EXISTENCE OF MINIMIZERS FOR POLYCONVEX AND NONPOLYCONVEX PROBLEMS

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Abstract. We study the existence of Lipschitz minimizers of integral functionals
\[ I(u) = \int_{\Omega} \varphi(x, \det Du(x)) \, dx, \]
where \( \Omega \) is an open subset of \( \mathbb{R}^N \) with Lipschitz boundary, \( \varphi : \Omega \times (0, +\infty) \to [0, +\infty) \) is a continuous function, and \( u \in W^{1,N}(\Omega, \mathbb{R}^N) \), \( u(x) = x \) on \( \partial \Omega \). We consider both the cases of \( \varphi \) convex and nonconvex with respect to the last variable. The attainment results are obtained passing through the minimization of an auxiliary functional and the solution of a prescribed Jacobian equation.

Key words. nonpolyconvex functional, existence of minimizers, Lipschitz regularity, prescribed Jacobian equation

AMS subject classifications. 49J10, 35J60

DOI. 10.1137/040611999

1. Introduction. In this paper we consider integral functionals
\[ I(u) = \int_{\Omega} \varphi(x, \det Du(x)) \, dx, \]
where \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \) with a Lipschitz boundary, \( N \geq 2 \), \( \varphi : \Omega \times (0, +\infty) \to [0, +\infty) \) is a continuous function, and \( u \in W^{1,N}(\Omega, \mathbb{R}^N) \).

We aim at proving the existence of Lipschitz solutions to the variational problem
\[ \min \{ I(u) : u \in W^{1,N}(\Omega, \mathbb{R}^N), \det Du > 0 \text{ a.e.}, u(x) = x \text{ on } \partial \Omega \}. \]
Noticing that even if a growth condition from below of the type \( t^p \leq \varphi(x, t) \) (which is common in the theory of calculus of variations) is assumed, no coercivity of \( I \) follows in any Sobolev space, preventing us from establishing the existence of minimizers via the direct method. Nevertheless many problems of this type have a solution, and the question of finding which conditions on \( \varphi \) ensure the existence of solutions is worthy of interest, as is its applications in physics, mainly in elasticity theory and in the problem of the equilibrium of gases (see [17], [5], [6], and [12]). For instance, (1.2) is the variational problem corresponding to a nonhomogeneous elastic material with reference configuration \( \Omega \) whose stored energy \( \varphi \) is a nonnegative, continuous function depending on the position \( x \) in the reference configuration and the size of the deformation of the volume element \( \det Du(x) > 0 \).

It is well known that an important role is played by the convexity of \( \varphi \) with respect to the last variable: when \( \varphi \) is convex, then \( I \) is said to be a polyconvex functional; if not, then \( I \) is nonpolyconvex. The polyconvex case \( \varphi = \varphi(t) \) has been studied by Dacorogna [5] and the nonpolyconvex case by Mascolo and Schianchi [14] and Cellina and Zagatti [4].
In order to solve (1.2) our strategy is the following: The first step is to look for solutions to the following problem (from now on referred to as the auxiliary problem)

\[
\begin{aligned}
\min \{ J(v) = \int_{\Omega} \varphi(x, v(x)) \, dx : v \in L^1(\Omega), \ v > 0 \ \text{a.e.}, \ \int_{\Omega} v(x) \, dx = |\Omega| \},
\end{aligned}
\]

where $|\Omega|$ stands for the $N$-dimensional Lebesgue measure of $\Omega$. Then, if $v$ is a solution to (1.3), the second step is to solve in $W^{1, N}(\Omega)$ the boundary value problem

\[
\begin{aligned}
\det Du(x) &= v(x) \quad \text{for a.e.} \ x \in \Omega, \\
u(x) &= x \quad \text{on} \ \partial \Omega.
\end{aligned}
\]

A solution $u$ to (1.4) is a solution to (1.2), too. In fact, if $w \in W^{1, N}(\Omega)$, $w(x) = x$ on $\partial \Omega$, then $\det Dw \in L^1(\Omega)$ and $\int_{\Omega} \det Dw(x) \, dx = |\Omega|$; therefore, if $\det Dw > 0$ a.e., then

\[
\mathcal{I}(u) = J(v) \leq J(\det Dw) = \mathcal{I}(w).
\]

Following the above scheme, Mascolo in [13] proves the existence of minimizers of (1.2) for smooth domains $\Omega$ and $\varphi \in C^2(\Omega \times (0, +\infty))$ strictly convex in the last variable.

