t = 1+\beta(\beta-1)$ if $\beta < 1$, we obtain

$$(1+\beta(\beta-1))^{\frac{1}{1-\beta}} = e^{\frac{\log(1+\beta(\beta-1))}{1-\beta}} \leq e^{\frac{\beta(\beta-1)}{1-\beta}} = e^{-\beta}$$

and (2.7) follows if $\beta \in [\beta_0, 1)$, since

$$c_\beta = \frac{\beta}{1-(1+\beta(\beta-1))^{\frac{1}{1-\beta}}} \leq \frac{\beta}{1-e^{-\beta}} \leq \frac{\beta}{1-e^{-\beta_0}}.$$

Finally, if $\beta = 1$, then $c_1 = \frac{e}{e-1} < \frac{1}{1-e^{-\beta_0}}$ holds, since it is equivalent to $1 < e^{1-\beta_0}$.

We now consider the case $\beta > 1$. By Taylor's formula we get

$$(1+\beta(\beta-1))^{\frac{1}{1-\beta}} = e^{\frac{\log(1+\beta(\beta-1))}{1-\beta}} = 1 + \frac{\log(1+\beta(\beta-1))}{1-\beta} + o\left(\frac{\log(1+\beta(\beta-1))}{1-\beta}\right)$$

and thus the quantity

$$\frac{c_\beta \log(1+\beta)}{\beta^2} = \frac{\log(1+\beta)}{\beta\left[\frac{\log(1+\beta(\beta-1))}{\beta-1} + o\left(\frac{\log(1+\beta(\beta-1))}{1-\beta}\right)\right]}$$

has a finite limit as $\beta \to +\infty$ (equal to $\frac{1}{2}$) and it is a bounded function for $\beta \in [1, +\infty)$, let us say bounded by c'. This proves (2.5). The other inequality (2.6) is a direct consequence of (2.5). □

Let now Ω be an open bounded subset of $\mathbb{R}^n$ for $n \geq 2$ and assume that f satisfies (1.7). We observe that we can transform $f(x,\xi)$ into $f(x, M_0\xi)$, which satisfies the same assumptions for $|\xi| \geq 1$ (with different constants depending on M_0). Then it is sufficient to obtain the a-priori bound and the regularity results for $v = \frac{1}{M_0}u$. Therefore, for clarity of exposition and without loss of generality, we assume $M_0 = 1$. Throughout the paper we will denote by B_ρ and B_R balls of radii, respectively, ρ and R (with $\rho < R$) compactly contained in Ω and with the same center, let us say, $x_0 \in \Omega$.

In this section we assume the following supplementary assumptions on f. Assume that $f \in \mathcal{C}^2(\Omega \times \mathbb{R}^n)$ and there exist two positive constants k and K such that for all $\xi \in \mathbb{R}^n$ and all $x \in \Omega$,

$$\begin{cases} k(1+|\xi|^2)^{\frac{q-2}{2}}|\lambda|^2 \leq \sum_{i,j} f_{\xi_i\xi_j}(x,\xi)\lambda_i\lambda_j, \\ |f_{\xi\xi}(x,\xi)| \leq K(1+|\xi|^2)^{\frac{q-2}{2}}, \\ |f_{\xi x}(x,\xi)| \leq K(1+|\xi|^2)^{\frac{q-1}{2}}. \end{cases} \tag{2.8}$$

In the next lemma, we obtain an a-priori estimate for the L^∞-norm of the gradient of u which is independent of k and K.

Lemma 2.3. *Let u be a local minimizer of the integral functional* (1.1) *with f satisfying* (1.7) *and* (2.8) *with*

$$\frac{q}{p} < 1 + \frac{2\alpha}{n-2} \quad \text{with } \alpha = 1 - \frac{n}{r} \tag{2.9}$$

if $n \geq 3$ and $p < q$ if $n = 2$. Then there exists a positive constants C depending only on n, r, p, q, M_1, M_2 (depending also on $|\Omega|$ when $n = 2$) such that

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^n)} \leq C\Big[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)}\Big]^{\theta\tilde{\beta}}\Big(\int_{B_R}\{1+|Du|^{qm}\}\,dx\Big)^{\frac{\theta}{qm}} \tag{2.10}$$

for every $0 < \rho < R \leq \rho + 1$, where

$$\tilde{\beta} := \frac{2^*}{p\frac{2^*}{2} - qm}, \qquad \theta := \frac{qm(\frac{2^*}{2m} - 1)}{p\frac{2^*}{2} - qm}, \qquad m := \frac{r}{r-2}. \tag{2.11}$$

Remark 2.4. We observe that

$$1 \leq m := \frac{r}{r-2} < \frac{n}{n-2} = \frac{2^*}{2}, \qquad \text{since } r > n; \tag{2.12}$$

the last inequality holds for $n > 2$, while we set 2^* equal to any fixed real number greater than 2 if $n = 2$. Moreover, we also have

$$\frac{1}{2m} - \frac{1}{2^*} = \frac{r-2}{2r} - \frac{n-2}{2n} = \frac{n(r-2) - r(n-2)}{2nr} = \frac{r-n}{nr} = \frac{1}{n} - \frac{1}{r} = \frac{\alpha}{n}, \tag{2.13}$$

therefore (2.9) can be equivalently expressed as

$$\frac{q}{p} < \frac{2^*}{2m} \tag{2.14}$$

because

$$1 + \frac{2\alpha}{n-2} = 1 + \frac{2^*\alpha}{n} \overset{(2.13)}{=} 1 + 2^*\Big(\frac{1}{2m} - \frac{1}{2^*}\Big) = \frac{2^*}{2m}.$$

Therefore, due to (2.14), in (2.11) we have $\tilde{\beta} > 0$ and $\theta > 1$.

Remark 2.5. The result obtained is sharp in the sense that if $m = 1$ ($r = +\infty$), then the relation between p and q reduces to the analogous one in [33, Theorem 2.1], i.e. $\frac{q}{p} < \frac{n}{n-2}$.

Proof. Let $u \in W^{1,q}(\Omega)$ be a local minimizer of (1.1). Then u satisfies the Euler first variation

$$\int_\Omega \sum_{i=1}^n f_{\xi_i}(x, Du)\varphi_{x_i}(x)\,dx = 0 \quad \text{for all } \varphi \in W_0^{1,q}(\Omega).$$

By (2.8), the technique of the difference quotients (see [18, 30], in particular [28, Chapter 8, Sections 8.1 and 8.2]) gives

$$u \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap W^{2,\min(2,q)}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad (1 + |Du|^2)^{\frac{q-2}{2}}|D^2u|^2 \in L^1_{\mathrm{loc}}(\Omega). \tag{2.15}$$

Let $\eta \in C_0^1(\Omega)$ and for any fixed $s \in \{1, \dots, n\}$ define

$$\varphi = \eta^2 u_{x_s}\Phi((|Du| - 1)_+)$$

for $\Phi : [0, +\infty) \to [0, +\infty)$ increasing, locally Lipschitz continuous function, with Φ and Φ' bounded on $[0, +\infty)$, such that $\Phi(0) = \Phi'(0) = 0$ and

$$\Phi'(s)s \leq c_\Phi\Phi(s) \tag{2.16}$$

for a suitable constant $c_\Phi \geq 1$. Here $(a)_+$ denotes the positive part of $a \in \mathbb{R}$; in the following we denote

$$\Phi((|Du| - 1)_+) = \Phi(|Du| - 1)_+.$$

We have then

$$\varphi_{x_i} = 2\eta\eta_{x_i}u_{x_s}\Phi(|Du| - 1)_+ + \eta^2 u_{x_s x_i}\Phi(|Du| - 1)_+ + \eta^2 u_{x_s}\Phi'(|Du| - 1)_+[(|Du| - 1)_+]_{x_i}. \tag{2.17}$$

Let $q \geq 2$. By (2.15) we have that $|D^2u|^2 \in L^1_{\text{loc}}(\Omega)$. Otherwise if $1 < q < 2$, we use the fact that $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ to infer that there exists $M = M(\operatorname{supp}\varphi)$ such that

$$|Du(x)| \leq M \quad \text{for a.e. } x \in \operatorname{supp}\varphi.$$

Now since $q - 2 < 0$, we have

$$(1 + M^2)^{\frac{q-2}{2}}|D^2u|^2 \leq (1 + |Du|^2)^{\frac{q-2}{2}}|D^2u|^2,$$

and by (2.15) we again get $|D^2u|^2 \in L^1(\operatorname{supp}\varphi)$. Therefore we can insert φ_{x_i} in the following second variation,

$$\int_\Omega \Big\{ \sum_{i,j=1}^n f_{\xi_i\xi_j}(x, Du)u_{x_j x_s}\varphi_{x_i} + \sum_{i=1}^n f_{\xi_i x_s}(x, Du)\varphi_{x_i} \Big\}\, dx = 0 \quad \text{for all } s = 1, \dots, n,$$

and we obtain

$$\begin{aligned}
0 = \sum_s \Big[& \int_\Omega 2\eta\Phi(|Du| - 1)_+ \sum_{i,j} f_{\xi_i\xi_j}(x, Du)\eta_{x_i}u_{x_s}u_{x_s x_j}\, dx \\
& + \int_\Omega \eta^2\Phi(|Du| - 1)_+ \sum_{i,j} f_{\xi_i\xi_j}(x, Du)u_{x_s x_i}u_{x_s x_j}\, dx \\
& + \int_\Omega \eta^2\Phi'(|Du| - 1)_+ \sum_{i,j} f_{\xi_i\xi_j}(x, Du)u_{x_s}u_{x_s x_j}[(|Du| - 1)_+]_{x_i}\, dx \\
& + \int_\Omega 2\eta\Phi(|Du| - 1)_+ \sum_i f_{\xi_i x_s}(x, Du)\eta_{x_i}u_{x_s}\, dx \\
& + \int_\Omega \eta^2\Phi(|Du| - 1)_+ \sum_i f_{\xi_i x_s}(x, Du)u_{x_s x_i}\, dx \\
& + \int_\Omega \eta^2\Phi'(|Du| - 1)_+ \sum_i f_{\xi_i x_s}(x, Du)u_{x_s}[(|Du| - 1)_+]_{x_i}\, dx \Big] \\
=: \sum_s & (I_1^s + I_2^s + I_3^s + I_4^s + I_5^s + I_6^s).
\end{aligned} \tag{2.18}$$

In the following, constants will be denoted by C, regardless of their actual value.

