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REGULARITY UNDER SHARP ANISOTROPIC GENERAL GROWTH CONDITIONS

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Abstract. We prove boundedness of minimizers of energy-functionals, for instance of the anisotropic type (1) below, under *sharp* assumptions on the exponents p_i in terms of $\overline{p}^*$: the *Sobolev conjugate exponent* of $\overline{p}$; i.e., $\overline{p}^* = \frac{n\overline{p}}{n-\overline{p}}$, $\frac{1}{\overline{p}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}$. As a consequence, by mean of regularity results due to Lieberman [21], we obtain the local Lipschitz-continuity of minimizers under sharp assumptions on the exponents of anisotropic growth.

1. **Introduction.**

Integrals of the calculus of variations of the form

$$\mathcal{F}(u) = \int_\Omega \sum_{i=1}^{n} |u_{x_i}(x)|^{p_i(x)}\, dx \tag{1}$$

for some bounded measurable functions $p_i(x)$ may have *not smooth*, even *unbounded*, minimizers. This happens also in the case of *constant* exponents p_i, $i = 1, \dots, n$, if they are spread out; i.e., if the ratio $\max\{p_i\}/\min\{p_i\}$ is not close enough to 1 in dependence on n. In fact integrals as in (1), with constant exponents p_i, may have unbounded minimizers ([18], [22], [23], see also [19]) for instance when $n > 3$ and

$$p_1 = \ldots = p_{n-1} = 2, \qquad p_n = q > \frac{2\,(n-1)}{n-3}\,. \tag{2}$$

However a large literature already exists on regularity of solutions under suitable assumptions on the exponents when these exponents are not spread out; see the end of this section for details.

Similar regularity questions can be posed for other integral-functionals, for instance of the form

$$\int_\Omega \left\{ |Du|^p \log(1+|Du|) + |u_{x_n}|^q \right\}\, dx \tag{3}$$

for some exponents p, q $(1 \le p < q)$, or

$$\int_\Omega \left\{ [g(|Du|)]^p + [g(|u_{x_n}|)]^q \right\}\, dx\,, \tag{4}$$

where $g = g(t)$ is a convex function satisfying the so-called Δ_2-*condition*, namely there exists $\mu > 1$ such that $g(\lambda t) \le \lambda^\mu g(t)$ for every $\lambda > 1$ and for every t

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sufficiently large (see Section 2). An example of such a function, with $a, b-$growth, is

$$g(t) = t^{[a+b+(b-a)\sin\log\log(e+t)]/2}.$$

The regularity results known in the literature seem not applicable to the integrals (3), (4) under *sharp* assumptions on the exponent p and q, as stated below.

Recently Lieberman [21] proved that integrals of the calculus of variations as in (1) may have Lipschitz continuous local minimizers u, independently of any condition on the $\{p_i\}$, if we assume *a priori* that u itself is bounded. This fact motivates the research proposed in this article.

To this aim and for the sake of exposition we deal again with integrals as in (1) and we consider exponents p_i, $i = 1, \ldots, n$, and q greater than or equal to 1, such that

$$\begin{cases} p_i \leq p_i(x), & \text{a.e. } x \in B_r \\ q \geq p_i(x), & \text{a.e. } x \in B_r, \quad 1 \leq i \leq n, \end{cases} \tag{5}$$

where B_r is a ball of radius $r > 0$ contained in Ω. Then let $\overline{p}$ be the *harmonic average* of the $\{p_i\}$; i.e.,

$$\frac{1}{\overline{p}} := \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}$$

and let $\overline{p}^*$ be the *Sobolev conjugate exponent* of $\overline{p}$; i.e., $\overline{p}^* = \frac{n\overline{p}}{n-\overline{p}}$ if $\overline{p} < n$, while $\overline{p}^*$ is any fixed real number greater than $\overline{p}$, if $\overline{p} \geq n$. The following regularity result holds.

Theorem 1.1. *Let u be a local minimizer of (1) and let $q < \overline{p}^*$. Then u is locally bounded in Ω and the following estimate holds*

$$\|u - u_r\|_{L^\infty(B_{r/(2\sqrt{n})}(x_0))} \leq c\left\{1 + \int_{B_r(x_0)}\sum_{i=1}^{n}|u_{x_i}(x)|^{p_i(x)}\,dx\right\}^{\frac{1+\theta}{p}},$$

for some constant $c > 0$, where $u_r = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u\,dx$, $p = \min\limits_{1\leq i\leq n}\{p_i\}$ and $\theta = \frac{\overline{p}^*(q-p)}{p(\overline{p}^*-q)}$.

Observe that if $p_1 = \ldots = p_{n-1} = 2$ and $p_n = q \geq 2$ then the assumption $q < \overline{p}^*$ gives $q < 2(n-1)/(n-3)$; this inequality is exactly the opposite of condition (2), apart from the equality which is not achieved, since the borderline case $q = \overline{p}^*$ is not included in Theorem 1.1. Thus, our regularity result is essentially sharp.

As a consequence of the previous theorem and of the quoted result by Lieberman [21] we get the following gradient estimate under a sharp assumption on the exponents of the anisotropic growth.

Corollary 1. *Let u be a local minimizer of the integral $\mathcal{F}$ in (1) with exponents $p_i(x)$, for $i = 1, \ldots, n$, locally Lipschitz continuous in Ω. Let $\overline{p}(x)$ be the harmonic average of the $\{p_i(x)\}$ and let $\overline{p}^*(x)$ be the Sobolev conjugate exponent of $\overline{p}(x)$. If $\overline{p}^*(x_0) > p_i(x_0)$ for some $x_0 \in \Omega$ and for every $i = 1, \ldots, n$, then u is Lipschitz continuous in a neighborhood of x_0.*

We emphasize that in fact in this paper we consider integrals more general than (1), (3) and (4). Precisely, we are able to consider general integrals with non-homogeneous densities of the form

$$\int_\Omega f(x, |u_{x_1}|, \dots, |u_{x_n}|)\, dx$$

with f satisfying some non-standard p_i-q growth conditions; precise assumptions and statements are in Section 2. We observe explicitly that, in the case of the functional in (4), the assumptions involve the exponents p and q, but they are independent of the function g.

The mathematical literature on the regularity in this context is very rich; energy functionals with anisotropic, non-standard or general growth have been studied by many authors and in different settings of applicability. Among the many related papers we quote, in a not exhaustive way, Marcellini [24], [25], Lieberman [20], Bhattacharya-Leonetti [5], Moscariello-Nania [27], Mascolo-Papi [26], Fan-Zhao [13], [14], Dall'Aglio-Mascolo-Papi [12] and, in the vectorial setting, Acerbi-Mingione [2], Coscia-Mingione [11], Cavaliere-D'Ottavio-Leonetti-Longobardi [8], Canale-D'Ottavio-Leonetti-Longobardi [7]. Specific regularity results addressed to the study of functionals with anisotropic growth under the sharp condition on the exponents $\bar{p}^* > q$, have been first obtained by Boccardo-Marcellini-Sbordone [6], see also a generalization due to Stroffolini [29]. Fusco-Sbordone [16] consider the borderline case $\overline{p}^* = q$ and, later, in [17] they study more general anisotropic integrands $f = f(x, u, Du)$ satisfying a growth of the form

$$\sum_{i=1}^n |u_{x_i}(x)|^{p_i} \le f(x, u, Du) \le \left(c + \sum_{i=1}^n |u_{x_i}(x)|^{p_i} \right),$$

obtaining a boundedness result by mean of De Giorgi's methods. More general functionals are considered in Cianchi [10], in which the study of the boundedness of minimizers is carried out using the optimal Sobolev conjugate of convex functions.

Because of the $p_i - q$ growth, we use a different approach based upon a variant of the classical Moser's iteration method, which has its starting in an inequality of Euler's type, see Theorem 5.1. Moreover, for the anisotropic behavior of the integrand, we base our estimates on an embedding result for anisotropic Sobolev spaces due to Troisi [31] (see also Acerbi-Fusco [1] and Fragalà-Gazzola-Kawhol [15]).

Our paper is organized as follows. In the next section we present the precise statement of our regularity theorem and few more examples of applicability. In Section 3 preliminary properties of convex functions are proved. Section 4 is devoted to higher integrability results for minimizers, Section 5 to the *Euler's inequality* and Section 6 to the proof of Theorem 2.1.

2. **Assumptions and statement of the main results.**

Let us define the integral functional

$$\mathcal{F}(u) := \int_\Omega f(x, Du(x))\, dx, \tag{6}$$

where Ω is an open bounded subset of $\mathbb{R}^n$, $n \ge 2$, and $u \in W^{1,1}(\Omega, \mathbb{R})$. For the sake of simplicity, and with a slight abuse of notation, we assume

$$f = f(x, |u_{x_1}|, \dots, |u_{x_n}|).$$

A more general case is considered in the last section. Denoting $\mathbb{R}^n_+$ the set $[0,+\infty)^n$, we assume

(H1) $f:\Omega\times\mathbb{R}^n_+\to\mathbb{R}$, $f(x,\xi)=f(x,\xi_1,...,\xi_n)$, is a Carathéodory function, convex and of class C^1 with respect to ξ and increasing with respect to each ξ_i,

(H2) there exist $\mu\geq 1$ and $t_0\geq 0$, such that

$$f(x,\lambda\xi)\leq\lambda^\mu f(x,\xi) \tag{7}$$

for every $\lambda>1$ and for a.e. x and every ξ, $|\xi|\geq t_0$.