As far as problem (1.3) is concerned, Ekeland and Temam in [8] prove a relaxation result and Ball and Knowles in [1] obtain an attainment result with the tool of the Young measures; see also Friezecke [10] for related results. The boundary value problem (1.4) may have no solution unless $v$ is sufficiently regular. For instance, the simple continuity of $v$ is not a sufficient condition to get Lipschitz solutions; see the counterexamples independently given by Burago and Kleiner [2] and McMullen [15]. Thus, also the regularity properties of minimizers of the auxiliary problem have to be studied. The pioneering papers on (1.4) are due to Moser [16] and Dacorogna and Moser [7]. In particular, in [7] the authors prove that if $v$ is in $C^{k, \alpha}(\Omega)$, $k \geq 0$, and $\partial \Omega \in C^{k+3, \alpha}$, then there exists a diffeomorphism of class $C^{k+1, \alpha}(\Omega)$ solution to (1.4).

Later results are due to Rivièrè and Ye, who prove in [18, Theorem 4] the existence of a bi-Lipschitz homeomorphism $u$ solution to (1.4) under less restrictive assumptions on $\Omega$ with $v$ satisfying a Dini-type continuity property. In [19] Ye proves existence results in the framework of the Sobolev spaces.

The plan of the paper is the following. In section 2 we introduce a class of open sets, invariant under bi-Lipschitz homeomorphisms, which is slightly larger than that of open sets with Lipschitz boundaries; see Definition 2.1. In Theorem 2.4 we state the existence of Lipschitz solutions to (1.4) with $\Omega$ in this class of open sets and Hölder continuous datum $v$. It is a variant of the above-cited Theorem 4 in [18], and in the appendix we give the details of the proof. In section 3 we deal with polyconvex functionals. We consider the class of functions $\varphi$ strictly convex in the last variable satisfying, as a substitute for the growth conditions,

\[
\begin{aligned}
&\lim_{t \to 0^+} D_t \varphi(x, t) = \lambda_0 \quad \text{with} \ \lambda_0 \in \mathbb{R} \cup \{-\infty\}, \\
&\lim_{t \to +\infty} D_t \varphi(x, t) = +\infty,
\end{aligned}
\]

uniformly with respect to $x$. In Proposition 3.1 we prove that a unique solution $v$ to (1.3) exists and that $v$ is in $L^\infty(\Omega)$. In Proposition 3.5, under more regularity assumptions on $\varphi$, we prove that $v$ is Hölder continuous. Therefore, the Lipschitz solution $u$ to (1.4), which exists by Theorem 2.4, is a minimizer of (1.2); see Theorem 3.6. In section 4 we deal with a function $\varphi$ nonconvex with respect to $t$, satisfying (1.5).
Denoting $\varphi^{**}$ the convex envelope of $\varphi$ with respect to $t$, we assume that there exist $\alpha, \beta \in L^\infty(\Omega)$, $\beta(x) > \alpha(x)$, $\inf \alpha > 0$ such that for every $x \in \Omega$,

$$t \mapsto \varphi^{**}(x, t)$$

is affine in $[\alpha(x), \beta(x)]$

and

$$\varphi(x, \cdot) \equiv \varphi^{**}(x, \cdot) \text{ and } \varphi(x, \cdot) \text{ is strictly convex in } (0, \alpha(x)] \text{ and } [\beta(x), +\infty].$$

Under these assumptions in Theorem 4.1 we prove the existence of a bounded solution $v$ to the auxiliary problem (1.3). In section 5 under regularity assumptions on $\varphi$ we get that $v$ is piecewise Hölder continuous; see Theorem 5.2. In section 6 we conclude the section considering functionals

$$\alpha, \beta$$

Denoting $\Omega$ has a covering of finitely many open sets $\phi$.