Let us start with the estimate of the first integral in (2.18). By the Cauchy–Schwarz inequality, the Young inequality and (1.7), we have

$$\begin{aligned}
\Big|\sum_s I_1^s\Big| &= \Big|\int_\Omega 2\eta\Phi(|Du| - 1)_+ \sum_{i,j,s} f_{\xi_i\xi_j}(x, Du)\eta_{x_i}u_{x_s}u_{x_s x_j}\, dx\Big| \\
&\leq \int_\Omega 2\eta\Phi(|Du| - 1)_+ \Big\{\sum_{i,j,s} f_{\xi_i\xi_j}(x, Du)\eta_{x_i}u_{x_s}\eta_{x_j}u_{x_s}\Big\}^{\frac12} \Big\{\sum_{i,j,s} f_{\xi_i\xi_j}(x, Du)u_{x_s x_i}u_{x_s x_j}\Big\}^{\frac12}\, dx \\
&\leq C\int_\Omega |D\eta|^2\Phi(|Du| - 1)_+|Du|^q\, dx + \frac12\int_\Omega \eta^2\Phi(|Du| - 1)_+ \sum_{i,j,s} f_{\xi_i\xi_j}(x, Du)u_{x_s x_i}u_{x_s x_j}\, dx.
\end{aligned}$$

Let us consider the third integral in (2.18). First of all we observe that

$$[(|Du| - 1)_+]_{x_i} \sum_s u_{x_s}u_{x_s x_j} = [(|Du| - 1)_+]_{x_i}|Du|[(|Du| - 1)_+]_{x_j}.$$

This entails using (1.7)

$$\sum_s I_3^s = \int_\Omega \eta^2 \Phi'(|Du|-1)_+ \sum_{i,j,s} f_{\xi_i \xi_j}(x, Du) u_{x_s} u_{x_s x_j} [(|Du-1|_+)]_{x_i}\, dx$$
$$\geq M_1 \int_\Omega \eta^2 \Phi'(|Du|-1)_+ |Du|^{p-1} |D(|Du|-1)_+|^2\, dx \geq 0.$$

We now deal with the fourth integral in (2.18). We have

$$\Big|\sum_s I_4^s\Big| = \Big|\int_\Omega 2\eta \Phi(|Du|-1)_+ \sum_{i,s} f_{\xi_i x_s}(x, Du) \eta_{x_i} u_{x_s}\, dx\Big|$$
$$\overset{(1.7)}{\leq} \int_\Omega 2\eta \Phi(|Du|-1)_+ h(x) |Du|^{\frac{p+q-2}{2}} \sum_{i,s} |\eta_{x_i} u_{x_s}|\, dx$$
$$\leq C \int_\Omega (\eta^2 + |D\eta|^2) h(x) \Phi(|Du|-1)_+ |Du|^q\, dx.$$

Consider now the fifth integral in (2.18). We have

$$\Big|\sum_s I_5^s\Big| = \Big|\int_\Omega \eta^2 \Phi(|Du|-1)_+ \sum_{i,s} f_{\xi_i x_s}(x, Du) u_{x_s x_j}\, dx\Big|$$
$$\overset{(1.7)}{\leq} \int_\Omega \eta^2 \Phi(|Du|-1)_+ h(x) |Du|^{\frac{p+q-2}{2}} |D^2 u|\, dx$$
$$\leq \int_\Omega [\eta^2 \Phi(|Du|-1)_+ |Du|^{p-2} |D^2 u|^2]^{\frac{1}{2}} [\eta^2 \Phi(|Du|-1)_+ |h(x)|^2 |Du|^q]^{\frac{1}{2}}\, dx$$
$$\leq \varepsilon \int_\Omega \eta^2 \Phi(|Du|-1)_+ |Du|^{p-2} |D^2 u|^2\, dx + C_\varepsilon \int_\Omega \eta^2 \Phi(|Du|-1)_+ |h(x)|^2 |Du|^q\, dx,$$

where in the last line we used the Young inequality. Finally, for any $0 < \delta < 1$,

$$\Big|\sum_s I_6^s\Big| = \Big|\int_\Omega \eta^2 \sum_{i,s} f_{\xi_i x_s}(x, Du) u_{x_s} \Phi'(|Du|-1)_+ [(|Du|-1)_+]_{x_i}\, dx\Big|$$
$$\overset{(1.7)}{\leq} \int_\Omega \eta^2 \Phi'(|Du|-1)_+ h(x) |Du|^{\frac{p+q-2}{2}} |Du| |D(|Du|-1)_+|\, dx$$
$$\leq \int_\Omega \eta^2 \Phi'(|Du|-1)_+ h(x) |Du|^{\frac{p+q}{2}} |D^2 u|\, dx$$
$$= \int_\Omega \eta^2 \Phi'(|Du|-1)_+ h(x) [(|Du|-1)_+ + \delta][(|Du|-1)_+ + \delta]^{-1} |Du|^{\frac{p+q}{2}} |D^2 u|\, dx$$
$$\leq \int_\Omega \eta^2 \Big\{\frac{1}{c_\Phi} \Phi'(|Du|-1)_+ [(|Du|-1)_+ + \delta] |Du|^{p-2} |D^2 u|^2\Big\}^{\frac{1}{2}}$$
$$\times \Big\{c_\Phi \Phi'(|Du|-1)_+ |h(x)|^2 |Du|^{q+2} [(|Du|-1)_+ + \delta]^{-1}\Big\}^{\frac{1}{2}}\, dx$$
$$\leq C_\varepsilon c_\Phi \int_\Omega \eta^2 \Phi'(|Du|-1)_+ |h(x)|^2 |Du|^{q+2} [(|Du|-1)_+ + \delta]^{-1}\, dx$$
$$+ \frac{\varepsilon}{c_\Phi} \int_\Omega \eta^2 \Phi'(|Du|-1)_+ [(|Du|-1)_+ + \delta] |Du|^{p-2} |D^2 u|^2\, dx.$$

Since $\Omega = \{x : |Du(x)| \geq 2\} \cup \{x : |Du(x)| < 2\}$ and $(|Du|-1)_+ \geq 1$ in $\{x : |Du(x)| \geq 2\}$, we also have

$$(|Du|-1)_+ + \delta \leq 2(|Du|-1)_+ \tag{2.19}$$

as long as we have chosen $\delta < 1$. Therefore, using (2.16), we can estimate the last integral as

$$\begin{aligned}
&\int_\Omega \eta^2 \Phi'(|Du|-1)_+[(|Du|-1)_+ + \delta]|Du|^{p-2}|D^2u|^2\,dx \\
&\quad = \int_{|Du|\geq 2} \eta^2 \Phi'(|Du|-1)_+[(|Du|-1)_+ + \delta]|Du|^{p-2}|D^2u|^2\,dx \\
&\qquad\qquad + \int_{1<|Du|<2} \eta^2 \Phi'(|Du|-1)_+[(|Du|-1)_+ + \delta]|Du|^{p-2}|D^2u|^2\,dx \\
&\quad \overset{(2.19)}{\leq} 2\int_{|Du|\geq 2} \eta^2 \Phi'(|Du|-1)_+(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx \\
&\qquad\qquad + \int_{1<|Du|<2} \eta^2 \Phi'(|Du|-1)_+(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx \\
&\qquad\qquad + \delta\int_{1<|Du|<2} \eta^2 \Phi'(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx \\
&\quad \overset{(2.16)}{\leq} 2c_\Phi \int_\Omega \eta^2 \Phi(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx + \delta\int_{1<|Du|<2} \eta^2 \Phi'(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx.
\end{aligned}$$

Therefore we finally have

$$\begin{aligned}
\Big|\sum_s I_6^s\Big| &\leq C_\varepsilon c_\Phi \int_\Omega \eta^2 \Phi'(|Du|-1)_+|h(x)|^2|Du|^{q+2}[(|Du|-1)_+ + \delta]^{-1}\,dx \\
&\qquad + 2\varepsilon\int_\Omega \eta^2 \Phi(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx + \delta\varepsilon\int_{1<|Du|<2} \eta^2 \Phi'(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx.
\end{aligned}$$

Now, for ε sufficiently small and putting together all the previous estimates, we deduce that there exists a constant C depending on n, r, p, q, M_1 such that

$$\begin{aligned}
&\int_\Omega \eta^2 \Phi(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx \\
&\quad \leq Cc_\Phi \int_\Omega (\eta^2 + |D\eta|^2)(1+h(x))^2|Du|^q[\Phi(|Du|-1)_+ + \Phi'(|Du|-1)_+|Du|^2[(|Du|-1)_+ + \delta]^{-1}]\,dx \\
&\qquad\qquad + \delta\int_{1<|Du|<2} \eta^2 \Phi'(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx. \qquad (2.20)
\end{aligned}$$

Let us now set

$$\Phi(s) := (1+s)^{\gamma-2}s^2, \quad \gamma \geq 0,$$

with

$$\Phi'(s) = (\gamma s + 2)s(1+s)^{\gamma-3}.$$

It is easy to check that Φ satisfies (2.16) with $c_\Phi = 2(1+\gamma)$.