A growth condition on f is assumed.

(H3) there exist $a>0$ and $1\leq p_i\leq q$, $1\leq i\leq n$, such that

$$\sum_{i=1}^n[g(\xi_i)]^{p_i}\leq f(x,\xi)\leq a\left\{1+\sum_{i=1}^n[g(\xi_i)]^q\right\} \tag{8}$$

for a.e. x and every $\xi\in\mathbb{R}^n_+$. Here $g:\mathbb{R}_+\to\mathbb{R}_+$ is of class C^1, convex, increasing, non-constant, $g(0)=0$ and

$$g(\lambda t)\leq\lambda^\mu g(t)\quad\text{for every}\quad\lambda>1\quad\text{and every}\quad t\geq t_0. \tag{9}$$

Without loss of generality, we assume t_0 large so that $g(t)>0$ and $f(x,\xi)>0$ for all $t>t_0$ and all ξ with $|\xi|\geq t_0$.

We denote $W^{1,\mathcal{F}}(\Omega)$ the space $W^{1,\mathcal{F}}(\Omega)=\{u\in W^{1,1}(\Omega):\mathcal{F}(u)<+\infty\}$ and we write $W^{1,\mathcal{F}}_0(\Omega)$ in place of $W^{1,1}_0(\Omega)\cap W^{1,\mathcal{F}}(\Omega)$. A function $u\in W^{1,1}(\Omega)$ is a local minimizer of (6) if $u\in W^{1,\mathcal{F}}(\Omega)$ and $\mathcal{F}(u)\leq\mathcal{F}(u+\varphi)$, for all $\varphi\in W^{1,\mathcal{F}}(\Omega)$ with $\operatorname{supp}\varphi\Subset\Omega$.

Our aim is to prove the local boundedness of local minimizers of (6). To do this, we need a restriction on the exponents $\{p_i\}$ and q. We will use the following notations: we write p in place of $\min\{p_i\}$ and, as in the introduction, we denote by $\overline{p}$ the harmonic average of $\{p_i\}$, i.e., $\frac{1}{\overline{p}}:=\frac{1}{n}\sum_{i=1}^n\frac{1}{p_i}$ and by $\overline{p}^*$ the Sobolev exponent of $\overline{p}$

$$\overline{p}^*:=\begin{cases}\frac{n\overline{p}}{n-\overline{p}} & \text{if } \overline{p}<n,\\ \text{any } \mu>\overline{p} & \text{if } \overline{p}\geq n.\end{cases} \tag{10}$$

Theorem 2.1. *Assume (H1), (H2) and (H3), and let $q<\overline{p}^*$. Then a local minimizer u of (6) is locally bounded. Moreover, for every $B_r(x_0)\Subset\Omega$ the following estimates hold true:*

(1) there exists $c>0$, depending on the data, such that

$$\|g(|u|)\|_{L^\infty(B_{r/2}(x_0))}\leq c\left\{1+\int_{B_r(x_0)}[g(|u|)]^q\,dx\right\}^{\frac{1+\theta}{q}}, \tag{11}$$

(2) there exists $c>0$, depending on the data, such that

$$\|g(|u-u_r|)\|_{L^\infty(B_{r/(2\sqrt{n})}(x_0))}\leq c\left\{1+\int_{B_r(x_0)}f(x,|u_{x_1}|,\dots,|u_{x_n}|)\,dx\right\}^{\frac{1+\theta}{p}}, \tag{12}$$

where $\theta=\frac{\overline{p}^*(q-p)}{p(\overline{p}^*-q)}$ and $u_r:=\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}u\,dx$.

For the sake of simplicity we wrote the growth condition (8) in place of

$$b\left\{\sum_{i=1}^{n}[g(\xi_i)]^{p_i}\right\}-c\leq f(x,\xi)\leq a\left\{1+\sum_{i=1}^{n}[g(\xi_i)]^{q}\right\},$$

with $b>0$, $c\in\mathbb{R}$. This is not a loss of generality since u is a local minimizer of (6) if and only if u is a local minimizer of the functional having the energy density f replaced by a_1f+a_2, with some constants $a_1>0$ and $a_2\in\mathbb{R}$. Taking this into account, it is not difficult to check that Theorem 2.1 applies to the functionals (1), (3) and (4) in Section 1. For instance, as far as (1) is concerned, we can take p_i and q as in (5), $\mu=q$, $g(t)=t$, $a=n2^{q-1}$, $b=2^{1-q}$, $c=n$.

Moreover Theorem 2.1 applies also to functionals $\mathcal{F}$ with different energy densities. We give below some more examples.

We can consider constants $\gamma>0$ and $\alpha\geq 1$ such that $\alpha\gamma\geq 1$, a measurable function $\beta:\Omega\to[\beta_1,\beta_2]$, with $\beta_1\geq 1$ and $\beta_1\gamma\geq 1$, and for instance the integrand

$$f(x,\xi)=(|\xi|^{\alpha}+|\xi_n|^{\beta(x)})^{\gamma}. \tag{13}$$

In this case $p=p_i:=\gamma\alpha$, if $1\leq i\leq n-1$, $p_n:=\gamma\cdot\max\{\alpha,\beta_1\}$ and $q:=\gamma\cdot\max\{\alpha,\beta_2\}$.

An other example can be exhibit through measurable functions $r_i:\Omega\to[p_i,q]$ and

$$f(x,\xi)=(\sum_{i=1}^{n}|\xi_i|^{r_i(x)})^{\gamma}, \tag{14}$$

with $p:=\min\{p_i\}\geq 1$ satisfying $1\leq\gamma p\leq\gamma q<(\overline{\gamma p})^{*}$. Here, $\overline{\gamma p}$ is the harmonic average of $\{\gamma p_1,\ldots,\gamma p_n\}$.

The previous example can be easily generalized to include integrands of the type

$$f(x,\xi)=F\left(\sum_{i=1}^{n}[h(|\xi_i|)]^{r_i(x)}\right); \tag{15}$$

or, more in general,

$$f(x,\xi)=F\left(\sum_{i=1}^{n}f_i\left(x,|\xi_i|\right)\right). \tag{16}$$

In particular in (16) we consider a convex function $f(x,\xi)$ of class C^1 with respect to ξ, functions $f_i\left(x,|\xi_i|\right)$ increasing with respect to each $|\xi_i|$ and satisfying (7), F increasing and satisfying (9). Finally the following growth condition holds

$$[g(t)]^{p_i}\leq F(f_i\left(x,t\right))\leq a\left\{1+[g(t)]^{q}\right\},$$

with g as in (H3).

3. **Preliminary results.** We begin clarifying the role played by (9).

Lemma 3.1. *Consider $h:\mathbb{R}_+\to\mathbb{R}_+$ of class C^1, convex and increasing, and fix $t_0>0$ and $\mu\geq 1$. The following two properties hold:*

(1) Suppose that for every $\lambda>1$ and $t\geq t_0$ we have

$$h(\lambda t)\leq\lambda^{\mu}h(t) \tag{17}$$

for all $\lambda>1$ and $t\geq t_0$. Then

$$h(\lambda t)\leq\lambda^{\mu}(h(t)+h(t_0))\quad\textit{and}\quad h'(t)t\leq\mu(h(t)+h(t_0))\qquad\textit{for all } t\geq 0. \tag{18}$$

(2) Suppose that $h(t) > 0$ for every $t > t_0$ and

$$h'(t)t \leq \mu h(t) \qquad \text{for all } t \geq t_0. \tag{19}$$

Then

$$h(\lambda t) \leq \lambda^{\mu}(h(t) + h(t_0)) \qquad \text{for all } t \geq 0. \tag{20}$$

Moreover, if (17) or (19) hold, then for every $(t_1, \dots, t_k) \in \mathbb{R}^k_+$ we have:

$$k^{-1} \sum_{i=1}^{k} h(t_i) \leq h\left(\sum_{i=1}^{k} t_i\right) \leq k^{\mu}\left\{h(t_0) + \sum_{i=1}^{k} h(t_i)\right\}. \tag{21}$$

The lemma deals with well known properties of the convex functions (see [28]), however for the sake of completeness we provide a proof.

Proof. Let us prove (1). The first inequality in (18) is trivial, since, by the monotonicity of h, we have $h(\lambda t) \leq h(\lambda t_0) \leq \lambda^{\mu} h(t_0)$ for every $t < t_0$.

Let us prove the other inequality in (18). By assumption, for every $\sigma > 0$ and $t > t_0$ we have

$$\frac{h(t+\sigma) - h(t)}{\sigma} = \frac{h(t(1+\frac{\sigma}{t})) - h(t)}{\sigma} \leq \left\{\left(1 + \frac{\sigma}{t}\right)^{\mu} - 1\right\} \frac{h(t)}{\sigma}$$

and for $\sigma \to 0$ we get $h'(t)t \leq \mu h(t)$ for all $t > t_0$ and, by continuity, for $t \geq t_0$. Since h' is increasing, if $t \leq t_0$ we have $h'(t)t \leq h'(t_0)t_0 \leq \mu h(t_0)$, which implies the last inequality in (18).

Now, let us prove (2). By (19), for every $t > t_0$ and $\lambda > 1$ we obtain

$$\int_t^{\lambda t} \frac{h'(s)}{h(s)}\, ds \leq \mu \int_t^{\lambda t} \frac{1}{s}\, ds\,,$$

so that $h(\lambda t) \leq \lambda^{\mu} h(t)$. From that, (20) follows.