Theorem 7.4 deals with a perturbation of these functionals; see problem (7.2). We conclude the section considering functionals with $\varphi$ satisfying the structure condition $\varphi(\cdot, t) = \tilde{\varphi}(|\cdot|, t)$. In this case the existence of bounded radial solutions to (1.3) directly implies the existence of Lipschitz solutions to (1.4).

2. Notation and preliminary results. In the following if $\Omega$ is a measurable subset of $\mathbb{R}^N$, then $|\Omega|$ stands for its $N$-dimensional Lebesgue measure. We write $Q$ in place of $(0, 1)^N$ and $B^r(x)$ denotes the ball in $\mathbb{R}^N$ with center at $x$ and radius $r$.

If $\varphi : \Omega \times (0, +\infty) \to (0, +\infty)$, then $\varphi^{**}$ is the convex envelope of $\varphi$ with respect to the second variable, i.e., $t \mapsto \varphi^{**}(x, t)$ is the greatest convex function lower than $t \mapsto \varphi(x, t)$. For the sake of simplicity we write $\varphi(x, \cdot)$ instead of $t \mapsto \varphi(x, t)$,

$$D^-\varphi(x, s) := \lim_{t \to s} \frac{\varphi(x, t) - \varphi(x, s)}{t - s}, \quad D^+\varphi(x, s) := \lim_{t \to s^+} \frac{\varphi(x, t) - \varphi(x, s)}{t - s},$$

and $\partial\varphi(x, s) := \{d \in \mathbb{R} : \varphi(x, t) \geq \varphi(x, s) + d(t - s) \text{ for every } t \in (0, +\infty)\}$.

We define a class of bounded open subsets of $\mathbb{R}^N$.

**Definition 2.1.** We say that a bounded open set $\Omega$ of $\mathbb{R}^N$ is of class (L) if $\Omega$ has a covering of finitely many open sets $\Omega^*_j$ such that for every $j$ there exists a bi-Lipschitz homeomorphism $\psi^*_j : \Omega^*_j \cap \Omega \to Q$ satisfying

- (a) $\psi^*_j(\Omega^*_j \cap \partial \Omega) = \{0\} \times [0, 1]^{N-1}$, whenever $\Omega^*_j \cap \partial \Omega$ is nonempty;
- (b) $\det D\psi^*_j$ is Lipschitz continuous and there exists $A \geq 1$ such that $\frac{1}{A} \leq \det D\psi^*_j \leq A$.

The above definition describes a larger class than that of open sets with Lipschitz boundary, i.e., with the boundary which locally is the graph of a Lipschitz function. This result can be proved in a way similar to that of Proposition A.1 in [7].

**Lemma 2.2.** If a bounded open set $\Omega$ of $\mathbb{R}^N$ has a Lipschitz boundary, then it is of class (L).
An easy consequence of the chain rule for Lipschitz functions is that Definition 2.1 is invariant under bi-Lipschitz homeomorphisms.

**Lemma 2.3.** Let \( u_0 : \mathbb{R}^N \to \mathbb{R}^N \) be a bi-Lipschitz homeomorphism, with \( \det Du_0 \) Lipschitz continuous, \( \frac{1}{A} \leq \det Du_0 \leq A \) for some \( A \). If \( \Omega \) is of class \((L)\), then \( u_0(\Omega) \) is of class \((L)\), too.

On the contrary, there are examples of bounded open sets of \( \mathbb{R}^N \) with Lipschitz boundary which are mapped by a bi-Lipschitz homeomorphism \( u : \mathbb{R}^N \to \mathbb{R}^N \) onto sets with a not (Lipschitz) continuous boundary; see, e.g., [11, pp. 8–9]. Therefore, the converse of Lemma 2.2 is not true.

Now, we state an existence result of Lipschitz solutions to

\[
\begin{align*}
\det Du &= f \quad \text{in } \Omega, \\
u(x) &= x \quad \text{on } \partial \Omega
\end{align*}
\]  

with \( f \) Hölder continuous.

