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Local Boundedness for Minimizers of Some Polyconvex Integrals

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Abstract

We give a regularity result for local minimizers $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ of a special class of polyconvex functionals. Under some structure assumptions on the energy density, we prove that local minimizers u are locally bounded. For each component u^α of u, we first prove a Caccioppoli's inequality and then apply De Giorgi's iteration method to get the boundedness of u^α. Our result can be applied to the polyconvex integral

$$\int_\Omega \left(\sum_{\alpha=1}^{3} |Du^\alpha|^p + |\operatorname{adj}_2 Du|^q + |\det Du|^r \right) \mathrm{d}x$$

with suitable $p, q, r > 1$.

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1. Introduction

We study the regularity of local minimizers for a special class of variational integrals

$$I(u, \Omega) = \int_\Omega f(Du)\mathrm{d}x \tag{1.1}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued map and Du is the $m \times n$ Jacobian matrix of its partial derivatives

$$u \equiv \left(u^1, u^2, \ldots, u^m\right), \qquad Du = \left(\frac{\partial u^\alpha}{\partial x_i}\right)_{i=1,2,\ldots,n}^{\alpha=1,2,\ldots,m}.$$

A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ is a local minimizer of I if $f(Du) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$I(u, \operatorname{supp}\varphi) \le I(u+\varphi, \operatorname{supp}\varphi),$$

for all $\varphi \in W^{1,1}(\Omega, \mathbb{R}^m)$ with $\operatorname{supp}\varphi \Subset \Omega$.

Motivated by the applications to nonlinear elasticity, J. Ball in 1977 pointed out in [1] that the convexity of f with respect to Du is unrealistic in the vectorial case. Indeed, it conflicts with, for the instance, the natural requirement that the elastic energy is frame-indifferent. The convexity must be replaced by different and more general assumptions, such as the so called *quasiconvexity* and *polyconvexity*, already introduced by Morrey [36] in an abstract setting. In particular, we are interested in the polyconvexity condition, which takes into account the constitutive hypothesis that the energy is invariant under the transformation $g \mapsto g + \varphi$, for every null Lagrangian φ.

A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$, $f = f(\xi)$, is said to be *polyconvex* if there exists a convex function $g : \mathbb{R}^{\tau(m,n)} \to \mathbb{R}$ such that

$$f(\xi) = g(T(\xi)), \tag{1.2}$$

where

$$\tau(m,n) := \sum_{i=1}^{\min(m,n)} \binom{n}{i}\binom{m}{i}$$

and $T(\xi)$ is the vector defined as follows:

$$T(\xi) := \left(\xi, \operatorname{adj}_2 \xi, \dots, \operatorname{adj}_i \xi, \dots, \operatorname{adj}_{\min\{m,n\}} \xi\right).$$

Here $\operatorname{adj}_i \xi$ denotes the adjugate matrix of order i. In particular, if $m = n$, then $\operatorname{adj}_n \xi = \det \xi$.

The polyconvexity assumption is commonly used as a structural assumption in mathematical models of elasticity, since, if $n = m = 3$, then ξ, $\operatorname{adj}_2 \xi$ and $\det \xi$ govern the deformations of line, surface and volume, respectively.

Our main purpose is to illustrate some ideas and methods which lead to local boundedness for local minimizers of some polyconvex functionals.

In our paper $n = m = 3$ and we assume that there exist $F_\alpha : \mathbb{R}^3 \to [0, +\infty)$, $G_\alpha : \mathbb{R}^3 \to [0, +\infty)$, with $\alpha \in \{1,2,3\}$, and $H : \mathbb{R} \to [0, +\infty)$ convex functions such that

$$f(\xi) := \sum_{\alpha=1}^{3} \left\{F_\alpha(\xi^\alpha) + G_\alpha\left(\left(\operatorname{adj}_2 \xi\right)^\alpha\right)\right\} + H(\det \xi), \tag{1.3}$$

where

$$\xi = \begin{pmatrix} \xi_1^1 & \xi_2^1 & \xi_3^1 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 \\ \xi_1^3 & \xi_2^3 & \xi_3^3 \end{pmatrix} = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix}, \qquad \xi^\alpha \in \mathbb{R}^3 \text{ for } \alpha \in \{1,2,3\},$$

and $\operatorname{adj}_2 \xi \in \mathbb{R}^{3\times 3}$ denotes the adjugate matrix of order 2 whose components are

$$(\operatorname{adj}_2 \xi)_{\gamma i} = (-1)^{\gamma+i} \det \begin{pmatrix} \xi_k^\alpha & \xi_l^\alpha \\ \xi_k^\beta & \xi_l^\beta \end{pmatrix} \qquad \gamma, i \in \{1,2,3\},$$

where $\alpha, \beta \in \{1, 2, 3\} \setminus \{\gamma\}$, $\alpha < \beta$, and $k, l \in \{1, 2, 3\} \setminus \{i\}$, $k < l$. Moreover,

$$(\operatorname{adj}_2 \xi)^\alpha = ((\operatorname{adj}_2 \xi)_{\alpha 1}, (\operatorname{adj}_2 \xi)_{\alpha 2}, (\operatorname{adj}_2 \xi)_{\alpha 3}).$$

Integrands (1.3) occur as stored energy densities for certain models from nonlinear elasticity (see Ball [1,2], Ogden [39]) and, by the results by Ball [1,2] and Müller [38], (see also the monograph by Dacorogna [8]), the corresponding minimization problems have a solution.

To have regular local minimizers some growth conditions have to be considered. We assume that $F_\alpha(\xi^\alpha)$ grows like $|\xi^\alpha|^p$, $G_\alpha(\big(\operatorname{adj}_2 \xi\big)^\alpha)$ grows like $\big|\big(\operatorname{adj}_2 \xi\big)^\alpha\big|^q$ and $H(\det \xi)$ grows like $|\det \xi|^r$.

In this paper, under suitable assumptions on the exponents p, q, r (see condition (2.5)) we prove that the local minimizers of I are locally bounded in Ω, see Theorem 2.1. We note that F_α, G_α, H may depend on x too: $F_\alpha(x, \xi^\alpha)$, $G_\alpha(x, \big(\operatorname{adj}_2 \xi\big)^\alpha)$, $H(x, \det \xi)$; see Theorem 2.1.

As an application of Theorem 2.1, let us consider the functional (1.1) with

$$f(Du) := \sum_{\alpha=1}^{3} \Big(|Du^\alpha|^{14/5} + |\operatorname{adj}_2 Du^\alpha|^2\Big) + |\det Du|^{3/2}.$$

By Theorem 2.1 every local minimizer $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ of I is locally bounded. Note that the existence of a minimizer of I in $\bar{u} + W_0^{1,\frac{14}{5}}(\Omega)$, with $\bar{u} \in W^{1,\frac{14}{5}}(\Omega)$, comes from Remark 8.32 in [8], see Theorem 3.1 below.

Partial regularity results, that is the regularity of solutions up to a set Ω_0 and the study of the properties of the singular set (see for example section 4.2 in [33] and section 1 in [34]) are contained in [7,12,14,15,17,22,40]. For the polyconvex case, only few everywhere regularity results are available; we mention those by Fusco and Hutchinson in [18], where the everywhere continuity is proved in the case $n = m = 2$, Fuchs and Seregin [16], where Hölder continuity for extremals is dealt with. Global pointwise bounds are in [9,24,26–29]. Interesting results are contained in [3–6,42]; see also [25].

The main novelty of our result is the technique used to obtain the regularity result. We prove the local boundedness of vector valued minimizers $u = (u^1, u^2, u^3)$ by using De Giorgi's iteration method, used until now only in the scalar case. Indeed, we first show that each component u^α satisfies a Caccioppoli inequality (see Proposition 2.3); then we apply De Giorgi's procedure, separately, to each u^α.

The special structure on f in (1.3) is in some sense necessary to treat this type of functional since in the vectorial framework minimizers can be unbounded in view of some counterexamples, see [10,43], section 3 in [33] and the recent [35].

The integrals considered can be inserted in the class of functionals with p, q-growth. The mathematical literature on the regularity under p, q-growth is very rich; energy functionals with anisotropic, non-standard or general growth have been studied by many authors and in different settings of applicability. Under p, q-growth it is now well known, as in our result, that a restriction between p and q must be imposed due to the counterexamples in [11,13,20,23,30–32]; we refer to [33] for a detailed survey on the subject.

