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Regularity of minimizers of vectorial integrals with $p - q$ growth

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Abstract

Local Lipschitz continuity of local minimizers of vectorial integrals

$$\int_\Omega f(x, Du(x))\,\mathrm{d}x$$

is proved when f satisfies $p - q$ growth condition and $\xi \mapsto f(x, \xi)$ is convex. The uniform convexity and the radial structure condition with respect to the last variable are assumed only at infinity.

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1. Introduction

In this paper we study the Lipschitz continuity of local minimizers of non-homogeneous integrals

$$\mathscr{F}(u; \Omega) = \int_\Omega f(x, Du(x))\,\mathrm{d}x, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, Du denotes the gradient of a vector-valued function $u : \Omega \to \mathbb{R}^N$ and $n,\ N > 1$.

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We say that $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local minimizer of (1.1) if $f(x, Du) \in L^1_{\mathrm{loc}}(\Omega)$ and $\mathscr{F}(u; \operatorname{spt}\varphi) \leqslant \mathscr{F}(u + \varphi; \operatorname{spt}\varphi)$ for any $\varphi \in W^{1,1}(\Omega, \mathbb{R}^N)$ with $\operatorname{spt}\varphi \subset\subset \Omega$.

Assume that $f : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$ is a Carathéodory function, convex with respect to the last variable, satisfying the following conditions

(A1) There exist $R > 0$ and a function $\tilde{f}$ such that for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{nN} \setminus B_R(0)$

$$f(x, \xi) = \tilde{f}(x, |\xi|),$$

(A2) f is *p-uniformly convex at infinity*, that is there exist $p > 1$ and $\nu > 0$ such that, for a.e. $x \in \Omega$ and for every $\xi_1, \xi_2 \in \mathbb{R}^{nN} \setminus B_R(0)$ endpoints of a segment contained in the complement of $B_R(0)$,

$$\frac{1}{2}[f(x, \xi_1) + f(x, \xi_2)] \geqslant f\left(x, \frac{\xi_1 + \xi_2}{2}\right) + \nu(1 + |\xi_1|^2 + |\xi_2|^2)^{(p-2)/2}|\xi_1 - \xi_2|^2, \tag{1.2}$$

(A3) There exist $L > 0$ and $q \geqslant p$ such that

$$f(x, \xi) \leqslant L(1 + |\xi|)^q,$$

for a.e. $x \in \Omega$ and $\xi \in \mathbb{R}^{nN}$,

(A4) For a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{nN} \setminus B_R(0)$ let $D_t^+ \tilde{f}(x, |\xi|)$ be the right side derivative of $\tilde{f}$ with respect to t and denote $D^+_{\xi_i^\alpha} f(x, \xi) = D_t^+ \tilde{f}(x, |\xi|)\xi_i^\alpha / |\xi|$. Then for every $\xi \in \mathbb{R}^{nN} \setminus B_R(0)$, the vector field $x \mapsto D^+_\xi f(x, \xi)$ is weakly differentiable and

$$|D_x D^+_\xi f(x, \xi)| \leqslant L(1 + |\xi|)^{q-1}.$$

Assumption (A2) was introduced by Fonseca–Fusco–Marcellini [7] in order to study the existence of minimizers of some non-convex variational problems. This condition, together with (A3), implies that there exist two positive constants c_0, c_1 such that

$$-c_0 + c_1|\xi|^p \leqslant f(x, \xi) \leqslant L(1 + |\xi|)^q$$

(see Theorem 2.5 (i) below), thus f satisfies the so called $p - q$ growth condition.

The main result of this paper is the following.

Theorem 1.1. *Let u be a local minimizer of* (1.1), *whose integrand f satisfies the assumptions* (A1)–(A4), $1 < p \leqslant q < p(n+1)/n$. *Then u is locally Lipschitz continuous and for all $B_r(x_0) \subset\subset \Omega$,*

$$\sup_{B_{r/4}(x_0)} |Du| \leqslant c\left[\int_{B_r(x_0)} (1 + f(x, Du))\,\mathrm{d}x\right]^\beta, \tag{1.3}$$

where $c = c(n, p, q, L, R, \nu)$, and $\beta = \beta(n, p, q)$.

When f is not convex, the previous theorem still holds provided

$$\mathrm{sc}^{-}\mathscr{F}(w;\Omega)=\int_{\Omega} f^{**}(x,Dw)\,\mathrm{d}x, \tag{1.4}$$

where $\mathrm{sc}^{-}\mathscr{F}$ is the relaxed functional of $\mathscr{F}$ and $\xi\mapsto f^{**}(x,\xi)$ is the convex envelope of $f(x,\cdot)$. In fact by Theorem 2.5(iv) f^{**} fulfills (A1)–(A4) with suitable constants v and R. We refer to [3] for some cases where (1.4) holds.

The result stated in Theorem 1.1, which obviously implies the non-occurrence of the Lavrentiev phenomenon, is proved via an approximation procedure, mainly using assumptions (A1) and (A2). It covers some interesting models in fluids mechanics and in nonlinear elasticity, see [12,19] and applies also to weak solutions of nonlinear elliptic systems in divergence form

$$\sum_{i=1}^{n} D_i a_i^{\alpha}(x,Du)=0 \quad \text{for } \alpha=1,\dots,N,$$

where $a_i^{\alpha}(x,\xi)=f_{\xi_i^{\alpha}}(x,\xi)$. Notice that by (A2) a_i^{α} may be degenerate elliptic for small deformations.

It is well known that, if $N>1$, generally only partial regularity of local minimizers of (1.1) may be achieved. However, starting from the seminal paper by Uhlenbeck [20], everywhere regularity results were proved under standard growth conditions, $p=q$, provided $f=f(|\xi|)$: in particular Giaquinta–Modica in [11] ($p\geqslant 2$) and Acerbi–Fusco in [1] ($1<p<2$) assume that f is of class C^2 and

$$v(1+|\xi|^2)^{(p-2)/2}|\lambda|^2\leqslant\langle D^2f(\xi)\lambda,\lambda\rangle\leqslant L(1+|\xi|^2)^{(p-2)/2}|\lambda|^2 \tag{1.5}$$

for all $\xi,\lambda\in\mathbb{R}^{nN}$. Using an approximation argument introduced by Fonseca–Fusco [6], Lipschitz continuity of local minimizers can still be achieved even if f is only continuous, see [8]. The ellipticity condition in (1.5) is replaced by the p-uniform convexity assumption, that is inequality (1.2) holds for any $x\in\Omega$ and $\xi_1,\xi_2\in\mathbb{R}^{nN}$.

The non-standard context $p<q$ is more recent. Firstly studied by Marcellini [14], various contributions to the subject have been provided in these last years, see [4,15,16] for an extensive list of references. The vectorial homogeneous framework was considered by Marcellini [17] when f is smooth, and by Esposito et al. [5] when f is continuous.

It is well known that in order to get regularity a restriction between p and q need to be imposed, namely $q\leqslant c(n)p$, with $c(n)$ close to 1, see [9,13] for counterexamples. In the non-homogeneous case the bound revealing the sharpness of our result can be found. Esposito et al. [4] provide an example from which it can be deduced that Theorem 1.1 does not hold if $q>p(n+1)/n$, since local minimizers may not belong to $W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$.

Also some regularity conditions with respect to x are needed. A functional $\mathscr{F}$ exists, with f only measurable with respect to x, such that its minimizers do not belong to $W^{1,q}(\Omega)$ (see [21]). On the other hand, when the same functional satisfies a suitable continuity assumption with respect to x, the minimizers are regular, see [2]. Recently, Mascolo and Migliorini [18] proved a Lipschitz continuity result for minimizers of smooth and strictly convex functionals, with $p-q$ growth and $p\geqslant 2$. There, also functionals with exponential growth are considered.

Let us briefly discuss the techniques used to prove Theorem 1.1. We construct a sequence of regular functions f_{kh} satisfying (A1)–(A4), p-growth condition, with constants independent of h, and (1.5), with suitable constants depending on h. Fixed a ball $B_r(x_0) \subset\subset \Omega$, we consider the following sequence of variational problems:

$$\min\left\{\int_{B_r(x_0)} f_{kh}(x, Dw)\,\mathrm{d}x : w \in u + W_0^{1,p}(B_r(x_0), \mathbb{R}^N)\right\},$$

where u is a local minimizer of (1.1). For the corresponding minimizers u_{kh} we prove a uniform (in k and h) estimate for the L^∞-norm of the gradient. The passage to the limit, with respect to h and k, is not influenced by the behaviour of f on bounded subsets of $\Omega \times \mathbb{R}^{nN}$. This and a suitable comparison argument, introduced in [7], permit to transfer the regularity properties of the minimizers u_{kh} to the local minimizer u.

The paper is organized as follows: in Section 2 we study some properties of functions uniformly convex at infinity. Section 3 is devoted to the proof of the a priori estimates for minimizers of regular functionals, while in Section 4 we construct the approximating variational problems and we prove Theorem 1.1.

2. Uniformly convex functions

In this section we prove some properties satisfied by functions uniformly convex at infinity. For the sake of clarity we first consider the case when $f : \mathbb{R}^m \to [0, +\infty)$ is a continuous function, $f = f(\xi)$, satisfying the following assumption:

(UC) there exist $p > 1$ and $\nu, R > 0$ such that for every $\xi_1, \xi_2 \in \mathbb{R}^m \backslash B_R(0)$, endpoints of a segment contained in the complement of $B_R(0)$,

$$\frac{1}{2}[f(\xi_1) + f(\xi_2)] \geqslant f\left(\frac{\xi_1 + \xi_2}{2}\right) + \nu(1 + |\xi_1|^2 + |\xi_2|^2)^{(p-2)/2}|\xi_1 - \xi_2|^2. \quad (2.1)$$

Some of the results are stated in [7]. Here we give alternative proofs, taking into account neither the growth of f nor its behavior in $B_R(0)$.

When f is smooth, uniform ellipticity and uniform convexity are equivalent conditions (see [7] for the proof).

Lemma 2.1. *Let $f : \mathbb{R}^m \to [0, +\infty)$ be of class C^2. Then f satisfies* (UC) *if and only if there exists $c(\nu) > 0$ such that for all $\xi \in \mathbb{R}^m \backslash B_R(0)$ and $\lambda \in \mathbb{R}^m$*

$$\langle D_{\xi\xi} f(\xi)\lambda, \lambda\rangle \geqslant c(1 + |\xi|^2)^{(p-2)/2}|\lambda|^2. \quad (2.2)$$

Moreover, (2.1) *holds for all ξ_1, ξ_2 in $\mathbb{R}^m$ if and only if* (2.2) *holds for all ξ and λ in $\mathbb{R}^m$.*

Now, we prove that the p-uniform convexity at infinity implies that f is p-coercive.