**Theorem 2.4.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded connected open set of class \((L)\). Let \( f \) be a Hölder continuous function, \( \inf f > 0 \), \( \int_{\Omega} f(x) \, dx = |\Omega| \). Then there exists a bi-Lipschitz homeomorphism \( u : \overline{\Omega} \to \overline{\Omega} \) solution to (2.1).

A similar result is proved in [18, Theorem 4], with a weaker assumption on \( v \), which is assumed to satisfy a Dini-type continuity property, and a regular domain \( \Omega \). In [18] the proof is given for cubes only. The proof of Theorem 2.4, based upon the application to open sets of class \((L)\) of the partition method due to Moser [16], is in the appendix.

3. Polyconvex problems: An attainment result. In this section we consider the variational problem

\[
\min \left\{ \int_{\Omega} \psi(x, \det Du(x)) \, dx : u \in W^{1,N}(\Omega, \mathbb{R}^N), \quad \det Du > 0 \text{ a.e.}, \quad u(x) = x \text{ on } \partial \Omega \right\},
\]

where \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \) with a Lipschitz boundary and \( \psi : \Omega \times (0, +\infty) \to [0, +\infty) \) is a continuous function.

To get solutions to (3.1), we first consider the following variational problem:

\[
\min \left\{ \int_{\Omega} \psi(x, v(x)) \, dx : v \in L^1(\Omega), \quad v > 0 \text{ a.e.}, \quad \int_{\Omega} v(x) \, dx = a \right\}, \quad a > 0.
\]

As far as the problem (3.2) is concerned, the Lipschitz regularity of the boundary of \( \Omega \) can be dropped.

We prove that there exists a (unique) bounded solution to (3.2) if

(H1) \( t \mapsto \psi(x, t) \) is strictly convex for all \( x \in \Omega \);

(H2) there exists \( \lambda_0 \in \mathbb{R} \cup \{-\infty\} \) such that

\[
\lim_{t \to 0^+} D^+ t \psi(x, t) = \lambda_0, \quad \lim_{t \to +\infty} D^- t \psi(x, t) = +\infty, \quad \text{uniformly in } x.
\]

**Proposition 3.1.** Assume that \( \psi : \Omega \times (0, +\infty) \to [0, +\infty) \) is a continuous function satisfying (H1) and (H2). Then for every \( \lambda > \lambda_0 \) there exists a unique \( u_\lambda \in L^\infty(\Omega) \), \( \inf u_\lambda > 0 \) such that

\[
\lambda \in \partial \psi(x, u_\lambda(x)) \quad \forall x \in \Omega.
\]
Moreover, there exists $\lambda_0 > 0$ such that $u_{\lambda_0}$ is the unique solution to (3.2).

Proof. We proceed as follows: At first we prove that for every $\lambda > \lambda_0$ there exists a function $u_\lambda$ such that (3.3) holds. Then, we prove that $u_\lambda$ is in $L^\infty(\Omega)$, $\inf u_\lambda > 0$, and there exists $\lambda_0$ such that $\int_\Omega u_{\lambda_0} \, dx = a$. Thus, it turns out that $u_{\lambda_0}$ is a solution to (3.2) and it is unique, because of the strict convexity of the functional.

Step 1. The definition of $u_\lambda$. Fixing $x \in \Omega$, we define the sets

$$C(x) := \{ s \in (0, +\infty) : D^-_t \psi(x, s) < D^+_t \psi(x, s) \}, \quad \Omega_C := \{ x \in \Omega : C(x) \neq \emptyset \}. $$

Notice that $\partial \psi(x, s) = [D^-_t \psi(x, s), D^+_t \psi(x, s)]$ for all $(x, s) \in \Omega \times (0, +\infty)$.