We now approximate this function Φ by a sequence of functions Φ_h, each of them being equal to Φ in the interval $[0, h]$, and then extended to $[h, +\infty)$ with the constant value $\Phi(h)$. Since Φ_h and Φ_h' converge monotonically to Φ and Φ', by inserting Φ_h in (2.20), it is possible to pass to the limit as $h \to +\infty$ by the Monotone Convergence Theorem.

Therefore, for every $0 < \delta < 1$, since

$$\frac{(|Du|-1)_+}{(|Du|-1)_+ + \delta} \leq 1 \quad \text{for all } \delta > 0$$

and $\Phi'(t-1)_+ \le C(\gamma)$ when $1 < t < 2$, we obtain

$$\int_\Omega \eta^2 (1 + (|Du| - 1)_+)^{\gamma-2} (|Du| - 1)_+^2 |Du|^{p-2} |D^2 u|^2 \, dx$$
$$\le C(1+\gamma)^2 \int_\Omega (\eta^2 + |D\eta|^2)(1 + h(x))^2 (1 + (|Du| - 1)_+)^{\gamma+q} \, dx + \delta C(\gamma) \int_{1<|Du|<2} \eta^2 |Du|^{p-2} |D^2 u|^2 \, dx. \quad (2.21)$$

Using [20, formula (3.51) of Lemma 3.3], namely the fact that $|Du|^{p-2} \le C(p)(1 + |Du|^2)^{\frac{p-2}{2}}$ when $|Du| > 1$, we have

$$\int_{1<|Du|<2} \eta^2 |Du|^{p-2} |D^2 u|^2 \, dx \le C \int_{1<|Du|<2} \eta^2 (1 + |Du|^2)^{\frac{p-2}{2}} |D^2 u|^2 \, dx < +\infty$$

by (2.15), and for $\delta \to 0$ and the last term in the previous inequality vanishes.

Since $h \in L^r(\Omega)$, by the Hölder inequality, since $\frac{1}{m} + \frac{2}{r} = 1$, by (2.21) we have

$$\int_\Omega \eta^2 (1 + (|Du| - 1)_+)^{\gamma-2} (|Du| - 1)_+^2 |Du|^{p-2} |D((|Du| - 1)_+)|^2 \, dx$$
$$\le C(1+\gamma)^2 \|1 + h\|_{L^r(\Omega)}^2 \left[\int_\Omega (\eta^2 + |D\eta|^2)^m (1 + (|Du| - 1)_+)^{(\gamma+q)m} \, dx \right]^{\frac{1}{m}}. \quad (2.22)$$

Let us introduce

$$G(t) = 1 + \int_0^t (1+s)^{\frac{\gamma}{2} + \frac{p}{2} - 2} s \, ds. \quad (2.23)$$

We obtain

$$[G(t)]^2 \le 4(1+t)^{\gamma+p} \le 4(1+t)^{\gamma+q}, \quad (2.24)$$

where we used the fact that $p \le q$. On the other hand

$$G'(t) = (1+t)^{\frac{\gamma}{2} + \frac{p}{2} - 2} t, \quad (2.25)$$

which in turn allows us to give the following estimate for the gradient of the function $w = \eta G((|Du| - 1)_+)$:

$$\int_\Omega |D(\eta G((|Du| - 1)_+))|^2 \, dx$$
$$\le 2 \int_\Omega |D\eta|^2 |G((|Du| - 1)_+)|^2 \, dx + 2 \int_\Omega \eta^2 [G_t((|Du| - 1)_+)]^2 [D((|Du| - 1)_+)]^2 \, dx$$
$$\le C(1+\gamma)^2 \|1 + h\|_{L^r(\Omega)}^2 \left[\int_\Omega (\eta^2 + |D\eta|^2)^m [1 + (|Du| - 1)_+]^{(\gamma+q)m} \, dx \right]^{\frac{1}{m}},$$

the second inequality by (2.22), (2.24), (2.25). By Sobolev's inequality there exists a constant C (depending also on $|\Omega|$ when $n = 2$) such that

$$\left\{ \int_\Omega [\eta G((|Du| - 1)_+)]^{2^*} \, dx \right\}^{\frac{2}{2^*}} \le C \int_\Omega |D(\eta G((|Du| - 1)_+))|^2 \, dx$$

and by the previous inequality we get (for a different constant)

$$\left\{ \int_\Omega [\eta G((|Du| - 1)_+)]^{2^*} \, dx \right\}^{\frac{2}{2^*}} \le C(1+\gamma)^2 \|1 + h\|_{L^r(\Omega)}^2 \left[\int_\Omega (\eta^2 + |D\eta|^2)^m [1 + (|Du| - 1)_+]^{(\gamma+q)m} \, dx \right]^{\frac{1}{m}}. \quad (2.26)$$

We take into account the definition of $G(t)$ in (2.23) and we use Lemma 2.2, and in particular formula (2.6) with $\beta = \frac{\gamma+p}{2}$. Being $\gamma \ge 0$, we have $\beta \ge \beta_0 := \frac{p}{2} > 0$ and

$$(1+t)^{\frac{\gamma+p}{2}} \le c'' \left(\frac{\gamma+p}{2} \right)^2 \left(1 + \int_0^t (1+s)^{\frac{\gamma+p}{2} - 2} s \, ds \right)$$

for every $\gamma \geq 0$ and every $t \in [0, +\infty)$. In terms of $G(t) = 1 + \int_0^t (1+s)^{\frac{\gamma}{2}+\frac{p}{2}-2} s\, ds$ equivalently

$$(1+t)^{\frac{\gamma+p}{2}} \leq c''\Big(\frac{\gamma+p}{2}\Big)^2 G(t) \quad \text{for all } \gamma \geq 0 \text{ and all } t \geq 0.$$

Therefore, if $t := (|Du|-1)_+$,

$$(1+(|Du|-1)_+)^{\frac{\gamma+p}{2}2^*} \leq (c'')^{2^*}\Big(\frac{\gamma+p}{2}\Big)^{2\cdot 2^*} [G((|Du|-1)_+)]^{2^*} \quad \text{for all } \gamma \geq 0,$$

and by (2.26) we finally get

$$\begin{aligned}\bigg\{\int_\Omega \eta^{2^*}[1+(|Du|-1)_+]^{\frac{\gamma+p}{2}2^*}\,dx\bigg\}^{\frac{2}{2^*}} &\leq (c'')^2\Big(\frac{\gamma+p}{2}\Big)^4 \bigg\{\int_\Omega [\eta G((|Du|-1)_+)]^{2^*}\,dx\bigg\}^{\frac{2}{2^*}}\\ &\leq C(\gamma+1)^6 \|1+h\|^2_{L^r(\Omega)}\bigg[\int_\Omega (\eta^2+|D\eta|^2)^m[1+(|Du|-1)_+]^{(\gamma+q)m}\,dx\bigg]^{\frac{1}{m}}\end{aligned}$$

with a new constant C and for every $\gamma \geq 0$.

As usual we consider a test function η equal to 1 in a ball B_ρ, with supp $\eta \subset B_R$ and such that $|D\eta| \leq \frac{2}{(R-\rho)}$. We get

$$\bigg[\int_{B_\rho} [1+(|Du|-1)_+]^{[(\gamma+p)m]\frac{2^*}{2m}}\,dx\bigg]^{\frac{2m}{2^*}} \leq C_0\|1+h\|^{2m}_{L^r(\Omega)} \frac{(\gamma+q)^{6m}}{(R-\rho)^{2m}} \int_{B_R} [1+(|Du|-1)_+]^{(\gamma+q)m}\,dx, \tag{2.27}$$

where the constant C_0 only depends on n, r, p, q, M_1, M_2 but is independent of γ.