Lemma 2.2. *Let $f : \mathbb{R}^m \to [0, +\infty)$ satisfy* (UC)*. Then there exist two positive constants $c_0(p, \nu, R, \max_{\partial B_R} f)$ and $c_1(\nu)$ such that*

$$f(\xi) \geqslant -c_0 + c_1|\xi|^p. \quad (2.3)$$

Proof. For every $\xi \in \mathbb{R}^m \setminus B_R(0)$ define $\xi_0 = R\xi/|\xi|$. From (UC) it follows

$$\frac{1}{2}[f(\xi)+f(\xi_0)] \geqslant f\left(\frac{\xi+\xi_0}{2}\right) + v(1+|\xi|^2+|\xi_0|^2)^{(p-2)/2}|\xi-\xi_0|^2.$$

On the other hand, Young inequality yields

$$(1+|\xi|^2+|\xi_0|^2)^{(p-2)/2}|\xi-\xi_0|^2 \geqslant \frac{1}{2}|\xi|^p - c(p,R).$$

Since $f \geqslant 0$, we get

$$f(\xi) \geqslant \frac{v}{2}|\xi|^p - 2c(p,R)v - \max_{\partial B_R} f$$

and (2.3) follows. □

The next proposition describes the "splitting" property of functions p-uniformly convex at infinity.

Proposition 2.3. *Let* $f:\mathbb{R}^m \to [0,+\infty)$. (UC) *is equivalent to*

(S) *There exist* c_0 *and* c_1 *positive constants and a function* $g:\mathbb{R}^m \to [-c_0,+\infty)$ *such that for every* $\xi \in \mathbb{R}^m$,

$$f(\xi) = c_1(1+|\xi|^2)^{p/2} + g(\xi)$$

and for every $\xi_1,\xi_2 \in \mathbb{R}^m$ *such that* $[\xi_1,\xi_2] \subseteq \mathbb{R}^m \setminus B_R(0)$

$$\frac{1}{2}[g(\xi_1)+g(\xi_2)] \geqslant g\left(\frac{\xi_1+\xi_2}{2}\right).$$

Proof. (UC) ⇒ (S). It is easy to check that there exists $c(p)>0$ such that for all $\xi_1,\xi_2 \in \mathbb{R}^m$,

$$(1+|\xi_1|^2)^{p/2} + (1+|\xi_2|^2)^{p/2} \leqslant 2\left(1+\left|\frac{\xi_1+\xi_2}{2}\right|^2\right)^{p/2} + c(p)(1+|\xi_1|^2+|\xi_2|^2)^{(p-2)/2}|\xi_1-\xi_2|^2. \quad (2.4)$$

Moreover, from Lemma 2.2,

$$f(\xi) \geqslant -c_0 + \tilde{c}_1(1+|\xi|^2)^{p/2}$$

for some positive constants c_0 and $\tilde{c}_1$.

Let $c_1 = \min\{\tilde{c}_1, 2v/c(p)\}$. The function

$$g(\xi) := f(\xi) - c_1(1+|\xi|^2)^{p/2}$$

is continuous and $g \geqslant -c_0$. (UC) and (2.4) imply

$$\frac{1}{2}[g(\xi_1)+g(\xi_2)] \geqslant g\left(\frac{\xi_1+\xi_2}{2}\right) + \left(v - c(p)\frac{c_1}{2}\right)(1+|\xi_1|^2+|\xi_2|^2)^{(p-2)/2}|\xi_1-\xi_2|^2,$$

for all ξ_1,ξ_2 with $[\xi_1,\xi_2] \subseteq \mathbb{R}^m \setminus B_R(0)$.

(S) $\Rightarrow$ (UC). From Lemma 2.1, the function $(1+|\xi|^2)^{p/2}$ is p-uniformly convex in $\mathbb{R}^m$, so (S) implies (UC). $\square$

The next result was proved in [7] for functions satisfying (UC) and with standard growth, but the proof seems not to work when f has non-standard growth, e.g. $p-q$ growth, with $q > p$. So we give an alternative proof which does not make use of any growth assumption on f.

Proposition 2.4. *Let* $f:\mathbb{R}^m \to [0,+\infty)$ *satisfy* (UC). *Then there exist* v_0, $R_0 > 0$ *depending only on* p, v, R *and* $\max_{\partial B_R(0)} f$, *such that for any* $\xi \in \mathbb{R}^m \setminus B_{R_0}(0)$ *there exists* $q_\xi \in \mathbb{R}^m$, *such that*

$$f(\eta) \geqslant f(\xi) + \langle q_\xi, \eta - \xi\rangle + v_0(1+|\xi|^2+|\eta|^2)^{(p-2)/2}|\xi-\eta|^2 \tag{2.5}$$

for all $\eta \in \mathbb{R}^m$.

Moreover, if f^{**} *is the convex envelope of* f, *then* $f = f^{**}$ *in* $\mathbb{R}^m \setminus B_{R_0}(0)$.

Proof. We divide the proof into two steps.

Step 1: Assume that f is a convex function. For $0<\varepsilon<1$, let us define $f_\varepsilon = \rho_\varepsilon * f$, with $\rho_\varepsilon(\eta) = \varepsilon^{-m}\rho(\eta/\varepsilon)$, where ρ is a positive radially symmetric mollifier with support in $B = B_1(0)$, such that $\int_B \rho = 1$.

Let $[\xi_1, \xi_2] \subseteq \mathbb{R}^m \setminus B_{R+1}$ and $\xi_0 = (\xi_1+\xi_2)/2$. Since

$$\int_B (1+|\xi_1+\varepsilon\eta|^2+|\xi_2+\varepsilon\eta|^2)^{(p-2)/2}\,\mathrm{d}\eta \geqslant c(p)(1+|\xi_1|^2+|\xi_2|^2)^{(p-2)/2},$$

then

$$\frac{1}{2}[f_\varepsilon(\xi_1)+f_\varepsilon(\xi_2)] \geqslant f_\varepsilon(\xi_0) + c(p,v)(1+|\xi_1|^2+|\xi_2|^2)^{(p-2)/2}|\xi_1-\xi_2|^2.$$

Thus, f_ε is p-uniformly convex at infinity and, by Lemma 2.1, it follows that for all $|\xi| \geqslant R+1$ and $\lambda \in \mathbb{R}^m$,

$$\langle D_{\xi\xi} f_\varepsilon(\xi)\lambda, \lambda\rangle \geqslant c(p,v)(1+|\xi|^2)^{(p-2)/2}|\lambda|^2. \tag{2.6}$$

Fix $\xi \in \mathbb{R}^m \setminus B_{R_0}$, with $R_0 = 2(R+1)$.

For all $\eta \in \mathbb{R}^m$, define $A_{\xi,\eta} = \{t : |\xi + t(\eta-\xi)| > R+1\}$. The convexity of f_ε implies

$$\begin{aligned} f_\varepsilon(\eta) &= f_\varepsilon(\xi) + \langle Df_\varepsilon(\xi), \eta-\xi\rangle \\ &\quad + \int_0^1 (1-t)\langle D^2 f_\varepsilon(\xi+t(\eta-\xi))(\eta-\xi), \eta-\xi\rangle\,\mathrm{d}t \\ &\geqslant f_\varepsilon(\xi) + \langle Df_\varepsilon(\xi), \eta-\xi\rangle \\ &\quad + \int_{A_{\xi,\eta}} (1-t)\langle D^2 f_\varepsilon(\xi+t(\eta-\xi))(\eta-\xi), \eta-\xi\rangle\,\mathrm{d}t. \end{aligned} \tag{2.7}$$

Let us prove that (2.5) holds for all $\eta \in \mathbb{R}^m$ such that $|\xi| > |\eta|$.

When $0 < t < \frac{1}{4}$,

$$|\xi + t(\eta-\xi)| \geqslant (1-t)|\xi| - t|\eta| > (1-2t)R_0 > R+1, \tag{2.8}$$

moreover there exists $c_1 > 0$ such that

$$c_1(|\xi|^2 + |\eta|^2) \leqslant |\xi + t(\eta - \xi)|^2 \leqslant 2(|\xi|^2 + |\eta|^2). \tag{2.9}$$

From (2.8), $(0, \frac{1}{4}) \subset A_{\xi,\eta}$ and, by (2.6) and (2.9),

$$\begin{aligned}\int_{A_{\xi,\eta}} &(1-t)\langle D^2 f_\varepsilon(\xi + t(\eta - \xi))(\eta - \xi), \eta - \xi\rangle \, dt \\ &\geqslant c(p, v) \int_0^{1/4} (1-t)(1 + |\xi + t(\eta - \xi)|^2)^{(p-2)/2} |\eta - \xi|^2 \, dt \\ &\geqslant c(p, v)(1 + |\xi|^2 + |\eta|^2)^{(p-2)/2} |\eta - \xi|^2.\end{aligned} \tag{2.10}$$

Collecting (2.7) and (2.10), we get

$$f_\varepsilon(\eta) \geqslant f_\varepsilon(\xi) + \langle D f_\varepsilon(\xi), \eta - \xi\rangle + c(p, v)(1 + |\xi|^2 + |\eta|^2)^{(p-2)/2} |\xi - \eta|^2. \tag{2.11}$$

Since f is locally Lipschitz continuous, then for all $\lambda \in B_1(\xi)$,

$$|f_\varepsilon(\lambda) - f_\varepsilon(\xi)| \leqslant \int_B |f(\lambda + \varepsilon y) - f(\xi + \varepsilon y)|\rho(y) \, dy \leqslant c(\xi)|\lambda - \xi|,$$

so that $(D_\varepsilon f(\xi))_\varepsilon$ is bounded with respect to ε. Thus, up to a subsequence, $D f_\varepsilon(\xi)$ converges to a certain $q_\xi \in \mathbb{R}^m$ when $\varepsilon \to 0$. (2.11) implies (2.5) when $|\xi| > |\eta|$.

Let now $\eta \in \mathbb{R}^m$ be such that $|\xi| \leqslant |\eta|$.

If $t > \frac{3}{4}$, then $|\xi + t(\eta - \xi)|$ is greater than $R + 1$. Hence $(\frac{3}{4}, 1) \subset A_{\xi,\eta}$ and

$$\begin{aligned}\int_{A_{\xi,\eta}} &(1-t)\langle D^2 f_\varepsilon(\xi + t(\eta - \xi))(\eta - \xi), \eta - \xi\rangle \, dt \\ &\geqslant c(p, v) \int_{3/4}^{1} (1-t)(1 + |\xi + t(\eta - \xi)|^2)^{(p-2)/2} |\eta - \xi|^2 \, dt.\end{aligned}$$

Reasoning as above, we get (2.5).

Step 2: Let now f be a continuous function satisfying (UC). Set $M = \max_{\partial B_R} f$ and define

$$h(\xi) := \begin{cases} M & \text{if } |\xi| \leqslant R, \\ \max\{M, f(\xi)\} & \text{if } |\xi| > R. \end{cases}$$

We claim that h is convex, i.e. given $\xi_1, \xi_2 \in \mathbb{R}^m$ and $\xi_0 = (\xi_1 + \xi_2)/2$, then

$$h(\xi_0) \leqslant \frac{1}{2}[h(\xi_1) + h(\xi_2)]. \tag{2.12}$$

If $\xi_0 \in B_R$, (2.12) is trivial being $h \geqslant M$. Let $|\xi_0| \geqslant R$. If $[\xi_1, \xi_2]$ is subset of $\mathbb{R}^m \backslash B_R$, then (2.12) follows from (UC) and $h \geqslant f$; if not, either the segment $[\xi_0, \xi_1] \subset \mathbb{R}^m \backslash B_R$ or else $[\xi_0, \xi_2] \subset \mathbb{R}^m \backslash B_R$. Suppose the last one, the other one being equivalent. Clearly, there exists $\gamma \in \partial B_R$ such that $\xi_0 \in [\gamma, \xi_2] \subset [\xi_1, \xi_2]$. Let $\tau \in (0, 1/2]$ be such that $\lambda = \tau\gamma + (1 - \tau)\xi_2$. Being h convex on $[\gamma, \xi_2]$ and $M = h(\gamma) = \min_{[\gamma, \xi_2]} h$, we get

$$h(\xi_0) \leqslant h\left(\frac{\gamma + \xi_2}{2}\right) \leqslant \frac{1}{2}[M + h(\xi_2)],$$

so (2.12) follows.