Suppose that $x \in \Omega \setminus \Omega_C$. From (H1) and the definition of $\Omega_C$, the function $D_t \psi(x, \cdot) : (0, +\infty) \to (\lambda_0, +\infty)$ is well defined, continuous, and strictly increasing. Moreover, it is a surjective function because of (H2). Let $u(x, \cdot)$ be the inverse function, i.e., $u(x, \cdot) : (\lambda_0, +\infty) \to (0, +\infty)$ is such that $u(x, \lambda)$ (from now on denoted by $u_\lambda(x)$) is the unique positive number such that $\lambda = D_t \psi(x, u_\lambda(x))$. $u(x, \cdot)$ is a well defined, strictly increasing, and continuous function.

Now let us consider $x \in \Omega_C$. From (H1), $C(x)$ is (at most) a countable set, so that we denote $C(x) = \{ t_n(x) \}_{n \in J(x)}$, where $J(x) \subseteq \mathbb{N}$. As in the above case, if $\lambda \not\in \cup_{n \in J(x)} \partial \psi(x, t_n(x))$, we define $u_\lambda(x)$ as the unique positive number such that $D^-_t \psi(x, u_\lambda(x)) = \lambda$. If instead $\lambda \in \partial \psi(x, t_n(x))$ for some $n \in J(x)$, then we set $u_\lambda(x) = t_n(x)$. Notice that if $u_\lambda(x)$ is chosen greater (less) than $t_n(x)$, then $\lambda < D^-_t \psi(x, u_\lambda(x))$ (respectively $\lambda > D^+_t \psi(x, u_\lambda(x))$). It is easy to prove that for each $x \in \Omega_C$ the function $u(x, \cdot) : (\lambda_0, +\infty) \to (0, +\infty)$ is well defined, increasing, and continuous.

Thus, $u_\lambda : \Omega \to (0, +\infty)$ is the unique function satisfying (3.3) and it is measurable, since

$$\{ x \in \Omega : u_\lambda(x) < \lambda \}$$

and $D^-_t \psi(x, t) = \sup_{h \to 0} (\psi(x, t+h) - \psi(x, t))/h$. By the second limit in (H2) for every $\lambda > \lambda_0$ there exists $R > 0$ such that $D^-_t \psi(x, R) > \lambda$ for every $x \in \Omega$, which implies $u_\lambda(x) < R$ for every $x \in \Omega$. In fact, if $u_\lambda(x) \geq R$ for some $x$, then by the convexity of $\psi$ with respect to the second variable it would be $D^-_t \psi(x, R) \leq D^-_t \psi(x, u_\lambda(x))$ and by (3.3) we would obtain $D^-_t \psi(x, R) \leq \lambda$, which is a contradiction. Thus, $u_\lambda$ is in $L^\infty(\Omega)$. The first limit in (H2) implies that for each $\lambda > \lambda_0$ there exists $c(\lambda) > 0$ such that $\sup_{y \in \Omega} D^+_t \psi(y, t) \leq \lambda$ for every $t < c(\lambda)$. Therefore, it cannot be $u_\lambda(x) < c(\lambda)$, because $\lambda < D^+_t \psi(x, u_\lambda(x))$, so that $\inf u_\lambda > 0$.

Step 2. The definition of $\lambda_0$. Define $\Psi : (\lambda_0, +\infty) \to (0, +\infty)$, $\Psi(\lambda) := \int_\Omega u_\lambda(x) \, dx$, where $u_\lambda(x) = u(x, \lambda)$ is defined as in Step 1. By the monotonicity of $u$ with respect to $\lambda$, $\Psi$ is increasing. It holds true that $\lim_{\lambda \to \lambda_0^+} u_\lambda(x) = 0$. In fact, suppose that $\lim_{\lambda \to \lambda_0} u_\lambda(x) = \delta(x) > 0$. By (H1), the first limit in (H2), and (3.3), we get

$$\lambda_0 < D^-_t \psi(x, \delta(x)) \leq D^-_t \psi(x, u_\lambda(x)) \leq \lambda.$$ 

Therefore, letting $\lambda$ go to $\lambda_0^+$ we get a contradiction. Analogously it can be proved that $\lim_{\lambda \to +\infty} u_\lambda(x) = +\infty$. Hence,

$$\lim_{\lambda \to \lambda_0^+} \Psi(\lambda) = 0, \quad \lim_{\lambda \to +\infty} \Psi(\lambda) = +\infty.$$ 

