LOCAL BOUNDEDNESS OF SOLUTIONS TO SOME ANISOTROPIC ELLIPTIC SYSTEMS

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This paper is dedicated to Patrizia Pucci in the occasion of her 60th birthday.

1. Introduction

We consider a map $u : \mathbb{R}^n \to \mathbb{R}^m$, $n, m > 1$ solution to a nonlinear system of partial differential equations, or minimizer of a functional of the calculus of variations. It is well known that either the global or the local boundedness of u cannot be obtained through truncation methods. This is due to the lack of the maximum principle for general systems. Nevertheless in this paper we present a method for local boundedness of u without assuming any condition on the boundary datum.

More precisely, we consider a minimizer $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $u \in W^{1,1}(\Omega; \mathbb{R}^m)$, with $n \geq 2$, $m \geq 1$, of the integral

$$I(v) = \int_\Omega f(x, Dv)\, dx \tag{1.1}$$

(the framework is similar for a solution to a nonlinear system in divergence form). We assume that the integrand $f = f(x, \xi)$, $x \in \Omega \subset \mathbb{R}^n$, $\xi \in \mathbb{R}^{m\times n}$, is a measurable function with respect to x, convex and of class C^1 with respect to ξ and satisfying the following *anisotropic behaviour*: for some exponents p_i , $i = 1, \dots, n$, and q with $1 \leq p_i \leq q$

$$\sum_{i=1}^{n} |\xi_i|^{p_i} \leq f(x, \xi) \leq c \left\{ 1 + \sum_{i=1}^{n} |\xi_i|^q \right\} \tag{1.2}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{m\times n}$ and for a constant $c > 0$. Here ξ_i, $i = 1, \dots, n$, is the i-column of the $m \times n$ matrix $\xi = (\xi_i^\alpha)$, $i = 1, \dots, n$, $\alpha = 1, \dots, m$; i.e.,

$$\xi = (\xi_1, \xi_2, \dots \xi_n) = \begin{pmatrix} \xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 \\ \xi_1^2 & \xi_2^2 & \cdots & \xi_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^m & \xi_2^m & \cdots & \xi_n^m \end{pmatrix} .$$

In particular, when $\xi = Du$, then $\xi_i = (u^1_{x_i}, \dots, u^m_{x_i})^T$.

The following result is a particular case of Theorem 2.1, proved in the next sections.

Theorem 1.1. *Let $f = f(x, \xi)$ satisfy (1.2) and the conditions*

$$f(x, \xi) = F(x, |\xi_1|, ..., |\xi_i|, ..., |\xi_n|), \tag{1.3}$$

$$f(x, \lambda\xi) \leq \lambda^\mu f(x, \xi), \qquad \text{for some } \mu > 1 \text{ and for every } \lambda > 1. \tag{1.4}$$

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If $q < \overline{p}^*$, where $\overline{p}^*$ is the Sobolev exponent of $\overline{p}$ ($\overline{p}$ is the harmonic average of $\{p_i\}$, i.e. $\frac{1}{\overline{p}} := \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}$), then every local minimizer u of (1.1) is locally bounded. Morover, for every ball $B_r(x_0)$ compactly contained in Ω, there exists $C > 0$, depending on the data, such that

$$\|u - u_r\|_{L^\infty(B_{r/(2\sqrt{n})}(x_0))} \leq C\left\{1 + \int_{B_r(x_0)} f(x, Du)\,dx\right\}^{\frac{1+\theta}{p}},$$

where u_r denotes the average of u in the ball $B_r(x_0)$ and $\theta = \frac{\overline{p}^*(q-p)}{p(\overline{p}^*-q)}$.

Notice that, by the examples in Giaquinta [13] and Marcellini [16], the condition $q < \overline{p}^*$ is nearly optimal, since the boundedness of minimizers may fail if $q > \overline{p}^*$. Actually the regularity result is proved under some more general assumptions on f, see Theorem 2.1. In particular, the convexity with respect to ξ and the structure assumptions (1.3),(1.4) are assumed only at infinity, i.e. for $|\xi| \geq t_0$.

This context of non-standard growth have been intensely investigated in recent years and it is quite impossible to give an exhaustive and comprehensive list of references; see e.g. [5], [12], [17], [18], [19] and Mingione [21] for an overview on the subject and a detailed bibliography. Anisotropic elliptic equations have been considered under many different aspects, for instance with respect to the maximum principle and to the multiplicity of solutions; see e.g. P. Pucci, V. Rădulesco *et al.* [6], [11] and [20].

In the vector-valued case, as suggested by well known counterexamples by de Giorgi [10], Giusti-Miranda [14] Nečas [22], Sverak-Yan [24], generally some structure conditions on the integrand, more specific than (1.3), are required for everywhere regularity. A boundedness result in the vectorial framework is proved by Dall'Aglio-Mascolo [9], assuming $f(x, Du) = g(x, |Du|)$. Recently in [8] the authors studied the boundedness of solutions for a class of quasilinear systems, which - in the variational case - may correspond to integrals as in (1.1) with a more restrictive growth than in (1.2). Other related results in the p, q case are in [8] and in Leonetti-Mascolo [15].

The main novelties of our Theorem 1.1 are the new form of the structure condition (1.3) and the anisotropic behaviour of the integrand (1.2). The main ingredients of the proof are the derivation of the Euler's equation and the Moser's iteration technique. This completes the study in [7] given for the scalar case $m = 1$. However we point out that the proof here in the vectorial case cannot be regarded as simple generalization of the scalar case, also for the lack of convexity near the origin. Moreover our analysis allows us to consider, as an assumption, only the asymptotic behaviour at infinity ($|\xi| \to +\infty$) of $f(x, \xi)$. In this context we quote Scheven-Schmidt [23].

It is worth to point out that in some recent paper by Bildhauer, Fuchs *et al.* (see [2],[3],[4]) regularity results are proved by assuming a-priori the local boundedness of minimizers, obtaining, for instance, the higher integrability of the gradient of u for the so called *splitting variational integrals*

$$f(Du) = (1 + |\tilde{D}u|^2)^{\frac{p}{2}} + (1 + |u_{x_n}|^2)^{\frac{q}{2}}$$

where $\tilde{D}u = (u_{x_1}, ..., u_{x_{n-1}})$, $1 < p < q$.

The paper is organized as follows. In the next section we state the regularity results. In Section 3 we prove some preliminary properties, mainly consequence of the convexity and of the Δ_2 condition and some higher integrability results. Section 4 is devoted to the proof of the Euler system, which is a main step in the proof of Theorem 2.1, given in the last section.

2. Assumptions and statement of the main results

Let us define the integral functional

$$\mathcal{F}(u) := \int_\Omega f(x, Du(x))\, dx, \tag{2.1}$$

where Ω is an open bounded subset of $\mathbb{R}^n$, $n \geq 2$, and $u \in W^{1,1}(\Omega; \mathbb{R}^m)$, $m \in \mathbb{N}$.

We denote $\mathbb{R}_+$ the set $[0, +\infty)$, $B_r(x_0)$ the ball in $\mathbb{R}^n$ centered at x_0 with radius r and $\mathbf{B}_t$ the ball in $\mathbb{R}^{mn}$ of radius t centered at the origin.
We need some notations. From now on, $i, j \in \{1, ..., n\}$ and $\alpha, \beta \in \{1, ..., m\}$. If $\xi \in \mathbb{R}^{mn}$ we write $\xi = (\xi_1, ..., \xi_n)$, where $\xi_i = (\xi_i^1, ..., \xi_i^m)^T \in \mathbb{R}^m$. In particular, $Du = (u_{x_1}, ..., u_{x_n})^T$ and $u_{x_i} = (u^1_{x_1}, ..., u^m_{x_n})^T$.

We assume that $f : \Omega \times \mathbb{R}^{mn} \to \mathbb{R}_+$ is a Carathéodory function, of class C^1 with respect to $\xi \in \mathbb{R}^{mn}$ and that there exists $t_0 \geq 0$ such that

- **(H1)** $f(x, \xi) = f^{**}(x, \xi)$ if $|\xi| \geq t_0$, where $f^{**}(x, \cdot)$ is the greatest convex function lower than $f(x, \cdot)$,
- **(H2)** there exists $F : \Omega \times (\mathbb{R}_+)^n \to \mathbb{R}_+$ such that $f(x, \xi) = F(x, |\xi_1|, ..., |\xi_i|, ..., |\xi_n|)$ if $|\xi| \geq t_0$,
- **(H3)** there exists $\mu > 1$ such that $f(x, \lambda\xi) \leq \lambda^\mu f(x, \xi)$ for every $\lambda > 1$ and for a.e. x and every $|\xi| \geq t_0$,
- **(H4)** $\sup_{|\xi| \leq t_0} \left| \frac{\partial f}{\partial \xi_i^\alpha}(\cdot, \xi) \right| \in L^\infty_{\mathrm{loc}}(\Omega)$ for every i and α.