The first inequality in (21) is implied by the monotonicity of h, since $h(t_j) \leq h(\sum_{i=1}^{k} t_i)$ for all j. To prove the second inequality, use the monotonicity of h again and (20), obtaining

$$h\left(\sum_{i=1}^{k} t_i\right) \leq h\left(k \max_{1 \leq i \leq k}\{t_i\}\right) \leq k^{\mu}\left\{h\left(\max_{1 \leq i \leq k}\{t_i\}\right) + h(t_0)\right\}$$

and the conclusion follows. □

Now, we consider the case of functions depending on more than one variable.

Lemma 3.2. *Let $f : \Omega \times \mathbb{R}^n_+ \to \mathbb{R}_+$ satisfy (H1), (H2) and (H3). Then there exists $c \geq 0$ such that*

- *(i) $f(x, \lambda\xi) \leq c\lambda^{n\mu}\{1 + f(x,\xi)\}$ for every $\xi \in \mathbb{R}^n_+$ and every $\lambda > 1$,*
- *(ii) $f(x, \xi + \zeta) \leq c\{1 + f(x,\xi) + f(x,\zeta)\}$ for every $\xi, \zeta \in \mathbb{R}^n_+$,*
- *(iii) $\frac{\partial f}{\partial \xi_i}(x,\xi)\xi_i \leq c\{1 + f(x,\xi)\}$ for every $\xi \in \mathbb{R}^n_+$.*

Proof. Fix $i = 1, ..., n$. By (H1) for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^n_+$, with $\xi_i \geq t_0$, we have

$$f(x, \xi_1, \dots, \xi_{i-1}, \lambda\xi_i, \xi_{i+1}, \dots, \xi_n) \leq f(x, \lambda\xi) \leq \lambda^{\mu} f(x,\xi) \qquad \text{for every } \lambda > 1. \tag{22}$$

Therefore, by Lemma 3.1 (1), then, for every $\xi \in \mathbb{R}^n_+$,

$$\begin{aligned} &f(x,\xi_1,\dots,\xi_{i-1},\lambda\xi_i,\xi_{i+1},\dots,\xi_n) \\ \le&\lambda^\mu \left\{f(x,\xi)+f(x,\xi_1,\dots,\xi_{i-1},t_0,\xi_{i+1},\dots,\xi_n)\right\}. \end{aligned} \tag{23}$$

Now, fix $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n_+$ and $k\in\mathbb{N}$, $1\le k\le n-1$. For each set of indexes $\{i_1,\dots,i_k\}$, with $1\le i_1<...<i_k\le n$, we define two vectors $a(i_1,\dots,i_k)$ and $b(i_1,\dots,i_k)$ in $\mathbb{R}^n$ with j-th component

$$a(i_1,\dots,i_k)_j=\begin{cases}\xi_j & \text{if } j\in\{i_1,\dots,i_k\}\\ 0 & \text{if } j\notin\{i_1,\dots,i_k\}\end{cases}$$

and, respectively,

$$b(i_1,\dots,i_k)_j=\begin{cases}0 & \text{if } j\in\{i_1,\dots,i_k\}\\ 2t_0 & \text{if } j\notin\{i_1,\dots,i_k\}.\end{cases}$$

An iterated use of (23) implies that for every $\lambda>1$ and every $\xi\in\mathbb{R}^n_+$

$$\begin{aligned} &f(x,\lambda\xi_1,\dots,\lambda\xi_n) \\ \le&(2\lambda)^{n\mu}\left\{f\left(x,\frac{\xi_1}{2},\dots,\frac{\xi_n}{2}\right)+f\left(x,t_0,\dots,t_0\right)\right\} \\ &+(2\lambda)^{n\mu}\sum_{k=1}^{n-1}\sum_{1\le i_1<\ldots<i_k\le n} f\left(x,\frac{1}{2}a(i_1,\dots,i_k)+\frac{1}{2}b(i_1,\dots,i_k)\right). \end{aligned} \tag{24}$$

Notice that by the monotonicity of f with respect to each variable ξ_j and the right inequality in (8)

$$\begin{aligned} f\left(x,\frac{\xi_1}{2},\dots,\frac{\xi_n}{2}\right)+f\left(x,t_0,\dots,t_0\right)&\le f(x,\xi)+f(x,2t_0,\dots,2t_0) \\ &\le c\{1+f(x,\xi)\}. \end{aligned} \tag{25}$$

To estimate the last sum in (24) we use the convexity of f and the monotonicity properties of f

$$\begin{aligned} &f\left(x,\frac{1}{2}a(i_1,\dots,i_k)+\frac{1}{2}b(i_1,\dots,i_k)\right) \\ \le&\frac{1}{2}f\left(x,a(i_1,\dots,i_k)\right)+\frac{1}{2}f\left(x,b(i_1,\dots,i_k)\right)\le\frac{1}{2}f(x,\xi)+\frac{1}{2}f\left(x,2t_0,\dots,2t_0\right) \end{aligned} \tag{26}$$

and apply (25). Thus, (i) is proved.

Claim (ii) is a trivial consequence of (i): fixed $\xi,\zeta\in\mathbb{R}^n_+$, by (7)

$$f(x,\xi+\zeta)=f\left(x,2\frac{\xi+\zeta}{2}\right)\le c\cdot 2^{n\mu}\left\{1+f\left(x,\frac{\xi+\zeta}{2}\right)\right\}$$

and the convexity of f gives the conclusion.

It remains to prove (iii). Fix $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n_+$. By (22) and Lemma 3.1 (1)

$$\frac{\partial f}{\partial\xi_i}(x,\xi)\xi_i\le\mu\left\{f(x,\xi)+f(x,\xi_1,\dots,\xi_{i-1},t_0,\xi_{i+1},...,\xi_n)\right\}.$$

The last term can be estimated using the monotonicity of f with respect to each variable ξ_j and (ii). In fact,

$$\begin{aligned} f(x,\xi_1,\dots,\xi_{i-1},t_0,\xi_{i+1},...,\xi_n)&\le f(x,\xi_1+t_0,\dots,\xi_i+t_0,\dots,\xi_n+t_0) \\ &\le c\left\{1+f(x,\xi)+f(x,t_0,\dots,t_0)\right\}. \end{aligned}$$

The last inequality in (8) implies (iii). □

4. **The space $W^{1,\mathcal{F}}(\Omega)$ and some higher integrability results.** Due to the assumptions on f in Section 2 the space $W^{1,\mathcal{F}}(\Omega)$ is a vector space.

Lemma 4.1. *Assume (H1), (H2) and (H3). Then $W^{1,\mathcal{F}}(\Omega)$ is a vector space.*

Proof. By the right inequality of (8), the function $u \equiv 0$ is in $W^{1,\mathcal{F}}(\Omega)$. Let us assume that u and v are both in $W^{1,\mathcal{F}}(\Omega)$ and $\gamma \in \mathbb{R}$. By Lemma 3.2 (ii) we immediately have that $u+v$ is in $W^{1,\mathcal{F}}(\Omega)$.
Let us prove that $\gamma u \in W^{1,\mathcal{F}}(\Omega)$. If $|\gamma| \leq 1$ the conclusion follows by the monotonicity of f, see (H1). If, instead, $|\gamma| > 1$ then the conclusion follows by Lemma 3.2 (i), which implies that there exists c independent of x and u, such that

$$f(x, |\gamma||u_{x_1}|, \ldots, |\gamma||u_{x_n}|) \leq |\gamma|^{n\mu} c \left\{1 + f(x, |u_{x_1}|, \ldots, |u_{x_n}|)\right\}.$$

□

To prove our result we use the following suitable anisotropic Sobolev space

$$W^{1,(p_1,\ldots,p_n)}(\Omega) := \left\{u \in W^{1,1}(\Omega) \, : \, u_{x_i} \in L^{p_i}(\Omega), \text{ for all } i = 1, \ldots, n\right\},$$

endowed with the norm

$$\|u\|_{W^{1,(p_1,\ldots,p_n)}(\Omega)} := \|u\|_{L^1(\Omega)} + \sum_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}(\Omega)}.$$

We write $W_0^{1,(p_1,\ldots,p_n)}(\Omega)$ in place of $W_0^{1,1}(\Omega) \cap W^{1,(p_1,\ldots,p_n)}(\Omega)$. These spaces are studied in [31], see also [1]. We remind an embedding theorem for this class of spaces (see [31]).

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and consider $u \in W_0^{1,(p_1,\ldots,p_n)}(\Omega)$, $p_i \geq 1$ for all $i = 1, \ldots, n$. Let $\max\{p_i\} < \overline{p}^*$, with $\overline{p}^*$ as in (10). Then $u \in L^{\overline{p}^*}(\Omega)$. Moreover, there exists c depending on $n, p_1, \ldots, p_n$ if $\overline{p} < n$, and also on Ω if $\overline{p} \geq n$, such that*

$$\|u\|^n_{L^{\overline{p}^*}(\Omega)} \leq c \prod_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}(\Omega)}.$$

The following embedding result, which holds for the cubes of $\mathbb{R}^n$, is proved in [1].