Fixed $0 < \rho_0 < R_0 \leq \rho_0 + 1$, we define the following decreasing sequence of radii $\{\rho_k\}_{k\geq 1}$:

$$\rho_k = \rho_0 + \frac{R_0-\rho_0}{2^k} \quad \text{for all } k \geq 1.$$

We define recursively a sequence α_k in the following way:

$$\alpha_1 := 0, \quad \alpha_{k+1} := (\alpha_k + pm)\frac{2^*}{2m} - qm. \tag{2.28}$$

Then we have the following representation formula for α_k which can easily be proved by induction:

$$\alpha_k = \Big(p\frac{2^*}{2} - qm\Big)\frac{[(\frac{2^*}{2m})^{k-1}-1]}{\frac{2^*}{2m}-1}. \tag{2.29}$$

We rewrite (2.27) with $R = \rho_k$, $\rho = \rho_{k+1}$, $\gamma = \frac{\alpha_k}{m}$ and observe that

$$R - \rho := \rho_k - \rho_{k+1} = \frac{R_0-\rho_0}{2^{k+1}}.$$

Set, for all $k \geq 1$,

$$A_k := \bigg(\int_{B_{\rho_k}} [1+(|Du|-1)_+]^{\alpha_k+qm}\,dx\bigg)^{\frac{1}{\alpha_k+qm}},$$

$$C_k := C_0\|1+h\|^{2m}_{L^r(\Omega)}\Big(\frac{(\alpha_k+qm)^3 2^{k+1}}{R_0-\rho_0}\Big)^{2m};$$

we obtain for every $k \geq 1$,

$$A_{k+1} \leq C_k^{\frac{1}{\alpha_k+pm}} A_k^{\frac{\alpha_k+qm}{\alpha_k+pm}}. \tag{2.30}$$

Let θ be defined by

$$\theta := \prod_{i=1}^\infty \frac{\alpha_i+qm}{\alpha_i+pm}. \tag{2.31}$$

We show that θ is finite and is given by

$$\theta = \frac{qm(\frac{2^*}{2m} - 1)}{p\frac{2^*}{2} - qm}.$$

Indeed by (2.31), the recursive definition of α_k, i.e. (2.28) and the representation formula for α_k, namely (2.29), we have

$$\theta_k := \prod_{i=1}^{k} \frac{\alpha_i + qm}{\alpha_i + pm} \overset{(2.28)}{=} \frac{qm}{\alpha_k + pm}\Big(\frac{2^*}{2m}\Big)^{k-1} \overset{(2.29)}{=} \frac{qm(\frac{2^*}{2m})^{k-1}}{\frac{(p\frac{2^*}{2}-qm)[(\frac{2^*}{2m})^{k-1}-1]}{\frac{2^*}{2m}-1} + pm}$$
$$= \frac{qm(\frac{2^*}{2m})^{k-1}(\frac{2^*}{2m} - 1)}{pm(\frac{2^*}{2m} - 1) + (p\frac{2^*}{2} - qm)[(\frac{2^*}{2m})^{k-1} - 1]},$$

which yields (2.31) once we pass to the limit as $k \to \infty$ as long as $\frac{2^*}{2m} > 1$ in view of (2.12). Note that θ makes sense due to (2.12) and the bound (2.14) in Remark 2.4.

Iterating (2.30), we deduce

$$A_{k+1} \le \tilde{C}\bigg(\Big[\frac{\|1 + h\|_{L^r(\Omega)}}{(R_0 - \rho_0)}\Big]^{\tilde{\beta}} A_1\bigg)^{\theta_k}, \quad k \ge 1, \tag{2.32}$$

where

$$\tilde{C} := C_0{}^{\tilde{\beta}\theta} \exp\bigg[\theta \sum_{i=1}^{\infty} \frac{\log[(\alpha_i + qm)^{6m} 2^{2m(i+1)}]}{\alpha_i + pm}\bigg] < +\infty,$$

which is finite because the series is convergent (α_i from the representation formula (2.29) grows exponentially) and

$$\sum_{i=1}^{\infty} \frac{2m}{\alpha_i + pm} \overset{(2.29)}{=} \sum_{i=1}^{\infty} \frac{2m}{(p\frac{2^*}{2} - qm)\frac{[(\frac{2^*}{2})^{i-1}-1]}{\frac{2^*}{2m}-1} + pm} \le \sum_{i=1}^{\infty} \frac{2m}{(p\frac{2^*}{2} - qm)\frac{(\frac{2^*}{2m})^{i-1}}{\frac{2^*}{2m}-1}}$$
$$\le \frac{2m(\frac{2^*}{2m} - 1)}{p\frac{2^*}{2} - qm} \sum_{i=0}^{\infty}\Big(\frac{2m}{2^*}\Big)^i = \frac{2m(\frac{2^*}{2m} - 1)}{p\frac{2^*}{2} - qm}\frac{1}{(1 - \frac{2m}{2^*})} = \frac{2^*}{p\frac{2^*}{2} - qm} =: \tilde{\beta},$$

where in the first inequality we used the fact that

$$\Big(p\frac{2^*}{2} - qm\Big)\frac{[(\frac{2^*}{2m})^{i-1} - 1]}{\frac{2^*}{2m} - 1} + pm \ge \Big(p\frac{2^*}{2} - qm\Big)\frac{(\frac{2^*}{2m})^{i-1}}{\frac{2^*}{2m} - 1} \iff -\frac{p\frac{2^*}{2} - qm}{\frac{2^*}{2m} - 1} + pm \ge 0 \iff q \ge p.$$

By letting $k \to +\infty$ in (2.32), we have (2.10). Therefore, the proof of Lemma 2.3 is complete. □

The a-priori estimate (1.12) in Theorem 1.1 follows by the classical interpolation inequality

$$\|v\|_{L^s(B_\rho)} \le \|v\|_{L^p(B_\rho)}^{\frac{p}{s}} \|v\|_{L^\infty(B_\rho)}^{1-\frac{p}{s}} \tag{2.33}$$

for any $s \ge p$, which permits to estimate the essential supremum of the gradient of the local minimizer in terms of its L^p-norm.

Proof of Theorem 1.1. Let us set

$$V(x) := 1 + (|Du|(x) - 1)_+$$

then estimate (2.10) becomes

$$\sup_{x\in B_\rho} |V(x)| \le C\bigg(\Big[\frac{\|1 + h\|_{L^r(\Omega)}}{R - \rho}\Big]^{\tilde{\beta}} \|V\|_{L^{qm}(B_R)}\bigg)^{\theta} \tag{2.34}$$

for every ρ, R such that $0 < \rho < R \le \rho + 1$ and where $C = C(n, r, p, q, M_1, M_2)$.

In the following we denote

$$s := qm.$$

At this point, (2.34) and (2.33) give

$$\|V\|_{L^s(B_\rho)} \le C^{1-\frac{p}{s}} \|V\|^{\frac{p}{s}}_{L^p(B_\rho)} \bigg(\bigg[\frac{\|1+h\|_{L^r(\Omega)}}{R-\rho} \bigg]^{\tilde\beta} \|V\|_{L^s(B_R)} \bigg)^{\theta(1-\frac{p}{qm})}. \tag{2.35}$$

We observe that

$$\tau := \theta\Big(1 - \frac{p}{qm}\Big) < 1, \tag{2.36}$$

because

$$\theta\Big(1-\frac{p}{qm}\Big) < 1 \iff \frac{q\frac{2^*}{2} - qm - \frac{p2^*}{2m} + p}{p\frac{2^*}{2} - qm} < 1 \iff q < p\Big(1 - \frac{2}{2^*} + \frac{1}{m}\Big) \overset{(2.13)}{=} p\Big(1 + \frac{2\alpha}{n}\Big).$$

For $0 < \rho < R$ and for every $k \ge 0$, let us define $\rho_k := R - (R-\rho)2^{-k}$. By inserting in (2.35) $\rho = \rho_k$ and $R = \rho_{k+1}$, (so that $R - \rho = (R-\rho)2^{-(k+1)}$) we have, for every $k \ge 0$,

$$\|V\|_{L^s(B_{\rho_k})} \le C^{1-\frac{p}{s}} \|V\|^{\frac{p}{s}}_{L^p(B_{\rho_k})} \bigg(2^{\tilde\beta(k+1)} \bigg[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)} \bigg]^{\tilde\beta} \|V\|_{L^s(B_{\rho_{k+1}})} \bigg)^{\tau}. \tag{2.37}$$

By iteration of (2.37), we deduce for $k \ge 0$,

$$\|V\|_{L^s(B_{\rho_0})} \le \bigg(C^{1-\frac{p}{s}} \bigg[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)} \bigg]^{\tilde\beta\tau} \|V\|^{\frac{p}{s}}_{L^p(B_{\rho_k})} \bigg)^{\sum_{i=0}^k \tau^i} 2^{\tilde\beta \sum_{i=0}^{k+1} i\tau^i} (\|V\|_{L^s(B_{\rho_{k+1}})})^{\tau^{k+1}}. \tag{2.38}$$

By (2.36), the series appearing in (2.38) are convergent. Since

$$\|V\|_{L^s(B_{\rho_k})} \le \|V\|_{L^s(B_R)},$$

we can pass to the limit as $k \to +\infty$ and we obtain for every $0 < \rho < R$ with a constant $C = C(n, r, p, q, M_1, M_2)$ independent of k,

$$\|V\|_{L^s(B_\rho)} \le C\bigg(\bigg[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)} \bigg]^{\tilde\beta\tau} \|V\|^{\frac{p}{s}}_{L^p(B_R)} \bigg)^{\frac{1}{1-\tau}}. \tag{2.39}$$