By Lemma 2.2, there exists $R_1 \geqslant R$, depending on p, v, R and M, such that $h(\xi) = f(\xi)$ when $|\xi| \geqslant R_1$, which implies that h is uniformly convex in $\mathbb{R}^m \backslash B_{R_1}(0)$. From step 1, there exist $R_2 \geqslant R_1$, $v_0(p,v) > 0$ and a vector q_ξ such that for $|\xi| > R_2$ and $\eta \in \mathbb{R}^m$,

$$\begin{aligned} h(\eta) &\geqslant h(\xi) + \langle q_\xi, \eta - \xi\rangle + v_0(1+|\xi|^2+|\eta|^2)^{(p-2)/2}|\xi-\eta|^2 \\ &= f(\xi) + \langle q_\xi, \eta - \xi\rangle + v_0(1+|\xi|^2+|\eta|^2)^{(p-2)/2}|\xi-\eta|^2, \end{aligned} \tag{2.13}$$

therefore (2.5) holds, when $|\eta| > R_1$.

Let $|\eta| \leqslant R_1$. Since for some $c_1, c_2 > 0$,

$$(1+|\xi|^2+|\eta|^2)^{(p-2)/2}|\xi-\eta|^2 \geqslant c_1(p,v)|\xi|^p - c_2(p,R_1,v), \tag{2.14}$$

taking into account (2.13), we have

$$h(\eta) - c_1|\xi|^p + c_2 \geqslant f(\xi) + \langle q_\xi, \eta-\xi\rangle + \frac{v_0}{2}(1+|\xi|^2+|\eta|^2)^{(p-2)/2}|\xi-\eta|^2.$$

We choose $R_0 \geqslant R_2$ such that, when $|\xi| > R_0$, for every $\eta \in \bar{B}_{R_1}(0)$ is $h(\eta) - c_1|\xi|^p + c_2 < 0$. Since $f \geqslant 0$, (2.5) follows.

Fixed $\xi \in \mathbb{R}^m \backslash B_{R_0}(0)$, for any $\eta \in \mathbb{R}^m$ define

$$v_\xi(\eta) := f(\xi) + \langle q_\xi, \eta - \xi\rangle.$$

v_ξ is convex and, from (2.5), $v_\xi(\eta) \leqslant f(\eta)$. Then $v_\xi(\eta) \leqslant f^{**}(\eta) \leqslant f(\eta)$ where f^{**} denotes the convex envelope of f. Therefore $f(\xi) = f^{**}(\xi)$. □

The previous results still hold when f is not homogeneous and satisfies (A3). The proof is straightforward and is left to the reader.

Theorem 2.5. *Let $f : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$, $f = f(x,\xi)$, be a Carathéodory function. Then* (A2) *and* (A3) *imply*

(i) *There exist $c_0(p,q,v,R,L)$ and $c_1(v)$ such that for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{nN}$,*

$$f(x,\xi) \geqslant -c_0 + c_1|\xi|^p,$$

(ii) *There exist $c_0(p,q,v,R,L)$, $c_1(p,v) > 0$ and a Carathéodory function $g : \Omega \times \mathbb{R}^{nN} \to [-c_0, +\infty)$ such that for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{nN}$,*

$$f(x,\xi) = c_1(1+|\xi|^2)^{p/2} + g(x,\xi). \tag{2.15}$$

Moreover, for a.e. $x \in \Omega$ and every $\xi_1, \xi_2 \in \mathbb{R}^{nN}$ such that $[\xi_1, \xi_2] \subseteq \mathbb{R}^{nN} \backslash B_R(0)$,

$$\frac{1}{2}[g(x,\xi_1) + g(x,\xi_2)] \geqslant g\left(x, \frac{\xi_1+\xi_2}{2}\right),$$

(iii) *There exist R_0, $v_0 > 0$ depending only on p, q, v, R and L, such that for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{nN} \backslash B_{R_0}(0)$ there exists $q_\xi(x) \in \mathbb{R}^{nN}$, $|q_\xi(x)| \leqslant c(q,L)(1+|\xi|)^{q-1}$, such that for all $\eta \in \mathbb{R}^{nN}$,*

$$f(x,\eta) \geqslant f(x,\xi) + \langle q_\xi(x), \eta - \xi\rangle + v_0(1+|\xi|^2+|\eta|^2)^{(p-2)/2}|\xi-\eta|^2.$$

Moreover, if $\xi \mapsto f(x,\xi)$ is $C^1(\mathbb{R}^{nN} \backslash B_R(0))$, then $q_\xi(x) = D_\xi f(x,\xi)$,

(iv) *If* $\xi \mapsto f^{**}(x,\xi)$ *is the convex envelope of* $\xi \mapsto f(x,\xi)$, *then* $f = f^{**}$ *in* $\Omega \times (\mathbb{R}^{nN} \backslash B_{R_0}(0))$.

Moreover, if f *satisfies* (ii) *for some* c_0, c_1 *and* R, *then* (A2) *holds, with the same* R *and with* ν *depending only on* p *and* c_1.

Theorem 2.5 (iii) enables us to compare different local minimizers of functionals whose densities satisfy the uniform convexity at infinity. The following proposition is essentially contained in the proof of Theorem 2.7 in [7]. For the sake of completeness we give a detailed proof.

Lemma 2.6. *Let* u *be a local minimizer of*

$$\mathscr{F}(w;\Omega) = \int_\Omega f(x, Dw(x))\,\mathrm{d}x,$$

where $f : \Omega \times \mathbb{R}^{nN} \to [0,+\infty)$ *is a Carathéodory function, convex with respect to the last variable, satisfying* (A2) *and* (A3). *Suppose that there exist* $B_r(x_0) \subset\subset \Omega$ *and* $v \in u + W_0^{1,1}(B_r(x_0), \mathbb{R}^N)$ *such that* $\mathscr{F}(u; B_r(x_0)) = \mathscr{F}(v; B_r(x_0))$. *Then there exists* $R_0(p,q,\nu,R,L) > R$ *such that the Lebesgue measure of the set*

$$\{x \in B_r(x_0) : |Du(x)+Dv(x)| > 2R_0 \text{ and } |Du(x)-Dv(x)| > 0\}$$

is zero.

Proof. From Theorem 2.5 (iii) there exist R_0, $\nu_0 > 0$ such that for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{nN} \backslash B_{R_0}(0)$

$$f(x,\eta) \geqslant f(x,\xi) + \langle q_\xi(x), \eta - \xi \rangle + \nu_0 (1 + |\xi|^2 + |\eta|^2)^{(p-2)/2} |\xi - \eta|^2 \tag{2.16}$$

for some $q_\xi(x) \in \mathbb{R}^{nN}$ and for all $\eta \in \mathbb{R}^{nN}$. Set $A = \{x \in B_r(x_0) : |Du + Dv| > 2R_0\}$. For a.e. $x \in A$ consider (2.16) with $\xi = (Du(x) + Dv(x))/2$ and $\eta = Du(x)$,

$$\begin{aligned} f(x,Du) - f\left(x, \frac{Du+Dv}{2}\right) &\geqslant \left\langle q_\xi(x), \frac{Du-Dv}{2} \right\rangle \\ &\quad + \nu_0 \left(1 + \left|\frac{Du+Dv}{2}\right|^2 + |Du|^2\right)^{(p-2)/2} \left|\frac{Du-Dv}{2}\right|^2 \end{aligned} \tag{2.17}$$

and (2.16) applied with $\eta = Dv(x)$,

$$f(x,Dv) - f\left(x, \frac{Du+Dv}{2}\right) \geqslant \left\langle q_\xi(x), \frac{Dv-Du}{2} \right\rangle$$

$$+\nu_0\left(1+\left|\frac{Du+Dv}{2}\right|^2+|Dv|^2\right)^{(p-2)/2}\left|\frac{Du-Dv}{2}\right|^2; \tag{2.18}$$

thus, summing (2.17) and (2.18) and integrating on A, we get

$$\begin{aligned}
&\int_A [f(x,Du)+f(x,Dv)]\,\mathrm{d}x\\
&\quad\geqslant 2\int_A f\left(x,\frac{Du+Dv}{2}\right)\mathrm{d}x+\nu_0\int_A\left(1+\left|\frac{Du+Dv}{2}\right|^2+|Du|^2\right)^{(p-2)/2}\\
&\qquad\times\left|\frac{Du-Dv}{2}\right|^2\mathrm{d}x\\
&\qquad+\nu_0\int_A\left(1+\left|\frac{Du+Dv}{2}\right|^2+|Dv|^2\right)^{(p-2)/2}\left|\frac{Du-Dv}{2}\right|^2\mathrm{d}x.
\end{aligned} \tag{2.19}$$

The convexity of f with respect to ξ and (2.19) give

$$\begin{aligned}
&\mathscr{F}\left(\frac{u+v}{2};B_r\right)-\frac{1}{2}\mathscr{F}(u;B_r)-\frac{1}{2}\mathscr{F}(v;B_r)\\
&\quad\leqslant-\frac{\nu_0}{2}\int_A\left(1+\left|\frac{Du+Dv}{2}\right|^2+|Du|^2\right)^{(p-2)/2}\left|\frac{Du-Dv}{2}\right|^2\mathrm{d}x\\
&\qquad-\frac{\nu_0}{2}\int_A\left(1+\left|\frac{Du+Dv}{2}\right|^2+|Dv|^2\right)^{(p-2)/2}\left|\frac{Du-Dv}{2}\right|^2\mathrm{d}x
\end{aligned}$$

and since $\mathscr{F}(u;B_r)=\mathscr{F}(v;B_r)$ the thesis follows. □

3. A priori estimates

In order to prove Theorem 1.1, we prove some regularity results for minimizers of functionals with smooth integrands.

Let us consider

$$\mathscr{F}(u;\Omega)=\int_\Omega f(x,Du)\,\mathrm{d}x, \tag{3.1}$$

where $f:\Omega\times\mathbb{R}^{nN}\to[0,+\infty)$, $f=f(x,\xi)$, is convex with respect to ξ, satisfies (A1)–(A4) and the supplementary assumptions

(H1) f is in $C^2(\Omega\times\mathbb{R}^{nN})$,

(H2) There exists $K>0$ such that for every $(x,\xi)\in\Omega\times\mathbb{R}^{nN}$,

$$f(x,\xi)\leqslant K(1+|\xi|^2)^{p/2},$$

(H3) There exist ε_0 and Λ positive constants such that for every $x \in \Omega$ and for every $\xi, \lambda \in \mathbb{R}^{nN}$,

$$\varepsilon_0(1+|\xi|^2)^{(p-2)/2}|\lambda|^2 \leqslant \langle D_{\xi\xi} f(x,\xi)\lambda,\lambda\rangle \leqslant \Lambda(1+|\xi|^2)^{(p-2)/2}|\lambda|^2,$$

(H4) For every $(x,\xi) \in \Omega \times \mathbb{R}^{nN}$.

$$|D_{\xi x} f(x,\xi)| \leqslant K(1+|\xi|^2)^{(p-1)/2}.$$

The following result holds:

Proposition 3.1. *Let u be a local minimizer of* (3.1), *where f satisfies* (A1), (A2) *and* (H1)–(H4). *Fixed $B_r(x_0) \subset\subset \Omega$, there exists $C = C(n,p,v,R,K,r) > 0$ such that*

$$\sup_{B_{r/4}(x_0)} |Du| \leqslant C\left[\int_{B_r(x_0)} (1+|Du|^2)^{p/2}\,\mathrm{d}x\right]^{1/p}. \tag{3.2}$$

We postpone the proof of Proposition 3.1 to some preliminary results.