Theorem 4.3. *Let $Q \subset \mathbb{R}^n$ be a cube with edges parallel to the coordinate axes and consider $u \in W^{1,(p_1,\ldots,p_n)}(Q)$, $p_i \geq 1$ for all $i = 1, \ldots, n$. Let $\max\{p_i\} < \overline{p}^*$, with $\overline{p}^*$ as in (10). Then $u \in L^{\overline{p}^*}(Q)$. Moreover, there exists c depending on $n, p_1, \ldots, p_n$ if $\overline{p} < n$, and also on Q if $\overline{p} \geq n$, such that*

$$\|u\|_{L^{\overline{p}^*}(Q)} \leq c \left\{ \|u\|_{L^1(Q)} + \sum_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}(Q)} \right\}.$$

A variant of the above lemma can be proved using Theorem 4.3 and a suitable Poincaré inequality proved in [3].

Proposition 1. *Let $u \in W^{1,1}(\Omega)$ and let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be of class C^1, convex, increasing, non-constant, $g(0) = 0$, $g(\lambda t) \leq \lambda^\mu g(t)$, for some $\mu \geq 1$ and every $\lambda > 1$ and every $t \geq t_0$. Suppose that $g(|u_{x_i}|) \in L^{p_i}_{loc}(\Omega)$ for every $i = 1, \ldots, n$, with $p_i \geq 1$.*

Let $\max\{p_i\} < \overline{p}^*$, *with* $\overline{p}^*$ *as in (10), then* $g(|u|) \in L^{\overline{p}^*}_{loc}(\Omega)$. *Moreover, if* $Q \Subset \Omega$ *is a cube with edges parallel to the coordinate axes, then*

$$\|g(|u|)\|_{L^{\overline{p}^*}(Q)} \leq c \left\{ 1 + \|g(|u|)\|_{L^1(Q)} + \sum_{i=1}^{n} \|g(|u_{x_i}|)\|_{L^{p_i}(Q)} \right\}. \tag{27}$$

Proof. We split the proof into steps.

Step 1. We claim that $g(|Du|) \in L^1_{\mathrm{loc}}(\Omega)$. In fact, since $|Du| \leq \sum_{i=1}^n |u_{x_i}|$, then by (21)

$$g(|Du|) \leq n^{\mu} \left\{ g(t_0) + \sum_{i=1}^{n} g(|u_{x_i}|) \right\}. \tag{28}$$

Step 2. Let us prove that $g(|u|) \in L^1_{\mathrm{loc}}(\Omega)$.

For every convex bounded open set $\Sigma \Subset \Omega$, by Lemma 3.2 (ii) we get

$$g(|u|) \leq g(|u - u_\Sigma| + |u_\Sigma|) \leq c\,\{1 + g(|u - u_\Sigma|) + g(|u_\Sigma|)\}$$

where $u_\Sigma = |\Sigma|^{-1} \int_\Sigma u\, dx$ and c is a positive constant independent of u and Σ. By Lemma 3.1 (1)

$$\int_\Sigma g\left(|u(x) - u_\Sigma|\right) dx \leq \max\left\{[\mathrm{diam}(\Sigma)]^{\mu}, 1\right\} \int_\Sigma \left\{ g\left(\frac{|u(x) - u_\Sigma|}{\mathrm{diam}(\Sigma)} \right) + g(t_0) \right\} dx \tag{29}$$

and a Poincaré inequality proved in [3] implies

$$\int_\Sigma g\left(\frac{|u(x) - u_\Sigma|}{\mathrm{diam}(\Sigma)} \right) dx \leq \left\{ \frac{\omega_n [\mathrm{diam}(\Sigma)]^n}{|\Sigma|} \right\}^{1 - \frac{1}{n}} \int_\Sigma g(|Du(x)|)\, dx, \tag{30}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. The conclusion follows by Step 1.

Step 3. Let a_k be an increasing sequence, $a_k \to +\infty$ as k goes to $+\infty$, such that the sets $\{|u| = a_k\}$ have zero measure. Define the increasing sequence of functions g_k defined as $g_k(t) = g(t)$ if $t < a_k$ and $g_k(t) = g(a_k)$ if $t \geq a_k$. We claim that $g_k(|u|) \in W^{1,(p_1,\ldots,p_n)}_{loc}(\Omega)$.

In fact, let Σ be an open subset, $\Sigma \Subset \Omega$. Since g_k is bounded then $g_k(|u|)$ is bounded, too. It remains to prove that $[g_k(|u|)]_{x_i} \in L^{p_i}(\Sigma)$. We notice that the following inequality holds: given two non-decreasing and non-negative functions h_1 and h_2, it holds true that

$$h_1(t_1)h_2(t_2) \leq h_1(t_1)h_2(t_1) + h_1(t_2)h_2(t_2) \qquad \text{for every } t_1, t_2. \tag{31}$$

Hence, we have that

$$\|[g_k(|u|)]_{x_i}\|_{L^{p_i}(\Sigma)} \leq \left\{ \int_{\Sigma \cap \{|u| \leq a_k\}} [g'(|u|)]^{p_i} |u|^{p_i}\, dx \right\}^{\frac{1}{p_i}} + \left\{ \int_{\Sigma \cap \{|u| \leq a_k\}} [g'(|u_{x_i}|)]^{p_i} |u_{x_i}|^{p_i}\, dx \right\}^{\frac{1}{p_i}}$$

and from Lemma 3.1 (1) we get

$$\|[g_k(|u|)]_{x_i}\|_{L^{p_i}(\Sigma)} \leq c\left\{1 + \|g_k(|u|)\|_{L^{p_i}(\Sigma)} + \|g(|u_{x_i}|)\|_{L^{p_i}(\Sigma)}\right\} < +\infty. \tag{32}$$

Thus, the claim is proved.

Step 4. Now, we conclude. Let $Q \Subset \Omega$ be a cube with edges parallel to the coordinate axes. Since $g_k(|u|) \in W^{1,(p_1,\dots,p_n)}(Q)$ we can apply Theorem 4.3, so that, using also (32), there exists $c_1 > 0$ such that

$$\begin{aligned}\|g_k(|u|)\|_{L^{\overline{p}^*}(Q)} &\leq c_1 \left\{1 + \|g_k(|u|)\|_{L^1(Q)}\right\} \\ &\quad + c_1 \left\{\sum_{i=1}^{n} \|g_k(|u|)\|_{L^{p_i}(Q)} + \sum_{i=1}^{n} \|g(|u_{x_i}|)\|_{L^{p_i}(Q)}\right\}.\end{aligned} \tag{33}$$

Notice that if $p_i > 1$ and being $\max\{p_i\} < \overline{p}^*$, then there exists $\alpha_i \in (0,1)$ such that $p_i^{-1} = (1-\alpha_i) + \alpha_i/\overline{p}^*$. Hence for every $\epsilon > 0$ and for every i there exists $c_{\epsilon,i} > 0$ such that

$$\begin{aligned}\|g_k(|u|)\|_{L^{p_i}(Q)} &\leq \|g_k(|u|)\|_{L^{\overline{p}^*}(Q)}^{\alpha_i} \|g_k(|u|)\|_{L^1(Q)}^{1-\alpha_i} \\ &\leq \epsilon \|g_k(|u|)\|_{L^{\overline{p}^*}(Q)} + c_{\epsilon,i} \|g_k(|u|)\|_{L^1(Q)}.\end{aligned}$$

Of course, if $p_i = 1$ the above inequality is trivial. Choosing $\epsilon = (2nc_1)^{-1}$ the above inequalities and (33) imply that a constant $c_2 > 0$ exists such that

$$\|g_k(|u|)\|_{L^{\overline{p}^*}(Q)} \leq c_2 \|g_k(|u|)\|_{L^1(Q)} + 2c_1 \left\{1 + \sum_{i=1}^{n} \|g(|u_{x_i}|)\|_{L^{p_i}(Q)}\right\}.$$

Using the monotone convergence theorem, inequality (27) follows. □

A consequence of the above result is the following corollary.

Corollary 2. *Assume (H1), (H2) and (H3), with $q < \overline{p}^*$. If $u \in W^{1,\mathcal{F}}(\Omega)$, then $g(|u|) \in L^{\overline{p}^*}_{loc}(\Omega)$.*

5. **The Euler's inequality.**

Since (H1) does not imply the C^1-regularity of $\xi \mapsto f(|\xi_1|, \dots, |\xi_n|)$, $\xi \in \mathbb{R}^n$, in place of the Euler's equation, we prove an inequality.

Theorem 5.1. *Assume that (H1), (H2) and (H3) hold true and let $u \in W^{1,\mathcal{F}}(\Omega)$ be a local minimizer of (6). Then*

$$\begin{aligned}&\sum_{i=1}^{n} \int_{\Omega \cap \{|u_{x_i}|>0\}} \frac{\partial f}{\partial \xi_i}(x, |u_{x_1}|, ..., |u_{x_n}|)\, sgn\,(u_{x_i})\, \varphi_{x_i}\, dx \\ &\qquad \leq \sum_{i=1}^{n} \int_{\Omega \cap \{|u_{x_i}|=0\}} \frac{\partial f}{\partial \xi_i}(x, |u_{x_1}|, ..., |u_{x_n}|)\, |\varphi_{x_i}|\, dx,\end{aligned} \tag{34}$$

for all $\varphi \in W^{1,\mathcal{F}}(\Omega)$, $supp\,\varphi \Subset \Omega$.