Combining (2.34) and (2.39), by setting $\rho' = \frac{(R+\rho)}{2}$ we have

$$\begin{aligned} \|V\|_{L^\infty(B_\rho)} &\le C\bigg(\bigg[\frac{\|1+h\|_{L^r(\Omega)}}{(\rho'-\rho)} \bigg]^{\tilde\beta} \|V\|_{L^s(B_{\rho'})} \bigg)^{\theta} \\ &\le C\bigg(\bigg[\frac{\|1+h\|_{L^r(\Omega)}}{(\rho'-\rho)} \bigg]^{\tilde\beta(1-\tau)} \bigg[\frac{1+\|h\|_{L^r(\Omega)}}{(R-\rho')} \bigg]^{\tilde\beta\tau} \|V\|^{\frac{p}{s}}_{L^p(B_R)} \bigg)^{\frac{\theta}{1-\tau}}; \end{aligned}$$

now, since

$$(\rho' - \rho) = (R - \rho') = \frac{R-\rho}{2},$$

we get

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^n)} \le C\bigg(\bigg[\frac{\|1+h\|_{L^r(\Omega)}}{(R-\rho)} \bigg]^{\beta} \bigg(\int_{B_R} \{1 + |Du|^p\}\,dx \bigg)^{\frac{1}{p}} \bigg)^{\gamma},$$

where

$$\beta := \tilde\beta \frac{qm}{p} = \frac{\frac{2^* q}{p} m}{p\frac{2^*}{2} - qm}, \quad \gamma := \frac{\theta p}{qm(1 - \theta(1 - \frac{p}{qm}))}, \quad \theta := \frac{qm(\frac{2^*}{2m} - 1)}{p\frac{2^*}{2} - qm},$$

so (1.12) follows. □

Let now f satisfy (1.7) and (1.14). Under these assumptions on f, we obtain the following result.

Theorem 2.6. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the integral functional* (1.1)*. Assume that $f = f(x,\xi)$ in* (1.1) *satisfies* (1.7), (1.14) *and* (2.8)*, with*

$$\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}. \tag{2.40}$$

Then there exist positive constants $C, \hat{\beta}, \hat{\gamma}$ depending on $n, r, p, q, M_1, M_2, \rho, R$ such that

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^n)} \le C[\|1+h\|_{L^r(\Omega)}]^{\hat{\beta}\hat{\gamma}}\Big(\int_{B_R}\{1+f(x,Du)\}\,dx\Big)^{\frac{\hat{\gamma}}{p}} \tag{2.41}$$

for every $0 < \rho < R \le \rho + 1$.

Proof. Let $t := 2q - p$. Then (1.14) can be written explicitly in the form

$$|f_{\xi x}(x,\xi)| \le h(x)|\xi|^{\frac{t+p-2}{2}}, \quad |\xi| \ge 1.$$

Moreover, (2.40) in terms of p and t is equivalent to

$$\frac{t}{p} < 1 + \frac{2\alpha}{n} \quad \text{with } \alpha = 1 - \frac{n}{r}.$$

Thus all the assumptions of Theorem 1.1 are satisfied with q replaced by t. In particular, the second inequality in (1.7) holds with q replaced by t since $t \ge q$. Then the conclusion of Theorem 1.1 holds with $q = t$ which corresponds to (2.41) with

$$\hat{\beta} := \frac{\frac{2^*}{p}(2q-p)m}{p\frac{2^*}{2} - (2q-p)m}, \quad \hat{\gamma} := \frac{\theta p}{(2q-p)m(1-\theta) + p\theta},$$

since $f(x,\xi) \ge C|\xi|^p$ for every $|\xi| \ge 1$. □

3 Extension of the integral energy

Let $f : \Omega \times \mathbb{R}^n \to [0,+\infty)$ be a continuous function, convex in ξ such that

$$|\xi|^p \le f(x,\xi) \le C(1+|\xi|^q) \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n. \tag{3.1}$$

For $u_0 \in W^{1,q}(\Omega)$, we define the extension to $W^{1,p}(\Omega)$ of the integral functional $\int_\Omega f(x,Du)\,dx$, i.e.

$$F(u) = \inf\Big\{\liminf_k \int_\Omega f(x,Du_k)\,dx : u_k \in W_0^{1,q}(\Omega) + u_0,\ u_k \overset{w}{\rightharpoonup} u \text{ in } W^{1,p}(\Omega)\Big\} \tag{3.2}$$

with

$$\int_\Omega f(x,Du_0)\,dx < +\infty.$$

It is easy to check that

$$F(u) = \int_\Omega f(x,Du)\,dx \quad \text{for } u \in W_0^{1,q}(\Omega) + u_0.$$

In fact, for $u_k = u$ for all k,

$$F(u) \le \int_\Omega f(x,Du)\,dx.$$

On the other hand, by the semicontinuity of $\int_\Omega f(x,Du)\,dx$ with respect to the weak topology of $W^{1,p}$, the inverse inequality also holds.

Lemma 3.1. *For each $v \in W_0^{1,p}(\Omega) + u_0$, there exists a sequence $v_k \in W_0^{1,q}(\Omega) + u_0$ such that $v_k \rightharpoonup v$ weakly in $W^{1,p}(\Omega)$ and*

$$F(v) = \lim_{k\to+\infty}\int_\Omega f(x,Dv_k)\,dx.$$

Proof. The proof follows similarly as in [5]. We give the sketch of the proof.

Let $v \in W_0^{1,p}(\Omega) + u_0$ such that $F(v) < \infty$. Then, for all k, there exists $v_h^{(k)} \in W_0^{1,q}(\Omega) + u_0$ such that $v_h^{(k)} \overset{w}{\rightharpoonup} v$, as $h \to +\infty$, weakly in $W^{1,p}(\Omega)$ and

$$F(v) \le \lim_{h\to+\infty} \int_\Omega f(x, Dv_h^{(k)})\,dx \le F(v) + \frac{1}{k}.$$

Moreover, by the weak convergence of $v_h^{(k)}$ in $W^{1,p}(\Omega)$ we get

$$\lim_{h\to+\infty} \|v_h^{(k)} - v\|_{L^p(\Omega)} = 0$$

and for h sufficiently large,

$$\int_\Omega |Dv_h^{(k)}|^p\,dx \le \int_\Omega f(x, Dv_h^{(k)})\,dx \le F(v) + 1.$$

Then for all k there exists h_k such that for all $h \ge h_k$,

$$\|v_h^{(k)} - v\|_{L^p(\Omega)} < \frac{1}{k}$$

and for $h = h_k$, by denoting $w_k = v_{h_k}^{(k)}$, we have

$$\|w_k - v\|_{L^p(\Omega)} < \frac{1}{k} \quad \text{and} \quad \int_\Omega |Dw_k|^p\,dx \le C;$$

then $w_k \overset{w}{\rightharpoonup} v$ as $k \to +\infty$ in the weak topology of $W^{1,p}(\Omega)$ and

$$F(v) \le \int_\Omega f(x, Dw_k)\,dx \le F(v) + \frac{1}{k},$$

i.e.

$$\lim_{k\to+\infty} \int_\Omega f(x, Dw_k)\,dx = F(v).$$

□

4 Existence and regularity

First of all we prove an approximation theorem for f through a suitable sequence of regular functions.

Proposition 4.1. *Let f be satisfying the growth conditions* (3.1), *$f_{\xi\xi}$ and $f_{\xi x}$ Carathéodory functions, satisfying* (1.7) *and* (1.14) *with $M_0 = 1$ and f strictly convex at infinity. Then there exists a sequence of $\mathcal{C}^2$-functions $f^{\ell k} : \Omega \times \mathbb{R}^n \to [0, +\infty)$, $f^{\ell k}$ convex in the last variable and strictly convex at infinity, such that $f^{\ell k}$ converges to f as $\ell \to \infty$ and $k \to \infty$ for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^n$ and uniformly in $\Omega_0 \times K$, where $\Omega_0 \subset\subset \Omega$ and K being a compact set of $\mathbb{R}^n$. Moreover:*

- *there exists $\tilde{C}$, independently of k, ℓ, such that*

$$|\xi|^p \le f^{\ell k}(x,\xi) \le \tilde{C}(1 + |\xi|^q) \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n, \tag{4.1}$$

- *there exists $\tilde{M}_1 > 0$ such that for $|\xi| > 2$ and a.e. $x \in \Omega$,*

$$\tilde{M}_1 |\xi|^{p-2} |\lambda|^2 \le \sum_{i,j} f^{\ell k}_{\xi_i \xi_j}(x,\xi)\lambda_i\lambda_j, \quad \lambda \in \mathbb{R}^n, \tag{4.2}$$

- *there exists $c(k) > 0$ such that for all $(x,\xi) \in \Omega \times \mathbb{R}^n$ and $\lambda \in \mathbb{R}^n$,*

$$c(k)(1 + |\xi|^2)^{\frac{q-2}{2}} |\lambda|^2 \le \sum_{i,j} f^{\ell k}_{\xi_i \xi_j}(x,\xi)\lambda_i\lambda_j, \tag{4.3}$$

- *there exists $\tilde{M}_2 > 0$ such that for $|\xi| > 2$ and a.e. $x \in \Omega$,*

$$|f^{\ell k}_{\xi\xi}(x,\xi)| \le \tilde{M}|\xi|^{q-2}, \tag{4.4}$$

- *there exists $C(k)$ such that for a.e. $x \in \Omega$ and $\xi \in \mathbb{R}^n$,*

$$|f^{\ell k}_{\xi\xi}(x,\xi)| \le C(k)(1+|\xi|^2)^{\frac{q-2}{2}}, \tag{4.5}$$

- *there exists a constant $C > 0$ such that for a.e. $x \in \Omega$ and $|\xi| > 2$,*

$$|f^{\ell k}_{\xi x}(x,\xi)| \le C h_\ell(x)|\xi|^{q-1}, \tag{4.6}$$

  where $h_\ell \in \mathcal{C}^\infty(\Omega)$ is the regularized function of h which converges to h in $L^r(\Omega)$,
- *for $\Omega_0 \subset\subset \Omega$, there exists a constant C such that for $x \in \Omega_0$ and $\xi \in \mathbb{R}^n$,*

$$|f^{\ell k}_{\xi x}(x,\xi)| \le C(k,\ell,\Omega_0)(1+|\xi|^2)^{\frac{q-1}{2}}. \tag{4.7}$$

Proof. We argue as in the proof of [25, Theorem 2.7 (Step 3)] and [20, Lemma 4.3]. For the sake of completeness, we give a sketch of the arguments of the proof.