Next lemma can be proved as Lemma 4.3 in [8]. From (A1), we define $\bar{f}$ such that, for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{nN} \backslash B_R(0)$, $f(x,\xi) = \bar{f}(x,|\xi|^2)$.

Lemma 3.2. *Let $f : \Omega \times \mathbb{R}^{nN} \to [0,+\infty)$ be of class C^2 and let* (A1) *and* (A2) *hold. Then for all $x \in \Omega$, $\xi \in \mathbb{R}^{nN} \backslash B_R(0)$ and $\lambda \in \mathbb{R}^n$,*

$$\sum_{\alpha,s,i} 4\bar{f}_{tt}(x,|\xi|^2)\xi_s^\alpha \xi_i^\alpha \lambda_s\lambda_i + 2\bar{f}_t(x,|\xi|^2)|\lambda|^2 \geqslant c(v)(1+|\xi|^2)^{(p-2)/2}|\lambda|^2.$$

The following result extends to our case some integrability properties for local minimizers proved in [11] ($p \geqslant 2$) and in [1] ($1 < p < 2$) when $f = f(\xi)$. We omit the proof which closely follows that of Lemma 2.5 in [1].

Lemma 3.3. *Let u be a local minimizer of* (3.1), *with f satisfying* (H1)–(H4). *Then $u \in W^{2,a}_{\mathrm{loc}}(\Omega)$, with $a = \min\{2,p\}$, and $(1+|Du|^2)^{(p-2)/2}|D^2u|^2 \in L^1_{\mathrm{loc}}(\Omega)$.*

Next lemma concerns the validity of Euler systems. It generalizes Lemma 4.1 in [5] to non-homogeneous energy densities, with a similar proof that we omit.

Lemma 3.4. *Let u be a local minimizer of* (3.1), *with f satisfying* (H1)–(H4). *Let $B_r(x_0) \subset\subset \Omega$ and suppose that Θ is a $W^{1,1}(B_r(x_0),\mathbb{R}^N)$ function with compact support in B_r, such that*

$$D_{\xi_i^\alpha} f(x,Du)\Theta^\alpha, \quad D_i(D_{\xi_i^\alpha} f(x,Du))\Theta^\alpha, \quad D_{\xi_i^\alpha} f(x,Du)D_i\Theta^\alpha$$

belong to $L^1(B_r)$, for all $i = 1,\dots,n$, $\alpha = 1,\dots,N$.

Then, for all α,

$$\sum_i \int_{B_r(x_0)} D_{\xi_i^\alpha} f(x,Du)D_i\Theta^\alpha\,\mathrm{d}x = 0.$$

Proof of Proposition 3.1. We divide it into two steps.

Step 1: Let us consider a cut-off function $\eta \in C_c^1(B_r)$, $0 \leqslant \eta \leqslant 1$. We prove that there exists c depending on p, ν, K such that for all $\delta > -1$ there exists

$$\int_{B_r} (1+|Du|^2)^{p/2+\delta-1} |D(|Du|^2 - R^2)_+|^2 \eta^2 \, dx$$
$$\leqslant c \frac{\max\{1, 1+\delta\}}{\min\{1, 1+\delta\}} \int_{B_r} (1+|Du|^2)^{p/2+\delta+1} (\eta^2 + |D\eta|^2) \, dx. \tag{3.3}$$

For any $M > 0$ let $\gamma_M \in C^\infty([0,+\infty),[0,1])$ be such that $\gamma_M \equiv 1$ in $[0, M]$, $\gamma_M(t) = 0$ if $t \geqslant 2M$, $|\gamma_M'(t)| \leqslant 2/M$.

Fixed $\delta > -1$ let us define $\Phi : [0,+\infty) \to [0,+\infty)$,

$$\Phi(t) := (1 + R^2 + t)^\delta t. \tag{3.4}$$

By a well-known property of summable functions there exists an increasing and divergent sequence (a_h) in $\mathbb{R}_+$ such that the Lebesgue measure of $\{x : (|Du(x)|^2 - R^2)_+ = a_h\}$ is zero.

For every $h \in \mathbb{N}$ let us define

$$\Phi_h(t) := \begin{cases} \Phi(t) & \text{if } t < a_h, \\ \Phi(a_h) & \text{otherwise.} \end{cases}$$

Notice that, for all h, $\Phi_h(0) = 0$, Φ_h and Φ_h' are non-negative, bounded and that if h goes to infinity they monotonically converge to Φ and Φ', respectively.

Recall that the difference quotient $\triangle_{s,h}$ in the direction e_s is defined by

$$\triangle_{s,h} v(x) := \frac{v(x + he_s) - v(x)}{h}.$$

For any h, M, $s = 1, \ldots, n$, $\alpha = 1, \ldots, N$ and $k > 0$, small enough, define

$$\Psi^\alpha_{h,M,s,k}(x) := \gamma_M(|Du(x)| + |Du(x + ke_s)|) \Phi_h((|Du(x)|^2 - R^2)_+) D_s u^\alpha(x)$$

and

$$\Theta^\alpha_{h,M,s,k}(x) := \eta^2(x) \triangle_{s,-k} \Psi^\alpha_{h,M,s,k}(x).$$

By Lemma 3.4, for all α,

$$\sum_i \int_{B_r} D_{\xi_i^\alpha} f(x, Du) D_i \Theta^\alpha_{h,M,s,k}(x) \, dx = 0. \tag{3.5}$$

For simplicity we shall write $\gamma_{M,k}$, γ_M, Φ_h in place of $\gamma_M(|Du(x)| + |Du(x + ke_s)|)$, $\gamma_M(2|Du|)$ and $\Phi_h((|Du(x)|^2 - R^2)_+)$, respectively. By (3.5), integrating by parts and summing on α and s we get

$$\sum_{i,\alpha,s} \int_{B_r} \triangle_{s,k}[D_{\xi_i^\alpha} f(x, Du)\eta^2] D_i \Psi^\alpha_{h,M,s,k} \, dx$$
$$= \sum_{i,\alpha,s} 2 \int_{B_r} D_{\xi_i^\alpha} f(x, Du) \eta D_i \eta \triangle_{s,-k} \Psi^\alpha_{h,M,s,k} \, dx. \tag{3.6}$$

From now on, we shall use the Einstein's convention for repeated indexes. Letting k go to 0 in (3.6) we get

$$\begin{aligned}
&\int_{B_r} D_{\xi_i^\alpha \xi_j^\beta} f(x,Du) D_{sj} u^\beta \eta^2 D_i[\gamma_M \Phi_h D_s u^\alpha]\,\mathrm{d}x \\
&\quad = -\int_{B_r} D_{x_s \xi_i^\alpha} f(x,Du)\eta^2 D_i[\gamma_M \Phi_h D_s u^\alpha]\,\mathrm{d}x \\
&\qquad -2\int_{B_r} D_{\xi_i^\alpha} f(x,Du)\eta D_s\eta D_i[\gamma_M \Phi_h D_s u^\alpha]\,\mathrm{d}x \\
&\qquad +2\int_{B_r} D_{\xi_i^\alpha} f(x,Du)\eta D_i\eta D_s[\gamma_M \Phi_h D_s u^\alpha]\,\mathrm{d}x
\end{aligned}$$

so that

$$\begin{aligned}
&\int_{B_r} D_{\xi_i^\alpha \xi_j^\beta} f(x,Du) D_{sj} u^\beta \eta^2 \gamma_M [\Phi_h D_{is} u^\alpha + \Phi_h' D_i((|Du|^2 - R^2)_+) D_s u^\alpha]\,\mathrm{d}x \\
&\quad = -2\int_{B_r} D_{\xi_i^\alpha \xi_j^\beta} f(x,Du) D_{sj} u^\beta \gamma_M' D_i(|Du|)\Phi_h D_s u^\alpha \eta^2\,\mathrm{d}x \\
&\qquad -2\int_{B_r} D_{x_s \xi_i^\alpha} f(x,Du)\Phi_h \gamma_M' D_i(|Du|) D_s u^\alpha \eta^2\,\mathrm{d}x \\
&\qquad -\int_{B_r} D_{x_s \xi_i^\alpha} f(x,Du)\gamma_M \eta^2 [\Phi_h' D_i((|Du|^2 - R^2)_+) D_s u^\alpha + \Phi_h D_{is} u^\alpha]\,\mathrm{d}x \\
&\qquad -2\int_{B_r} D_{\xi_i^\alpha} f(x,Du)\gamma_M \eta D_s\eta [\Phi_h D_{is} u^\alpha + \Phi_h' D_i((|Du|^2 - R^2)_+) D_s u^\alpha]\,\mathrm{d}x \\
&\qquad -2\int_{B_r} D_{\xi_i^\alpha} f(x,Du)\Phi_h \gamma_M' D_i(|Du|) D_s u^\alpha \eta D_s\eta\,\mathrm{d}x \\
&\qquad +2\int_{B_r} D_{\xi_i^\alpha} f(x,Du)\gamma_M \eta D_i\eta [\Phi_h D_{ss} u^\alpha + \Phi_h' D_s((|Du|^2 - R^2)_+) D_s u^\alpha]\,\mathrm{d}x \\
&\qquad +2\int_{B_r} D_{\xi_i^\alpha} f(x,Du)\Phi_h \gamma_M' D_s u^\alpha D_s(|Du|)\eta D_i\eta\,\mathrm{d}x.
\end{aligned} \tag{3.7}$$

Recalling that Φ_h stands for $\Phi_h((|Du(x)|^2 - R^2)_+)$ and that $\Phi_h(0)=0$, (A2) and Lemma 2.1 imply

$$\begin{aligned}
&\int_{B_r} D_{\xi_i^\alpha \xi_j^\beta} f(x,Du) D_{js} u^\beta D_{is} u^\alpha \Phi_h \gamma_M \eta^2\,\mathrm{d}x \\
&\quad \geqslant c(v)\int_{B_r} (1+|Du|^2)^{(p-2)/2} |D^2 u|^2 \Phi_h \gamma_M \eta^2\,\mathrm{d}x.
\end{aligned}$$

On the other hand, since

$$D_{\xi_i^\alpha \xi_j^\beta} f(x,\xi) = 4\bar{f}_{tt}(x,|\xi|^2)\xi_i^\alpha \xi_j^\beta + 2\bar{f}_t(x,|\xi|^2)\delta^{ij}\delta_{\alpha\beta},$$

by Lemma 3.2 it follows

$$\int_{B_r} D_{\xi_i^\alpha \xi_j^\beta} f(x,Du) D_{sj}u^\beta D_i((|Du|^2-R^2)_+) D_s u^\alpha \Phi_h' \gamma_M \eta^2 \,\mathrm{d}x$$

$$= 2\int_{B_r} \bar f_{tt}(x,|Du|^2) D_s u^\alpha D_i u^\alpha D_s(|Du|^2) D_i(|Du|^2-R^2)_+ \Phi_h' \gamma_M \eta^2 \,\mathrm{d}x$$

$$+ \int_{B_r} \bar f_t(x,|Du|^2) D_i(|Du|^2) D_i(|Du|^2-R^2)_+ \Phi_h' \gamma_M \eta^2 \,\mathrm{d}x$$

$$\geqslant c(v)\int_{B_r} (1+|Du|^2)^{(p-2)/2} |D(|Du|^2-R^2)_+|^2 \Phi_h' \gamma_M \eta^2 \,\mathrm{d}x. \tag{3.8}$$

In the following we denote by c any constant which may take different values from line to line.