Our paper is organized as follows. In the next section we present the precise statement of our local boundedness result (Theorem 2.1) and we describe our strategy for proving it; eventually we provide the proof. In Section 3 we recall an existence result for a suitable class of polyconvex functionals (Theorem 3.1); using this result and Theorem 2.1 we obtain the existence of locally bounded minimizers for a class of functionals satisfying the assumptions of these two results (Theorem 3.2). Section 4 is the Appendix, devoted to two technical results used to prove Theorem 2.1.

2. Local Boundedness

We consider $\Omega \subseteq \mathbb{R}^3$ open set, a function $f : \Omega \times \mathbb{R}^{3\times 3} \to [0, +\infty)$, and the functional

$$I(u) := \int_\Omega f(x, Du(x))\, \mathrm{d}x$$

where $u : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3$,

$$Du := \begin{pmatrix} Du^1 \\ Du^2 \\ Du^3 \end{pmatrix} = \begin{pmatrix} u^1_{x_1} & u^1_{x_2} & u^1_{x_3} \\ u^2_{x_1} & u^2_{x_2} & u^2_{x_3} \\ u^3_{x_1} & u^3_{x_2} & u^3_{x_3} \end{pmatrix}.$$

We assume that there exist Carathéodory functions $F_\alpha : \Omega \times \mathbb{R}^3 \to [0, +\infty)$, $G_\alpha : \Omega \times \mathbb{R}^3 \to [0, +\infty)$, $\alpha \in \{1, 2, 3\}$ and $H : \Omega \times \mathbb{R} \to [0, +\infty)$, such that $\lambda \to F_\alpha(x, \lambda)$, $\lambda \to G_\alpha(x, \lambda)$, $t \to H(x, t)$ are convex, with

$$f(x, \xi) := \sum_{\alpha=1}^{3} \left\{ F_\alpha(x, \xi^\alpha) + G_\alpha(x, \left(\operatorname{adj}_2 \xi\right)^\alpha) \right\} + H(x, \det \xi). \tag{2.1}$$

Here

$$\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix}, \qquad \xi^\alpha \in \mathbb{R}^3 \text{ for } \alpha \in \{1, 2, 3\}$$

and $\operatorname{adj}_2 \xi \in \mathbb{R}^{3\times 3}$ denotes the adjugate matrix of order 2 whose components are

$$(\operatorname{adj}_2 \xi)_{\gamma i} = (-1)^{\gamma+i} \det \begin{pmatrix} \xi^\alpha_k & \xi^\alpha_l \\ \xi^\beta_k & \xi^\beta_l \end{pmatrix} \qquad \gamma, i \in \{1, 2, 3\},$$

where $\alpha, \beta \in \{1, 2, 3\} \setminus \{\gamma\}$, $\alpha < \beta$, and $k, l \in \{1, 2, 3\} \setminus \{i\}$, $k < l$. Moreover,

$$(\operatorname{adj}_2 \xi)^\alpha = ((\operatorname{adj}_2 \xi)_{\alpha 1}, (\operatorname{adj}_2 \xi)_{\alpha 2}, (\operatorname{adj}_2 \xi)_{\alpha 3}).$$

We assume that there exist exponents $1 < p \leq 3$, $1 < q$, $1 \leq r$, constants $k_1, k_3 > 0$, $k_2 \geq 0$ and functions $a, b, c : \Omega \to [0, +\infty)$ such that, for all $\alpha \in \{1, 2, 3\}$,

$$k_1|\lambda|^p - k_2 \leq F_\alpha(x, \lambda) \leq k_3 \left(|\lambda|^p + 1\right) + a(x) \qquad \forall \lambda \in \mathbb{R}^3 \tag{2.2}$$

$$k_1 |\lambda|^q - k_2 \leq G_\alpha(x, \lambda) \leq k_3 \left(|\lambda|^q + 1\right) + b(x) \qquad \forall \lambda \in \mathbb{R}^3 \tag{2.3}$$

$$0 \leq H(t) \leq k_3 \left(|t|^r + 1\right) + c(x) \qquad \forall t \in \mathbb{R}, \tag{2.4}$$

where $a, b, c \in L^\sigma(\Omega)$, $\sigma > 1$.

Our main result is the following.

Theorem 2.1. *Let f satisfy (2.1) and growth conditions (2.2), (2.3), (2.4), with $1 \leq r < q < p \leq 3$. Assume*

$$\frac{p}{p^*} < \min\left\{1 - \frac{qp^*}{p(p^* - q)}, 1 - \frac{rp^*}{q(p^* - r)}, 1 - \frac{1}{\sigma}\right\}, \tag{2.5}$$

where $p^* = \frac{3p}{3-p}$, if $p < 3$, and, if $p = 3$, then p^* is any $\nu > 3$.

Then all the local minimizers $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^3)$ of I are locally bounded.

Remark 2.2. If $\sigma = \infty$ then $\frac{1}{\sigma}$ must be read as 0. Moreover, we remark that if $p = 3$, then p^* can be chosen large enough so that (2.5) is implied by the assumptions $1 \leq r < q < p$ and $\sigma > 1$.

For the sake of simplicity we prove the theorem in the case with no dependence on x, that is, $f(\xi)$, $F_\alpha(\xi^\alpha)$, $G^\alpha(\left(\mathrm{adj}_2\, \xi\right)^\alpha)$, $H(t)$, with $a(x) = b(x) = c(x) = 0$ in the growth conditions. See also Remark 2.6.

Sketch of the Proof

We now provide a sketch of the proof of Theorem 2.1. For a local minimizer $u = (u^1, u^2, u^3)$ we will prove that each component is locally bounded. In the following we consider the first component u^1 (we can argue similarly for the other components u^2, u^3).

STEP 1: Caccioppoli inequality for u^1. We use the minimality condition with a suitable test function; such a test function and the particular structure (2.1) of the density f guarantee a Caccioppoli inequality for u^1 on every superlevel set $\{u^1 > k\}$. More precisely, with fixed $x_0 \in \Omega$ and a ball $B_{R_0}(x_0) \Subset \Omega$ (we will not write the center x_0 if no confusion may arise) we have that there exists $c > 0$ such that for all $s, t > 0$, $s < t \leq R_0$,

$$\int_{\{u^1>k\}\cap B_s} |Du^1|^p \,\mathrm{d}x \leq c \int_{\{u^1>k\}\cap B_t} \left(\frac{u^1 - k}{t - s}\right)^{p^*} \mathrm{d}x + c|\{u^1 > k\} \cap B_t|^\vartheta \tag{2.6}$$

with a suitable $\vartheta > 0$. The Caccioppoli inequality (2.6) permits us to apply the classical methods to get the regularity in the scalar case. Observe that on the right

hand side of (2.6) we do not get the same exponent p as in the left hand side, but the larger p^*; it still allows us to prove the local boundedness of u^1, see also [19,37].
STEP 2: Decay of the "excess" on superlevel sets. For a suitable radius $R < R_0$ and a suitable level d, we define a sequence ρ_h of radii starting from R and decreasing to $\frac{R}{2}$, another sequence k_h of levels starting from $\frac{d}{2}$ and increasing to d. We define the "excess" on the superlevel set as follows:

$$J_h := \int_{\{u^1 > k_h\} \cap B_{\rho_h}} \left(u^1 - k_h\right)^{p^*} \, \mathrm{d}x. \tag{2.7}$$

Note that J_h is a decreasing sequence. Using Sobolev inequality and Caccioppoli estimate (2.6) we are able to show that

$$J_{h+1} \leqq c Q^h J_h^{\vartheta p^*/p} \tag{2.8}$$

for some constants c, $Q > 1$.
STEP 3: Iteration. On the right hand side of (2.8) there is competition between the increasing Q^h and the decreasing $J_h^{\vartheta p^*/p}$; if $\vartheta p^*/p > 1$ and the initial value J_0 is small, then

$$J_h \leqq Q^{\frac{-h}{\vartheta p^*/p - 1}} J_0 \tag{2.9}$$

so that

$$\lim_{h \to +\infty} J_h = 0, \tag{2.10}$$

which implies

$$u^1 \leqq d \qquad \text{a. e. in } B_{R/2}. \tag{2.11}$$

Since assumption (2.5) guarantees $\vartheta p^*/p > 1$ we get (2.11). Lower bounds for u^1 can be obtained by showing that $-u$ is a minimizer for a similar functional.