Proof. Let $\varphi \in W^{1,\mathcal{F}}(\Omega)$ be a function with compact support and $\lambda \in (-1,0)$. For every $i \in \{1, ..., n\}$ define $H_i : \Omega \times (-1,0) \times \mathbb{R}_+ \to \mathbb{R}_+$,

$$\begin{aligned}H_i(x,\lambda,s) := f(x, &|u_{x_1}(x) + \lambda\varphi_{x_1}(x)|, ... \\ &..., |u_{x_{i-1}}(x) + \lambda\varphi_{x_{i-1}}(x)|, s, |u_{x_{i+1}}(x)|, ..., |u_{x_n}(x)|).\end{aligned}$$

Notice that if $i \leq n-1$ then

$$H_i(x, \lambda, |u_{x_i}(x) + \lambda\varphi_{x_i}(x)|) = H_{i+1}(x, \lambda, |u_{x_{i+1}}(x)|).$$

By the minimality of u and the convexity of f with respect to each variable ξ_j, we get

$$\begin{aligned} 0 &\geq \frac{1}{\lambda}\{\mathcal{F}(u+\lambda\varphi)-\mathcal{F}(u)\} \\ &= \frac{1}{\lambda}\int_\Omega \{H_n(x,\lambda,|u_{x_n}+\lambda\varphi_{x_n}|)-H_1(x,\lambda,|u_{x_1}|)\}\,dx \\ &= \frac{1}{\lambda}\sum_{i=1}^n\int_\Omega \{H_i(x,\lambda,|u_{x_i}+\lambda\varphi_{x_i}|)-H_i(x,\lambda,|u_{x_i}|)\}\,dx \\ &\geq \sum_{i=1}^n\int_\Omega \frac{\partial H_i}{\partial s}(x,\lambda,|u_{x_i}+\lambda\varphi_{x_i}|)\,\frac{|u_{x_i}+\lambda\varphi_{x_i}|-|u_{x_i}|}{\lambda}\,dx. \end{aligned} \tag{35}$$

Since $\frac{\partial H_i}{\partial s}=\frac{\partial f}{\partial \xi_i}$, by Lemma 3.2 (iii) we obtain

$$\begin{aligned} &\left|\frac{\partial H_i}{\partial s}(x,\lambda,|u_{x_i}(x)+\lambda\varphi_{x_i}(x)|)\,\frac{|u_{x_i}(x)+\lambda\varphi_{x_i}(x)|-|u_{x_i}(x)|}{\lambda}\right| \\ &\leq \frac{\partial H_i}{\partial s}(x,\lambda,|u_{x_i}(x)|+|\varphi_{x_i}(x)|)(|u_{x_i}(x)|+|\varphi_{x_i}(x)|) \\ &\leq c\{1+f(x,|u_{x_1}(x)+\lambda\varphi_{x_1}(x)|,\ldots \\ &\qquad \ldots,|u_{x_{i-1}}(x)+\lambda\varphi_{x_{i-1}}(x)|,|u_{x_i}(x)|+|\varphi_{x_i}(x)|,|u_{x_{i+1}}(x)|,\ldots,|u_{x_n}(x)|)\} \end{aligned}$$

and, using the monotonicity property in (H1) and Lemma 3.2 (ii),

$$\begin{aligned} &f(x,|u_{x_1}(x)+\lambda\varphi_{x_1}(x)|,\ldots \\ &\qquad \ldots,|u_{x_{i-1}}(x)+\lambda\varphi_{x_{i-1}}(x)|,|u_{x_i}(x)|+|\varphi_{x_i}(x)|,|u_{x_{i+1}}(x)|,\ldots,|u_{x_n}(x)|) \\ &\quad\leq f\left(x,|u_{x_1}(x)|+|\varphi_{x_1}(x)|,\ldots,|u_{x_n}(x)|+|\varphi_{x_n}(x)|\right) \\ &\quad\leq c\left\{1+f(x,|u_{x_1}(x)|,\ldots,|u_{x_n}(x)|)+f(x,|\varphi_{x_1}(x)|,\ldots,|\varphi_{x_n}(x)|)\right\}. \end{aligned}$$

Now, notice that the right hand side is in $L^1(\Omega)$, being $u,\varphi\in W^{1,\mathcal{F}}(\Omega)$. Moreover, by the regularity C^1 of $f(x,\cdot)$,

$$\lim_{\lambda\to 0^-}\frac{\partial H_i}{\partial s}(x,\lambda,|u_{x_i}(x)+\lambda\varphi_{x_i}(x)|)=\frac{\partial f}{\partial \xi_i}(x,|u_{x_1}(x)|,\ldots,|u_{x_n}(x)|).$$

Thus, by the dominated convergence theorem and (35) we get

$$\sum_{i=1}^n\int_\Omega \frac{\partial f}{\partial \xi_i}(x,|u_{x_1}(x)|,\ldots,|u_{x_n}(x)|)\lim_{\lambda\to 0^-}\frac{|u_{x_i}(x)+\lambda\varphi_{x_i}(x)|-|u_{x_i}(x)|}{\lambda}\,dx\leq 0.$$

The conclusion follows. □

6. **Proof of the boundedness of local minimizers.** Fixed $i\in\{1,\ldots,n\}$ and $\beta\geq 1$, let $\Phi:\mathbb{R}\to\mathbb{R}$ be the odd function defined as follows

$$\Phi^{(i,\beta)}(t):=\int_0^t [g(|s|)]^{p_i(\beta-1)}\,ds. \tag{36}$$

In a first step, we deal with an approximating sequence of odd functions $\Phi_k^{(i,\beta)}$. Fixed $k\in\mathbb{N}$, the function $\Phi_k^{(i,\beta)}:\mathbb{R}\to\mathbb{R}$ is defined in $\mathbb{R}_+$ as

$$\Phi_k^{(i,\beta)}(t):=\begin{cases}\Phi^{(i,\beta)}(t) & \text{if } 0\leq t\leq k\\ t\,(\Phi^{(i,\beta)})'(k)+\Phi^{(i,\beta)}(k)-k\,(\Phi^{(i,\beta)})'(k) & \text{if } t>k.\end{cases} \tag{37}$$

From now on, we do not write explicitly the dependence on i and β. Notice that the restriction of Φ_k to $\mathbb{R}_+$ is C^1, increasing and convex. Moreover, its first order derivative is bounded and

$$|\Phi_k(t)| \le \Phi_k'(t)|t| \qquad \text{for all } t \in \mathbb{R}. \tag{38}$$

In the following lemma we define φ_k, an admissible test function for the Euler's inequality (34).

Lemma 6.1. *Assume (H1), (H2) and (H3), with $q < \overline{p}^*$. Let $u \in W^{1,\mathcal{F}}(\Omega)$, fix a ball $B_R(x_0) \Subset \Omega$ and let $\eta \in C_c^\infty(B_R(x_0))$ be a cut-off function, satisfying the following assumptions*

$$0 \le \eta \le 1, \qquad \eta \equiv 1 \text{ in } B_\rho(x_0) \text{ for some } \rho < R, \qquad |D\eta| \le \frac{2}{R-\rho}. \tag{39}$$

Fixed $k \in \mathbb{N}$, define

$$\varphi_k(x) := \Phi_k(u(x))[\eta(x)]^\alpha \qquad \text{for every } x \in B_R(x_0), \tag{40}$$

with $\alpha \ge 1$. Then φ_k is in $W_0^{1,\mathcal{F}}(B_R(x_0))$.

Proof. By Lemma 4.1, Lemma 3.2 and the definition of Φ_k we get the thesis if we prove that

$$A := \int_{B_R \cap \{|u|<k\}} f\left(x, |[\Phi(u)]_{x_1}|, \dots, |[\Phi(u)]_{x_n}|\right) dx < +\infty$$
$$B := \int_{B_R \cap \{|u|<k\}} f\left(x, |\Phi(u)|\,|\eta_{x_1}|, \dots, |\Phi(u)|\,|\eta_{x_n}|\right) dx < +\infty$$
$$C := \int_{B_R \cap \{|u|\ge k\}} f\left(x, [g(k)]^{p_i(\beta-1)}|u_{x_1}|, \dots, [g(k)]^{p_i(\beta-1)}|u_{x_n}|\right) dx < +\infty$$
$$D := \int_{B_R \cap \{|u|\ge k\}} f\left(x, |\Phi_k(u)|\,|\eta_{x_1}|, \dots, |\Phi_k(u)|\,|\eta_{x_n}|\right) dx < +\infty.$$

Let us deal with A.
By the monotonicity of g, $|[\Phi(u)(x)]_{x_j}| \le [g(k)]^{p_i(\beta-1)}|u_{x_j}(x)|$, for a.e. $x \in \{|u| < k\}$. Then, by (H1) and Lemma 3.2 (ii) we get

$$A \le c\left\{\max\{[g(k)]^{p_i(\beta-1)}, 1\}\right\}^{n\mu} \cdot \left\{1 + \int_{B_R \cap \{|u|<k\}} f\left(x, |u_{x_1}|, \dots, |u_{x_n}|\right) dx\right\},$$

which is finite being $u \in W^{1,\mathcal{F}}(B_R)$. The boundedness of C follows similarly.

As far as B is concerned, from (H1), the assumptions on η and the monotonicity of g we obtain

$$B \le \int_{B_R \cap \{|u|<k\}} f\left(x, \frac{2\alpha}{R-\rho}k[g(k)]^{p_i(\beta-1)}, \dots, \frac{2\alpha}{R-\rho}k[g(k)]^{p_i(\beta-1)}\right) dx,$$

which is finite because of the growth condition (8).