(H2) and the convexity of f with respect to ξ give

$$|D_\xi f(x,\xi)| \leqslant c(K)(1+|\xi|)^{p-1} \tag{3.9}$$

for all ξ in $\mathbb{R}^{nN}$. (H3), (H4), (3.7)–(3.9) imply

$$\int_{B_r} (1+|Du|^2)^{(p-2)/2} |D^2u|^2 \Phi_h \gamma_M \eta^2 \,\mathrm{d}x$$

$$+ \int_{B_r} (1+|Du|^2)^{(p-2)/2} |D(|Du|^2-R^2)_+|^2 \Phi_h' \gamma_M \eta^2 \,\mathrm{d}x$$

$$\leqslant c(p,v,K,\Lambda)\int_{B_r} (1+|Du|^2)^{(p-1)/2} |D^2u|^2 \Phi_h |\gamma_M'| \eta^2 \,\mathrm{d}x$$

$$+ c(p,v,K)\int_{B_r} (1+|Du|^2)^{p/2} |D(|Du|^2-R^2)_+| \Phi_h' \gamma_M \eta(\eta+|D\eta|) \,\mathrm{d}x$$

$$+ c(p,v,K)\int_{B_r} (1+|Du|^2)^{(p-1)/2} |D^2u| \Phi_h \gamma_M \eta(\eta+|D\eta|) \,\mathrm{d}x$$

$$+ c(p,v,K)\int_{B_r} (1+|Du|^2)^{p/2} |D^2u| \Phi_h |\gamma_M'| \eta(\eta+|D\eta|) \,\mathrm{d}x. \tag{3.10}$$

By Young inequality, it follows

$$\int_{B_r} (1+|Du|^2)^{(p-2)/2} |D^2u|^2 \Phi_h \eta^2 \,\mathrm{d}x$$

$$+ \int_{B_r} (1+|Du|^2)^{(p-2)/2} |D(|Du|^2-R^2)_+|^2 \Phi_h' \eta^2 \,\mathrm{d}x$$

$$\leqslant c(\Lambda)\int_{B_r} (1+|Du|^2)^{(p-1)/2} |D^2u|^2 \Phi_h |\gamma_M'| \eta^2 \,\mathrm{d}x$$

$$+ c\int_{B_r} (1+|Du|^2)^{p/2} \Phi_h \gamma_M (\eta^2+|D\eta|^2) \,\mathrm{d}x$$

$$+ c \int_{B_r} (1+|Du|^2)^{p/2+1} \Phi_h' \gamma_M (\eta^2 + |D\eta|^2)\,\mathrm{d}x$$

$$+ c \int_{B_r} (1+|Du|^2)^{p/2} |D^2u| \Phi_h |\gamma_M'| \eta(\eta + |D\eta|)\,\mathrm{d}x. \tag{3.11}$$

Since Lemma 3.3 implies that $(1+|Du|^2)^{(p-1)/2}|D^2u| \in L^1_{\mathrm{loc}}$, taking into account that $\gamma_M'(x) = 0$ if $|Du(x)| > 2M$, and that $(1+|Du|^2)^{1/2}|\gamma_M'| \leqslant C$, with C independent of M, then every term containing the derivative of γ_M in (3.11) tends to zero when $M \to +\infty$. Setting

$$V(x) := 1 + R^2 + (|Du(x)|^2 - R^2)_+, \tag{3.12}$$

we get

$$\int_{B_r} V^{(p-2)/2} |D^2u|^2 \Phi_h((|Du|^2 - R^2)_+)\eta^2\,\mathrm{d}x$$

$$+ \int_{B_r} V^{(p-2)/2} |D(|Du|^2 - R^2)_+|^2 \Phi_h'((|Du|^2 - R^2)_+)\eta^2\,\mathrm{d}x$$

$$\leqslant c \int_{B_r} V^{p/2+1} \Phi_h'((|Du|^2 - R^2)_+)(\eta^2 + |D\eta|^2)\,\mathrm{d}x$$

$$+ c \int_{B_r} V^{p/2} \Phi_h((|Du|^2 - R^2)_+)(\eta^2 + |D\eta|^2)\,\mathrm{d}x. \tag{3.13}$$

Passing to the limit as $h \to +\infty$, by the monotone convergence theorem it follows that (3.13) holds with Φ and Φ' in place of Φ_h and Φ_h', respectively:

$$\int_{B_r} V^{p/2+\delta-1} (|Du|^2 - R^2)_+ |D^2u|^2 \eta^2\,\mathrm{d}x$$

$$+ \int_{B_r} V^{p/2+\delta-2} (1 + R^2 + (1+\delta)(|Du|^2 - R^2)_+)|D(|Du|^2 - R^2)_+|^2 \eta^2\,\mathrm{d}x$$

$$\leqslant c \int_{B_r} V^{p/2+\delta} (1 + R^2 + (1+\delta)(|Du|^2 - R^2)_+)(\eta^2 + |D\eta|^2)\,\mathrm{d}x$$

$$+ c \int_{B_r} V^{p/2+\delta} (|Du|^2 - R^2)_+ (\eta^2 + |D\eta|^2)\,\mathrm{d}x.$$

Hence (3.3) holds

$$\int_{B_r} V^{p/2+\delta-1} |D(|Du|^2 - R^2)_+|^2 \eta^2\,\mathrm{d}x$$

$$\leqslant c\,\frac{\max\{1, 1+\delta\}}{\min\{1, 1+\delta\}} \int_{B_r} V^{p/2+\delta+1} (\eta^2 + |D\eta|^2)\,\mathrm{d}x,$$

with $c = c(p, v, K)$.

Step 2: Let $\delta > -1$ be as in step 1. Define $\gamma = p/4 + (\delta+1)/2$. Let χ be equal to $n/(n-2)$ if $n > 2$, or to any number greater than 1 if $n = 2$, from the above estimate we have

$$\int_{B_r} |D(V^\gamma \eta)|^2 \,\mathrm{d}x \leqslant c\left(1 + \gamma^2 \frac{\max\{1, 2\gamma - p/2\}}{\min\{1, 2\gamma - p/2\}}\right) \int_{B_r} V^{2\gamma}(\eta^2 + |D\eta|^2)\,\mathrm{d}x$$

and by Sobolev inequality we get

$$\left[\int_{B_r} (V^{2\gamma}\eta^2)^\chi \,\mathrm{d}x\right]^{1/\chi} \leqslant c\left(1 + \gamma^2 \frac{\max\{1, 2\gamma - p/2\}}{\min\{1, 2\gamma - p/2\}}\right) \times \int_{B_r} V^{2\gamma}(\eta^2 + |D\eta|^2)\,\mathrm{d}x. \tag{3.14}$$

For any $i \in \mathbb{N}$ set $r_i = r/4 + r/4^i$. Let η_i be a cut-off function in $C_c^1(B_{r_i})$, such that $\eta_i = 1$ in $B_{r_{i+1}}$ and $\eta_i^2 + |D\eta_i|^2 \leqslant c(r)4^{2i}$. Given $\theta > 1$, to be chosen later, consider the sequence (γ_i) defined by $\gamma_{i+1} = \gamma_i \chi$, where $\gamma_1 = (p/4)\theta$. Then γ_i goes to $+\infty$ when $i \to +\infty$. From (3.14) it follows that for all i,

$$\left[\int_{B_{r_{i+1}}} V^{2\gamma_{i+1}} \,\mathrm{d}x\right]^{1/\chi} \leqslant c4^{2i}\left(1 + \gamma_i^2 \frac{\max\{1, 2\gamma_i - p/2\}}{\min\{1, 2\gamma_i - p/2\}}\right) \int_{B_{r_i}} V^{2\gamma_i}\,\mathrm{d}x.$$

Setting

$$c_i = \left[c4^{2i}\left(1 + \gamma_i^2 \frac{\max\{1, 2\gamma_i - p/2\}}{\min\{1, 2\gamma_i - p/2\}}\right)\right]^{1/(2\gamma_i)},$$

the previous inequality can be rewritten as

$$\|V\|_{L^{2\gamma_{i+1}}(B_{r_{i+1}})} \leqslant c_i \|V\|_{L^{2\gamma_i}(B_{r_i})}. \tag{3.15}$$

Iterating (3.15), we get

$$\|V\|_{L^{2\gamma_{i+1}}(B_{r_{i+1}})} \leqslant \prod_{j=1}^{i} c_j \|V\|_{L^{2\gamma_1}(B_{r_1})}.$$

Since $\prod_{j=1}^{i} c_j$ is uniformly bounded in i, passing to the limit with respect to i we get

$$\sup_{B_{r/4}} |V| \leqslant C\left(\int_{B_{r/2}} V^{p\theta/2}(x)\,\mathrm{d}x\right)^{2/(p\theta)},$$

where $C = C(n, p, v, K, \theta, r)$. On the other hand, since $c(R)V(x) \leqslant 1 + |Du(x)|^2 \leqslant V(x)$, we have

$$\sup_{B_{r/4}} |Du| \leqslant c\left[\int_{B_{r/2}} (1 + |Du|^2)^{p\theta/2}\,\mathrm{d}x\right]^{1/(p\theta)}, \tag{3.16}$$

with c depending also on R.

By Theorem 2.5 (i), there exist $\tilde{c}_0, \tilde{c}_1$ such that for all $(x,\xi) \in \Omega \times \mathbb{R}^{nN}$,

$$-\tilde{c}_0 + \tilde{c}_1|\xi|^p \leqslant f(x,\xi) \leqslant K(1+|\xi|)^p,$$

therefore the higher summability property of local minimizers (see [10]), implies that there exists $\theta_0 > 1$, depending only on n, p, v, R and K, such that Du is locally in $L^{p\theta_0}$ and

$$\left[\int_{B_{r/2}} (1+|Du|^2)^{p\theta_0/2}\,\mathrm{d}x\right]^{1/(p\theta_0)} \leqslant c\left[\int_{B_r} (1+|Du|^2)^{p/2}\,\mathrm{d}x\right]^{1/p}. \tag{3.17}$$

Choosing $\theta = \theta_0$, (3.16) and (3.17) give (3.2). □

Suppose now that u is a local minimizer of (3.1), with f satisfying (A1)–(A4) and (H1)–(H4). From Proposition 3.1 u is locally Lipschitz continuous. This fact, together with a suitable iteration argument, allows us to obtain an estimate similar to (3.2), with constants depending only on the constants in (A2)–(A4). Precisely, the following proposition holds:

Proposition 3.5. *Let u be a local minimizer of* (3.1), *where f satisfies* (A1)–(A4) *and* (H1)–(H4), *with $q < p(n+1)/n$. For any $B_r(x_0) \subset\subset \Omega$, there exist $C = C(n, p, q, v, R, L, r)$ and $\beta = \beta(n, p, q)$ positive constants, such that*

$$\sup_{B_{r/4}(x_0)} |Du|^p \leqslant C\left[\int_{B_r(x_0)} (1+|Du|^2)^{p/2}\,\mathrm{d}x\right]^{\beta}. \tag{3.18}$$

Proof. By Proposition 3.1 and Lemma 3.3, u is in $W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap W^{2,2}_{\mathrm{loc}}(\Omega)$.