Let B be the unit ball of $\mathbb{R}^n$ centered in the origin and consider a positive decreasing sequence $\varepsilon_\ell \to 0$. We introduce

$$f^\ell(x,\xi) = \int_{B\times B} \rho(y)\rho(\eta)f(x+\varepsilon_\ell y, \xi+\varepsilon_\ell \eta)\,d\eta\,dy,$$

where ρ is a positive symmetric mollifier, and

$$f^{\ell k}(x,\xi) = f^\ell(x,\xi) + \frac{1}{k}(1+|\xi|^2)^{\frac{q}{2}}. \tag{4.8}$$

It is easy to check that the sequence $f^{\ell k}$ satisfies conditions (4.1), (4.2), (4.3), (4.4), (4.5), (4.7). Let us verify (4.6). For $|\xi| > 2$ we have

$$|f^{\ell k}_{\xi x}(x,\xi)| \le \int_{B\times B} \rho(y)\rho(\eta)|\xi+\varepsilon_\ell\eta|^{q-1}h(x+\varepsilon_\ell y)\,dy\,d\eta \le C h_\ell(x)|\xi|^{q-1},$$

where

$$h_\ell(x) = \int_B \rho(y)h(x+\varepsilon_\ell y)\,dy,$$

h_ℓ is a smooth function and it converges to h in $L^r(\Omega)$. Moreover,

$$|f^{\ell k}_{\xi x}(x,\xi)| \le C(k,\Omega_0)[\|1+h_\ell\|_{L^\infty(\Omega_0)}](1+|\xi|^2)^{\frac{q-1}{2}}.$$

This concludes the proof. □

Proof of Theorem 1.2. For $u_0 \in W^{1,q}(\Omega)$, let us consider the variational problems

$$\inf\left\{\int_\Omega f^{\ell k}(x,Dv)\,dx : v \in W^{1,q}_0(\Omega)+u_0\right\}, \tag{4.9}$$

where $f^{\ell k}$ are defined in (4.8). By semicontinuity arguments, there exists $v^{\ell k} \in u_0 + W^{1,q}_0(\Omega)$, a solution to (4.9). By the growth conditions and the minimality of $v^{\ell k}$, we get

$$\begin{aligned}\int_\Omega |Dv^{\ell k}|^p\,dx &\le \int_\Omega f^{\ell k}(x,Dv^{\ell k})\,dx\\ &\le \int_\Omega f^{\ell k}(x,Du_0)\,dx\\ &= \int_\Omega f^\ell(x,Du_0)\,dx + \frac{1}{k}\int_\Omega (1+|Du_0|^2)^{\frac{q}{2}}\,dx.\end{aligned}$$

Moreover, the properties of the convolutions imply that

$$f^\ell(x, Du_0) \xrightarrow{\ell\to\infty} f(x, Du_0) \quad \text{a.e. in } \Omega,$$

and since

$$\int_\Omega f^\ell(x, Du_0)\,dx \le C\int_\Omega (1+|Du_0|^2)^{\frac{q}{2}}\,dx,$$

by the Lebesgue Dominated Convergence Theorem we deduce therefore

$$\begin{aligned}\lim_{\ell\to\infty}\int_\Omega |Dv^{\ell k}|^p\,dx &\le \lim_{\ell\to\infty}\int_\Omega f^\ell(x, Du_0)\,dx + \frac{1}{k}\int_\Omega (1+|Du_0|^2)^{\frac{q}{2}}\,dx\\ &= \int_\Omega f(x, Du_0)\,dx + \frac{1}{k}\int_\Omega (1+|Du_0|^2)^{\frac{q}{2}}\,dx.\end{aligned}$$

By Proposition 4.1, the functions $f^{\ell k}$ satisfy (1.7), (1.14) and (2.8), so we can apply the a-priori estimate (2.41) to $v^{\ell k}$ and obtain, by standard covering arguments for all $\Omega' \subset\subset \Omega$,

$$\|Dv^{\ell k}\|_{L^\infty(\Omega';\mathbb{R}^n)} \le C(\Omega')[\|1+h_\ell\|_{L^r(\Omega)}]^{\hat\beta\hat\gamma}\Big[\int_\Omega (1+f^{\ell k}(x, Dv^{\ell k}))\,dx\Big]^{\frac{\hat\gamma}{p}}.$$

Since $\|1+h_\ell\|_{L^r(\Omega)} = \|(1+h)_\ell\|_{L^r(\Omega)} \le \|1+h\|_{L^r(\Omega)}$, we obtain

$$\begin{aligned}\|Dv^{\ell k}\|_{L^\infty(\Omega';\mathbb{R}^n)} &\le C(\Omega')[\|1+h\|_{L^r(\Omega)}]^{\hat\beta\hat\gamma}\Big[\int_\Omega (1+f^{\ell k}(x, Dv^{\ell k}))\,dx\Big]^{\frac{\hat\gamma}{p}}\\ &\le C(\Omega')[\|1+h\|_{L^r(\Omega)}]^{\hat\beta\hat\gamma}\Big[\int_\Omega 1+f^\ell(x, Du_0)+\frac{1}{k}(1+|Du_0|^2)^{\frac{q}{2}}\,dx\Big]^{\frac{\hat\gamma}{p}},\end{aligned}$$

where $C, \hat\gamma, \hat\beta$ depend on $n, r, p, q, M_1, M_2, \rho, R$ but are independent of ℓ, k. Therefore we conclude that

$$\begin{aligned}v^{\ell k} &\xrightarrow{\ell\to\infty} v^k \quad \text{weakly in } W_0^{1,p}(\Omega)+u_0,\\ v^{\ell k} &\xrightarrow{\ell\to\infty} v^k \quad \text{weakly star in } W^{1,\infty}_{\mathrm{loc}}(\Omega),\end{aligned}$$

and by the previous estimates

$$\begin{aligned}\|Dv^k\|_{L^p(\Omega;\mathbb{R}^n)} &\le \liminf_{\ell\to\infty}\|Dv^{\ell k}\|_{L^p(\Omega;\mathbb{R}^n)}\\ &\le \int_\Omega f(x, Du_0)\,dx + \int_\Omega (1+|Du_0|^2)^{\frac{q}{2}}\,dx\end{aligned}$$

and

$$\begin{aligned}\|Dv^k\|_{L^\infty(\Omega';\mathbb{R}^n)} &\le \liminf_{\ell\to\infty}\|Dv^{\ell k}\|_{L^\infty(\Omega';\mathbb{R}^n)}\\ &\le C(\Omega')[\|1+h\|_{L^r(\Omega)}]^{\hat\beta\hat\gamma}\Big[\int_\Omega 1+f(x, Du_0)\,dx + \int_\Omega (1+|Du_0|^2)^{\frac{q}{2}}\,dx\Big]^{\frac{\hat\gamma}{p}}.\end{aligned}$$

Thus we can deduce that there exists, up to subsequences, $\bar u \in u_0 + W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}v^k &\to \bar u \quad \text{weakly in } W_0^{1,p}(\Omega)+u_0,\\ v^k &\to \bar u \quad \text{weakly star in } W^{1,\infty}_{\mathrm{loc}}(\Omega).\end{aligned}$$

Now, for any fixed $k \in \mathbb{N}$, using the uniform convergence of f^ℓ to f in $\Omega_0 \times K$ (for any K compact subset of $\mathbb{R}^n$)

and the minimality of $v^{\ell k}$, we get for all $v \in W_0^{1,q}(\Omega) + u_0$,

$$\begin{aligned}\int_{\Omega_0} f(x, Dv^k)\,dx &\le \liminf_{\ell\to\infty} \int_{\Omega_0} f(x, Dv^{\ell k})\,dx\\ &= \liminf_{\ell\to\infty} \int_{\Omega_0} f^\ell(x, Dv^{\ell k})\,dx\\ &\le \liminf_{\ell\to\infty} \int_{\Omega_0} f^\ell(x, Dv^{\ell k})\,dx + \frac{1}{k}\int_{\Omega} (1 + |Dv^{\ell k}|^2)^{\frac{q}{2}}\,dx\\ &\le \liminf_{\ell\to\infty} \int_{\Omega} f^\ell(x, Dv^{\ell k})\,dx + \frac{1}{k}\int_{\Omega} (1 + |Dv^{\ell k}|^2)^{\frac{q}{2}}\,dx\\ &\le \liminf_{\ell\to\infty} \int_{\Omega} f^\ell(x, Dv)\,dx + \frac{1}{k}\int_{\Omega} (1 + |Dv|^2)^{\frac{q}{2}}\,dx.\end{aligned}$$