Let us prove the boundedness of D.
From (38) we obtain $|\Phi_k(u(x))| \le [g(k)]^{p_i(\beta-1)}|u(x)|$ for a.e. x in the integration domain. Thus,

$$|\Phi_k(u(x))| \cdot |\eta_{x_j}(x)| \le \frac{2[g(k)]^{p_i(\beta-1)}}{R-\rho}|u(x)|.$$

Using the assumptions on f and the right inequality in (8) we get

$$D \leq c \left\{ \max \left\{ \frac{2[g(k)]^{p_i(\beta-1)}}{R-\rho}, 1 \right\} \right\}^{n\mu} \cdot \left\{ 1 + \int_{B_R \cap \{|u| \geq k\}} [g(|u|)]^q \, dx \right\}. \tag{41}$$

Since $q < \overline{p}^*$, Corollary 2 implies that the last term in (41) is finite. □

The lemma below is a simple consequence of the Hölder inequality. We omit the proof.

Lemma 6.2. *Let Ω be a bounded measurable set. Suppose that $1 \leq p \leq q$, $\beta \geq 1$ and $v \in L^{q\beta}(\Omega)$. Then*

$$\int_\Omega |v|^{q+p(\beta-1)} \, dx \leq \left\{ \int_\Omega (|v|+1)^q \, dx \right\}^{1-\frac{p}{q}} \cdot \left\{ \int_\Omega (|v|^\beta + 1)^q \, dx \right\}^{\frac{p}{q}}.$$

Now, we turn to the proof of our main result.

Proof of Theorem 2.1. Let u be a local minimizer of (6) and consider $x_0 \in \Omega$ and $R_0 > 0$, such that $B_{R_0} := B_{R_0}(x_0) \Subset \Omega$. In particular, by Corollary 2 $g(|u|) \in L^{\overline{p}^*}(B_{R_0})$. Fix also $0 < \rho < R \leq R_0$. We split the proof into steps.

Step 1. Assume that $g(|u|) \in L^{q\beta}(B_R)$ for some $\beta \geq 1$. Fixed $i \in \{1, \ldots, n\}$ we prove that if η is a cut-off function satisfying (39), then

$$\begin{aligned} &\int_{B_R} \left\{ [g(|u|)]^{(\beta-1)} g'(|u_{x_i}|) |u_{x_i}| \eta^\mu \right\}^{p_i} \, dx \\ \leq &\frac{c}{(R-\rho)^\mu} \left\{ \|g(|u|)\|_{L^q(B_{R_0})} + 1 \right\}^{q-p} \cdot \left\{ \int_{B_R} \left(g^\beta(|u|) + 1 \right)^q \, dx \right\}^{\frac{p_i}{q}}, \end{aligned} \tag{42}$$

for some c depending on n, μ, p, q, a, $g(t_0)$ and R_0, but independent of i, β, u, R and ρ.

We begin using Theorem 5.1 with the test function $\varphi_k^{(i,\beta)} := \Phi_k^{(i,\beta)} \eta^\mu$ with $\Phi_k^{(i,\beta)}$ as in (37). From now on, we write φ_k and Φ_k in place of $\varphi_k^{(i,\beta)}$ and $\Phi_k^{(i,\beta)}$, respectively. We obtain

$$\begin{aligned} &\sum_{j=1}^n \int_{B_R \cap \{|u_{x_j}|>0\}} \frac{\partial f}{\partial \xi_j} (x, |u_{x_1}|, ..., |u_{x_n}|) \, |u_{x_j}| \Phi_k'(u) \eta^\mu \, dx \\ \leq &\mu \sum_{j=1}^n \int_{B_R} \frac{\partial f}{\partial \xi_j} (x, |u_{x_1}|, ..., |u_{x_n}|) \, |\Phi_k(u)| \eta^{\mu-1} |\eta_{x_j}| \, dx. \end{aligned}$$

Thus, using (38),

$$\begin{aligned} &\sum_{j=1}^n \int_{B_R} \frac{\partial f}{\partial \xi_j} (x, |u_{x_1}|, ..., |u_{x_n}|) \, |u_{x_j}| \Phi_k'(u) \eta^\mu \, dx \\ \leq &\frac{2\mu}{R-\rho} \sum_{j=1}^n \int_{B_R} \frac{\partial f}{\partial \xi_j} (x, |u_{x_1}|, ..., |u_{x_n}|) \, \Phi_k'(u) |u| \eta^{\mu-1} \, dx. \end{aligned} \tag{43}$$

We estimate from below the left hand side using the convexity of $f(x,\cdot)$, obtaining

$$\begin{aligned}&\sum_{j=1}^{n}\int_{B_R}\frac{\partial f}{\partial \xi_j}\left(x,|u_{x_1}|,...,|u_{x_n}|\right)|u_{x_j}|\ \Phi_k'(u)\ \eta^{\mu}\,dx\\ \geq&\int_{B_R}\left\{f\left(x,|u_{x_1}|,...,|u_{x_n}|\right)-f\left(x,0,...,0\right)\right\}\Phi_k'(u)\ \eta^{\mu}\,dx.\end{aligned}\tag{44}$$

Now, let us estimate from above the right hand side in (43). For a.e. $x\in\Omega$ and every $s\geq 0$ define

$$H_j(x,s):=f(x,|u_{x_1}(x)|,\dots,|u_{x_{j-1}}(x)|,s,|u_{x_{j+1}}(x)|,\dots,|u_{x_n}(x)|).$$

Let $L>0$ to be chosen later. Since f is convex we have that $\frac{\partial H_j}{\partial s}(x,\cdot)$ is increasing, then by (31) and Lemma 3.2 (iii), the following chain of inequalities holds true for a.e. $x\in\{\eta\neq 0\}$

$$\begin{aligned}&\frac{\partial f}{\partial \xi_j}\left(x,|u_{x_1}|,\dots,|u_{x_n}|\right)\frac{2\mu|u|}{\eta(R-\rho)}\\ \leq&\frac{1}{L}\frac{\partial H_j}{\partial s}\left(x,|u_{x_j}|\right)|u_{x_j}|+\frac{1}{L}\frac{\partial H_j}{\partial s}\left(x,\frac{2\mu L|u|}{\eta(R-\rho)}\right)\frac{2\mu L|u|}{\eta(R-\rho)}\\ \leq&\frac{c_1}{L}\left\{1+f\left(x,|u_{x_1}|,\dots,|u_{x_n}|\right)+H_j\left(x,\frac{2\mu L|u|}{\eta(R-\rho)}\right)\right\},\end{aligned}$$

with c_1 depending only on $n,\mu,q,a,g(t_0)$.

Now, denote with e_j the vector $(0,\dots,0,\underbrace{1}_{j},0,\dots,0)$. Using Lemma 3.2 (ii) with

$$\xi:=(|u_{x_1}|,\dots,|u_{x_{j-1}}|,\underbrace{0}_{j},|u_{x_{j+1}}|,\dots,|u_{x_n}|),\qquad \zeta:=\frac{2\mu L|u|}{\eta\,(R-\rho)}\ \mathsf{e}_j,$$

and the monotonicity property in (H1), we have that there exists c_2 such that

$$H_j\left(x,\frac{2\mu L|u|}{\eta(R-\rho)}\right)\leq c_2\left\{1+f(x,|u_{x_1}|,\dots,|u_{x_n}|)+f\left(x,\frac{2\mu L|u|}{\eta(R-\rho)}\ \mathsf{e}_j\right)\right\}.$$

Thus,

$$\begin{aligned}&\sum_{j=1}^{n}\frac{\partial f}{\partial \xi_j}\left(x,|u_{x_1}|,\dots,|u_{x_n}|\right)\frac{2\mu|u|}{\eta(R-\rho)}\\ \leq&\frac{c_3}{L}\left\{1+f(x,|u_{x_1}|,\dots,|u_{x_n}|)+\sum_{j=1}^{n}f\left(x,\frac{2\mu L|u|}{\eta(R-\rho)}\ \mathsf{e}_j\right)\right\}\end{aligned}$$

with c_3 depending only on $n,\mu,q,a,g(t_0)$. Choosing $L>\max\{2c_3,(2\mu)^{-1}R_0\}$, which implies $2\mu L>\eta\,(R-\rho)$, and using Lemma 3.1 (1), the above inequality implies

$$\begin{aligned}&\sum_{j=1}^{n}\frac{\partial f}{\partial \xi_j}\left(x,|u_{x_1}|,\dots,|u_{x_n}|\right)\frac{2\mu|u|}{\eta\,(R-\rho)}\\ \leq&\frac{1}{2}f(x,|u_{x_1}|,\dots,|u_{x_n}|)+\frac{c_4}{\eta^{\mu}(R-\rho)^{\mu}}\left\{1+\sum_{j=1}^{n}f(x,|u|\ \mathsf{e}_j)\right\}\end{aligned}\tag{45}$$

for some positive c_4. Collecting (43), (44) and (45), we obtain

$$\begin{aligned}&\int_{B_R} f(x,|u_{x_1}|,\dots,|u_{x_n}|)\Phi_k'(u)\ \eta^\mu\,dx\\ \leq&2\int_{B_R} f(x,0,\dots,0)\Phi_k'(u)\,dx\\ &+\frac{2c_4}{(R-\rho)^\mu}\int_{B_R}\{1+\sum_{j=1}^n f(x,|u|\ \mathsf{e}_j)\}\Phi_k'(u)\,dx.\end{aligned}\tag{46}$$