To accomplish this program, we will use two technical lemmas; their statements and proofs can be found in the Appendix.

STEP 1: Caccioppoli Inequality

The particular structure (2.1) of the density f guarantees a Caccioppoli inequality for any component u^α of u on every superlevel set $\{u^\alpha > k\}$. In the next proposition we state this result in the case of the first component u^1.

Proposition 2.3. *Let f be as in (2.1), satisfying the growth conditions (2.2), (2.3) and (2.4), with*

$$q < \frac{p^* p}{p^* + p} \qquad \textit{and} \qquad r < \frac{p^* q}{p^* + q}. \tag{2.12}$$

Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^3)$ be a local minimizer of I.

Let $B_R(x_0) \Subset \Omega$, $|B_R| < 1$, $R < 1$, *and, fixed* $k \in \mathbb{R}$, *denote*

$$A^1_{k,\tau} := \{x \in B_\tau(x_0) \,:\, u^1(x) > k\} \qquad 0 < \tau \le R.$$

Then there exists $c > 0$, *independent of* k, *such that for every* $0 < s < t \le R$*:*

$$\begin{aligned}\int_{A^1_{k,s}} |Du^1|^p \,\mathrm{d}x &\le c \int_{A^1_{k,t}} \left(\frac{u^1 - k}{t-s}\right)^{p^*} \mathrm{d}x \\ &+ c \left\{ 1 + ||a+b+c||_{L^\sigma(B_R)} + \left(\int_{B_R} (|Du^2| + |Du^3|)^p \,\mathrm{d}x\right)^{\frac{qp^*}{(p^*-q)p}} \right. \\ &\left. + \left(\int_{B_R} \left|\left(\mathrm{adj}_2\, Du\right)^1\right|^q \,\mathrm{d}x\right)^{\frac{rp^*}{(p^*-r)q}} \right\} |A^1_{k,t}|^\vartheta, \end{aligned} \tag{2.13}$$

where $\vartheta := \min\left\{1 - \frac{qp^*}{p(p^*-q)}, 1 - \frac{rp^*}{q(p^*-r)}, 1 - \frac{1}{\sigma}\right\}$.

Proof. For the sake of simplicity we will give a proof assuming that the integrand function f is independent on x, and consequently, that a, b and c in (2.2), (2.3) and (2.4) are equal to 0.

Let $B_R(x_0) \Subset \Omega$, $|B_R| < 1$ and $R < 1$. Let s, t be such that $s < t \le R$. Consider a cut-off function $\eta \in C^\infty_0(B_t)$ satisfying the following assumptions:

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ in } B_s(x_0), \quad |D\eta| \le \frac{2}{t-s}. \tag{2.14}$$

Fixing $k \in \mathbb{R}$, define $w \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^3)$,

$$w^1 := \max(u^1 - k, 0), \quad w^2 := 0, \quad w^3 := 0,$$

and, for $\mu \ge p^*$,

$$\varphi := -\eta^\mu w.$$

For almost every x in $\Omega \setminus (\{\eta > 0\} \cap \{u^1 > k\})$ we have $\varphi = 0$, thus

$$f(Du + D\varphi) = f(Du) \qquad \text{almost everywhere in } \Omega \setminus (\{\eta > 0\} \cap \{u^1 > k\}). \tag{2.15}$$

For almost every x in $\{\eta > 0\} \cap \{u^1 > k\}$ denote

$$A := \begin{pmatrix} \mu\eta^{-1}(k - u^1)D\eta \\ Du^2 \\ Du^3 \end{pmatrix}. \tag{2.16}$$

We notice that

$$Du + D\varphi = \begin{pmatrix} (1-\eta^\mu)Du^1 + \mu\eta^{\mu-1}(k-u^1)D\eta \\ Du^2 \\ Du^3 \end{pmatrix} = (1-\eta^\mu)Du + \eta^\mu A.$$

Since

$$\det(Du + D\varphi) = (1 - \eta^\mu)\det Du + \eta^\mu \det A$$

and

$$\operatorname{adj}_2(Du + D\varphi) = (1 - \eta^\mu)\operatorname{adj}_2 Du + \eta^\mu \operatorname{adj}_2 A,$$

and using that f is polyconvex, we get

$$f(Du + D\varphi) \le (1 - \eta^\mu) f(Du) + \eta^\mu f(A) \quad \text{almost everywhere in } \{\eta > 0\} \cap \{u^1 > k\}. \tag{2.17}$$

By the minimality of u, $f(Du) \in L^1_{\text{loc}}(\Omega)$. Lemma 4.2 in the Appendix ensures that

$$\eta^\mu f(A) \in L^1(\{u^1 > k\} \cap \{\eta > 0\}).$$

Therefore (2.15) and (2.17) imply $f(Du + D\varphi) \in L^1_{\text{loc}}(\Omega)$.

By the local minimality of u, (2.15) and (2.17) we have

$$\int_{A^1_{k,t} \cap \{\eta > 0\}} f(Du)\,\mathrm{d}x \le \int_{A^1_{k,t} \cap \{\eta > 0\}} \left\{(1 - \eta^\mu) f(Du) + \eta^\mu f(A)\right\} \mathrm{d}x.$$

The inequality above implies

$$\int_{A^1_{k,t} \cap \{\eta > 0\}} \eta^\mu f(Du)\,\mathrm{d}x \le \int_{A^1_{k,t} \cap \{\eta > 0\}} \eta^\mu f(A)\,\mathrm{d}x. \tag{2.18}$$

Taking into account (2.16) and the particular structure of f (see (2.1)) we obtain

$$F_2(A^2) = F_2(Du^2), \quad F_3(A^3) = F_3(Du^3), \quad G_1((\operatorname{adj}_2 A)^1) = G_1((\operatorname{adj}_2 Du)^1), \tag{2.19}$$

then, by (2.18),

$$\begin{aligned}\int_{A^1_{k,t} \cap \{\eta > 0\}} \eta^\mu \left\{ F_1(Du^1) + \sum_{\alpha=2}^{3} G_\alpha(\left(\operatorname{adj}_2 Du\right)^\alpha) + H(\det Du) \right\} \mathrm{d}x \\ \le \int_{A^1_{k,t} \cap \{\eta > 0\}} \eta^\mu \left\{ F_1(\mu\eta^{-1}(k - u^1)D\eta) + \sum_{\alpha=2}^{3} G_\alpha((\operatorname{adj}_2 A)^\alpha) + H(\det A) \right\} \mathrm{d}x.\end{aligned} \tag{2.20}$$

By the growth assumption (2.2),

$$\eta^\mu F_1(\mu\eta^{-1}(k - u^1)D\eta) \le c\eta^\mu + c\mu^p \eta^{\mu - p} \left(\frac{u^1 - k}{t - s}\right)^p \quad \text{almost everywhere in } A^1_{k,t} \cap \{\eta > 0\}.$$

Therefore, recalling $\mu > p$ and the inequality $z^p \le z^{p^*} + 1$ if $z \ge 0$, we obtain

$$\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu F_1(\mu\eta^{-1}(k-u^1)D\eta)\,\mathrm{d}x \le c\int_{A^1_{k,t}} \left\{1 + \left(\frac{u^1-k}{t-s}\right)^{p^*}\right\} \mathrm{d}x. \tag{2.21}$$

Moreover, by (2.3) and Lemma 4.1-(c) in the Appendix,

$$\eta^\mu \sum_{\alpha=2}^{3} G_\alpha((\mathrm{adj}_2\, A)^\alpha) \le \eta^\mu k_3 \sum_{\alpha=2}^{3} \left(|(\mathrm{adj}_2\, A)^\alpha|^q + 1\right)$$
$$\le c\eta^\mu + c\mu^q \eta^{\mu-q} \left(\frac{u^1-k}{t-s}\right)^q (|Du^2| + |Du^3|)^q.$$