From the previous step $\lambda \mapsto u_\lambda(x)$ is continuous and increasing for all $x$ and $u_\lambda \in L^\infty(\Omega)$ for all $\lambda$, and therefore $\Psi$ is a continuous function. Thus, there exists $\lambda_0 > \lambda_0$
thus that $\Psi(\lambda_0) = a$. We claim that $u_{\lambda_0}$ is a solution to (3.2). In fact, from (H1) and (3.3) for every $w \in L^1(\Omega)$ such that $w > 0$ and $\int_{\Omega} w(x) \, dx = a$, we have that

$$
\psi(x, w(x)) \geq \psi(x, u_{\lambda_0}(x)) + \lambda_0 (w(x) - u_{\lambda_0}(x)) \quad \forall x \in \Omega.
$$

Thus,

$$
\int_{\Omega} \psi(x, w(x)) \, dx \geq \int_{\Omega} \psi(x, u_{\lambda_0}(x)) \, dx + \lambda_0 \int_{\Omega} (w(x) - u_{\lambda_0}(x)) \, dx
$$

$$
= \int_{\Omega} \psi(x, u_{\lambda_0}(x)) \, dx.
$$

\[ \Box \]

**Remark 3.2.** The growth conditions

$$
\lim_{t \to -\infty} \inf_{y \in \Omega} \psi(y, t) = +\infty, \quad \lim_{t \to +\infty} \inf_{y \in \Omega} \frac{\psi(y, t)}{t} = +\infty
$$

imply (H2). If the first limit in (H2) is not uniform with respect to $x$, then maybe $\inf u_{\lambda_0} = 0$. Moreover, the proof of Proposition 3.1 works also if we replace $\lim_{t \to +\infty} D^{-}_t \psi(x, t) = +\infty$ with the more general

$$
\lim_{t \to +\infty} D^{-}_t \psi(x, t) = \lambda_\infty, \quad \lambda_\infty \in \mathbb{R} \cup \{+\infty\}.
$$

It is easy to prove the following refinement of Proposition 3.1.

**Proposition 3.3.** Let $\psi : \Omega \times (0, +\infty) \to [0, +\infty)$ be a continuous function, differentiable with respect to the last variable, $D_t \psi \in C(\Omega \times (0, +\infty))$. If (H1) and (H2) hold, then the functions $u_{\lambda}$ in Proposition 3.1 are continuous for every $\lambda > \lambda_0$.

**Proof.** For every $\lambda > \lambda_0$ let $u_{\lambda} \in L^\infty(\Omega)$ be as in Proposition 3.1. $u_{\lambda}$ is lower semicontinuous. In fact, if

$$
\liminf_{x \to x_0} u_{\lambda}(x) < \alpha < u_{\lambda}(x_0),
$$

then (H1) and (3.3) imply $D_t \psi(x_0, \alpha) < \lambda$. By continuity of $D_t \psi$ there exists $\delta > 0$ such that $D_t \psi(x, \alpha) < \lambda$ for every $x \in (x_0 - \delta, x_0 + \delta)$. Then, from (3.3) again we have that $D_t \psi(x, \alpha) < D_t \psi(x, u_{\lambda}(x))$ for every $x \in (x_0 - \delta, x_0 + \delta)$, which implies $\alpha < u_{\lambda}(x)$, in contradiction with (3.5). Analogously the upper semicontinuity of $u_{\lambda}$ can be proved.

To get Hölder continuous solutions to (3.2) we require more regularity on $\psi$:

(H3) there exists $0 < \sigma \leq 1$ such that for every compact $K \subset (0, +\infty)$ and for every $t \in K$ the function $x \mapsto D_t \psi(x, t)$ is of class $C^{0, \sigma}(\Omega)$ with $[D_t \psi(\cdot, t)]_{0, \sigma} \leq k_K$;

(H4) for every $m > 0$ there exists $c_m > 0$ such that

$$
\psi(x, t) \geq \psi(x, s) + D_t \psi(x, s)(t - s) + c_m |t - s|^{2+\varepsilon}
$$

for every $t > s \geq m$, for every $x \in \Omega$, and for some $\varepsilon \geq 0$.