By (H3) and Lemma 3.1 (1) applied with $h=g^{p_i}$

$$\begin{aligned}f(x,|u_{x_1}|,\dots,|u_{x_n}|)&\geq\sum_{j=1}^n[g(|u_{x_j}|)]^{p_j}\geq g(|u_{x_i}|)]^{p_i}\\ &\geq\frac{1}{\mu q}(g^{p_i})'(|u_{x_i}|)|u_{x_i}|-[g(t_0)+1]^q.\end{aligned}\tag{47}$$

Moreover, by (8)

$$1+f(x,0,\dots,0)+\sum_{j=1}^n f(x,|u|\ \mathsf{e}_j)\leq c_5\left\{[g(|u|)]^q+1\right\}.\tag{48}$$

Inequalities (46), (47) and (48) give

$$\int_{B_R}(g^{p_i})'(|u_{x_i}|)|u_{x_i}|\ \Phi_k'(u)\ \eta^\mu\,dx\leq\frac{c_6}{(R-\rho)^\mu}\int_{B_R}\left\{[g(|u|)]^q+1\right\}\Phi_k'(u)\,dx.$$

We recall that $\Phi_k=\Phi_k^{(i,\beta)}$ and we explicitly notice that c_6 is independent of β, ρ and R. Using the monotone convergence theorem we let k go to $+\infty$ and by the definition of Φ we obtain

$$\begin{aligned}&\int_{B_R}[g(|u|)]^{p_i(\beta-1)}[g(|u_{x_i}|)]^{p_i-1}g'(|u_{x_i}|)|u_{x_i}|\ \eta^\mu\,dx\\ \leq&\frac{c_6}{(R-\rho)^\mu}\int_{B_R}\left\{[g(|u|)]^{p_i(\beta-1)}+[g(|u|)]^{q+p_i(\beta-1)}\right\}\,dx\,.\end{aligned}$$

Now, by the Hölder inequality there exists c, depending on R_0, such that

$$\begin{aligned}\int_{B_R}[g(|u|)]^{p_i(\beta-1)}\,dx&\leq\int_{B_R}\left([g(|u|)]^\beta+1\right)^{p_i}\,dx\\ &\leq c\left\{\int_{B_R}\left([g(|u|)]^\beta+1\right)^q\,dx\right\}^{\frac{p_i}{q}}.\end{aligned}\tag{49}$$

Moreover, by Lemma 6.2 applied to $v=g(|u|)$, with p replaced by p_i, we get the existence of a positive constant c, independent of β, such that

$$\int_{B_R}[g(|u|)]^{q+p_i(\beta-1)}\,dx\leq c\left\{\kappa+1\right\}^{q-p_i}\left\{\int_{B_R}\left([g(|u|)]^\beta+1\right)^q\,dx\right\}^{\frac{p_i}{q}},$$

where $\kappa:=\|g(|u|)\|_{L^q(B_{R_0})}$ is finite by Corollary 2 and the assumption $q<\overline{p}^*$. So, it follows that

$$\begin{aligned}&\int_{B_R}[g(|u|)]^{p_i(\beta-1)}[g(|u_{x_i}|)]^{p_i-1}g'(|u_{x_i}|)|u_{x_i}|\eta^\mu\,dx\\ \leq&c_7\frac{\{\kappa+1\}^{q-p_i}}{(R-\rho)^\mu}\left\{\int_{B_R}\left([g(|u|)]^\beta+1\right)^q\,dx\right\}^{\frac{p_i}{q}}.\end{aligned}\tag{50}$$

Now, by (9) and by the first step of the proof of Lemma 3.1 (1) we get that for a.e. $x \in \{|u_{x_i}| > t_0\}$ the inequality $g(|u_{x_i}|) \geq \frac{1}{\mu} g'(|u_{x_i}|)|u_{x_i}|$ holds true. Moreover, being $\mu \geq 1$ and $p_i \leq q$ we get $\mu^{p_i - 1} \leq \mu^{q-1}$. Thus, (50) implies

$$\begin{aligned} &\int_{B_R \cap \{|u_{x_i}|>t_0\}} [g(|u|)]^{p_i(\beta-1)} [g'(|u_{x_i}|)]^{p_i} |u_{x_i}|^{p_i} \eta^\mu \, dx \\ \leq & c_8 \frac{\{\kappa+1\}^{q-p_i}}{(R-\rho)^\mu} \left\{ \int_{B_R} \left([g(|u|)]^\beta + 1 \right)^q \, dx \right\}^{\frac{p_i}{q}}, \end{aligned} \tag{51}$$

with c_8 independent of i. Filling the hole, that is adding to both sides

$$\int_{B_R \cap \{|u_{x_i}| \leq t_0\}} [g(|u|)]^{p_i(\beta-1)} [g'(|u_{x_i}|)]^{p_i} |u_{x_i}|^{p_i} \eta^\mu \, dx,$$

and noticing that, due to the convexity of g, (9), the first step of the proof of Lemma 3.1 (1) and (49) imply

$$\begin{aligned} &\int_{B_R \cap \{|u_{x_i}| \leq t_0\}} [g(|u|)]^{p_i(\beta-1)} [g'(|u_{x_i|}|)|u_{x_i|}|]^{p_i} \eta^\mu \, dx \\ \leq & \int_{B_R \cap \{|u_{x_i}| \leq t_0\}} [g(|u|)]^{p_i(\beta-1)} [g'(t_0) t_0]^{p_i} \eta^\mu \, dx \\ \leq & \mu^q \int_{B_R} [g(|u|)]^{p_i(\beta-1)} [g(t_0)]^{p_i} \eta^\mu \, dx \\ \leq & \mu^q [g(t_0)+1]^q \int_{B_R} [g(|u|)]^{p_i(\beta-1)} \eta^\mu \, dx \leq c_9 \left\{ \int_{B_R} \left([g(|u|)]^\beta + 1 \right)^q \, dx \right\}^{\frac{p_i}{q}}, \end{aligned}$$

we obtain that

$$\begin{aligned} &\int_{B_R} \left\{ [g(|u|)]^{(\beta-1)} g'(|u_{x_i}|)|u_{x_i}| \right\}^{p_i} \eta^\mu \, dx \\ \leq & c_{10} \frac{\{\kappa+1\}^{q-p_i}}{(R-\rho)^\mu} \left\{ \int_{B_R} \left([g(|u|)]^\beta + 1 \right)^q \, dx \right\}^{\frac{p_i}{q}}. \end{aligned}$$

Since $\eta^{\mu p_i} \leq \eta^\mu$ and $p_i \geq p$ we get (42).

Step 2. In this step we prove that if $g(|u|) \in L^{q\beta}(B_R)$ for some $\beta \geq 1$, then there exists c, independent of β, R and ρ, such that

$$\begin{aligned} &\int_{B_R} \left| \left[\eta^\mu ([g(|u|)]^\beta + 1) \right]_{x_i} \right|^{p_i} \, dx \\ \leq & \frac{c \, \beta^\gamma}{(R-\rho)^\gamma} \left\{ \|g(|u|)\|_{L^q(B_{R_0})} + 1 \right\}^{q-p} \cdot \left\{ \int_{B_R} ([g(|u|)]^\beta + 1)^q \, dx \right\}^{\frac{p_i}{q}}, \end{aligned} \tag{52}$$

with $\gamma = \max\{\mu, q\}$.
We begin noticing that

$$\begin{aligned} &\int_{B_R} \left| \left[\eta^\mu ([g(|u|)^\beta + 1) \right]_{x_i} \right|^{p_i} \, dx \\ \leq & 2^{q-1} \int_{B_R} \left\{ \left| [\eta^\mu]_{x_i} \right| \; ([g(|u|)]^\beta + 1) \right\}^{p_i} \, dx \\ & + 2^{q-1} \beta^q \int_{B_R} \left\{ [g(|u|)]^{\beta-1} g'(|u|)|u_{x_i}| \; \eta^\mu \right\}^{p_i} \, dx = I_1 + I_2. \end{aligned} \tag{53}$$

By (39) and the Hölder inequality we have that

$$I_1 \le \frac{c}{(R-\rho)^q}\left\{\int_{B_R}([g(|u|)]^\beta+1)^q\,dx\right\}^{\frac{p_i}{q}}. \tag{54}$$

Let us consider I_2. Use (31) with $h_1 = g'^{p_i}$, $h_2 = \mathrm{id}^{p_i}$, $t_1 = |u(x)|$ and $t_2 = |u_{x_i}(x)|$, obtaining

$$[g'(|u(x)|)]^{p_i}|u_{x_i}(x)|^{p_i} \le [g'(|u_{x_i}(x)|)]^{p_i}|u_{x_i}(x)|^{p_i} + [g'(|u(x)|)]^{p_i}|u(x)|^{p_i}.$$

Therefore

$$\begin{aligned} I_2 \le\, & 2^{q-1}\beta^q\int_{B_R}\left\{[g(|u|)]^{(\beta-1)}g'(|u_{x_i}|)|u_{x_i}|\ \eta^\mu\right\}^{p_i}\,dx \\ & + 2^{q-1}\beta^q\int_{B_R}\left\{[g(|u|)]^{\beta-1}g'(|u|)|u|\ \eta^\mu\right\}^{p_i}\,dx. \end{aligned}$$

The first term in the right hand is estimated by (42). To estimate the second term, use Lemma 3.1 (1), which implies $g'(|u|)|u| \le c\,\{g(|u|)+1\}$, obtaining

$$\begin{aligned} & \int_{B_R}\left\{[g(|u|)]^{\beta-1}g'(|u|)|u|\ \eta^\mu\right\}^{p_i}\,dx \\ \le & c\int_{B_R}[g(|u|)]^{p_i(\beta-1)}([g(|u|)]^{p_i}+1)\,dx \\ \le & c\int_{B_R}\left([g(|u|)]^\beta+1\right)^{p_i}\,dx \le c\left\{\int_{B_R}\left([g(|u|)]^\beta+1\right)^q\,dx\right\}^{\frac{p_i}{q}}. \end{aligned}$$

Thus,

$$I_2 \le \frac{c\ \beta^q}{(R-\rho)^\mu}\left\{\|g(|u|)\|_{L^q(B_{R_0})}+1\right\}^{q-p}\cdot\left\{\int_{B_R}\left([g(|u|)]^\beta+1\right)^q\,dx\right\}^{\frac{p_i}{q}}. \tag{55}$$

The inequalities (53), (54) and (55) imply (52).