Then, for $\Omega_0 \to \Omega$,

$$\int_{\Omega} f(x, Dv^k)\,dx \le \int_{\Omega} f(x, Dv)\,dx + \frac{1}{k}\int_{\Omega} (1 + |Dv|^2)^{\frac{q}{2}}\,dx.$$

By definition (3.2), we have

$$F(\bar u) \le \liminf_{k\to\infty} \int_{\Omega} f(x, Dv^k)\,dx \le \int_{\Omega} f(x, Dv)\,dx \quad \text{for all } v \in W_0^{1,q}(\Omega) + u_0. \tag{4.10}$$

Let $w \in W_0^{1,p}(\Omega) + u_0$. By Lemma 3.1, there exists $v_k \in W_0^{1,q}(\Omega) + u_0$ such that $v_k \rightharpoonup w$ weakly in $W^{1,p}(\Omega)$ and

$$\lim_{k\to\infty} \int_{\Omega} f(x, Dv_k)\,dx = F(w).$$

By (4.10),

$$F(\bar u) \le \int_{\Omega} f(x, Dv_k)\,dx,$$

and we can conclude that

$$F(\bar u) \le \lim_{k\to\infty} \int_{\Omega} f(x, Dv_k)\,dx = F(w) \quad \text{for all } w \in W_0^{1,p}(\Omega) + u_0.$$

Then $\bar u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is a solution to the problem $\min\{F(u) : u \in W_0^{1,p}(\Omega) + u_0\}$. □

5 Regularity of local minimizers in a special case

Let us consider now the case of a special form of integrand

$$f(x, \xi) = \sum_{i=1}^{N} a_i(x) g_i(\xi) \tag{5.1}$$

with $a_i(x) > 0$ a.e. in Ω, $a_i \in W^{1,r}(\Omega)$, $r > n$, $g_i : \mathbb{R}^n \to [0, +\infty)$ convex in ξ and strictly convex for ξ such that $|\xi| \ge M_0$. The following regularity result holds.

Theorem 5.1. *Assume that $f = f(x, \xi)$ as in (5.1) satisfies the assumptions of Theorem 2.6. Then every local minimizer $u \in W^{1,p}(\Omega)$ of the integral functional*

$$\int_{\Omega} f(x, Dv)\,dx = \int_{\Omega} \sum_{i=1}^{N} a_i(x) g_i(Du)\,dx \tag{5.2}$$

is locally Lipschitz continuous in Ω.

Proof. Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the integral functional (5.2). For a suitable φ_σ mollifier, consider $u_\sigma = u * \varphi_\sigma \in W^{1,q}_{\mathrm{loc}}(\Omega)$. Consider the following sequence of problems in $B_R \subset\subset \Omega$:

$$\inf\left\{ \int_{B_R} f^{\ell k}(x, Dv)\, dx : v \in W^{1,q}_0(B_R) + u_\sigma \right\}, \tag{5.3}$$

where $f^{\ell k}$ are defined in Proposition 4.1.

For fixed σ, ℓ, k, problem (5.3) has a unique solution $v^{\ell k}_\sigma \in W^{1,q}_0(B_R) + u_\sigma$. By proceeding as in the previous theorem, we have that for each fixed σ, by the minimality of $v^{\ell k}_\sigma$,

$$\begin{aligned} v^{\ell k}_\sigma &\xrightarrow{\ell\to\infty} v^k_\sigma \quad \text{weakly in } W^{1,p}_0(B_R) + u_\sigma,\\ v^{\ell k}_\sigma &\xrightarrow{\ell\to\infty} v^k_\sigma \quad \text{weakly star in } W^{1,\infty}_{\mathrm{loc}}(B_R). \end{aligned}$$

We also have

$$\begin{aligned} v^k_\sigma &\xrightarrow{k\to\infty} v_\sigma \quad \text{weakly in } W^{1,p}_0(B_R) + u_\sigma,\\ v^k_\sigma &\xrightarrow{k\to\infty} v_\sigma \quad \text{weakly star in } W^{1,\infty}_{\mathrm{loc}}(B_R) \end{aligned}$$

and

$$\begin{aligned} \|Dv_\sigma\|_{L^\infty(B_\rho;\mathbb{R}^n)} &\le C \liminf_{k\to\infty} \left[1 + \int_{B_R} f(x, Du_\sigma)\, dx + \frac{1}{k} \int_{B_R} (1 + |Du_\sigma|^2)^{\frac{q}{2}}\, dx \right]^{\frac{\hat{\gamma}}{p}}\\ &= C \liminf_{k\to\infty} \left[1 + \int_{B_R} f(x, Du_\sigma)\, dx \right]^{\frac{\hat{\gamma}}{p}} \end{aligned} \tag{5.4}$$

for any $0 < \rho < R$ and where C is independent of k, σ. For fixed k, by proceeding as in the previous theorem we have

$$\int_{B_R} f(x, Dv^k_\sigma)\, dx \le \int_{B_R} f(x, Du_\sigma)\, dx + \frac{1}{k} \int_{B_R} (1 + |Du_\sigma|^2)^{\frac{q}{2}}\, dx.$$

Then, by semicontinuity,

$$\begin{aligned} \int_{B_R} f(x, Dv_\sigma)\, dx &\le \liminf_{k\to\infty} \int_{B_R} f(x, Du_\sigma)\, dx + \frac{1}{k} \int_{B_R} (1 + |Du_\sigma|^2)^{\frac{q}{2}}\, dx\\ &\le \int_{B_R} f(x, Du_\sigma)\, dx. \end{aligned} \tag{5.5}$$

Now we claim that, by the particular form of f, we may deduce

$$\liminf_{\sigma\to 0} \int_{B_R} f(x, Du_\sigma)\, dx \le \int_{B_R} f(x, Du)\, dx. \tag{5.6}$$

Since g_i is convex, for $i = 1, \dots, N$, Jensen's inequality (applied to each g_i) yields

$$\begin{aligned} \int_{B_R} a_i(x) g_i(Du_\sigma)\, dx &= \int_{B_R} a_i(x) g_i\left(\int_{B_\sigma} Du(y)\varphi_\sigma(x-y)\, dy \right) dx\\ &\le \int_{B_R} a_i(x) \int_{B_\sigma} g_i(Du(y))\varphi_\sigma(x-y)\, dy\, dx\\ &= \int_{B_R}\int_{B_\sigma} a_i(x)\varphi_\sigma(x-y)\, dy\, g_i(Du(y))\, dx\\ &\le \int_{B_{R+\sigma}} (a_i)_\sigma(y) g_i(Du(y))\, dy. \end{aligned}$$

Then

$$\sum_{i=1}^{N} \int_{B_R} a_i(x) g_i(Du_\sigma)\,dx \leq \sum_{i=1}^{N} \int_{B_{R+\sigma}} (a_i)_\sigma(x) g_i(Du)\,dx$$

so that passing to the limit as $\sigma \to 0$,

$$\liminf_{\sigma\to 0} \sum_{i=1}^{N} \int_{B_R} a_i(x) g_i(Du_\sigma)\,dx \leq \sum_{i=1}^{N} \int_{B_R} a_i(x) g_i(Du)\,dx$$

because $(a_i)_\sigma \to a_i$ in $L^\infty(B_R)$, $g_i(Du) \in L^1(B_R)$, the Dominated Convergence Theorem may be applied, and (5.6) holds.

By collecting (5.5) and (5.6),

$$\liminf_{\sigma\to 0} \int_{B_R} f(x, Dv_\sigma)\,dx \leq \int_{B_R} f(x, Du)\,dx. \tag{5.7}$$

On the other hand, the growth assumption on f yields, since u is a local minimizer of (5.2),

$$\liminf_{\sigma\to 0} \int_{B_R} |Dv_\sigma|^p\,dx \leq \liminf_{\sigma\to 0} \int_{B_R} f(x, Dv_\sigma)\,dx \overset{(5.7)}{\leq} \int_{B_R} f(x, Du)\,dx < +\infty.$$

Thus there exists $\bar v \in u + W_0^{1,p}(B_R)$ such that, up to a subsequence,

$$v_\sigma \rightharpoonup \bar v \quad \text{weakly in } W^{1,p}(B_R).$$

By the semicontinuity of the functional, using (5.5) and (5.7),

$$\int_{B_R} f(x, D\bar v)\,dx \leq \liminf_{\sigma\to 0} \int_{B_R} f(x, Dv_\sigma)\,dx \leq \int_{B_R} f(x, Du)\,dx. \tag{5.8}$$

Moreover, since (5.4) holds, Dv_σ converges to $D\bar v$ as $\sigma \to 0$ in the weak star topology of L^∞ and there exists a constant C such that, for any $0 < \rho < R$,

$$\|D\bar v\|_{L^\infty(B_\rho;\mathbb{R}^n)} \leq C\left[1 + \int_{B_R} f(x, Du)\,dx\right]^{\frac{\hat\gamma}{p}}.$$

Consider the following problem in $B_R \subset\subset \Omega$:

$$\inf\left\{\int_{B_R} f(x, Dv)\,dx : v \in W_0^{1,p}(B_R) + u\right\}. \tag{5.9}$$

Then (5.8) implies that $\bar v$ and u are solutions to (5.9) and $\bar v \in W^{1,\infty}_{\mathrm{loc}}(B_R)$.