The first inequality in (2.12) implies $q < p^*$. Using the Young inequality with exponents $\frac{p^*}{q}$ and $\frac{p^*}{p^*-q}$ we get that, almost everywhere in $A^1_{k,t} \cap \{\eta > 0\}$,

$$c\mu^q \eta^{\mu-q} \left(\frac{u^1-k}{t-s}\right)^q (|Du^2| + |Du^3|)^q$$
$$\le c\left(\frac{u^1-k}{t-s}\right)^{p^*} + c\,(|Du^2| + |Du^3|)^{\frac{qp^*}{p^*-q}}.$$

We have thus proved that

$$\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu \sum_{\alpha=2}^{3} G_\alpha((\mathrm{adj}_2\, A)^\alpha)\,\mathrm{d}x$$
$$\le c\int_{A^1_{k,t}} \left\{1 + \left(\frac{u^1-k}{t-s}\right)^{p^*} + (|Du^2| + |Du^3|)^{\frac{qp^*}{p^*-q}}\right\} \mathrm{d}x. \tag{2.22}$$

By (2.4), and computing $\det(A)$ with respect to the first row (see Lemma 4.1-(b)),

$$\eta^\mu H(\det A) \le c\eta^\mu + c\mu^r \eta^{\mu-r} \left(\frac{u^1-k}{t-s}\right)^r \left|(\mathrm{adj}_2\, Du)^1\right|^r.$$

Notice that, by (2.12), $r < p^*$. By the Young inequality with exponents $\frac{p^*}{r}$ and $\frac{p^*}{p^*-r}$ we get

$$c\mu^r \eta^{\mu-r} \left(\frac{u^1-k}{t-s}\right)^r \left|(\mathrm{adj}_2\, Du)^1\right|^r \le c\left(\frac{u^1-k}{t-s}\right)^{p^*} + \left|(\mathrm{adj}_2\, Du)^1\right|^{\frac{rp^*}{p^*-r}}.$$

Therefore

$$\int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu H(\det A)\,\mathrm{d}x$$
$$\le c\int_{A^1_{k,t}} \left\{1 + \left(\frac{u^1-k}{t-s}\right)^{p^*} + \left|(\mathrm{adj}_2\, Du)^1\right|^{\frac{rp^*}{p^*-r}}\right\} \mathrm{d}x. \tag{2.23}$$

Taking into account that the inequalities in (2.12) are equivalent to $\frac{qp^*}{p^*-q} < p$ and $\frac{rp^*}{p^*-r} < q$, by the Hölder inequality we obtain

$$
\begin{aligned}
&\int_{A^1_{k,t}} \left\{ (|Du^2| + |Du^3|)^{\frac{qp^*}{p^*-q}} + \left| \left(\operatorname{adj}_2 Du\right)^1 \right|^{\frac{rp^*}{p^*-r}} \right\} \mathrm{d}x \\
&\quad \le \left(\int_{B_R} (|Du^2| + |Du^3|)^p \,\mathrm{d}x \right)^{\frac{qp^*}{(p^*-q)p}} |A^1_{k,t}|^{1-\frac{qp^*}{p(p^*-q)}} \\
&\qquad + \left(\int_{B_R} \left| \left(\operatorname{adj}_2 Du\right)^1 \right|^q \mathrm{d}x \right)^{\frac{rp^*}{(p^*-r)q}} |A^1_{k,t}|^{1-\frac{rp^*}{q(p^*-r)}}. \qquad (2.24)
\end{aligned}
$$

Since $|A^1_{k,t}| \le |B_R| \le 1$, by (2.20), (2.21), (2.22), (2.23) and (2.24) we get

$$
\begin{aligned}
&\int_{A^1_{k,s}} \left\{ F_1(Du^1) + \sum_{\alpha=2}^{3} G_\alpha\left(\left(\operatorname{adj}_2 Du\right)^\alpha\right) + H(\det Du) \right\} \mathrm{d}x \\
&\le c \int_{A^1_{k,t}} \left(\frac{u^1 - k}{t - s} \right)^{p^*} \mathrm{d}x \\
&\quad + c \left\{ 1 + \left(\int_{B_R} (|Du^2| + |Du^3|)^p \,\mathrm{d}x \right)^{\frac{qp^*}{(p^*-q)p}} \right. \\
&\quad \left. + \left(\int_{B_R} \left| \left(\operatorname{adj}_2 Du\right)^1 \right|^q \mathrm{d}x \right)^{\frac{rp^*}{(p^*-r)q}} \right\} |A^1_{k,t}|^\vartheta, \qquad (2.25)
\end{aligned}
$$

where

$$
\vartheta := \min\left\{ 1 - \frac{qp^*}{p(p^*-q)}, 1 - \frac{rp^*}{q(p^*-r)} \right\}.
$$

Since $G_2, G_3, H \ge 0$ and

$$
F_1(Du^1) \ge k_1 |Du^1|^p - k_2,
$$

we have that (2.25) implies (2.13). □

STEP 2: Decay of the "Excess" on Superlevel Sets

In this step we consider a *scalar* Sobolev function $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $n \ge 2$.

Let us assume that Ω is an open set in $\mathbb{R}^n$ and v is a *scalar* function $v \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R})$, $p \ge 1$. Fix $B_{R_0}(x_0) \Subset \Omega$ with $R_0 < 1$ small enough so that

$$
|B_{R_0}(x_0)| < 1 \quad \text{and} \quad \int_{B_{R_0}} |v|^{p^*} \,\mathrm{d}x < 1. \qquad (2.26)
$$

Here $p^* = \frac{np}{n-p}$ if $p < n$ and p^* is any $\nu > p$ if $p = n$.

For every $R \in (0, R_0]$ we define the decreasing sequences

$$\rho_h := \frac{R}{2} + \frac{R}{2^{h+1}} = \frac{R}{2}\left(1 + \frac{1}{2^h}\right), \qquad \bar{\rho}_h := \frac{\rho_h + \rho_{h+1}}{2} = \frac{R}{2}\left(1 + \frac{3}{4 \cdot 2^h}\right).$$

Fix a positive constant $d \geqq 1$ and define the increasing sequence of positive real numbers

$$k_h := d\left(1 - \frac{1}{2^{h+1}}\right), \;\; h \in \mathbb{N}.$$

Moreover, define the sequence $(J_{v,h})$ as

$$J_{v,h} := \int_{A_{k_h,\rho_h}} (v - k_h)^{p^*}\, \mathrm{d}x,$$

where $A_{k,\rho} = \{v > k\} \cap B_\rho$. The following result holds:

Proposition 2.4. *Let $v \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R})$, $p \geq 1$. Fix $B_{R_0}(x_0) \Subset \Omega$ with $R_0 < 1$ small enough such that (2.26) holds. If there exist $0 \leq \vartheta \leq 1$ and $c_0 > 0$ such that for every $0 < s < t \leq R_0$ and for every $k \in \mathbb{R}$*

$$\int_{A_{k,s}} |Dv|^p\, \mathrm{d}x \leq c_0 \left\{ \int_{A_{k,t}} \left(\frac{v-k}{t-s}\right)^{p^*} \mathrm{d}x + |A_{k,t}|^{\vartheta} \right\}, \tag{2.27}$$

then, for every $R \in (0, R_0]$,

$$J_{v,h+1} \leq c(\vartheta, R)\left(2^{\frac{p^* p^*}{p}}\right)^h J_{v,h}^{\vartheta \frac{p^*}{p}},$$

with the positive constant c independent of h.

Proof. In the following we write J_h in place of $J_{v,h}$.

Notice that (J_h) is a decreasing sequence, since the following chain of inequalities holds:

$$J_{h+1} \leq \int_{A_{k_{h+1},\rho_h}} (v - k_{h+1})^{p^*}\, \mathrm{d}x \leq \int_{A_{k_{h+1},\rho_h}} (v - k_h)^{p^*}\, \mathrm{d}x \leq J_h \qquad \forall\, h. \tag{2.28}$$

Let us now define a sequence (ζ_h) of cut-off functions in $C_c^\infty(B_{\bar{\rho}_h}(x_0))$ such that $0 \leq \zeta_h \leq 1$ and $\zeta_h \equiv 1$ in $B_{\rho_{h+1}}$, $|D\zeta_h| \leqq \frac{2^{h+4}}{R}$.