Moreover, a growth condition on f is assumed:

- **(H5)** there exist $k_1, k_2 > 0$ and $1 \leq p_i \leq q$, $i = 1, \ldots, n$, such that

$$-k_1 + \sum_{i=1}^n |\xi_i|^{p_i} \leq f(x, \xi) \leq k_2 \left\{ 1 + \sum_{i=1}^n |\xi_i|^q \right\} \quad \text{for a.e. } x \text{ and every } \xi \in \mathbb{R}^{mn}. \tag{2.2}$$

We define

$$W^{1,f}(\Omega; \mathbb{R}^m) := \{ u \in W^{1,1}(\Omega; \mathbb{R}^m) \, : \, \mathcal{F}(u) < +\infty \}$$

and we denote $W^{1,f}_0(\Omega; \mathbb{R}^m)$ the space $W^{1,1}_0(\Omega; \mathbb{R}^m) \cap W^{1,f}(\Omega; \mathbb{R}^m)$.

A function u is a local minimizer of (2.1) if $u \in W^{1,f}(\Omega; \mathbb{R}^m)$ and $\mathcal{F}(u) \leq \mathcal{F}(u + \varphi)$, for all $\varphi \in W^{1,f}(\Omega; \mathbb{R}^m)$ with $\operatorname{supp} \varphi \Subset \Omega$.

To prove the local boundedness of local minimizers of (2.1) we need a restriction on the exponents $\{p_i\}$ and q. Let p denote $\min\{p_i\}$ and, as in the introduction, let $\overline{p}$ be the harmonic average of $\{p_i\}$, i.e., $\frac{1}{\overline{p}} := \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$ and $\overline{p}^*$ be the Sobolev exponent of $\overline{p}$, i.e.

$$\overline{p}^* := \begin{cases} \frac{n\overline{p}}{n - \overline{p}} & \text{if } \overline{p} < n, \\ \text{any } \mu > \overline{p} & \text{if } \overline{p} \geq n. \end{cases} \tag{2.3}$$

Our main theorem is the following.

Theorem 2.1. *Assume (H1)–(H5) and let $q < \overline{p}^*$. Then a local minimizer u of (2.1) is locally bounded. Moreover, for every $B_r(x_0) \Subset \Omega$ the following estimates hold true:*

(1) *there exists $c > 0$, depending on the data, such that*

$$\|u\|_{L^\infty(B_{r/2}(x_0))} \leq c \left\{ 1 + \int_{B_r(x_0)} |u|^q \, dx \right\}^{\frac{1+\theta}{q}}, \tag{2.4}$$

(2) *there exists $c > 0$, depending on the data, such that*

$$\|u - u_r\|_{L^\infty(B_{r/(2\sqrt{n})}(x_0))} \leq c \left\{ 1 + \int_{B_r(x_0)} f(x, Du)\, dx \right\}^{\frac{1+\theta}{p}}, \tag{2.5}$$

where $\theta = \frac{\overline{p}^*(q-p)}{p(\overline{p}^*-q)}$ *and* $u_r := \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u\, dx$.

3. Preliminary results

Trivial consequences of (H1), (H2) and (H5) are the following properties (that hold true possibly with a larger t_0):

$$r \mapsto f(x, r\xi) \quad \text{is increasing in } (1, +\infty) \text{ for every } |\xi| = t_0, \tag{3.1}$$

$$F(x, |\xi_1|, ..., |\xi_i|, ..., |\xi_n|) \quad \text{is increasing w.r.t. each variable } |\xi_i| \text{ when } |\xi| \geq t_0, \tag{3.2}$$

and $f(x, \xi) > 0$ for all ξ with $|\xi| \geq t_0$.

The following elementary lemma, whose proof is trivial, holds true.

Lemma 3.1. *Consider $h : \mathbb{R}_+ \to \mathbb{R}_+$ of class C^1. Suppose that there exist $t_0 \geq 0$ and $\gamma > 0$ such that*

$$h(\lambda t) \leq \lambda^\gamma h(t) \qquad \text{for all } \lambda > 1 \text{ and } t \geq t_0. \tag{3.3}$$

Then

$$h'(t)t \leq \gamma h(t) \qquad \text{for all } t \geq t_0. \tag{3.4}$$

If f is as in the previous section, then $W^{1,f}(\Omega; \mathbb{R}^m)$ is a vector space; this is a consequence of the following lemma.

Lemma 3.2. *By (H1), (H3) and (H5) we have that*

(i) $f(x, \lambda\xi) \leq \max\{1, \lambda^\mu\} \{\kappa + f(x, \xi)\}$ *for every $\lambda > 0$ and every $\xi \in \mathbb{R}^{mn}$,*

(ii) $f(x, \xi + \eta) \leq 2^{\mu-1} \{2\kappa + f(x, \xi) + f(x, \eta)\}$ *for every $\xi, \eta \in \mathbb{R}^{mn}$*

with $\kappa = k_2 \{1 + nt_0^q\}$.

Proof. Let us prove (i). If $\xi \in \mathbb{R}^{mn}$ and $|\lambda\xi| \leq t_0$ then (2.2) gives $f(x, \lambda\xi) \leq \kappa$ and the conclusion follows.

Assume $|\lambda\xi| > t_0$. We separately consider the case $\lambda > 1$ and $\lambda \leq 1$.

Let $\lambda > 1$. If $|\xi| \leq t_0$ then (3.1), (H3) and (2.2) imply

$$f(x, \lambda\xi) \leq f(x, \lambda t_0 \frac{\xi}{|\xi|}) \leq \lambda^\mu f(x, t_0 \frac{\xi}{|\xi|}) \leq \lambda^\mu \kappa.$$

If instead $|\xi| > t_0$ then (H3) implies $f(x, \lambda\xi) \leq \lambda^\mu f(x, \xi)$.

Let us consider $\lambda \leq 1$. By $|\lambda\xi| > t_0$ and (3.1), we get $f(x, \lambda\xi) \leq f(x, \xi)$ and the conclusion follows.

Let us prove (ii).

If $|\xi + \eta| \leq t_0$ then $f(x, \xi + \eta) \leq \kappa$ by (2.2).

Suppose $|\xi + \eta| > t_0$. Then

$$f(x, \xi + \eta) = f^{**}(x, \xi + \eta) \leq \frac{1}{2} [f^{**}(x, 2\xi) + f^{**}(x, 2\eta)] \leq \frac{1}{2} [f(x, 2\xi) + f(x, 2\eta)].$$

By (i) $f(x, 2\xi) + f(x, 2\eta) \leq 2^\mu \{2\kappa + f(x, \xi) + f(x, \eta)\}$ and we conclude. □

By Lemma 3.2 it easily follows that $W^{1,f}(\Omega; \mathbb{R}^m)$ is a vector space.

Consider now the anisotropic Sobolev space

$$W^{1,(p_1,\dots,p_n)}(\Omega;\mathbb{R}^m) := \left\{u \in W^{1,1}(\Omega;\mathbb{R}^m) \,:\, u_{x_i} \in L^{p_i}(\Omega;\mathbb{R}^m), \text{ for all } i = 1,\dots,n\right\}$$

endowed with the norm

$$\|u\|_{W^{1,(p_1,\dots,p_n)}(\Omega;\mathbb{R}^m)} := \|u\|_{L^1(\Omega;\mathbb{R}^m)} + \sum_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}(\Omega;\mathbb{R}^m)}.$$

Sometimes, when no misunderstanding may arise, we will not indicate the target space $\mathbb{R}^m$. Denote $W_0^{1,(p_1,\dots,p_n)}(\Omega;\mathbb{R}^m)$ in place of $W_0^{1,1}(\Omega;\mathbb{R}^m) \cap W^{1,(p_1,\dots,p_n)}(\Omega;\mathbb{R}^m)$. These spaces are studied in [25], see also [1]. We remind an embedding theorem for this class of spaces (see [25]).