**Remark 3.4.** Assumption (H4) is equivalent to assuming that for every $m > 0$ there exists $c_m > 0$ such that

$$
D_t \psi(x, t) - D_t \psi(x, s) \geq c_m |t - s|^{1+\varepsilon} \quad \forall t > s \geq m \quad \forall x \in \Omega.
$$

Roughly speaking, if $\psi \in C^2$ satisfies (H4), then $D_t \psi$ may vanish provided that a suitable growth near the zeros is satisfied; see (3)(a) below.

Notice that if $\psi_0$ satisfies (H4) and $\psi_1 = \psi_1(x, t)$ is such that $\psi_1(x, \cdot)$ is convex and $C^1$, then $\psi = \psi_0 + \psi_1$ satisfies (H4), too. Examples of functions $\psi_0$ satisfying (H4) are as follows.
(1) \( \psi_0(t) := (1 + t^2)^{p/2}, p \geq 2. \) See [9] for details.
(2) \( \psi_0(x, t) := |t - a(x)|^p \) with \( a : \Omega \to \mathbb{R} \) and \( p \geq 2. \)
(3) \( \psi_0 : \Omega \times (0, +\infty) \to [0, +\infty) \) of class \( C^2, \) strictly convex with respect to \( t \) such that for every \( x \) there exist at most finitely many positive numbers \( \{s_i(x)\} \)

such that \( D_{tt}\psi_0(x, s_i(x)) = 0 \) and the following hold:
(a) there exist \( \varepsilon, c > 0 \) such that \( D_{tt}\psi_0(x, t) \geq c(t - s_i(x))^\varepsilon \) for every \( t \) in a neighborhood of \( s_i(x); \)
(b) there exists \( M > 0 \) such that \( \inf \{D_{tt}\psi_0(x, t) : (x, t) \in \Omega \times [M, +\infty)\} > 0. \)

PROPOSITION 3.5. Let \( \psi : \Omega \times (0, +\infty) \to [0, +\infty) \) be a continuous function, differentiable with respect to the last variable, satisfying (H1)–(H4). Then for every \( \lambda > \lambda_0, \) the function \( u_\lambda \) in Proposition 3.1 is in \( C^{0,\sigma/(1+\varepsilon)}(\Omega). \) In particular, for every \( a > 0 \) the unique solution \( u_\lambda \) to (3.2) is H"older continuous.

Proof. Fix \( \lambda \) and let \( u_\lambda, \) from now on referred to as \( u, \) be the correspondent function as described in Proposition 3.1. From the strict convexity of \( \psi \) with respect to the last variable and since \( \lambda = D_t\psi(x, u(x)) \) for every \( x \in \Omega \) it is easy to check that \( u = \gamma \)-H"older continuous with H"older constant \( |u|_\gamma, \) if and only if
\[
D_t\psi(y, u(x)) + [u]_{0, \gamma} |x - y|^\gamma - D_t\psi(x, u(x)) \geq 0 \quad \forall x, y \in \Omega.
\]

Fix \( x, y \in \Omega. \) By (H4) and (3.6) there exist \( \varepsilon \geq 0 \) and \( \tilde{c} > 0 \) such that
\[
D_t\psi(y, x) - D_t\psi(x, s) \geq \tilde{c}(t - s)^{1+\varepsilon} \quad \forall t > s \geq \inf u > 0 \quad \forall x \in \Omega.
\]
Consider the compact interval \( K = [\inf u, \|u\|_\infty] \) and let \( s \) and \( t \) be equal to \( u(x) \) and \( u(x) + (\frac{k}{\varepsilon^2}|x - y|^\sigma)^{1/(1+\varepsilon)} \), respectively, with \( \sigma \) and \( k_K \) as in (H3). Using (3.8) and (H3) to estimate \( D_t\psi(y, t) - D_t\psi(y, s) \) and \( D_t\psi(y, s) - D_t\psi(x, s) \), respectively, we get
\[
D_t\psi(y, t) - D_t\psi(x, s) = D_t\psi(y, t) - D_t\psi(y, s) + D_t\psi(y, s) - D_t\psi(x, s) \geq 0.
\]
Then \( u \) is \( \gamma \)-H"older continuous with \( \gamma = \frac{\sigma}{1+\varepsilon}. \)