In the present case the functional is not strictly convex; we proceed as in [20, Theorem 2.1] (see also [25]) and we have that $u \in W^{1,\infty}_{\mathrm{loc}}(B_R)$. Indeed, set

$$E_0 := \left\{x \in B_R : \left|\frac{Du(x) + D\bar v(x)}{2}\right| > M_0,\ Du(x) \neq D\bar v(x)\right\} \quad \text{and} \quad w := \frac{u + \bar v}{2}.$$

If E_0 has positive measure, then from the convexity of $f(x,\cdot)$ we have

$$\int_{B_R\setminus E_0} f(x, Dw)\,dx \leq \frac{1}{2}\int_{B_R\setminus E_0} f(x, Du)\,dx + \frac{1}{2}\int_{B_R\setminus E_0} f(x, D\bar v)\,dx. \tag{5.10}$$

Now, by the strict convexity of $f(x,\xi)$ for ξ such that $|\xi| \geq M_0$ and applying two times the inequality

$$f(x,\eta) > f(x,\xi) + \langle f_\xi(x,\xi), \eta - \xi\rangle \quad \text{for } \xi \text{ such that } |\xi| \geq M_0$$

first with $\xi = Dw$ and $\eta = Du$, then for $\xi = Dw$ and $\eta = D\bar{v}$, finally by adding up the two inequalities obtained, we have

$$\int_{B_R \cap E_0} f(x, Dw)\,dx < \frac{1}{2} \int_{B_R \cap E_0} f(x, Du)\,dx + \frac{1}{2} \int_{B_R \cap E_0} f(x, Dv)\,dx. \tag{5.11}$$

Adding (5.10) and (5.11), we get a contradiction with the minimality of u and $\bar{v}$. Therefore the set E_0 has zero measure, which implies that

$$\sup_{B_\rho} |Du(x)| \le \sup_{B_\rho} |Du(x) + D\bar{v}(x)| + \sup_{B_\rho} |D\bar{v}(x)| \le 2M_0 + \sup_{B_\rho} |D\bar{v}(x)|$$

and this yields the thesis. □

Acknowledgment: The authors wish to express their gratitude to the referee for carefully reading the manuscript providing useful comments and remarks. In particular, the referee pointed out a technical problem which has been modified correctly, which however did not influenced the method and the details in the proof of the a-priori estimates.

Funding: The authors are members of GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica).

References

[1] A. L. Baisón, A. Clop, R. Giova, J. Orobitg and A. Passarelli di Napoli, Fractional differentiability for solutions of nonlinear elliptic equations, *Potential Anal.* **46** (2017), no. 3, 403–430.
[2] P. Baroni, M. Colombo and G. Mingione, Harnack inequalities for double phase functionals, *Nonlinear Anal.* **121** (2015), 206–222.
[3] P. Baroni, M. Colombo and G. Mingione, Nonautonomous functionals, borderline cases and related function classes, *Algebra i Analiz* **27** (2015), no. 3, 6–50.
[4] P. Baroni, M. Colombo and G. Mingione, Regularity for general functionals with double phase, preprint (2017), https://arxiv.org/abs/1708.09147.
[5] L. Boccardo and P. Marcellini, Sulla convergenza delle soluzioni di disequazioni variazionali, *Ann. Mat. Pura Appl. (4)* **110** (1976), 137–159.
[6] V. Bögelein, F. Duzaar and P. Marcellini, Parabolic systems with p, q-growth: A variational approach, *Arch. Ration. Mech. Anal.* **210** (2013), no. 1, 219–267.
[7] V. Bögelein, F. Duzaar and P. Marcellini, Existence of evolutionary variational solutions via the calculus of variations, *J. Differential Equations* **256** (2014), no. 12, 3912–3942.
[8] V. Bögelein, F. Duzaar and P. Marcellini, A time dependent variational approach to image restoration, *SIAM J. Imaging Sci.* **8** (2015), no. 2, 968–1006.
[9] G. Buttazzo and M. Belloni, A survey on old and recent results about the gap phenomenon in the calculus of variations, in: *Recent Developments in Well-Posed Variational Problems*, Math. Appl. 331, Kluwer Academic Publisher, Dordrecht (1995), 1–27.
[10] M. Chipot and L. C. Evans, Linearisation at infinity and Lipschitz estimates for certain problems in the calculus of variations, *Proc. Roy. Soc. Edinburgh Sect. A* **102** (1986), no. 3–4, 291–303.
[11] M. Colombo and G. Mingione, Bounded minimisers of double phase variational integrals, *Arch. Ration. Mech. Anal.* **218** (2015), no. 1, 219–273.
[12] M. Colombo and G. Mingione, Regularity for double phase variational problems, *Arch. Ration. Mech. Anal.* **215** (2015), no. 2, 443–496.
[13] G. Cupini, F. Giannetti, R. Giova and A. Passarelli di Napoli, Higher integrability for minimizers of asymptotically convex integrals with discontinuous coefficients, *Nonlinear Anal.* **154** (2017), 7–24.
[14] G. Cupini, M. Guidorzi and E. Mascolo, Regularity of minimizers of vectorial integrals with p-q growth, *Nonlinear Anal.* **54** (2003), no. 4, 591–616.
[15] G. Cupini, P. Marcellini and E. Mascolo, Existence and regularity for elliptic equations under p, q-growth, *Adv. Differential Equations* **19** (2014), no. 7–8, 693–724.
[16] E. De Giorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)* **3** (1957), 25–43.

[17] E. De Giorgi and T. Franzoni, Su un tipo di convergenza variazionale, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **58** (1975), no. 6, 842–850.
[18] E. DiBenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.* **7** (1983), no. 8, 827–850.
[19] M. Eleuteri, P. Marcellini and E. Mascolo, Lipschitz continuity for energy integrals with variable exponents, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **27** (2016), no. 1, 61–87.
[20] M. Eleuteri, P. Marcellini and E. Mascolo, Lipschitz estimates for systems with ellipticity conditions at infinity, *Ann. Mat. Pura Appl. (4)* **195** (2016), no. 5, 1575–1603.
[21] M. Eleuteri and A. Passarelli di Napoli, Higher differentiability for solutions to a class of obstacle problems, preprint (2017).
[22] A. Esposito, F. Leonetti and P. V. Petricca, Absence of Lavrentiev gap for non-autonomous functionals with (p,q)-growth, *Adv. Nonlinear Anal.* (2017), DOI 10.1515/anona-2016-0198.
[23] L. Esposito, F. Leonetti and G. Mingione, Regularity results for minimizers of irregular integrals with (p,q) growth, *Forum Math.* **14** (2002), no. 2, 245–272.
[24] L. Esposito, F. Leonetti and G. Mingione, Sharp regularity for functionals with (p,q) growth, *J. Differential Equations* **204** (2004), no. 1, 5–55.
[25] I. Fonseca, N. Fusco and P. Marcellini, An existence result for a nonconvex variational problem via regularity, *ESAIM Control Optim. Calc. Var.* **7** (2002), 69–95.
[26] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Ann. of Math. Stud. 105, Princeton University Press, Princeton, 1983.
[27] M. Giaquinta, Growth conditions and regularity, a counterexample, *Manuscripta Math.* **59** (1987), no. 2, 245–248.
[28] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific Publishing, River Edge, 2003.
[29] A. D. Ioffe and V. M. Tihomirov, Extension of variational problems (in Russian), *Trudy Moskov. Mat. Obšč.* **18** (1968), 187–246.
[30] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.
[31] P. Marcellini, Un example de solution discontinue d'un problème variationnel dans le cas scalaire, Preprint 11, Istituto Matematico "U.Dini", Università di Firenze, 1987.
[32] P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions, *Arch. Ration. Mech. Anal.* **105** (1989), no. 3, 267–284.
[33] P. Marcellini, Regularity and existence of solutions of elliptic equations with p,q-growth conditions, *J. Differential Equations* **90** (1991), no. 1, 1–30.
[34] P. Marcellini, Regularity for elliptic equations with general growth conditions, *J. Differential Equations* **105** (1993), no. 2, 296–333.
[35] P. Marcellini, Regularity for some scalar variational problems under general growth conditions, *J. Optim. Theory Appl.* **90** (1996), no. 1, 161–181.
[36] G. Mingione, Regularity of minima: an invitation to the dark side of the calculus of variations, *Appl. Math.* **51** (2006), no. 4, 355–426.
[37] A. Passarelli di Napoli, Higher differentiability of minimizers of variational integrals with Sobolev coefficients, *Adv. Calc. Var.* **7** (2014), no. 1, 59–89.
[38] J. Serrin, A new definition of the integral for nonparametric problems in the calculus of variations, *Acta Math.* **102** (1959), 23–32.
[39] V. V. Zhikov, On Lavrentiev's phenomenon, *Russian J. Math. Phys.* **3** (1995), no. 2, 249–269.