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and consider $u \in W_0^{1,(p_1,\dots,p_n)}(\Omega;\mathbb{R}^m)$, $p_i \geq 1$ for all $i = 1,\dots,n$. Let $\max\{p_i\} \leq \overline{p}^*$, with $\overline{p}^*$ as in (2.3). Then $u \in L^{\overline{p}^*}(\Omega;\mathbb{R}^m)$. Moreover, there exists c depending on $n, p_1,\dots,p_n$ if $\overline{p} < n$, and also on Ω if $\overline{p} \geq n$, such that*

$$\|u\|^n_{L^{\overline{p}^*}(\Omega;\mathbb{R}^m)} \leq c \prod_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}(\Omega;\mathbb{R}^m)}.$$

The following embedding result is proved in [1].

Theorem 3.4. *Let $Q \subset \mathbb{R}^n$ be a cube with edges parallel to the coordinate axes and consider $u \in W^{1,(p_1,\dots,p_n)}(Q;\mathbb{R}^m)$, $p_i \geq 1$ for all $i = 1,\dots,n$. Let $\max\{p_i\} < \overline{p}^*$, with $\overline{p}^*$ as in (2.3). Then $u \in L^{\overline{p}^*}(Q;\mathbb{R}^m)$. Moreover, there exists c depending on $n, p_1,\dots,p_n$ if $\overline{p} < n$, and also on Q if $\overline{p} \geq n$, such that*

$$\|u\|_{L^{\overline{p}^*}(Q)} \leq c \left\{ \|u\|_{L^1(Q)} + \sum_{i=1}^{n} \|u_{x_i}\|_{L^{p_i}(Q)} \right\}.$$

A consequence of the above result is the following corollary.

Corollary 3.5. *Assume (H5), with $q < \overline{p}^*$. If $u \in W^{1,f}(\Omega;\mathbb{R}^m)$, then $|u| \in L^{\overline{p}^*}_{loc}(\Omega;\mathbb{R}^m)$.*

4. The Euler's equation

In this section we prove the Euler's equation, our starting point of the proof of Theorem 2.1.

Theorem 4.1. *Assume (H1)-(H3) and (H5) and let u be a local minimizer of (2.1). Then*

$$\int_\Omega \sum_{i=1}^{n} \sum_{\alpha=1}^{m} \frac{\partial f}{\partial \xi_i^\alpha}(x, Du)\, (\varphi^\alpha)_{x_i}\, dx = 0$$

for all $\varphi \in W^{1,f}(\Omega;\mathbb{R}^m)$, $\operatorname{supp} \varphi \Subset \Omega$.

Proof. Let $\varphi \in W^{1,f}(\Omega;\mathbb{R}^m)$, $\operatorname{supp} \varphi \Subset \Omega$. We aim to prove that

$$\frac{d}{dt}\mathcal{F}(u + t\varphi)\Big|_{t=0} = \int_\Omega \frac{d}{dt} f(x, Du(x) + tD\varphi(x))\Big|_{t=0} dx.$$

To prove this, we need to prove that

$$\Big|\sum_{i=1}^{n} \sum_{\alpha=1}^{m} f_{\xi_i^\alpha}(x, Du + tD\varphi)\varphi^\alpha_{x_i}\Big| \leq H(x) \qquad \forall\, t \in (-1, 1) \tag{4.1}$$

with $H \in L^1(\Omega)$. By the convexity,

$$f^{**}(x,\xi_0) - f^{**}(x,2\xi_0-\xi) \le \sum_{i=1}^{n}\sum_{\alpha=1}^{m}(f^{**})_{\xi_i^\alpha}(x,\xi_0)(\xi_i^\alpha - (\xi_0)_i^\alpha) \le f^{**}(x,\xi) - f^{**}(x,\xi_0).$$

If $\xi_0 = Du(x) + tD\varphi(x)$, $\xi = Du(x) + (1+t)D\varphi(x)$, we have $2\xi_0 - \xi = Du(x) + (t-1)D\varphi(x)$ and

$$\begin{aligned} f^{**}(x,Du+tD\varphi) - f^{**}(x,Du+(t-1)D\varphi) &\le \sum_{i=1}^{n}\sum_{\alpha=1}^{m}(f^{**})_{\xi_i^\alpha}(x,Du+tD\varphi)\varphi_{x_i}^\alpha \\ &\le f^{**}(x,Du+(1+t)D\varphi) - f^{**}(x,Du+tD\varphi). \end{aligned}$$

Therefore, since f^{**} is non-negative,

$$|\sum_{i=1}^{n}\sum_{\alpha=1}^{m}(f^{**})_{\xi_i^\alpha}(x,Du+tD\varphi)\varphi_{x_i}^\alpha| \le f^{**}(x,Du+(1+t)D\varphi) + f^{**}(x,Du+(t-1)D\varphi).$$

Using again the convexity we get

$$\begin{aligned} f^{**}(x,Du+(1+t)D\varphi) &\le tf^{**}(x,Du+2D\varphi) + (1-t)f^{**}(x,Du+D\varphi) \\ &\le f^{**}(x,Du+2D\varphi) + f^{**}(x,Du+D\varphi) \end{aligned}$$

and

$$\begin{aligned} f^{**}(x,Du+(t-1)D\varphi) &\le tf^{**}(x,Du) + (1-t)f^{**}(x,Du-D\varphi) \\ &\le f^{**}(x,Du) + f^{**}(x,Du-D\varphi). \end{aligned}$$

Lemma 3.2 obviously holds true also with f replaced by f^{**}, therefore

$$f^{**}(x,Du-D\varphi) \le f^{**}(x,Du) + f^{**}(x,-D\varphi).$$

If $|D\varphi(x)| \le t_0$ then $f^{**}(x,-D\varphi(x)) \le \kappa$ (see Lemma 3.2 for the definition of κ); if instead $|D\varphi(x)| > t_0$ then $f^{**}(x,-D\varphi(x)) = f^{**}(x,D\varphi(x))$ by (H2).
Thus, the above inequalities, (H1), (H2), (H5), and Lemma 3.2 imply

$$|\sum_{i=1}^{n}\sum_{\alpha=1}^{m}(f^{**})_{\xi_i^\alpha}(x,Du+tD\varphi)\varphi_{x_i}^\alpha| \le c(n,k_2,q,\mu,t_0)\,(1+f^{**}(x,Du)+f^{**}(x,D\varphi)) =: h_1(x)$$

with $h_1 \in L^1(\Omega)$ since $f^{**} \le f$ and $u,\varphi \in W^{1,f}(\Omega;\mathbb{R}^m)$.

Now, if $x \in \{|Du+tD\varphi| \ge t_0\}$ then by (H1) $f_{\xi_i^\alpha}(x,Du+tD\varphi) = (f^{**})_{\xi_i^\alpha}(x,Du+tD\varphi)$ and if $x \in \{|Du+tD\varphi| < t_0\}$

$$|\sum_{i=1}^{n}\sum_{\alpha=1}^{m} f_{\xi_i^\alpha}(x,Du+tD\varphi)\varphi_{x_i}^\alpha| \le \sum_{i=1}^{n}\sum_{\alpha=1}^{m}\sup_{\xi\in\mathbf{B}_{t_0}}|f_{\xi_i^\alpha}(x,\xi)|\cdot|\varphi_{x_i}^\alpha| =: h_2(x)$$

with $h_2 \in L^1(\Omega)$ since $\varphi \in W^{1,1}(\Omega)$, supp $\varphi \Subset \Omega$ and (H4) holds. We have so proved that (4.1) holds true with $H = h_1 + h_2$. □

5. Proof of the boundedness of local minimizers

In the following we define a class of suitable test functions for the Euler's equation (4.1).

Let us approximate the identity function id : $\mathbb{R}_+ \to \mathbb{R}_+$ with an *increasing* sequence of C^1 functions $h_k : \mathbb{R}_+ \to \mathbb{R}_+$, with the following properties:

$$h_k(t) = 0 \quad \forall t \in [0, \frac{1}{k}], \qquad h_k(t) = k \quad \forall t \in [k+1, +\infty], \qquad 0 \le h_k'(t) \le 2 \text{ in } \mathbb{R}_+. \tag{5.1}$$

Fixed $k, i \in \mathbb{N}$, $i \le n$, and $\gamma \ge 0$, let $\Phi_k^{(i,\gamma)} : \mathbb{R}_+ \to \mathbb{R}_+$ be the increasing function defined as follows

$$\Phi_k^{(i,\gamma)}(t) := h_k(t^{p_i\gamma}). \tag{5.2}$$

The following lemma holds.