Step 3. Now, we prove the boundedness of u and the estimate (1). If $g(|u|) \in L^{q\beta}(B_R)$ for some $\beta \ge 1$, then Step 2 implies that $x \mapsto \eta^\mu(x)\{[g(|u(x)|)]^\beta+1\}$ is in $W_0^{1,(p_1,\dots,p_n)}(B_R)$. Multiplying (52) on i and being $p_i \ge p$, we get

$$\begin{aligned} & \prod_{i=1}^n\left\{\int_{B_R}\left|D_i\left(\eta^\mu([g(|u|)]^\beta+1)\right)\right|^{p_i}\,dx\right\}^{\frac{1}{p_i}} \\ \le & c_{11}\left\{\frac{\beta}{R-\rho}\right\}^{\frac{n\gamma}{p}}\{\|g(|u|)\|_{L^q(B_{R_0})}+1\}^{n\frac{q-p}{p}}\left\{\int_{B_R}([g(|u|)]^\beta+1)^q\,dx\right\}^{\frac{n}{q}}, \end{aligned}$$

with c_{11} independent of β, R and ρ. By Theorem 4.2 we get

$$\begin{aligned} & \left\{\int_{B_\rho}\left([g(|u|)]^\beta+1\right)^{\overline{p}^*}\,dx\right\}^{\frac{1}{\overline{p}^*}} \\ \le & c_{12}\left\{\frac{\beta}{R-\rho}\right\}^{\frac{\gamma}{p}}\{\|g(|u|)\|_{L^q(B_{R_0})}+1\}^{\frac{q-p}{p}}\left\{\int_{B_R}([g(|u|)]^\beta+1)^q\,dx\right\}^{\frac{1}{q}} \end{aligned}$$

and, defining $G(x) := \max\{1, g(|u(x)|)\}$, we obtain

$$\begin{aligned}&\left\{\int_{B_\rho}[G(x)]^{\beta\overline{p}^*}\,dx\right\}^{\frac{1}{\overline{p}^*}}\\ \le& 2c_{12}\left\{\frac{\beta}{R-\rho}\right\}^{\frac{\gamma}{p}}\left\{\|g(|u|)\|_{L^q(B_{R_0})}+1\right\}^{\frac{q-p}{p}}\left\{\int_{B_R}[G(x)]^{\beta q}\,dx\right\}^{\frac{1}{q}}.\end{aligned} \tag{56}$$

For all $h \in \mathbb{N}$ define $\beta_h = \left(\frac{\overline{p}^*}{q}\right)^{h-1}$, $\rho_h = R_0/2 + R_0/2^{h+1}$ and $R_h = R_0/2 + R_0/2^h$. By (56), replacing β, R and ρ with β_h, R_h and ρ_h, respectively, we have that $G \in L^{\beta_h q}(B_{R_h})$ implies $G \in L^{\beta_{h+1} q}(B_{R_{h+1}})$. Precisely,

$$\begin{aligned}&\|G\|_{L^{\beta_{h+1}q}(B_{R_{h+1}})}\\ \le&\left\{2c_{12}\left\{\frac{2^{h+1}}{R_0}\left(\frac{\overline{p}^*}{q}\right)^{h-1}\right\}^{\frac{\gamma}{p}}\left\{\|g(|u|)\|_{L^q(B_{R_0})}+1\right\}^{\frac{q-p}{p}}\right\}^{\frac{1}{\beta_h}}\|G\|_{L^{\beta_h q}(B_{R_h})}\end{aligned} \tag{57}$$

holds true for every h. Corollary 2 and the inequality $q < \overline{p}^*$ imply $G \in L^q(B_{R_0})$. An iterated use of (57) implies the existence of a constant c_{13} such that

$$\|G\|_{L^\infty(B_{R_0/2}(x_0))} \le c_{13}\left\{\|g(|u|)\|_{L^q(B_{R_0})}+1\right\}^{\frac{\overline{p}^*(q-p)}{p(\overline{p}^*-q)}}\|G\|_{L^q(B_{R_0}(x_0))}.$$

Therefore, by the very definition of G,

$$\|g(|u|)\|_{L^\infty(B_{R_0/2}(x_0))} \le c_{14}\left\{\|g(|u|)\|_{L^q(B_{R_0}(x_0))}+1\right\}^{\frac{\overline{p}^*(q-p)}{p(\overline{p}^*-q)}+1}.$$

From (H3), which implies that $g(t) \to +\infty$ as $t \to +\infty$, the above inequality implies that u is in $L^\infty(B_{R_0/2}(x_0))$.

Step 4. Here we prove estimate (12). Fix $B_r(x_0) \Subset \Omega$. Notice that if $Q_s(x_0)$ denotes the cube with edges parallel to the coordinate axes, centered at x_0 and with side length $2s$, then $B_{r/\sqrt{n}}(x_0) \subseteq Q_{r/\sqrt{n}}(x_0) \subseteq B_r(x_0)$.

Let $u \in W^{1,\mathcal{F}}(\Omega)$ be a local minimizer of $\mathcal{F}$ and define $u_r := \fint_{B_r(x_0)} u\,dx$. Since $u - u_r$ is a local minimizer, too, then by (11) and the Hölder inequality

$$\|g(|u-u_r|)\|_{L^\infty(B_{r/(2\sqrt{n})}(x_0))} \le c\left\{1+\|g(|u-u_r|)\|_{L^{\overline{p}^*}(B_{r/\sqrt{n}}(x_0))}\right\}^{\frac{\overline{p}^*(q-p)}{p(\overline{p}^*-q)}+1}. \tag{58}$$

By (27) in Proposition 1

$$\begin{aligned}&\|g(|u-u_r|)\|_{L^{\overline{p}^*}(B_{r/\sqrt{n}}(x_0))}\\ \le&\|g(|u-u_r|)\|_{L^{\overline{p}^*}(Q_{r/\sqrt{n}}(x_0))}\\ \le& c\left\{1+\|g(|u-u_r|)\|_{L^1(B_r(x_0))}+\sum_{i=1}^n\|g(|u_{x_i}|)\|_{L^{p_i}(B_r(x_0))}\right\}.\end{aligned} \tag{59}$$

and by the Poincaré inequality proved in [3], (see (28), (29) and (30))

$$\|g(|u-u_r|)\|_{L^1(B_r(x_0))} \le c\left\{1+\sum_{i=1}^n\|g(|u_{x_i}|)\|_{L^1(B_r(x_0))}\right\}.$$

Therefore, using (59) we get

$$\|g(|u-u_r|)\|_{L^{\overline{p}^*}(B_{r/\sqrt{n}}(x_0))} \leq c\left\{1+\sum_{i=1}^{n}\|g(|u_{x_i}|)\|_{L^{p_i}(B_r(x_0))}\right\}. \tag{60}$$

Now, (8) implies

$$\sum_{i=1}^{n}\|g(|u_{x_i}|)\|_{L^{p_i}(B_r(x_0))} \leq c\left\{1+\|f(x,|u_{x_1}|,\ldots,|u_{x_n}|)\|_{L^1(B_r(x_0))}\right\}^{\frac{1}{p}}. \tag{61}$$

The final estimate (12) follows collecting (58), (59), (60) and (61). □

Remark 1. It is not difficult to see that similar results to those stated in Theorem 2.1 can be proved for functionals (6) with more general Lagrangians f. For instance, few and straightforward changes in the proof of Theorem 2.1 allow to obtain the local boundedness of local minimizers of (6), together with estimates similar to (11) and (12), under the following set of assumptions:

$f:\Omega\times\mathbb{R}^n\to\mathbb{R}_+$ is a Carathéodory function, convex and of class C^1 with respect to ξ, satisfying the growth assumption

$$\bar{f}(x,|\xi_1|,\ldots,|\xi_n|) \leq f(x,\xi) \leq M\left\{1+\bar{f}(x,|\xi_1|,\ldots,|\xi_n|)\right\}, \qquad M>0,$$

with $\bar{f}:\Omega\times\mathbb{R}^n_+\to\mathbb{R}_+$, $\bar{f}=\bar{f}(x,z_1,...,z_n)$, satisfying (H1), (H2) and (H3), and such that, for some $\Lambda>0$,

$$\left|\frac{\partial f}{\partial \xi_i}(x,\xi)\right| \leq \Lambda\frac{\partial \bar{f}}{\partial z_i}(x,|\xi_1|,\ldots,|\xi_n|) \qquad \text{for all } \xi\in\mathbb{R}^n.$$

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