If we denote $(v - k_{h+1})_+ = \max\{v - k_{h+1}, 0\}$ we get

$$\begin{aligned} J_{h+1} &= \int_{A_{k_{h+1},\rho_{h+1}}} (v - k_{h+1})^{p^*} \zeta_h^{p^*}\, \mathrm{d}x \leqq \int_{A_{k_{h+1},\bar{\rho}_h}} (v - k_{h+1})^{p^*} \zeta_h^{p^*}\, \mathrm{d}x \\ &= \int_{B_R} (\zeta_h (v - k_{h+1})_+)^{p^*}\, \mathrm{d}x. \end{aligned} \tag{2.29}$$

By the Sobolev embedding Theorem and the properties of ζ_h we get

$$\begin{aligned}
&\int_{B_R} (\zeta_h (v - k_{h+1})_+)^{p^*} \, \mathrm{d}x \\
&\quad \leqq c \left(\int_{B_R} |D(\zeta_h (v - k_{h+1})_+)|^p \, \mathrm{d}x \right)^{\frac{p^*}{p}} \\
&\quad \leq c \left\{ \left(\int_{B_R} |Dv \zeta_h|^p \chi_{\{v > k_{h+1}\}} \, \mathrm{d}x \right)^{\frac{1}{p}} + \left(\int_{B_R} |(v - k_{h+1})_+ D\zeta_h|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \right\}^{p^*} \\
&\quad \leq c \left\{ \left(\int_{A_{k_{h+1}, \bar{\rho}_h}} |Dv|^p \, \mathrm{d}x \right)^{\frac{1}{p}} + \left(\left(\frac{2^h}{R} \right)^p \int_{A_{k_{h+1}, \rho_h}} (v - k_{h+1})^p \, \mathrm{d}x \right)^{\frac{1}{p}} \right\}^{p^*} .
\end{aligned} \tag{2.30}$$

Using (2.27) with $k = k_{h+1}$, $t = \rho_h$, $s = \bar{\rho}_h$ we obtain

$$\int_{A_{k_{h+1}, \bar{\rho}_h}} |Dv|^p \, \mathrm{d}x \leq c \left\{ \left(\frac{2^h}{R} \right)^{p^*} \int_{A_{k_{h+1}, \rho_h}} |v - k_{h+1}|^{p^*} \, \mathrm{d}x + |A_{k_{h+1}, \rho_h}|^{\vartheta} \right\} . \tag{2.31}$$

Collecting (2.29), (2.30) and (2.31) we obtain

$$\begin{aligned}
J_{h+1} \leq c \Bigg\{ &\left(\frac{2^h}{R} \right)^{p^*} \int_{A_{k_{h+1}, \rho_h}} (v - k_{h+1})^{p^*} \, \mathrm{d}x + |A_{k_{h+1}, \rho_h}|^{\vartheta} \\
&+ \left(\frac{2^h}{R} \right)^p \int_{A_{k_{h+1}, \rho_h}} (v - k_{h+1})^p \, \mathrm{d}x \Bigg\}^{\frac{p^*}{p}} .
\end{aligned} \tag{2.32}$$

Since $z^p \leq z^{p^*} + 1$ for every $z \geq 0$,

$$\left(\frac{2^h}{R} \right)^p \int_{A_{k_{h+1}, \rho_h}} (v - k_{h+1})^p \, \mathrm{d}x \leq \left(\frac{2^h}{R} \right)^{p^*} \int_{A_{k_{h+1}, \rho_h}} (v - k_{h+1})^{p^*} \, \mathrm{d}x + |A_{k_{h+1}, \rho_h}|,$$

thus obtaining

$$J_{h+1} \leq c \left\{ \left(\frac{2^h}{R} \right)^{p^*} \int_{A_{k_{h+1}, \rho_h}} (v - k_{h+1})^{p^*} \, \mathrm{d}x + |A_{k_{h+1}, \rho_h}|^{\vartheta} + |A_{k_{h+1}, \rho_h}| \right\}^{\frac{p^*}{p}} . \tag{2.33}$$

Since

$$|A_{k_{h+1}, \rho_h}| (k_{h+1} - k_h)^{p^*} \leq \int_{A_{k_{h+1}, \rho_h}} (v - k_h)^{p^*} \, \mathrm{d}x \leq J_h,$$

we have

$$|A_{k_{h+1},\rho_h}| \leq \frac{J_h}{(k_{h+1}-k_h)^{p^*}} = \left(\frac{2^{h+2}}{d}\right)^{p^*} J_h.$$

Taking into account that

$$\int_{A_{k_{h+1},\rho_h}} (v-k_{h+1})^{p^*}\,\mathrm{d}x \leqq \int_{A_{k_{h+1},\rho_h}} (v-k_h)^{p^*}\,\mathrm{d}x \leq J_h,$$

the inequality (2.33) gives

$$J_{h+1} \leq c\left\{\left(\frac{2^h}{R}\right)^{p^*} J_h + \left(\frac{2^h}{d}\right)^{\vartheta p^*} J_h^{\vartheta} + \left(\frac{2^h}{d}\right)^{p^*} J_h\right\}^{\frac{p^*}{p}}. \tag{2.34}$$

Since $J_h \leq 1$ for every h and recalling that $d \geq 1 > R_0 \geq R$, we get

$$\begin{aligned}\left(\frac{2^h}{R}\right)^{p^*} J_h + \left(\frac{2^h}{d}\right)^{\vartheta p^*} J_h^{\vartheta} + \left(\frac{2^h}{d}\right)^{p^*} J_h &\leq \left\{2\frac{2^{hp^*}}{R^{p^*}} + \frac{2^{h\vartheta p^*}}{R^{\vartheta p^*}}\right\} J_h^{\vartheta} \\ &\leq \left(\frac{2}{R^{p^*}} + \frac{1}{R^{\vartheta p^*}}\right) 2^{hp^*} J_h^{\vartheta}.\end{aligned}$$

By (2.34) it follows that

$$J_{h+1} \leq c\left\{\left(\frac{2}{R^{p^*}} + \frac{1}{R^{\vartheta p^*}}\right) 2^{hp^*} J_h^{\vartheta}\right\}^{\frac{p^*}{p}} \leq c(\vartheta, R)\left(2^{\frac{p^* p^*}{p}}\right)^h J_h^{\vartheta\frac{p^*}{p}}.$$

□

STEP 3: Iteration and Proof of Theorem 2.1

We now resume the proof of Theorem 2.1. As in the proof of Proposition 2.3, we will consider an integrand function f independent on x; consequently, a, b and c in (2.2), (2.3) and (2.4) have to be considered 0.

We need the following classical result, see for example [21].

Lemma 2.5. *Let $\gamma > 0$ and let (J_h) be a sequence of real positive numbers such that*

$$J_{h+1} \leqq A\,\lambda^h J_h^{1+\gamma} \qquad \forall h \in \mathbb{N}\cup\{0\}, \tag{2.35}$$

with $A > 0$ and $\lambda > 1$. If $J_0 \leqq A^{-\frac{1}{\gamma}}\lambda^{-\frac{1}{\gamma^2}}$, then $J_h \leq \lambda^{-\frac{h}{\gamma}} J_0$ and $\lim_{h\to\infty} J_h = 0$.

Fix $B_{R_0}(x_0) \Subset \Omega$ with $R_0 < 1$ small enough such that $|B_{R_0}(x_0)| < 1$ and $\int_{B_{R_0}} |u|^{p^*}\,\mathrm{d}x \leqq 1$. By Proposition 2.3 we have that u^1 satisfies, for every $0 < s < t \leq R_0$ and every $k \in \mathbb{R}$,

$$\begin{aligned}\int_{A^1_{k,s}} |Du^1|^p\,\mathrm{d}x &\leq c\int_{A^1_{k,t}} \left(\frac{u^1-k}{t-s}\right)^{p^*}\mathrm{d}x\\ &\quad + c\left\{1+\left(\int_{B_{R_0}} (|Du^2|+|Du^3|)^p\,\mathrm{d}x\right)^{\frac{qp^*}{(p^*-q)p}}\right.\\ &\quad \left. +\left(\int_{B_{R_0}} \left|\left(\operatorname{adj}_2 Du\right)^1\right|^q\,\mathrm{d}x\right)^{\frac{rp^*}{(p^*-r)q}}\right\}|A^1_{k,t}|^{\vartheta},\end{aligned}\tag{2.36}$$

where $c > 0$ is independent of s, t, k and $\vartheta := \min\{1-\frac{qp^*}{p(p^*-q)}, 1-\frac{rp^*}{q(p^*-r)}\}$. Therefore the scalar function u^1 satisfies (2.27) of Proposition 2.4 with constant c_0 depending on

$$\int_{B_{R_0}} (|Du^2|+|Du^3|)^p\,\mathrm{d}x \qquad \text{and} \qquad \int_{B_{R_0}} \left|\left(\operatorname{adj}_2 Du\right)^1\right|^q\,\mathrm{d}x.$$

Note that these integrals are finite by (2.2) and (2.3). Moreover, they are independent of u^1; indeed $\left(\operatorname{adj}_2 Du\right)^1$ depends only on u^2 and u^3.