Lemma 5.1. *Assume (H1)-(H3) and (H5), with $q < \overline{p}^*$. Let $u \in W^{1,f}(\Omega)$, fix a ball $B_R(x_0) \Subset \Omega$ and let $\eta \in C_c^\infty(B_R(x_0))$ be a cut-off function, satisfying the following assumptions*

$$0 \le \eta \le 1, \qquad \eta \equiv 1 \text{ in } B_\rho(x_0) \text{ for some } \rho < R, \qquad |D\eta| \le \frac{2}{R-\rho}. \tag{5.3}$$

Fixed $k \in \mathbb{N}$ and $\gamma \ge 0$, define $\varphi_k : B_R(x_0) \to \mathbb{R}^m$,

$$\varphi_k(x) := \Phi_k^{(i,\gamma)}(|u(x)|)u(x)[\eta(x)]^\delta \qquad \text{for every } x \in B_R(x_0), \tag{5.4}$$

with $\delta \ge 1$. Then φ_k is in $W^{1,f}(B_R(x_0))$, supp $\varphi \Subset B_R(x_0)$.

Proof. From now on, we omit the dependence of Φ_k on i and γ, i.e. $\Phi_k = \Phi_k^{(i,\gamma)}$. We have that Φ_k is in $C^1(\mathbb{R}_+)$, bounded and with bounded derivative. Precisely, define a_k and b_k positive, such that $a_k^{p_i\gamma} = \frac{1}{k}$ and $b_k^{p_i\gamma} = k+1$. In particular,

$$\Phi_k'(s) = \begin{cases} 0 & \text{if } s \in \mathbb{R}_+ \setminus [a_k, b_k] \\ p_i\gamma h_k'(s^{p_i\gamma})s^{p_i\gamma-1} & \text{if } s \in [a_k, b_k] \end{cases}, \quad \|\Phi_k'\|_{L^\infty(\mathbb{R}_+)} \le 2p_i\gamma \max\{a_k^{p_i\gamma-1}, b_k^{p_i\gamma-1}\} < \infty. \tag{5.5}$$

As a consequence, taking into account that $u \in W^{1,1}(\Omega;\mathbb{R}^m)$ we have that $\Phi_k(|u|)u$ is in $W^{1,1}(\Omega;\mathbb{R}^m)$ which implies that $\varphi_k(x) \in W^{1,1}(\Omega;\mathbb{R}^m)$, too.

By Lemma 3.2 (i) we conclude if we prove that

$$\begin{aligned}
A &:= \int_{B_R\cap\{a_k<|u|<b_k\}} f\left(x, \Phi_k'(|u|)\,\frac{u(x)}{|u(x)|}\langle u, u_{x_1}\rangle\,\eta^\delta, ..., \Phi_k'(|u|)\,\frac{u(x)}{|u(x)|}\langle u, u_{x_n}\rangle\,\eta^\delta\right) dx < +\infty \\
B &:= \int_{B_R} f\left(x, \Phi_k(|u|)Du\,\eta^\delta\right) dx < +\infty \\
C &:= \int_{B_R} f\left(x, \Phi_k(|u|)\,u\,\delta\eta^{\delta-1}\eta_{x_1}, ..., \Phi_k(|u|)\,u\,\delta\eta^{\delta-1}\eta_{x_n}\right) dx < +\infty.
\end{aligned}$$

Of course, by (H5) $\int_{B_R\cap\{|u|\notin[a_k,b_k]\}} f(x,0)\,dx \le k_2|B_R| < +\infty$.

First we estimate A. By Lemma 3.2 (i) we get

$$\begin{aligned}
A &\le \int_{B_R\cap\{a_k<|u|<b_k\}} \max\left\{1, [\Phi_k'(|u|)\eta^\delta]^\mu\right\}\left\{\kappa + f\left(x, \frac{u}{|u|}\langle u, u_{x_1}\rangle, ..., \frac{u}{|u|}\langle u, u_{x_n}\rangle\right)\right\} dx \\
&\le \max\left\{1, \|\Phi_k'\|^\mu_{L^\infty(\mathbb{R}_+)}\right\}\int_{B_R\cap\{a_k<|u|<b_k\}} \left\{\kappa + f\left(x, \frac{u}{|u|}\langle u, u_{x_1}\rangle, ..., \frac{u}{|u|}\langle u, u_{x_n}\rangle\right)\right\} dx.
\end{aligned}$$

Now, denoting $v(x) = (\frac{u(x)}{|u(x)|}\langle u, u_{x_1}\rangle, ..., \frac{u(x)}{|u(x)|}\langle u, u_{x_n}\rangle)$ and using (H2)

$$\int_{B_R\cap\{a_k<|u|<b_k\}} f\left(x, \frac{u}{|u|}\langle u, u_{x_1}\rangle, ..., \frac{u}{|u|}\langle u, u_{x_n}\rangle\right) dx$$

$$\leq \int_{B_R\cap\{a_k<|u|<b_k\}\cap\{|v|\leq t_0\}} f\left(x, \frac{u}{|u|}\langle u, u_{x_1}\rangle, ..., \frac{u}{|u|}\langle u, u_{x_n}\rangle\right) dx$$
$$+ \int_{B_R\cap\{a_k<|u|<b_k\}\cap\{|v|>t_0\}} F\left(x, |\langle u, u_{x_1}\rangle|, ..., |\langle u, u_{x_n}\rangle|\right) dx.$$

The first integral is obviously bounded by the continuity of f; as far as the second one is concerned we use the monotonicity property on F, see (3.2), (H2) and Lemma 3.2 (i) obtaining

$$\int_{B_R\cap\{a_k<|u|<b_k\}\cap\{|v|>t_0\}} F\left(x, |\langle u, u_{x_1}\rangle|, ..., |\langle u, u_{x_n}\rangle|\right) dx$$
$$\leq \int_{B_R\cap\{a_k<|u|<b_k\}\cap\{|v|>t_0\}} F\left(x, b_k|u_{x_1}|, ..., b_k|u_{x_n}|\right) dx$$
$$\leq \int_{B_R\cap\{|v|>t_0\}} f\left(x, b_k Du\right) dx \leq \max\{1, b_k^\mu\} \int_{B_R} \{\kappa + f(x, Du)\}\, dx$$

and this integral is bounded by the assumption $u \in W^{1,f}$.

To prove that B is bounded we use Lemma 3.2 (i) obtaining

$$\int_{B_R} f\left(x, \Phi_k(|u|)\, \eta^\delta Du\right) dx \leq \int_{B_R} \max\{1, [\Phi_k(|u|)\eta^\delta]^\mu\}\, \{\kappa + f(x, Du)\}\, dx$$
$$\leq k^\mu \int_{B_R} \{\kappa + f(x, Du)\}\, dx$$

and the last integral is finite.

Let us consider C. Using Lemma 3.2 (i) once more,

$$\int_{B_R} f\left(x, \Phi_k(|u|)\, u\, \delta\eta^{\delta-1}\eta_{x_1}, ..., \Phi_k(|u|)\, u\, \delta\eta^{\delta-1}\eta_{x_n}\right) dx$$
$$\leq \max\{1, [k\delta]^\mu\} \int_{B_R} \left(\kappa + f\left(x, u\eta_{x_1}, ..., u\eta_{x_n}\right)\right) dx.$$

Now, by (H5) and (H2)

$$\int_{B_R} f\left(x, u\eta_{x_1}, ..., u\eta_{x_n}\right) dx \leq \kappa |B_R \cap \{|u|\cdot|D\eta| \leq t_0\}| + \int_{B_R\cap\{|u|\cdot|D\eta|>t_0\}} F\left(x, |u||\eta_{x_1}|, ..., |u||\eta_{x_n}|\right) dx.$$

The last integral can be majorized, using (5.3) and (3.2), as follows:

$$\int_{B_R\cap\{|u|\cdot|D\eta|>t_0\}} F\left(x, |u||\eta_{x_1}|, ..., |u||\eta_{x_n}|\right) dx \leq \int_{B_R\cap\{|u|\cdot|D\eta|>t_0\}} F\left(x, \frac{2|u|}{R-\rho}, ..., \frac{2|u|}{R-\rho}\right) dx.$$

Therefore, by (H2), Lemma 3.2 (i) and (H5) we get

$$\int_{B_R\cap\{|u|\cdot|D\eta|>t_0\}} F\left(x, \frac{2|u|}{R-\rho}, ..., \frac{2|u|}{R-\rho}\right) dx \leq \max\left\{1, \left(\frac{2}{R-\rho}\right)^\mu\right\} \int_{B_R} \left(\kappa + k_2\{n|u|^q + 1\}\right) dx,$$

which is finite by Corollary 3.5 since $q < \overline{p}^*$. □

Now, we turn to the proof of our main result.