Step 1: Given $\eta \in C^1_c(B_r)$, $0 \leqslant \eta \leqslant 1$, we will show that if $\delta > -1$ then

$$\int_{B_r} (1+|Du|^2)^{p/2+\delta-1}|D(|Du|^2-R^2)_+|^2\eta^2\,\mathrm{d}x$$
$$\leqslant c\,\frac{\max\{1, 1+\delta\}}{\min\{1, 1+\delta\}}\int_{B_r} (1+|Du|^2)^{q-p/2+\delta+1}(\eta^2+|D\eta|^2)\,\mathrm{d}x, \tag{3.19}$$

with c depending on q, v and L. The proof of this claim can be obtained following closely step 1 of Proposition 3.1, to which we address for formulas and notations. Since u is locally Lipschitz continuous then the Euler system (3.5) is satisfied with $\Phi_h = \Phi$ and $\gamma_M = 1$. Using assumption (A3) instead of (H2), (3.9) becomes

$$|D_\xi f(x,\xi)| \leqslant c(L)(1+|\xi|)^{q-1}$$

and, using (A4) in place of (H3), we obtain (3.10) with p replaced by q on the right-hand side, namely

$$\int_{B_r} (1+|Du|^2)^{(p-2)/2}|D^2u|^2\Phi\eta^2\,\mathrm{d}x,$$
$$+\int_{B_r} (1+|Du|^2)^{(p-2)/2}|D(|Du|^2-R^2)_+|^2\Phi'\eta^2\,\mathrm{d}x$$

$$\leqslant c(q,v,L)\int_{B_r}(1+|Du|^2)^{q/2}|D(|Du|^2-R^2)_+|\Phi'\eta(\eta+|D\eta|)\,\mathrm{d}x$$

$$+c(q,v,L)\int_{B_r}(1+|Du|^2)^{(q-1)/2}|D^2u|\Phi\eta(\eta+|D\eta|)\,\mathrm{d}x.$$

Using the definition of Φ and V, see (3.4) and (3.12), Young inequality yields

$$\int_{B_r}V^{p/2+\delta-1}(|Du|^2-R^2)_+|D^2u|^2\eta^2\,\mathrm{d}x$$

$$+\int_{B_r}V^{p/2+\delta-2}(1+R^2+(1+\delta)(|Du|^2-R^2)_+)|D(|Du|^2-R^2)_+|^2\eta^2\,\mathrm{d}x$$

$$\leqslant c\int_{B_r}V^{q-p/2+\delta}(1+R^2+(1+\delta)(|Du|^2-R^2)_+)(\eta^2+|D\eta|^2)\,\mathrm{d}x$$

$$+c\int_{B_r}V^{q-p/2+\delta}(|Du|^2-R^2)_+(\eta^2+|D\eta|^2)\,\mathrm{d}x$$

and (3.19) easily follows.

Step 2: Define $\gamma=p/4+(\delta+1)/2$. Setting $\tau=q-p$ and $\chi=n/(n-2)$ if $n>2$, or any number greater than $p/(p-2\tau)$ if $n=2$, from (3.19)

$$\int_{B_r}|D(V^\gamma\eta)|^2\,\mathrm{d}x\leqslant c(q,v,L)\left(1+\gamma^2\frac{\max\{1,2\gamma-p/2\}}{\min\{1,2\gamma-p/2\}}\right)$$

$$\times\int_{B_r}V^{2\gamma+\tau}(\eta^2+|D\eta|^2)\,\mathrm{d}x$$

and the Sobolev inequality yields

$$\left[\int_{B_r}(V^{2\gamma}\eta^2)^\chi\,\mathrm{d}x\right]^{1/\chi}\leqslant c\left(1+\gamma^2\frac{\max\{1,2\gamma-p/2\}}{\min\{1,2\gamma-p/2\}}\right)$$

$$\times\int_{B_r}V^{2\gamma+\tau}(\eta^2+|D\eta|^2)\,\mathrm{d}x. \tag{3.20}$$

For every $i\in\mathbb{N}$ let us define $r_i=r/4+r/4^i$. Let η_i be a cut-off function in $C_c^1(B_{r_i})$, such that $\eta_i=1$ in $B_{r_{i+1}}$ and $\eta_i^2+|D\eta_i|^2\leqslant c(r)4^{2i}$. Let $\theta>1$ and consider the sequence (γ_i) defined as $2\gamma_{i+1}=2\gamma_i\chi-\tau$, with $\gamma_1=(p/4)\theta$.

Since $2\tau/(p(\chi-1))<1<\theta$ then γ_i goes to $+\infty$ when $i\to+\infty$. From (3.20) it follows that for all i,

$$\left[\int_{B_{r_{i+1}}}V^{2\gamma_i\chi}\,\mathrm{d}x\right]^{1/\chi}\leqslant c4^{2i}\left(1+\gamma_i^2\frac{\max\{1,2\gamma_i-p/2\}}{\min\{1,2\gamma_i-p/2\}}\right)\int_{B_{r_i}}V^{2\gamma_{i-1}\chi}\,\mathrm{d}x.$$

Setting

$$c_i=\left[c4^{2i}\left(1+\gamma_i^2\frac{\max\{1,2\gamma_i-p/2\}}{\min\{1,2\gamma_i-p/2\}}\right)\right]^{1/(2\gamma_i)},$$

the previous inequality can be rewritten as

$$\|V\|_{L^{2\gamma_i\chi}(B_{r_{i+1}})} \leqslant c_i \|V\|^{\gamma_{i-1}\chi/\gamma_i}_{L^{2\gamma_{i-1}\chi}(B_{r_i})}. \tag{3.21}$$

Iterating (3.21), we get

$$\|V\|_{L^{2\gamma_i\chi}(B_{r_{i+1}})} \leqslant \prod_{j=0}^{i-1} c_{i-j}^{[(\gamma_{i-j})/\gamma_i]\chi^j} \|V\|^{(\gamma_0/\gamma_i)\chi^i}_{L^{2\gamma_1+\tau}(B_{r_1})},$$

where $\gamma_0 = (2\gamma_1 + \tau)/(2\chi)$. Since $\prod_{j=0}^{i-1} c_{i-j}^{[(\gamma_{i-j})/\gamma_i]\chi^j}$ is uniformly bounded, passing to the limit with respect to i we get

$$\sup_{B_{r/4}} |V| \leqslant c \left(\int_{B_{r/2}} V^{(p/2)\theta+\tau}\,\mathrm{d}x \right)^{\mu}, \tag{3.22}$$

where $c = c(n, p, q, v, L, r)$ and $\mu = (\chi - 1)(p\theta(\chi - 1)/2 - \tau)^{-1}$.

Step 3: In order to get the result we need to estimate the second term of (3.22). By an iteration technique we shall show that there exist $c > 0$ and $\alpha > 1$, depending only on the constants in (A2)–(A4), such that

$$\int_{B_{r/2}} V^{(p/2)\theta+\tau}\,\mathrm{d}x \leqslant c \left[\int_{B_r} V^{p/2}\,\mathrm{d}x \right]^{\alpha}. \tag{3.23}$$

Fixed $\rho < \sigma < r$, let η be a cut-off function such that $\eta = 1$ on B_ρ, $\eta = 0$ outside B_σ, $|D\eta| \leqslant 2/(\sigma - \rho)$. From (3.20)

$$\left[\int_{B_\rho} V^{(p/2)\theta\chi}\,\mathrm{d}x \right]^{1/\chi} \leqslant c\, \frac{1}{(\sigma-\rho)^2} \int_{B_\sigma} V^{p\theta/2+\tau}\,\mathrm{d}x, \tag{3.24}$$

where $c = c(n, p, q, v, L)$. Setting $s = 1 + 2\tau/(p\theta)$, $\delta = \chi/s$ and $H = V^{(p/2)\theta s}$, (3.24) may be rewritten as

$$\left[\int_{B_\rho} H^{\delta}\,\mathrm{d}x \right]^{1/\chi} \leqslant c\, \frac{1}{(\sigma-\rho)^2} \int_{B_\sigma} H\,\mathrm{d}x.$$

Let $t > \chi$ to be chosen later. By Hölder inequality

$$\int_{B_\sigma} H\,\mathrm{d}x \leqslant \left[\int_{B_\sigma} H^{\delta}\,\mathrm{d}x \right]^{1/t} \left[\int_{B_\sigma} H^{(t-\delta)/(t-1)}\,\mathrm{d}x \right]^{(t-1)/t},$$

therefore

$$\left[\int_{B_\rho} H^{\delta}\,\mathrm{d}x \right]^{1/\chi} \leqslant c\, \frac{1}{(\sigma-\rho)^2} \left[\int_{B_\sigma} H^{\delta}\,\mathrm{d}x \right]^{1/t} \left[\int_{B_r} H^{(t-\delta)/(t-1)}\,\mathrm{d}x \right]^{(t-1)/t}. \tag{3.25}$$

For all $i \in \mathbb{N}$, consider the last inequality with $\sigma = \rho_i$, $\rho = \rho_{i-1}$, where $\rho_i = r - r/2^i$. Iterating (3.25) we get

$$\int_{B_{r/2}} H^{\delta} \leqslant \left[\int_{B_{\rho_{i+1}}} H^{\delta} \right]^{(\chi/t)^i} \prod_{j=1}^{i} (c4^j)^{\chi(\chi/t)^{j-1}} \left[\int_{B_r} H^{(t-\delta)/(t-1)} \right]^{(t-1)(\chi/t)^j}$$

$$\leqslant \left[\int_{B_r} H^\delta\right]^{(\chi/t)^i} \prod_{j=1}^{i} (c4^j)^{\chi(\chi/t)^{j-1}} \left[\int_{B_r} H^{(t-\delta)/(t-1)}\right]^{(t-1)(\chi/t)^j}. \qquad (3.26)$$

Since $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ the first term on the right-hand side of (3.26) tends to 1 when $i \to +\infty$, while

$$\prod_{j=1}^{\infty} (c4^{j+1})^{\chi(\chi/t)^j} \left[\int_{B_r} H^{(t-\delta)/(t-1)}\right]^{(t-1)(\chi/t)^j} \leqslant c\left[\int_{B_r} H^{(t-\delta)/(t-1)}\right]^{\chi(t-1)/(t-\chi)}$$

$$= c\left[\int_{B_r} V^{(p/2)\theta s(t-\delta)/(t-1)}\right]^{\chi(t-1)/(t-\chi)}.$$

where $c = c(n, p, q, t, v, R, L, r)$.