As above, let us define

$$k_h := d\left(1-\frac{1}{2^{h+1}}\right),\quad h \in \mathbb{N}$$

with $d \geqq 1$ (d will be fixed later) and, for every $R \in (0, R_0]$, define

$$\rho_h := \frac{R}{2}+\frac{R}{2^{h+1}} = \frac{R}{2}\left(1+\frac{1}{2^h}\right),\qquad \bar\rho_h := \frac{\rho_h+\rho_{h+1}}{2} = \frac{R}{2}\left(1+\frac{3}{4\cdot 2^h}\right)$$

and

$$J_{u^1,h} := \int_{A^1_{k_h,\rho_h}} (u^1-k_h)^{p^*}\,\mathrm{d}x.$$

Proposition 2.4, applied to u^1, gives

$$J_{u^1,h+1} \leq c(\vartheta, R)\left(2^{\frac{p^*p^*}{p}}\right)^h J_{u^1,h}^{\vartheta\frac{p^*}{p}},\tag{2.37}$$

with the positive constant c independent of h and, by (2.5), with the exponent $\vartheta\frac{p^*}{p}$ greater than 1.

Indeed, since

$$J_{u^1,0} = \int_{A^1_{\frac{d}{2},R}} \left(u^1 - \frac{d}{2}\right)^{p^*} \mathrm{d}x \to_{d\to+\infty} 0,$$

we can choose $d \geqq 1$ large enough so that

$$J_{u^1,0} < c(\vartheta, R)^{-\frac{1}{\vartheta \frac{p^*}{p}-1}} \left(2^{\frac{p^* p^*}{p}}\right)^{-\frac{1}{(\vartheta \frac{p^*}{p}-1)^2}}.$$

Therefore, by Lemma 2.5, $\lim_{h\to\infty} J_{u^1,h} = 0$. Thus, $u^1 \leq d$ almost everywhere in $B_{\frac{R_0}{2}}$. We have proved that u^1 is locally bounded from above.

To prove that u^1 is locally bounded from below, we notice that $-u$ is a local minimizer of $\int_\Omega \tilde{f}(Dv)\,\mathrm{d}x$ where

$$\tilde{f}(\xi) := \sum_{\alpha=1}^{3} \left\{F_\alpha(-\xi^\alpha) + G_\alpha((\mathrm{adj}_2\, \xi)^\alpha)\right\} + H(-\det \xi).$$

If we denote

$$\tilde{F}_\alpha(\lambda) := F_\alpha(-\lambda), \qquad \tilde{H}(t) = H(-t), \qquad \lambda \in \mathbb{R}^3,\ t \in \mathbb{R},$$

the functions $\tilde{F}_\alpha$, $\tilde{H}_\alpha$ are convex and satisfy (2.2) and (2.4).

The function $\tilde{f}$ satisfies the assumptions of Theorem 2.1, so we obtain that there exists d' such that $-u^1 \leq d'$ almost everywhere in $B_{\frac{R_0}{2}}$. We have therefore proved that $u^1 \in L^\infty(B_{\frac{R_0}{2}}(x_0))$. Due to the arbitrariness of x_0 and R_0, we get $u^1 \in L^\infty_{\mathrm{loc}}(\Omega)$.

The symmetric structure of the energy density f allows us to obtain an analogous statement to Proposition 2.3 also for u^2 and u^3. Therefore, reasoning as for u^1 (see also Remark 4.3), we obtain that $u^2, u^3 \in L^\infty_{\mathrm{loc}}(\Omega)$, too.

Remark 2.6. We proved Theorem 2.1 by assuming that the integrand function is independent of x. In the general case, with f depending on x and satisfying the general growth conditions (2.2)–(2.4), with a, b, c belonging to L^σ, $\sigma > 1$, the proof goes in a similar way, with the additional condition that $1 - \frac{1}{\sigma} > \frac{p}{p^*}$.

3. Existence and Regularity

Consider an open, bounded set $\Omega \subseteq \mathbb{R}^3$ and a Caratheodory function $f : \Omega \times \mathbb{R}^{3\times3} \to [0, +\infty)$, $f(x, \xi) := g(x, T(\xi))$ with

$$T(\xi) := (\xi, \mathrm{adj}_2\, \xi, \det \xi) \in \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R}$$

where $z \mapsto g(x, z)$ is convex.

Let $\bar{u} \in W^{1,p}(\Omega;\mathbb{R}^3)$ be a function such that $\int_\Omega f(x, D\bar{u}(x))\,\mathrm{d}x < +\infty$. Consider the minimization problem

$$\min\left\{I(u) := \int_\Omega f(x, Du(x))\,\mathrm{d}x \;:\; u \in \bar{u} + W_0^{1,p}(\Omega;\mathbb{R}^3)\right\}. \tag{P}$$

We suppose that there exist constants $c_1 > 0$, $c_2 \geq 0$ and real exponents $p, q, \tilde{r} > 1$ such that

$$c_1\left(|\xi|^p + |\operatorname{adj}_2 \xi|^q + |\det \xi|^{\tilde{r}}\right) - c_2 \leq f(x,\xi). \tag{3.1}$$

The following existence result holds (see Remark 8.32 (iii) in [8]):

Theorem 3.1. *Consider the variational problem* (P). *If f satisfies* (3.1) *with exponents*

$$2 \leq p < +\infty, \qquad \frac{p}{p-1} \leq q < +\infty, \qquad 1 < \tilde{r} < +\infty,$$

then (P) *has a solution.*

As a consequence of Theorem 3.1 and Theorem 2.1 we have the following:

Theorem 3.2. *Consider the variational problem* (P)*, where Ω is an open bounded set in $\mathbb{R}^3$ and f satisfies* (2.1)*, the growth conditions* (2.2)*,* (2.3)*,* (2.4) *and*

$$k_4|\det\xi|^{\tilde{r}} - k_5 \leq H(\det\xi) \tag{3.2}$$

with $p \in \left(\frac{3+\sqrt{45}}{4}, 3\right)$, $\frac{p}{p-1} \leq q < \frac{p^*(p^*-p)p}{(p^*)^2+(p^*-p)p}$, $1 < \tilde{r} \leq r < \frac{p^*(p^*-p)q}{(p^*)^2+(p^*-p)q}$, $k_4 > 0$, $k_5 \geq 0$ *and* $\sigma > \frac{3}{p}$.

Then there exists a minimizer u of (P)*, with $u \in L^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^3)$.*

Proof. It suffices to notice that $\frac{p}{p^*} < 1 - \frac{qp^*}{p(p^*-q)}$ is equivalent to $q < \frac{p^*(p^*-p)p}{(p^*)^2+(p^*-p)p}$ and that $p \in \left(\frac{3+\sqrt{45}}{4}, 3\right)$ implies that $\frac{p}{p-1} < \frac{p^*(p^*-p)p}{(p^*)^2+(p^*-p)p}$. The thesis immediately follows by Theorem 3.1 and, taking into account that a minimizer is also a local minimizer, by Theorem 2.1. □

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4. Appendix

Given a vector $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$ we write $|v| := \sqrt{\sum_{i=1}^n v_i^2}$. Analogously, given a matrix $A = (a_{ij})$, $i, j \in \{1, \cdots, n\}$, A^i is its i-th row and $|A| := \sqrt{\sum_{i,j=1}^n a_{ij}^2}$.