Proof of Theorem 2.1. Let u be a local minimizer of (2.1) and consider $x_0 \in \Omega$ and $R_0 > 0$, such that $B_{R_0} := B_{R_0}(x_0) \Subset \Omega$. In particular, by Corollary 3.5 $|u| \in L^{\overline{p}^*}(B_{R_0})$. Fix also $0 < \rho < R \leq R_0$. We split the proof into steps.

Step 1. Assume that $|u| \in L^{q(\gamma+1)}(B_R)$ for some $\gamma \geq 0$. Fixed $i \in \{1,\dots,n\}$ we prove the following estimate

$$\int_{B_R} |u_{x_i}|^{p_i}\, |u|^{p_i\gamma}\, \eta^\mu\, dx \leq \frac{c}{(R-\rho)^\mu} \int_{B_R} (|u|^q+1)|u|^{p_i\gamma}\, dx \tag{5.6}$$

for some c depending on n, μ, p, q, k_1, k_2, t_0 and R_0, but independent of i, γ, u, R and ρ. The function η is a cut-off function satisfying (5.3),

Let us consider the Euler's equation (4.1) with test function

$$\varphi_k^{(i,\gamma)}(x) := \Phi_k^{(i,\gamma)}(|u(x)|)u(x)\eta^\mu$$

with $\Phi_k^{(i,\gamma)}$ as in (5.2), $k \in \mathbb{N}$. From now on, we write φ_k and Φ_k in place of $\varphi_k^{(i,\gamma)}$ and $\Phi_k^{(i,\gamma)}$, respectively. We obtain

$$\sum_{j=1}^n\sum_{\alpha=1}^m \int_{B_R} \frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\, u_{x_j}^\alpha\, \Phi_k(|u|)\,\eta^\mu\, dx + \sum_{j=1}^n\sum_{\alpha,\beta=1}^m \int_{B_R} \frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\, u^\alpha \frac{u^\beta}{|u|}\, u_{x_j}^\beta\, \Phi_k'(|u|)\,\eta^\mu\, dx$$
$$\leq \mu \left| \sum_{j=1}^n\sum_{\alpha=1}^m \int_{B_R} \frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\, \Phi_k(|u|)\, u^\alpha\, \eta^{\mu-1}\eta_{x_j}\, dx \right|$$

that implies

$$\begin{aligned} I_1+I_2 &:= \sum_{j=1}^n\sum_{\alpha=1}^m \int_{B_R} \frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\, u_{x_j}^\alpha\, \Phi_k(|u|)\,\eta^\mu\, dx \\ &+ \sum_{j=1}^n\sum_{\alpha,\beta=1}^m \int_{B_R\cap\{|Du|>t_0\}} \frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\, u^\alpha \frac{u^\beta}{|u|}\, u_{x_j}^\beta\, \Phi_k'(|u|)\,\eta^\mu\, dx \\ &\leq \mu \left| \sum_{j=1}^n\sum_{\alpha=1}^m \int_{B_R} \frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\, \Phi_k(|u|)\, u^\alpha\, \eta^{\mu-1}\eta_{x_j}\, dx \right| \\ &+ \left| \sum_{j=1}^n\sum_{\alpha,\beta=1}^m \int_{B_R\cap\{|Du|\leq t_0\}} \frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\, u^\alpha \frac{u^\beta}{|u|}\, u_{x_j}^\beta\, \Phi_k'(|u|)\,\eta^\mu\, dx \right| =: I_3+I_4. \end{aligned} \tag{5.7}$$

Now, we separately estimate I_1,...,I_4.

Estimate of I_1

To estimate I_1 we separately consider the case $\{|Du| \leq t_0\}$ and $\{|Du| > t_0\}$. Precisely,

$$\begin{aligned} I_1 &= \sum_{j=1}^n\sum_{\alpha=1}^m \int_{B_R\cap\{|Du|\leq t_0\}} \frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\, u_{x_j}^\alpha\, \Phi_k(|u|)\,\eta^\mu\, dx \\ &+ \sum_{j=1}^n\sum_{\alpha=1}^m \int_{B_R\cap\{|Du|>t_0\}} \frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\, u_{x_j}^\alpha\, \Phi_k(|u|)\,\eta^\mu\, dx =: I_1^1+I_1^2. \end{aligned} \tag{5.8}$$

Of course,

$$I_1^1 \geq -\nu \int_{B_R\cap\{|Du|\leq t_0\}} |Du|\, \Phi_k(|u|)\,\eta^\mu\, dx \geq -t_0\nu \int_{B_R\cap\{|Du|\leq t_0\}} \Phi_k(|u|)\, dx$$

with $\nu = \sum_{j=1}^n \sum_{\alpha=1}^m \|\frac{\partial f}{\partial \xi_j^\alpha}(x,\xi)\|_{L^\infty(B_{R_0}\times\mathbf{B}_{t_0})}$.

As far as I_1^2 is concerned, we use that $f(x,\cdot)$ is convex in any radial direction outside the ball centered at 0 and radius t_0, see (H1). Thus,

$$\begin{aligned} I_1^2 \geq & \sum_{j=1}^{n}\sum_{\alpha=1}^{m}\int_{B_R\cap\{|Du|>t_0\}} \frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\,\frac{t_0 u^\alpha_{x_j}}{|Du|}\,\Phi_k(|u|)\,\eta^\mu\,dx \\ & + \int_{B_R\cap\{|Du|>t_0\}} f(x,Du)\,\Phi_k(|u|)\,\eta^\mu\,dx - \int_{B_R\cap\{|Du|>t_0\}} f\left(x,\frac{Du}{|Du|}t_0\right)\,\Phi_k(|u|)\,\eta^\mu\,dx. \end{aligned} \tag{5.9}$$

The first integral in the right hand side is non-negative since $\sum_{j,\alpha}\frac{\partial f}{\partial \xi_j^\alpha}(x,Du)u^\alpha_{x_j}$ is equal to $\frac{\partial f}{\partial \zeta}(x,Du)|Du|$ with $\zeta=\frac{Du}{|Du|}$, and this last quantity is non-negative by (3.1).

By (2.2) we can majorize the last integral in (5.9) taking into account that

$$f\left(x,\frac{Du}{|Du|}t_0\right)\leq k_2\left\{1+\sum_{i=1}^{n}\left(\frac{|u_{x_i}|t_0}{|Du|}\right)^q\right\}\leq \kappa.$$

Thus, (5.9) implies

$$I_1^2 \geq \int_{B_R\cap\{|Du|>t_0\}} f(x,Du)\,\Phi_k(|u|)\,\eta^\mu\,dx - \kappa\int_{B_R}\Phi_k(|u|)\,dx.$$

We have so proved that

$$I_1 \geq \int_{B_R\cap\{|Du|>t_0\}} f(x,Du)\,\Phi_k(|u|)\,\eta^\mu\,dx - (\kappa+t_0\nu)\int_{B_R}\Phi_k(|u|)\,dx. \tag{5.10}$$

Estimate of I_2

We claim that $I_2\geq 0$. Indeed, by (H2), if $|\xi|\geq t_0$ then $f(x,\xi)=F(x,|\xi_1|,...,|\xi_n|)$ and, using (3.2),

$$\sum_{j=1}^{n}\sum_{\alpha,\beta=1}^{m}\frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\,u^\alpha u^\beta\,u^\beta_{x_j} = \sum_{j=1}^{n}\frac{\partial F}{\partial z_j}(x,|u_{x_1}|,...,|u_{x_n}|)\frac{\left(\sum_{\alpha=1}^m u^\alpha\,u^\alpha_{x_j}\right)^2}{|u_{x_j}|}\geq 0.$$

Thus, by the monotonicity of Φ_k we have

$$I_2 = \sum_{j=1}^{n}\int_{B_R\cap\{|Du|>t_0\}}\frac{\partial F}{\partial z_j}(x,|u_{x_1}|,...,|u_{x_n}|)\frac{\left(\sum_{\alpha=1}^m u^\alpha\,u^\alpha_{x_j}\right)^2}{|u_{x_j}||u|}\,\Phi_k'(|u|)\,\eta^\mu\,dx\geq 0. \tag{5.11}$$

Estimate of I_3

As above, we split I_3 into two integrals:

$$\begin{aligned} I_3 = & \,\mu\sum_{j=1}^{n}\sum_{\alpha=1}^{m}\int_{B_R\cap\{|Du|\leq t_0\}}\frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\,\Phi_k(|u|)\,u^\alpha\,\eta^{\mu-1}\eta_{x_j}\,dx \\ & + \mu\sum_{j=1}^{n}\sum_{\alpha=1}^{m}\int_{B_R\cap\{|Du|>t_0\}}\frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\,\Phi_k(|u|)\,u^\alpha\,\eta^{\mu-1}\eta_{x_j}\,dx =: I_3^1+I_3^2. \end{aligned} \tag{5.12}$$

Let us consider I_3^1. Defining ν as above, that is $\nu=\sum_{j=1}^n\sum_{\alpha=1}^m\|\frac{\partial f}{\partial \xi_j^\alpha}(x,\xi)\|_{L^\infty(B_{R_0}\times \mathbf{B}_{t_0})}$, using (5.3) and (H4) we get

$$I_3^1 \leq \frac{2\mu\nu}{R-\rho}\int_{B_R\cap\{|Du|\leq t_0\}}\Phi_k(|u|)\,|u|\,dx \leq \frac{2\mu\nu}{R-\rho}\int_{B_R}\Phi_k(|u|)\,|u|\,dx\leq \frac{c_1}{R-\rho}\int_{B_R}\Phi_k(|u|)\,(|u|^q+1)\,dx \tag{5.13}$$

with c_1 depending on μ, ν and t_1.