We choose t in such a way that $\theta ps(t-\delta)/(2(t-1)) = p/2$, that is $\theta s(t-\delta)/(t-1) = 1$. Such a t exists, since by the choice of s, χ and the bound on q ($q < p((n+1)/n)$)

$$\lim_{t\to+\infty} \theta s\,\frac{t-\delta}{t-1} = \theta s > 1, \qquad \theta s\,\frac{\chi-\delta}{\chi-1} < 1.$$

With this choice, since

$$\int_{B_{r/2}} H\,\mathrm{d}x \leqslant \left[\int_{B_{r/2}} H^\delta\,\mathrm{d}x\right]^{1/\delta} |B_{r/2}|^{1-1/\delta},$$

(3.23) follows with $\alpha = s(t-1)/(t-\chi)$. Since $c(R)V(x) \leqslant 1 + |Du(x)|^2 \leqslant V(x)$ we finally have (3.18) with $\beta = p\mu\alpha/2$. $\square$

Remark 3.6. Obvious changes in the proof of Proposition 3.5 imply that estimate (3.18) can be rewritten in such a way that for all $\rho < r$ there exists c_0 depending on $n, p, q, v, R, L, r, \rho$ such that

$$\sup_{B_\rho(x_0)} |Du|^p \leqslant c_0\left(\int_{B_r(x_0)} (1+|Du|^p)\,\mathrm{d}x\right)^\beta.$$

4. Approximation and proof of Theorem 1.1

The main tools to prove Theorem 1.1 are two approximation lemmas. In the first lemma we define a sequence of functions (f_k) with p-growth, which monotonically converges to f. The second lemma gives a smooth approximation for each f_k, with functions satisfying (H1)–(H4).

Lemma 4.1. *Let $f : \Omega \times \mathbb{R}^{nN} \to [0,+\infty)$, $f = f(x,\xi)$, be a Carathéodory function, convex with respect to ξ, which satisfies assumptions* (A1)–(A4). *Then there exists a sequence (f_k) of Carathéodory functions $f_k : \Omega \times \mathbb{R}^{nN} \to [0,+\infty)$, convex with* *respect to the last variable, monotonically convergent to* f, *such that*

(I) *For a.e.* $x \in \Omega$ *and every* $\xi \in \mathbb{R}^{nN} \backslash B_R(0)$, $f_k(x,\xi) = \tilde{f}_k(x,|\xi|)$,

(II) *For a.e.* $x \in \Omega$, *for every* $\xi \in \mathbb{R}^{nN}$, *and for every* k, $f_k(x,\xi) \leqslant f_{k+1}(x,\xi) \leqslant f(x,\xi)$,

(III) f_k *is* p*-uniformly convex at infinity with respect to* ξ, *with* $\bar{v}$ *depending only on* p *and* v,

(IV) *For a.e.* $x \in \Omega$ *and for every* $\xi \in \mathbb{R}^{nN}$, *there exist* L_1, *independent of* k, *and* $\bar{L}_1$, *depending on* k, *such that*

$$f_k(x,\xi) \leqslant L_1(1+|\xi|)^q,$$

$$f_k(x,\xi) \leqslant \bar{L}_1(k)(1+|\xi|)^p,$$

(V) *Using the same notations as in* (A4), *for every* $\xi \mapsto f_k(x,\xi)$ *the vector field* $x \mapsto D_\xi^+ f_k(x,\xi)$ *is weakly differentiable and for a.e.* $x \in \Omega$ *and for every* $\xi \in \mathbb{R}^{nN} \backslash B_R(0)$,

$$|D_x D_\xi^+ f_k(x,\xi)| \leqslant L_1(1+|\xi|)^{q-1},$$

$$|D_x D_\xi^+ f(x,\xi)| \leqslant \bar{L}_1(k)(1+|\xi|)^{p-1}.$$

Proof. From Theorem 2.5(ii) there exist $c_0 = c_0(p,q,v,R,L)$ and $c_1 = c_1(p,v)$ positive constants and a function $g : \Omega \times \mathbb{R}^{nN} \to [-c_0,+\infty)$ such that

$$f(x,\xi) = c_1(1+|\xi|^2)^{p/2} + g(x,\xi),$$

with g convex at infinity, and there exists $\tilde{g} : \Omega \times [R,+\infty) \to [-c_0,+\infty)$ such that $\tilde{g}(x,|\xi|) = g(x,\xi)$ for any $\xi \in \mathbb{R}^{nN} \backslash B_R(0)$. Since $nN > 1$, for a.e. $x \in \Omega$, $t \mapsto \tilde{g}(x,t)$ is convex and increasing. For any $k \in \mathbb{N}$, $k > R$, using the same notations as in (A4), define $\tilde{g}_k : \Omega \times [R,+\infty) \to [-c_0,+\infty)$ as follows:

$$\tilde{g}_k(x,t) = \tilde{g}(x,t) \quad \forall (x,t) \in \Omega \times [R,k],$$

$$\tilde{g}_k(x,t) = \tilde{g}(x,k) + D_t^+ \tilde{g}(x,k)(t-k) \quad \forall (x,t) \in \Omega \times (k,+\infty).$$

For a.e. $x \in \Omega$, $t \mapsto \tilde{g}_k(x,t)$ is convex and increasing in $[R,+\infty)$ and $\tilde{g}_k(x,t) \leqslant \tilde{g}_{k+1}(x,t) \leqslant \tilde{g}(x,t)$. (A3) and the definition of $\tilde{g}_k(x,t)$ give

$$\tilde{g}_k(x,t) \leqslant L(1+t)^q,$$

$$\tilde{g}_k(x,t) \leqslant c(q,L,k)(1+t)^p.$$

If $t \in (k,+\infty)$ $D_t \tilde{g}_k(x,t) = D_t^+ \tilde{g}(x,k)$ and (A4) implies that $|D_x D_t^+ \tilde{g}(x,k)| \leqslant L(1+k)^{q-1}$, therefore for a.e. x and every $t > R$

$$|D_x D_t^+ \tilde{g}(x,t)| \leqslant c(1+t)^{q-1}$$

$$|D_x D_t^+ \tilde{g}(x,t)| \leqslant c(k)(1+t)^{p-1}.$$

For any k, define $g_k : \Omega \times \mathbb{R}^{nN} \to [-c_0,+\infty)$, $g_k(x,\xi) = g(x,\xi)$ if $|\xi| < R$ and $g_k(x,\xi) = \tilde{g}_k(x,|\xi|)$ otherwise. Setting $f_k : \Omega \times \mathbb{R}^{nN} \to [0,+\infty)$

$$f_k(x,\xi) := c_1(1+|\xi|^2)^{p/2} + g_k(x,\xi),$$

the result easily follows. □

Lemma 4.2. *Let f_k be as in Lemma* 4.1. *Fixed an open set $A \subset\subset \Omega$, there exists a sequence of C^2-functions $f_{kh} : A \times \mathbb{R}^{nN} \to [0, +\infty)$ with the following property. For any compact set $K \in \mathbb{R}^{nN}$ and for any $\delta > 0$ there exists $A_\delta \subset A$ with $|A_\delta| < \delta$ such that*

$$\lim_{h\to\infty} f_{kh} = f_k \quad \text{uniformly on } (A \backslash A_\delta) \times K. \tag{4.1}$$

Moreover f_{kh} satisfies (A1)–(A4) *and* (H1)–(H4). *More precisely,*

($\mathscr{A}1$) *For every $x \in A$ and $\xi \in \mathbb{R}^{nN} \backslash B_{R+1}(0)$, $f_{kh}(x, \xi) = \tilde{f}_{kh}(x, |\xi|)$,*

($\mathscr{A}2$) *There exists $\tilde{\nu}$ depending on p, ν and R, but neither on k nor h, such that for every $x \in A$ and for every $\xi_1, \xi_2 \in \mathbb{R}^{nN} \backslash B_{R+1}(0)$ endpoints of a segment contained in the complement of $B_{R+1}(0)$,*

$$\tfrac{1}{2}[f_{kh}(x, \xi_1) + f_{kh}(x, \xi_2)]$$
$$\geqslant f_{kh}\left(x, \frac{\xi_1 + \xi_2}{2}\right) + \tilde{\nu}(1 + |\xi_1|^2 + |\xi_2|^2)^{(p-2)/2}|\xi_1 - \xi_2|^2,$$

($\mathscr{A}3$) *There exists L_2, depending neither on k nor on h, such that for all $(x, \xi) \in A \times \mathbb{R}^{nN}$*

$$f_{kh}(x, \xi) \leqslant L_2(1 + |\xi|)^q,$$

($\mathscr{A}4$) *For every $x \in A$ and $\xi \in \mathbb{R}^{nN} \backslash B_{R+1}(0)$*

$$|D_{\xi x} f_{kh}(x, \xi)| \leqslant L_2(1 + |\xi|)^{q-1},$$

($\mathscr{H}1$) *f_{kh} is in $C^2(A \times \mathbb{R}^{nN})$,*

($\mathscr{H}2$) *There exists $\bar{L}_2(k)$, independent of h, such that for all $(x, \xi) \in A \times \mathbb{R}^{nN}$*

$$f_{kh}(x, \xi) \leqslant \bar{L}_2(k)(1 + |\xi|)^p,$$

($\mathscr{H}3$) *There exist $\varepsilon_h > 0$, independent of k, and $\Lambda_1(h)$ such that for all $(x, \xi) \in A \times \mathbb{R}^{nN}$ and for all $\lambda \in \mathbb{R}^{nN}$,*

$$\varepsilon_h(1 + |\xi|^2)^{(p-2)/2}|\lambda|^2 \leqslant \langle D_{\xi\xi} f_{kh}(x, \xi)\lambda, \lambda\rangle \leqslant \Lambda_1(h)(1 + |\xi|^2)^{(p-2)/2},$$

($\mathscr{H}4$) *There exists $\bar{\Lambda}_1(k)$, independent of h, such that for every $x \in A$ and $\xi \in \mathbb{R}^{nN} \backslash B_{R+1}(0)$,*

$$|D_{\xi x} f_{kh}(x, \xi)| \leqslant \bar{\Lambda}_1(k)(1 + |\xi|^2)^{(p-1)/2}.$$

Proof. Let f_k be as in Lemma 4.1, that is

$$f_k(x, \xi) = c_1(1 + |\xi|^2)^{p/2} + g_k(x, \xi). \tag{4.2}$$

Fixed an open set $A \subset\subset \Omega$ and an infinitesimal positive sequence ε_h, with h large enough, consider $B_1^n \subset \mathbb{R}^n$ and $B_1^{nN} \subset \mathbb{R}^{nN}$ the unit balls centered in 0. Let $\sigma : B_1^n \to [0, +\infty)$ and $\rho : B_1^{nN} \to [0, +\infty)$ be two radially symmetric mollifiers such that $\int_{B_1^n} \sigma = 1$ and $\int_{B_1^{nN}} \rho = 1$. Define $F_{kh} : A \times \mathbb{R}^{nN} \to [0, +\infty)$,

$$F_{kh}(x, \xi) := \int_{B_1^n} \int_{B_1^{nN}} \sigma(y)\rho(\eta) f_k(x + \varepsilon_h y, \xi + \varepsilon_h \eta)\, \mathrm{d}\eta\, \mathrm{d}y.$$

Since $g_k(x,\xi)=\tilde{g}_k(x,|\xi|)$ if $|\xi|\geqslant R$, then there exists $\tilde{F}_{kh}: A\times[R+1,+\infty)\to[0,+\infty)$ such that $F_{kh}(x,\xi)=\tilde{F}_{kh}(x,|\xi|)$ for every $x\in A$ and $\xi\in\mathbb{R}^{nN}\setminus B_{R+1}(0)$. Moreover, F_{kh} is convex with respect to ξ in $\mathbb{R}^{nN}$.