Lemma 4.1. *Consider the matrices $A, B \in \mathbb{R}^{3\times 3}$*

$$A = \begin{pmatrix} A^1 \\ B^2 \\ B^3 \end{pmatrix}, \qquad B = \begin{pmatrix} B^1 \\ B^2 \\ B^3 \end{pmatrix}.$$

Then the following estimates hold:

(a) $|A| \le |A^1| + |B^2| + |B^3|$,
(b) $|\det A| \le |A^1||(\mathrm{adj}_2\, B)^1|$,
(c) $|(\mathrm{adj}_2\, A)_{2j}| \le |A^1||B^3|$ *and* $|(\mathrm{adj}_2\, A)_{3j}| \le |A^1||B^2|$, *for all* $j \in \{1, 2, 3\}$.

Proof. The first estimate is trivial, because

$$|A| = \sqrt{|A^1|^2 + |B^2|^2 + |B^3|^2} \le |A^1| + |B^2| + |B^3|.$$

To prove the second one, notice that

$$|\det A| \le \sum_{j=1}^{3} |A_{1j}||(\mathrm{adj}_2\, A)_{1j}|.$$

Since the second and third rows of A and B coincide,

$$(\mathrm{adj}_2\, A)_{1j} = (\mathrm{adj}_2\, B)_{1j} \qquad j \in \{1, 2, 3\};$$

moreover,

$$\left|\left(\mathrm{adj}_2\, B\right)^1\right| = \left|\left((\mathrm{adj}_2\, B)_{11}, (\mathrm{adj}_2\, B)_{12}, (\mathrm{adj}_2\, B)_{13}\right)\right|,$$

so we have

$$\sum_{j=1}^{3} |A_{1j}||(\mathrm{adj}_2\, A)_{1j}| = \sum_{j=1}^{3} |A_{1j}||(\mathrm{adj}_2\, B)_{1j}| \le |A^1||(\mathrm{adj}_2\, B)^1|$$

and we conclude.
To prove (c), notice that, with fixed $j \in \{1, 2, 3\}$,

$$|(\mathrm{adj}_2\, A)_{2j}| \le |A_{1i}||B_{3k}| + |A_{1k}||B_{3i}| \qquad i, k \in \{1, 2, 3\} \setminus \{j\},\ i \ne k;$$

so the first inequality in (c) follows. Analogously, the second inequality follows. □

Lemma 4.2. *Let Ω be an open subset of $\mathbb{R}^3$. Consider a Caratheodory function $f : \Omega \times \mathbb{R}^{3\times 3} \to [0, +\infty)$. Assume that there exists $c_1, c_3 > 0$ and $c_2 \geq 0$ such that for every $\xi \in \mathbb{R}^9$*

$$\begin{aligned} c_1 \left(|\xi|^p + |\operatorname{adj}_2 \xi|^q \right) - c_2 \leq f(x, \xi) \qquad\qquad\qquad\qquad\qquad\qquad\qquad & (4.1)\\ \leq c_3 \left(|\xi|^p + |\operatorname{adj}_2 \xi|^q + |\det \xi|^r + 1 + \omega(x) \right), & \end{aligned}$$

with $1 \leq p$, $1 \leq q$, $1 \leq r$, $\omega(x) \geqq 0$.
Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^3)$ be such that $x \to f(x, Du(x)) \in L^1_{\mathrm{loc}}(\Omega)$. Fix $\eta \in C^1_c(\Omega)$, $\eta \geq 0$ and $k \in \mathbb{R}$, and denote, for almost every $x \in \{u^1 > k\} \cap \{\eta > 0\}$,

$$A := \begin{pmatrix} \mu\eta^{-1}(k - u^1)D\eta \\ Du^2 \\ Du^3 \end{pmatrix}.$$

For the sake of simplicity, we write $f(A)$ instead of $f(x, A)$ and $f(Du)$ instead of $f(x, Du(x))$. If (2.12) holds and $\omega \in L^1_{\mathrm{loc}}(\Omega)$, then

$$\eta^t f(A) \in L^1(\{u^1 > k\} \cap \{\eta > 0\}) \quad \forall t \geq p^*.$$

Proof. Denote $\hat{u} := (u^2, u^3)$ and

$$D\hat{u} := \begin{pmatrix} Du^2 \\ Du^3 \end{pmatrix}.$$

By the growth condition (4.1) and Lemma 4.1 we have, almost everywhere in $\{u^1 > k\} \cap \{\eta > 0\}$,

$$\begin{aligned} f(A) \leq & c \left\{ \left(\mu\eta^{-1}(u^1 - k)|D\eta| \right)^p + |D\hat{u}|^p + \left| (\operatorname{adj}_2 Du)^1 \right|^q + 1 + \omega \right\} \\ & + c \left\{ \mu\eta^{-1}(u^1 - k)|D\eta||D\hat{u}| \right\}^q + c \left\{ \mu\eta^{-1}(u^1 - k)|D\eta| \left| (\operatorname{adj}_2 Du)^1 \right| \right\}^r. \end{aligned} \tag{4.2}$$

Since (2.12) holds, $q < p$ and $r < q$; thus there exist $\alpha > 1$ and $\beta > 1$ such that

$$q\alpha < p^*, \ q\alpha' = p, \quad \text{and} \quad r\beta < p^*, \ r\beta' = q.$$

Therefore, by the Young inequality, there exists $c > 0$ such that, almost everywhere in $\{u^1 > k\} \cap \{\eta > 0\}$,

$$\begin{aligned} & \left\{ \mu\eta^{-1}(u^1 - k)|D\eta||D\hat{u}| \right\}^q + \left\{ \mu\eta^{-1}(u^1 - k)|D\eta| \left| (\operatorname{adj}_2 Du)^1 \right| \right\}^r \\ & \leq c \left(\mu\eta^{-1}(u^1 - k)|D\eta| \right)^{q\alpha} + c \left(\mu\eta^{-1}(u^1 - k)|D\eta| \right)^{r\beta} \\ & \quad + c \left(|D\hat{u}|^p + \left| (\operatorname{adj}_2 Du)^1 \right|^q \right). \end{aligned} \tag{4.3}$$

Denote $\tilde{q} := \max\{p, q\alpha, r\beta\}$. We have

$$\begin{aligned} & \left(\mu\eta^{-1}(u^1 - k)|D\eta| \right)^p + \left(\mu\eta^{-1}(u^1 - k)|D\eta| \right)^{q\alpha} + \left(\mu\eta^{-1}(u^1 - k)|D\eta| \right)^{r\beta} \\ & \leq \mu^{\tilde{q}} \eta^{-\tilde{q}} (u^1 - k)^{\tilde{q}} |D\eta|^{\tilde{q}} + 3. \end{aligned} \tag{4.4}$$

Therefore, by (4.2), (4.3) and (4.4), almost everywhere in $\{u^1 > k\} \cap \{\eta > 0\}$, we have

$$\eta^t f(A) \leq c \left\{ \mu^{\tilde{q}} \eta^{t-\tilde{q}} (u^1 - k)^{\tilde{q}} |D\eta|^{\tilde{q}} + \eta^t |D\hat{u}|^p + \eta^t \left| (\operatorname{adj}_2 Du)^1 \right|^q + \eta^t + \eta^t \omega \right\}.$$

By (4.1) and $f(Du) \in L^1_{\mathrm{loc}}(\Omega)$ we obtain

$$\begin{aligned} \eta^t |D\hat{u}|^p + \eta^t \left| (\operatorname{adj}_2 Du)^1 \right|^q &\leq \eta^t \left(|Du|^p + |\operatorname{adj}_2 Du|^q \right) \\ &\leq \frac{\eta^t}{c_1} (f(Du) + c_2) \in L^1(\Omega). \end{aligned}$$

Since $u \in L^{p^*}_{\mathrm{loc}}(\Omega; \mathbb{R}^3)$ and $t - \tilde{q} > t - p^* \geq 0$, we have $\eta^{t-\tilde{q}} (u^1 - k)^{\tilde{q}} |D\eta|^{\tilde{q}} \in L^1(\{u^1 > k\} \cap \{\eta > 0\})$. We have thus proved that $\eta^t f(A) \in L^1(\{u^1 > k\} \cap \{\eta > 0\})$ for all $t \geq p^*$. □

Remark 4.3. Analogous inequalities to those in Lemma 4.1 hold true if

$$A = \begin{pmatrix} B^1 \\ A^2 \\ B^3. \end{pmatrix}, \qquad \text{or} \qquad A = \begin{pmatrix} B^1 \\ B^2 \\ A^3 \end{pmatrix}.$$

Therefore, a statement similar to Lemma 4.2 can be given for u^2 and u^3, with

$$A := \begin{pmatrix} Du^1 \\ \mu \eta^{-1} (k - u^2) D\eta \\ Du^3 \end{pmatrix} \qquad \text{and} \qquad A := \begin{pmatrix} Du^1 \\ Du^2 \\ \mu \eta^{-1} (k - u^3) D\eta \end{pmatrix},$$

respectively.