Consider now I_3^2. Notice that if $|\xi| > t_0$ then $\frac{\partial f}{\partial \xi_j^\alpha}(x,\xi) = \frac{\partial F}{\partial z_j}(x,|\xi_1|,...,|\xi_n|)\frac{\xi_i^\alpha}{|\xi_j|}$. Moreover, $\frac{\partial F}{\partial z_j}(x,|\xi_1|,...,|\xi_n|)$ is positive by (H2), thus, using also (5.3) we have

$$I_3^2 \leq \frac{2m\mu}{R-\rho}\sum_{j=1}^n \int_{A_{R,j}^- \cup A_{R,j}^+} \frac{\partial F}{\partial z_j}(x,|u_{x_1}|,...,|u_{x_n}|)|u|\Phi_k(|u|)\,\eta^{\mu-1}\,dx \tag{5.14}$$

where

$$A_{R,j}^- := B_R \cap \{|Du| > t_0\} \cap \left\{\eta \neq 0, |u_{x_j}| \leq \frac{2m\mu L|u|}{\eta(R-\rho)}\right\}$$

and

$$A_{R,j}^+ := B_R \cap \{|Du| > t_0\} \cap \left\{\eta \neq 0, |u_{x_j}| > \frac{2m\mu L|u|}{\eta(R-\rho)}\right\}$$

with $L > 0$ to be chosen later.
For a.e. $x \in A_{R,j}^-$ define $H_j(x,\cdot) : \left[\sqrt{t_0^2 + |Du(x)|^2 - |u_{x_j}(x)|^2}, +\infty\right) \to \mathbb{R}_+$,

$$H_j(x,s) := F(x,|u_{x_1}(x)|,\ldots,|u_{x_{j-1}}(x)|,s,|u_{x_{j+1}}(x)|,\ldots,|u_{x_n}(x)|),$$

of class C^1 w.r.t. s. By (H1), (H2) and the assumption $x \in A_{R,j}^-$ the following inequality follows:

$$\frac{\partial F}{\partial z_j}(x,|u_{x_1}|,...,|u_{x_n}|)\frac{2m\mu|u|}{\eta(R-\rho)} \leq \frac{1}{L}\frac{\partial H_j}{\partial s}\left(x, \frac{2m\mu L|u|}{\eta(R-\rho)}\right)\frac{2m\mu L|u|}{\eta(R-\rho)}. \tag{5.15}$$

Since (3.3) holds, by (3.4) we get

$$\frac{1}{L}\frac{\partial H_j}{\partial s}\left(x, \frac{2m\mu L|u|}{\eta(R-\rho)}\right)\frac{2m\mu L|u|}{\eta(R-\rho)} \leq \frac{\mu}{L}H_j\left(x, \frac{2m\mu L|u|}{\eta(R-\rho)}\right). \tag{5.16}$$

Now, denote with e_1 the vector $(1,0,\ldots,0)$ in $\mathbb{R}^m$.
By definition of H_j, (3.2) and (H2)

$$\begin{aligned} H_j\left(x, \frac{2m\mu L|u|}{\eta(R-\rho)}\right) &\leq F\left(x,|u_{x_1}|,...,|u_{x_{j-1}}|,|u_{x_j}| + \max\left\{t_0, \frac{2m\mu L|u|}{\eta(R-\rho)}\right\}, |u_{x_{j+1}}|,...,|u_{x_n}|\right) \\ &= f^{**}\left(x, u_{x_1},...,u_{x_{j-1}}, \left(|u_{x_j}| + \max\left\{t_0, \frac{2m\mu L|u|}{\eta(R-\rho)}\right\}\right)\mathsf{e}_1, u_{x_{j+1}},...,u_{x_n}\right). \end{aligned} \tag{5.17}$$

Since

$$\begin{aligned} &\left(u_{x_1},...,u_{x_{j-1}}, \left(|u_{x_j}| + \max\left\{t_0, \frac{2m\mu L|u|}{\eta(R-\rho)}\right)\right\}\mathsf{e}_1, u_{x_{j+1}},...,u_{x_n}\right) \\ &= \frac{1}{2}(2u_{x_1},...,2u_{x_{j-1}}, 2|u_{x_j}|\mathsf{e}_1, 2u_{x_{j+1}},...,2u_{x_n}) \\ &+ \frac{1}{2}\Bigg(\underbrace{0}_{\in\mathbb{R}^m},\ldots,\underbrace{0}_{\in\mathbb{R}^m}, \max\left\{2t_0, \frac{4m\mu L|u|}{\eta(R-\rho)}\right\}\mathsf{e}_1, \underbrace{0}_{\in\mathbb{R}^m},\ldots,\underbrace{0}_{\in\mathbb{R}^m}\Bigg) =: \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}, \end{aligned}$$

by the convexity of f^{**} we get

$$\begin{aligned} &f^{**}\left(x, u_{x_1},...,u_{x_{j-1}}, \left(|u_{x_j}| + \max\left\{2t_0, \frac{4m\mu L|u|}{\eta(R-\rho)}\right\}\right)\mathsf{e}_1, u_{x_{j+1}},...,u_{x_n}\right) \\ &\leq \frac{1}{2}f^{**}(x,\mathbf{v}) + \frac{1}{2}f^{**}(x,\mathbf{w}). \end{aligned} \tag{5.18}$$

Of course, by (H1), $x \in A^-_{R,j}$ and (H3)

$$f^{**}(x, \mathbf{v}) = f(x, \mathbf{v}) = f(x, 2Du(x)) \le 2^\mu f(x, Du(x)). \tag{5.19}$$

Let us deal with $f^{**}(x, \mathbf{w})$. Since $|\mathbf{w}| > 2t_0$ and (H2) and (H5) hold,

$$\begin{aligned} f^{**}(x, \mathbf{w}) &= F(x, \underbrace{0, ..., 0}_{j-1}, \max\left\{2t_0, \frac{4m\mu L|u|}{\eta(R-\rho)}\right\}, \underbrace{0, ..., 0}_{n-j}) \\ &\le F\left(x, \underbrace{\max\left\{2t_0, \frac{4m\mu L|u|}{\eta(R-\rho)}\right\}, ..., \max\left\{2t_0, \frac{4m\mu L|u|}{\eta(R-\rho)}\right\}}_{n}\right) \le k_2 n \left\{[2t_0]^q + \left[\frac{4m\mu L|u|}{\eta(R-\rho)}\right]^q + 1\right\}. \end{aligned}$$

Without loss of generality we can assume L large so that $\frac{4m\mu L}{R_0} > 1$, therefore

$$f^{**}(x, \mathbf{w}) \le k_2 n \left\{[2t_0]^q + \left[\max\left\{1, \frac{4m\mu L}{\eta(R-\rho)}\right\}\right]^\mu |u|^q + 1\right\} \le c_2 + c_2 \left[\frac{L}{\eta(R-\rho)}\right]^\mu |u|^q. \tag{5.20}$$

Collecting (5.15)-(5.20) we get

$$\begin{aligned} &\frac{2m\mu}{R-\rho} \sum_{j=1}^n \int_{A^-_{R,j}} \frac{\partial F}{\partial z_j}(x, |u_{x_1}|, ..., |u_{x_n}|)|u|\Phi_k(|u|)\, \eta^{\mu-1}\, dx \\ &\le c_2 \sum_{j=1}^n \int_{A^-_{R,j}} \frac{1}{L} f(x, Du)\Phi_k(|u|)\eta^\mu\, dx + c_3 \frac{L^{\mu-1}}{(R-\rho)^\mu} \sum_{j=1}^n \int_{A^-_{R,j}} \{|u|^q + 1\}\, \Phi_k(|u|)\, dx \\ &\le \frac{nc_2}{L} \int_{B_R} f(x, Du)\Phi_k(|u|)\eta^\mu\, dx + c_3 \frac{L^{\mu-1}}{(R-\rho)^\mu} \sum_{j=1}^n \int_{B_R} \{|u|^q + 1\}\, \Phi_k(|u|)\, dx. \end{aligned} \tag{5.21}$$