Fixed a compact subset K in $\mathbb{R}^{nN}$, for any $m\in\mathbb{N}$ there exists a finite number of open balls $B_{1/m}(\xi_j^m)$ such that $K\subset\bigcup_{j=1}^{i(m)}B_{1/m}(\xi_j^m)$. For every m and $j=1,\dots,i(m)$, $x\mapsto\int_{B_1^n}\sigma(y)f_k(x+\varepsilon_h y,\xi_j^m)\,\mathrm{d}y$ converges in L^1 to $x\mapsto f_k(x,\xi_j^m)$, as h goes to $+\infty$. By a diagonal argument, eventually passing to subsequences, we have that for every $m\in\mathbb{N}$, $j=1,\dots,i(m)$ and for a.e. $x\in A$

$$\lim_{h\to\infty}\int_{B_1^n}\sigma(y)[f_k(x+\varepsilon_h y,\xi_j^m)-f_k(x,\xi_j^m)]\,\mathrm{d}y=0.$$

By Egorov Theorem, for any $\delta>0$ there exists $A_\delta\subset A$ with $|A_\delta|<\delta$ such that for any m,

$$\lim_{h\to\infty}\sup_{x\in A\setminus A_\delta}\int_{B_1^n}\sigma(y)[f_k(x+\varepsilon_h y,\xi_j^m)-f_k(x,\xi_j^m)]\,\mathrm{d}y=0, \tag{4.3}$$

for every $j=1,\dots,i(m)$.

We claim that F_{kh} converges uniformly to f_k in $(A\setminus A_\delta)\times K$ as $h\to+\infty$, that is

$$\lim_{h\to\infty}\sup_{(x,\xi)\in(A\setminus A_\delta)\times K}|F_{kh}(x,\xi)-f_k(x,\xi)|=0. \tag{4.4}$$

In fact, given $\xi\in K$, for every m there exists $j=1,\dots,i(m)$ such that $\xi\in B_{1/m}(\xi_j^m)$. Consider

$$\begin{aligned}
&F_{kh}(x,\xi)-f_k(x,\xi)\\
&\quad=\int_{B_1^n}\int_{B_1^{nN}}\sigma(y)\rho(\eta)\,[f_k(x+\varepsilon_h y,\xi+\varepsilon_h\eta)-f_k(x+\varepsilon_h y,\xi)]\,\mathrm{d}\eta\,\mathrm{d}y\\
&\qquad+\int_{B_1^n}\sigma(y)[f_k(x+\varepsilon_h y,\xi)-f_k(x+\varepsilon_h y,\xi_j^m)]\,\mathrm{d}y\\
&\qquad+\int_{B_1^n}\sigma(y)\,[f_k(x+\varepsilon_h y,\xi_j^m)-f_k(x,\xi_j^m)]\,\mathrm{d}y\\
&\qquad+(f_k(x,\xi_j^m)-f_k(x,\xi))\\
&\quad=I_1^{h,m}(x,\xi)+I_2^{h,m}(x,\xi)+I_3^{h,m}(x,\xi)+I_4^{h,m}(x,\xi).
\end{aligned}\tag{4.5}$$

The first inequality in (IV) of the previous lemma and the convexity of f_k imply that there exists $c>0$ such that for a.e. $x\in\Omega$ and for every $\lambda,\eta\in\mathbb{R}^{nN}$,

$$|f_k(x,\lambda)-f_k(x,\eta)|\leqslant c(1+|\lambda|+|\eta|)^{q-1}|\lambda-\eta|;$$

therefore

$$|I_1^{h,m}(x,\xi)+I_2^{h,m}(x,\xi)+I_4^{h,m}(x,\xi)|\leqslant c\left(\varepsilon_h+\frac{1}{m}\right)\sup_{\xi\in K}(1+|\xi|)^{q-1}.$$

This estimate, together with (4.3), implies (4.4).

Define

$$f_{kh}(x,\xi) := F_{kh}(x,\xi) + \varepsilon_h(1+|\xi|^2)^{p/2}.$$

For each h, $f_{kh} \in C^2$ and f_{kh} is radial with respect to ξ if $|\xi| \geqslant R+1$.

It is easy to check that Theorem 2.5 gives ($\mathscr{A}2$) and that property (IV) implies ($\mathscr{A}3$) and ($\mathscr{H}2$). ($\mathscr{A}4$) and ($\mathscr{H}4$) are an easy consequence of (V). The ellipticity condition in ($\mathscr{H}3$) follows by the convexity of $F_{kh}(x,\xi)$, while the right inequality in ($\mathscr{H}3$) is a consequence of the definition of $\tilde{g}_k$. $\square$

Now we are ready to prove the main result.

Proof of Theorem 1.1. Let u be a local minimizer of $\mathscr{F}$ in (1.1). Fixed $A \subset\subset \Omega$, let f_k, f_{kh} be defined as in Lemmas 4.1 and 4.2. From ($\mathscr{A}2$) and Theorem 2.5(i), there exists $c_1 > 0$ such that

$$|\xi|^p \leqslant c_1(1 + f_{kh}(x,\xi)), \quad \forall k, h. \tag{4.6}$$

Fixed $B_r(x_0)$ in A, let v_{kh} be the solution of the problem

$$\min\left\{\mathscr{F}_{kh}(w; B_r(x_0)) := \int_{B_r(x_0)} f_{kh}(x, Dw)\,\mathrm{d}x : w \in u + W_0^{1,p}(B_r(x_0))\right\}.$$

From (4.6), the minimality of v_{kh} implies

$$\int_{B_r(x_0)} |Dv_{kh}|^p\,\mathrm{d}x \leqslant c_1 \int_{B_r(x_0)} (1 + f_{kh}(x, Du))\,\mathrm{d}x. \tag{4.7}$$

By ($\mathscr{H}2$), the dominated convergence theorem, recalling that $f_k(x,\xi) \leqslant f(x,\xi)$, we have

$$\lim_h \int_{B_r(x_0)} f_{kh}(x, Du)\,\mathrm{d}x = \int_{B_r(x_0)} f_k(x, Du)\,\mathrm{d}x \leqslant \int_{B_r(x_0)} f(x, Du)\,\mathrm{d}x. \tag{4.8}$$

From (4.7) and (4.8)

$$\liminf_h \int_{B_r(x_0)} |Dv_{kh}|^p\,\mathrm{d}x \leqslant c_1 \int_{B_r(x_0)} (1 + f(x, Du))\,\mathrm{d}x. \tag{4.9}$$

Therefore, up to subsequences, v_{kh} weakly converges in $W^{1,p}$ to a function $v_k \in u + W_0^{1,p}(B_r(x_0))$. For any k and h, $\mathscr{F}_{kh}$ satisfies the assumptions of Proposition 3.5 with $R+1$ in place of R. Remark 3.6, the minimality of v_{kh} and (4.8) imply that for any $\rho < r$ there exists $\beta = \beta(n,p,q)$ and c_2, independent of h and k, such that

$$\begin{aligned}\liminf_h \sup_{B_\rho(x_0)} |Dv_{kh}|^p &\leqslant c_2 \liminf_h \left[\int_{B_r(x_0)} (1 + f_{kh}(x, Dv_{kh}))\,\mathrm{d}x\right]^\beta \\ &\leqslant c_2 \left[\int_{B_r(x_0)} (1 + f(x, Du))\,\mathrm{d}x\right]^\beta =: M.\end{aligned} \tag{4.10}$$

Up to a subsequence, v_{kh} converges to v_k $w^* - W^{1,\infty}(B_\rho(x_0))$ for any $\rho < r$ and

$$\int_{B_r(x_0)} |Dv_k|^p \, dx \leqslant c_1 \int_{B_r(x_0)} (1 + f(x, Du)) \, dx, \tag{4.11}$$

$$\sup_{B_\rho(x_0)} |Dv_k|^p \leqslant c_2 \left[\int_{B_r(x_0)} (1 + f(x, Du)) \, dx \right]^\beta. \tag{4.12}$$

Therefore, up to subsequences, v_k weakly converges in $W^{1,p}$ and weakly* in $W^{1,\infty}_{\mathrm{loc}}$ $(B_r(x_0))$ to a function $v \in u + W^{1,p}_0(B_r(x_0))$.

Let us show that v is a minimizer of $\mathscr{F}(w; B_r(x_0))$. Fix $\rho < r$ and $k_0 \in \mathbb{N}$. Since if $k > k_0$, $f_k(x,\xi) \geqslant f_{k_0}(x,\xi)$ for any x and ξ, by the lower semicontinuity of $w \mapsto \int_{B_r} f_{k_0}(x, Dw) \, dx$,

$$\int_{B_\rho(x_0)} f_{k_0}(x, Dv_k) \, dx \leqslant \liminf_h \int_{B_\rho(x_0)} f_{k_0}(x, Dv_{kh}) \, dx$$
$$\leqslant \liminf_h \int_{B_\rho(x_0)} f_k(x, Dv_{kh}) \, dx$$

By Lemma 4.2, fixed $K = \{\xi \in \mathbb{R}^{nN} : |\xi| \leqslant M+1\}$, for every $\delta > 0$ there exists A_δ with $|A_\delta| < \delta$ such that f_{kh} converges to f_k uniformly in $(A \backslash A_\delta) \times K$. Thus

$$\limsup_h \int_{B_\rho(x_0) \backslash A_\delta} f_k(x, Dv_{kh}) \, dx = \limsup_h \int_{B_\rho(x_0) \backslash A_\delta} f_{kh}(x, Dv_{kh}) \, dx$$

and, from (IV) and (4.10), there exists c independent of δ, such that

$$\limsup_h \int_{B_\rho(x_0) \cap A_\delta} f_k(x, Dv_{kh}) \, dx \leqslant c\delta \bar{L}_1(k)(1 + M).$$

Therefore,

$$\liminf_h \int_{B_\rho(x_0)} f_k(x, Dv_{kh}) \, dx \leqslant \limsup_h \int_{B_r(x_0)} f_{kh}(x, Dv_{kh}) \, dx + c\delta \bar{L}_1(k)(1 + M).$$

Letting δ go to 0, from the previous estimates, the minimality of v_{kh} and (4.8) we have

$$\int_{B_\rho(x_0)} f_{k_0}(x, Dv_k) \, dx \leqslant \int_{B_r(x_0)} f(x, Du) \, dx$$

and, by the lower semicontinuity,

$$\int_{B_\rho(x_0)} f_{k_0}(x, Dv) \, dx \leqslant \liminf_k \int_{B_\rho(x_0)} f_{k_0}(x, Dv_k) \, dx \leqslant \int_{B_r(x_0)} f(x, Du) \, dx.$$

Letting $k_0 \to \infty$ and $\rho \to r$ we finally have

$$\int_{B_r(x_0)} f(x, Dv) \, dx \leqslant \int_{B_r(x_0)} f(x, Du) \, dx.$$

Finally, passing to the limit in (4.12) with respect to k,

$$\sup_{B_{r/4}(x_0)} |Dv|^p \leqslant c \left[\int_{B_r(x_0)} (1 + f(x, Du)) \, dx \right]^\beta,$$

and from Lemma 2.6 there exists $R_0 = R_0(p, q, v, R, L)$ such that

$$\sup_{B_{r/4}(x_0)} |Du| \leqslant \sup_{B_{r/4}(x_0)} |Dv| + \sup_{B_{r/4}(x_0)} |Du + Dv| \leqslant 3 \sup_{B_{r/4}(x_0)} |Dv| + 2R_0$$

and the thesis follows. □

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