References

1. Ball, J.M.: Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Ration. Mech. Anal.* **63**, 337–403 1977
2. Ball, J.M.: Constitutive inequalities and existence theorems in elastostatics. In: *Nonlinear Analysis and Mechanics, Proceedings, Research Notes*, Vol. 17. (Ed. R.J. Knops) Pitman, London, 13–25, 1977
3. Bauman, P., Owen, N., Phillips, D.: Maximum principles and a priori estimates for a class of problems from nonlinear elasticity. *Ann. Inst. Henri Poincare Anal. Non Lineaire* **8**, 119–157 1991
4. Bauman, P., Owen, N., Phillips, D.: Maximal smoothness of solutions to certain Euler–Lagrange equations from nonlinear elasticity. *Proc. R. Soc. Edinb. Sect. A* **119**, 241–263 1991
5. Bauman, P., Owen, N., Phillips, D.: Maximum principles and a priori estimates for an incompressible material in nonlinear elasticity. *Commun. Partial Differ. Equ.* **17**, 1185–1212 1992
6. Bauman, P., Phillips, D.: Univalent minimizers of polyconvex functionals in two dimensions. *Arch. Ration. Mech. Anal.* **126**, 161–181 1994
7. Carozza, M., Passarelli di Napoli, A.: Model problems from nonlinear elasticity: partial regularity results. *ESAIM Control Opt. Calc. Var.* **13**, 120–134 2007

8. DACOROGNA, B.: *Direct Methods in the Calculus of Variations. Applied Mathematical Sciences*, Vol. 78, 2nd edn, Springer, New York 2008
9. D'OTTAVIO, A., LEONETTI, F., MUSCIANO, C.: Maximum principle for vector-valued mappings minimizing variational integrals. *Atti Sem. Mat. Fis. Modena* **46** Suppl. 677–683 1998
10. DE GIORGI, E.: Un esempio di estremali discontinue per un problema variazionale di tipo ellittico. *Boll. Un. Mat. Ital. (4)* **1**, 135–137 1968
11. ESPOSITO, L., LEONETTI, F., MINGIONE, G.: Sharp regularity results for functionals with (p,q) growth. *J. Differ. Equ.* **204**, 5–55 2004
12. ESPOSITO, L., MINGIONE, G.: Partial regularity for minimizers of degenerate polyconvex energies. *J. Convex Anal.* **8**, 1–38 2001
13. FONSECA, I., MALY, J., MINGIONE, G.: Scalar minimizers with fractal singular sets. *Arch. Ration. Mech. Anal.* **172**, 295–307 2004
14. FUCHS, M., REULING, J.: Partial regularity for certain classes of polyconvex functionals related to non linear elasticity. *Manuscr. Math.* **87**, 13–26 1995
15. FUCHS, M., SEREGIN, G.: Partial regularity of the deformation gradient for some model problems in nonlinear twodimensional elasticity. *Algebra Anal.* **6**, 128–153 1994
16. FUCHS, M., SEREGIN, G.: Holder continuity for weak estremals of some two-dimensional variational problems related to non linear elasticity. *Adv. Math. Sci. Appl.* **7**, 413–425 1997
17. FUSCO, N., HUTCHINSON, J.E.: Partial regularity in problems motivated by nonlinear elasticity. *SIAM J. Math. Anal.* **22**(6), 1516–1551 1991
18. FUSCO, N., HUTCHINSON, J.E.: Partial regularity and everywhere continuity for a model problem from non-linear elasticity. *J. Aust. Math. Soc. Ser. A* **57**(2), 158–169 1994
19. FUSCO, N., SBORDONE, C.: Some remarks on the regularity of minima of anisotropic integrals. *Commun. Partial Differ. Equ.* **18**, 153–167 1993
20. GIAQUINTA, M.: Growth conditions and regularity, a counterexample. *Manuscr. Math.* **59**, 245–248 1987
21. GIUSTI, E.: *Direct Methods in the Calculus of Variations*. World Scientific, River Edge 2003
22. HAMBURGER, C.: Partial regularity of minimizers of polyconvex variational integrals. *Calc. Var. Partial Differ. Equ.* **18**, 221–241 2003
23. HONG, M.C.: Some remarks on the minimizers of variational integrals with non standard growth conditions. *Boll. Un Mat. Ital. A* **6**, 91–101 1992
24. LEONETTI, F.: Maximum principle for vector-valued minimizers of some integral functionals. *Boll. Un Mat. Ital. A* **5**, 51–56 1991
25. LEONETTI, F.: Pointwise estimates for a model problem in nonlinear elasticity. *Forum Math.* **18**, 529–534 2006
26. LEONETTI, F., PETRICCA, P.V.: Bounds for some minimizing sequences of functionals. *Adv. Calc. Var.* **4**, 83–100 2011
27. LEONETTI, F., PETRICCA, P.V.: Bounds for vector valued minimizers of some relaxed functionals. *Complex Var. Elliptic Equ.* **58**, 221–230 2013
28. LEONETTI, F., SIEPE, F.: Maximum principle for vector-valued minimizers of some integral functionals. *J. Convex Anal.* **12**, 267–278 2005
29. LEONETTI, F., SIEPE, F.: Bounds for vector valued minimizers of some integral functionals. *Ricerche Mat.* **54**(1), 303–312 2005
30. MARCELLINI, P.: Un example de solution discontinue d'un problème variationnel dans le cas scalaire, *Preprint 11, Istituto Matematico "U.Dini"*, Università di Firenze, 1987
31. MARCELLINI, P.: Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions. *Arch. Ration. Mech. Anal.* **105**, 267–284 1989
32. MARCELLINI, P.: Regularity and existence of solutions of elliptic equations with $p-q$-growth conditions. *J. Differ. Equ.* **90**, 1–30 1991
33. MINGIONE, G.: Regularity of minima: an invitation to the dark side of the calculus of variations. *Appl. Math.* **51**, 355–426 2006

34. Mingione, G.: Singularities of minima: a walk in the wild side of the calculus of variations. *J. Glob. Optim.* **40**, 209–223 2008
35. Mooney, C., Savin, O.: Some singular minimizers in low dimensions in the calculus of variations. *Arch. Rational Mech. Anal.* **221**, 1–22 2016
36. Morrey, C.B.: *Multiple Integrals in the Calculus of Variations*. Springer, Berlin 1966
37. Moscariello, G., Nania, L.: Hölder continuity of minimizers of functionals with non standard growth conditions. *Ricerche Mat.* **40**, 259–273 1991
38. Muller, S.: Higher integrability of determinants and weak convergence in L^1. *J. Reine Angew. Math.* **412**, 20–34 1990
39. Ogden, R.W.: *Nonlinear Elastic Deformations, Ellis Horwood Series: Mathematics and its Applications*. Ellis Horwood Ltd., Chichester; Halsted Press, New York 1984
40. Passarelli di Napoli, A.: A regularity result for a class of polyconvex functionals. *Ricerche Mat.* **48**, 379–393 1999
41. Reshetnyak, Y.: Stability theorems for mappings with bounded distortion. *Sib. Mat. Zh.* **9**, 667–684 1968
42. Šverák, V.: Regularity properties for deformations with finite energy. *Arch. Ration. Mech. Anal.* **100**, 105–127 1988
43. Šverák, V., Yan, X.: A singular minimizer of a smooth strongly convex functional in three dimensions. *Calc. Var. Partial Differ. Equ.* **10**, 213–221 2000

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