Let us now deal with $A^+_{R,j}$. For a.e. $x \in A^+_{R,j}$, by (H2) it follows

$$\frac{2m\mu|u|}{\eta(R-\rho)} \frac{\partial F}{\partial z_j}(x, |u_{x_1}|, ..., |u_{x_n}|) \le \frac{1}{L} \frac{\partial F}{\partial z_j}(x, |u_{x_1}|, ..., |u_{x_n}|)|u_{x_j}| \le \frac{1}{L} f(x, Du),$$

thus

$$\frac{2m\mu}{R-\rho} \sum_{j=1}^n \int_{A^+_{R,j}} \frac{\partial F}{\partial z_j}(x, |u_{x_1}|, ..., |u_{x_n}|)|u|\Phi_k(|u|)\, \eta^{\mu-1}\, dx \le \frac{n}{L} \int_{B_R} f(x, Du)\Phi_k(|u|)\eta^\mu\, dx. \tag{5.22}$$

By (5.14), (5.21) and (5.22) we obtain

$$I_3^2 \le \frac{c_4}{L} \int_{B_R} f(x, Du)\Phi_k(|u|)\eta^\mu\, dx + c_4 \frac{L^{\mu-1}}{(R-\rho)^\mu} \int_{B_R} \{|u|^q + 1\}\, \Phi_k(|u|)\, dx. \tag{5.23}$$

By (5.12), (5.13) and (5.23) and using $R - \rho \ge \frac{(R-\rho)^\mu}{R_0^{\mu-1}}$ we get

$$\begin{aligned} I_3 &\le c_1 \frac{2mn\mu}{R-\rho} \int_{B_R} \Phi_k(|u|)\, \{|u|^q + 1\}\, dx + \frac{c_4}{L} \int_{B_R} f(x, Du)\Phi_k(|u|)\eta^\mu\, dx \\ &+ c_4 \frac{L^{\mu-1}}{(R-\rho)^\mu} \int_{B_R} \{|u|^q + 1\}\, \Phi_k(|u|)\, dx \\ &\le \frac{c_5(L^{\mu-1} + 1)}{(R-\rho)^\mu} \int_{B_R} \Phi_k(|u|)\, \{|u|^q + 1\}\, dx + \frac{c_5}{L} \int_{B_R} f(x, Du)\Phi_k(|u|)\eta^\mu\, dx. \end{aligned} \tag{5.24}$$

Estimate of I_4

As in the proof of Lemma 5.1, for every $k \in \mathbb{N}$ define a_k and b_k positive, such that $a_k^{p_i\gamma} = \frac{1}{k}$ and $b_k^{p_i\gamma} = k+1$. By (5.5)

$$I_4 \leq t_0\nu \int_{B_R} \Phi_k'(|u|)\,|u|\,\eta^\mu\,dx = t_0\nu \int_{B_R\cap\{a_k\leq|u|\leq b_k\}} \Phi_k'(|u|)\,|u|\,\eta^\mu\,dx \tag{5.25}$$

where $\nu = \sum_{j=1}^n \sum_{\alpha=1}^m \|\frac{\partial f}{\partial \xi_j^\alpha}(x,\xi)\|_{L^\infty(B_{R_0}\times\mathbf{B}_{t_0})}$.

For a.e. $x \in B_R \cap \{a_k \leq |u| \leq b_k\}$ we have that $\Phi_k'(|u(x)|)\,|u(x)| \leq p_i\gamma h_k'(|u(x)|^{p_i\gamma})|u(x)|^{p_i\gamma}$.

Therefore, since $p_i \leq q$ and recalling that $|h'(s)| \leq 2$ by (5.1) we obtain

$$\Phi_k'(|u(x)|)\,|u(x)|\eta^\mu \leq 2\mu q\gamma|u(x)|^{p_i\gamma} \qquad \text{for all } k \in \mathbb{N}.$$

Thus, by (5.25)

$$I_4 \leq c_6 \int_{B_R} |u|^{p_i\gamma}\,dx \tag{5.26}$$

with $c_6 = 2\mu q\gamma t_0\nu$.

Collecting (5.10), (5.11), (5.24) and (5.26) we obtain

$$\begin{aligned}&\int_{B_R\cap\{|Du|>t_0\}} f(x,Du)\,\Phi_k(|u|)\,\eta^\mu\,dx - (\kappa + t_0\nu)\int_{B_R} \Phi_k(|u|)\,dx \\ &\leq \frac{c_5}{L}\int_{B_R} f(x,Du)\Phi_k(|u|)\,dx + \frac{c_5(L^{\mu-1}+1)}{(R-\rho)^\mu}\int_{B_R} \Phi_k(|u|)\,\{|u|^q+1\}\,dx + c_6\int_{B_R}|u|^{p_i\gamma}\,dx.\end{aligned}$$

that implies

$$\begin{aligned}\int_{B_R\cap\{|Du|>t_0\}} f(x,Du)\,\Phi_k(|u|)\,\eta^\mu\,dx &\leq \frac{c_5}{L}\int_{B_R} f(x,Du)\Phi_k(|u|)\,\eta^\mu\,dx \\ &+ \frac{c_7(L^{\mu-1}+1)}{(R-\rho)^\mu}\int_{B_R} \Phi_k(|u|)\,\{|u|^q+1\}\,dx + c_6\int_{B_R}|u|^{p_i\gamma}\,dx.\end{aligned} \tag{5.27}$$

Taking into account that by definition (5.2) the increasing sequence $(\Phi_\kappa(t))_k$ converges to $t^{p_i\gamma}$, by the monotone convergence theorem we obtain

$$\begin{aligned}\int_{B_R\cap\{|Du|>t_0\}} f(x,Du)\,|u|^{p_i\gamma}\,\eta^\mu\,dx &\leq \frac{c_5}{L}\int_{B_R} f(x,Du)\,|u|^{p_i\gamma}\,\eta^\mu\,dx \\ &+ \frac{c_7(L^{\mu-1}+1)}{(R-\rho)^\mu}\int_{B_R} |u|^{p_i\gamma}\,\{|u|^q+1\}\,dx + c_6\int_{B_R}|u|^{p_i\gamma}\,dx.\end{aligned}$$

Filling the hole and using (2.2) we obtain

$$\begin{aligned}\int_{B_R} f(x,Du)\,|u|^{p_i\gamma}\,\eta^\mu\,dx &\leq \frac{c_5}{L}\int_{B_R} f(x,Du)\,|u|^{p_i\gamma}\,\eta^\mu\,dx \\ &+ \frac{c_7(L^{\mu-1}+1)}{(R-\rho)^\mu}\int_{B_R} |u|^{p_i\gamma}\,\{|u|^q+1\}\,dx + (c_6+\kappa)\int_{B_R}|u|^{p_i\gamma}\,dx.\end{aligned}$$

Choosing L greater than $\max\{1, 2c_5\}$ and noticing that $1 \leq \frac{R_0^\mu}{(R-\rho)^\mu}$ we get

$$\int_{B_R} f(x,Du)\,|u|^{p_i\gamma}\,\eta^\mu\,dx \leq \frac{c_8 L^{\mu-1}}{(R-\rho)^\mu}\int_{B_R} |u|^{p_i\gamma}\,\{|u|^q+1\}\,dx. \tag{5.28}$$

By the first inequality in (2.2)

$$f(x,Du) \geq -k_1 + \sum_{j=1}^{n} |u_{x_j}|^{p_j} \geq -k_1 + |u_{x_i}|^{p_i}$$

and we get (5.6). From now on, the proof goes as in the proof of Theorem 2.1 in [7], even if there scalar valued minimizers where considered. However, we sketch the remain steps for the reader's convenience.

Step 2. We prove now that

$$\int_{B_R} \{|u|^\gamma |u_{x_i}|\,\eta^\mu\}^{p_i}\,dx \leq c_9 \frac{\{\||u|\|_{L^q(B_{R_0})}+1\}^{q-p_i}}{(R-\rho)^\mu} \left\{\int_{B_R} \left(|u|^{\gamma+1}+1\right)^q\,dx\right\}^{\frac{p_i}{q}} \tag{5.29}$$

for some c_9 independent of γ.

Indeed, (5.6) implies

$$\int_{B_R} \{|u|^\gamma |u_{x_i}|\,\eta^\mu\}^{p_i}\,dx \leq \int_{B_R} \{|u|^\gamma |u_{x_i}|\}^{p_i}\,\eta^\mu\,dx \leq \frac{c}{(R-\rho)^\mu}\int_{B_R} \{|u|^q+1\}|u|^{p_i\gamma}\,dx$$

where we used that $\eta \leq 1$. As far as the right hand side is concerned, notice that by the Hölder inequality there exists c, depending on R_0, such that

$$\int_{B_R} |u|^{p_i\gamma}\,dx \leq \int_{B_R} \left(|u|^{\gamma+1}+1\right)^{p_i}\,dx \leq c\left\{\int_{B_R} \left(|u|^{\gamma+1}+1\right)^q\,dx\right\}^{\frac{p_i}{q}}. \tag{5.30}$$

Moreover, using the Hölder inequality once more, see [7, Lemma 6.2], we get the existence of a positive constant c, independent of γ, such that

$$\int_{B_R} |u|^{q+p_i\gamma}\,dx \leq c\,\{\Lambda+1\}^{q-p_i}\left\{\int_{B_R} \left(|u|^{\gamma+1}+1\right)^q\,dx\right\}^{\frac{p_i}{q}},$$

where $\Lambda := \|u\|_{L^q(B_{R_0})}$ is finite by Corollary 3.5 and the assumption $q < \overline{p}^*$. So, (5.29) follows.

Step 3. From Step 2, it follows that if $|u| \in L^{q\beta}(B_R)$ for some $\beta \geq 1$, then there exists c, independent of β, R and ρ, such that

$$\int_{B_R} \left|\left[\eta^\mu(|u|^\beta+1)\right]_{x_i}\right|^{p_i} dx \leq \frac{c_{10}\,\beta^\lambda}{(R-\rho)^\lambda}\left\{\|u\|_{L^q(B_{R_0})}+1\right\}^{q-p}\cdot\left\{\int_{B_R}(|u|^\beta+1)^q\,dx\right\}^{\frac{p_i}{q}}, \tag{5.31}$$

with $\lambda = \max\{\mu, q\}$. We refer to Step 2, proof of Theorem 2.1 in [7] for the details.

Step 4. We claim that if $G(x) := \max\{1, |u(x)|\}$, and $|u| \in L^{q\beta}(B_R)$ for some $\beta \geq 1$, then

$$\left\{\int_{B_\rho} [G(x)]^{\beta\overline{p}^*}\,dx\right\}^{\frac{1}{\overline{p}^*}} \leq c\left\{\frac{\beta}{R-\rho}\right\}^{\frac{\lambda}{p}} \{\|u\|_{L^q(B_{R_0})}+1\}^{\frac{q-p}{p}} \left\{\int_{B_R} [G(x)]^{\beta q}\,dx\right\}^{\frac{1}{q}}. \tag{5.32}$$

Indeed, the assumption $|u| \in L^{q\beta}(B_R)$ for some $\beta \geq 1$ and Step 3 imply that $x \mapsto \eta^\mu(x)\{|u(x)|^\beta + 1\}$ is in $W_0^{1,(p_1,\dots,p_n)}(B_R)$. Multiplying (5.31) on i and using $p_i \geq p$, we get

$$\prod_{i=1}^{n}\left\{\int_{B_R}\left|\left(\eta^\mu(|u|^\beta+1)\right)_{x_i}\right|^{p_i} dx\right\}^{\frac{1}{p_i}} \leq c_{11}\left\{\frac{\beta}{R-\rho}\right\}^{\frac{n\lambda}{p}} \{\|u\|_{L^q(B_{R_0})}+1\}^{n\frac{q-p}{p}} \left\{\int_{B_R}(|u|^\beta+1)^q\,dx\right\}^{\frac{n}{q}},$$

with c_{11} independent of β, R and ρ.

By Theorem 3.3 we get

$$\left\{\int_{B_\rho}\left(|u|^\beta+1\right)^{\overline{p}^*}\,dx\right\}^{\frac{1}{\overline{p}^*}}\le c_{12}\left\{\frac{\beta}{R-\rho}\right\}^{\frac{\lambda}{p}}\{\|u\|_{L^q(B_{R_0})}+1\}^{\frac{q-p}{p}}\left\{\int_{B_R}(|u|^\beta+1)^q\,dx\right\}^{\frac{1}{q}}$$

and, defining $G(x):=\max\{1,|u(x)|\}$, we obtain (5.32).

Step 5. Now, we prove the boundedness of u and the estimate (1), using the Moser's iteration technique.

For all $h\in\mathbb{N}$ define $\beta_h=\left(\frac{\overline{p}^*}{q}\right)^{h-1}$, $\rho_h=R_0/2+R_0/2^{h+1}$ and $R_h=R_0/2+R_0/2^h$. By (5.32), replacing β, R and ρ with β_h, R_h and ρ_h, respectively, we have that $G\in L^{\beta_h q}(B_{R_h})$ implies $G\in L^{\beta_{h+1}q}(B_{R_{h+1}})$. Precisely,

$$\|G\|_{L^{\beta_{h+1}q}(B_{R_{h+1}})}\le\left\{2c_{12}\left\{\frac{2^{h+1}}{R_0}\left(\frac{\overline{p}^*}{q}\right)^{h-1}\right\}^{\frac{\gamma}{p}}\{\|u\|_{L^q(B_{R_0})}+1\}^{\frac{q-p}{p}}\right\}^{\frac{1}{\beta_h}}\|G\|_{L^{\beta_h q}(B_{R_h})}\tag{5.33}$$

holds true for every h. Corollary 3.5 and the inequality $q<\overline{p}^*$ imply $G\in L^q(B_{R_0})$. An iterated use of (5.33) implies the existence of a constant c_{13} such that

$$\|G\|_{L^\infty(B_{R_0/2}(x_0))}\le c_{13}\,\{\|u\|_{L^q(B_{R_0})}+1\}^{\frac{\overline{p}^*(q-p)}{p(\overline{p}^*-q)}}\,\|G\|_{L^q(B_{R_0}(x_0))}.$$

Therefore, by the very definition of G,

$$\|u\|_{L^\infty(B_{R_0/2}(x_0))}\le c_{14}\left\{\|u\|_{L^q(B_{R_0}(x_0))}+1\right\}^{\frac{\overline{p}^*(q-p)}{p(\overline{p}^*-q)}+1}.$$

The inequality above implies that u is in $L^\infty(B_{R_0/2}(x_0);\mathbb{R}^m)$ and estimate (2.4).

Step 6. Here we prove estimate (2.5). Fix $B_r(x_0)\Subset\Omega$. Notice that if $Q_s(x_0)$ denotes the cube with edges parallel to the coordinate axes, centered at x_0 and with side length $2s$, then $B_{r/\sqrt{n}}(x_0)\subseteq Q_{r/\sqrt{n}}(x_0)\subseteq B_r(x_0)$.

Let $u\in W^{1,f}(\Omega;\mathbb{R}^m)$ be a local minimizer of $\mathcal{F}$ and define $u_r:=\fint_{B_r(x_0)}u\,dx$. Since $u-u_r$ is a local minimizer, too, then by (2.4) and the Hölder inequality

$$\|u-u_r\|_{L^\infty(B_{r/(2\sqrt{n})}(x_0))}\le c\left\{1+\|u-u_r\|_{L^{\overline{p}^*}(B_{r/\sqrt{n}}(x_0))}\right\}^{\frac{\overline{p}^*(q-p)}{p(\overline{p}^*-q)}+1}.$$

By Theorem 3.4

$$\begin{aligned}\|u-u_r\|_{L^{\overline{p}^*}(B_{r/\sqrt{n}}(x_0))}&\le\|u-u_r\|_{L^{\overline{p}^*}(Q_{r/\sqrt{n}}(x_0))}\le\\ &\le c\left\{1+\|u-u_r\|_{L^1(B_r(x_0))}+\sum_{i=1}^n\|u_{x_i}\|_{L^{p_i}(B_r(x_0))}\right\}\end{aligned}$$

and by the Poincaré inequality

$$\|u-u_r\|_{L^1(B_r(x_0))}\le c\left\{1+\sum_{i=1}^n\|u_{x_i}\|_{L^1(B_r(x_0))}\right\}.$$

Thus, using the above estimates and (2.2) we get (2.5). □

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