Thus, for fixed \( a > 0, \) the solution \( u_\lambda \) to (3.2), which exists by Proposition 3.1, is H"older continuous.

Now we are ready to state an existence result of Lipschitz solutions to the polyconvex problem (3.1).

THEOREM 3.6. Suppose that \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \) with Lipschitz boundary and let \( \psi : \Omega \times (0, +\infty) \to [0, +\infty) \) be a continuous function, differentiable with respect to the last variable, satisfying (H1)–(H4). Then there exists a Lipschitz continuous solution to (3.1).

Proof. Set \( a = |\Omega| \) and consider the variational problem (3.2). From Propositions 3.1 and 3.5 such a problem has a (unique) solution \( u_\lambda \in C^{0,\gamma}(\Omega), \gamma > 0, \) and \( \inf u_\lambda > 0. \) Hence, from Theorem 2.4 there exists a bi-Lipschitz homeomorphism \( u \)
solving
\[
\left\{ \begin{array}{l}
\det Du = u_\lambda \quad \text{in } \Omega, \\
u(x) = x \quad \text{on } \partial \Omega,
\end{array} \right.
\]
and \( u \) is a solution to (3.1), too. \( \square \)

4. Nonpolyconvex problems: Attainment result for the auxiliary problem. In this section we consider the variational problem
\[
(4.1) \quad \min \left\{ \int_\Omega \varphi(x, v(x)) \, dx : v \in L^1(\Omega), v > 0 \text{ a.e.}, \int_\Omega v(x) \, dx = a \right\}, \quad a > 0,
\]
where $\Omega$ is a bounded open subset of $\mathbb{R}^N$, and $\varphi: \Omega \times (0, +\infty) \to [0, +\infty)$ is a continuous function, nonconvex with respect to the last variable $t$.

Let $\varphi^{**}$ be the convex envelope of $\varphi$ with respect to the second variable and define

$$\Omega_A := \{ x \in \Omega : t \to \varphi(x, t) \text{ is not strictly convex} \}.$$ 

We assume that the following assumptions hold:

(K1) $\Omega_A$ is a (not empty) measurable set and there exist $\alpha, \beta \in L^\infty(\Omega_A)$, $\beta(x) > \alpha(x)$ for all $x$, and $\alpha > 0$, such that $\varphi(x, \cdot)$ and $\varphi^{**}(x, \cdot)$ both coincide and are strictly convex in $(0, \alpha(x)]$ and $(\beta(x), +\infty)$ for every $x \in \Omega_A$;

(K2) $\varphi^{**}(x, \cdot)$ is affine in $[\alpha(x), \beta(x)]$ for all $x \in \Omega_A$, i.e., for every $\alpha(x) \leq t \leq \beta(x)$,

$$\varphi^{**}(x, t) = h(x)t + q(x) \text{ with } h(x) = \frac{\varphi(x, \beta(x)) - \varphi(x, \alpha(x))}{\beta(x) - \alpha(x)};$$

(K3) there exists $\lambda_0 \in \mathbb{R} \cup \{-\infty\}$ such that

$$\lim_{t \to 0^+} D^+_t \varphi(x, t) = \lambda_0, \quad \lim_{t \to +\infty} D^-_t \varphi(x, t) = +\infty, \quad \text{uniformly in } x.$$ 

**Theorem 4.1.** Assume (K1), (K2), and (K3). Then there exist $\lambda > \lambda_0$ and $v_{\lambda_0} \in L^\infty(\Omega)$, inf $v_{\lambda_0} > 0$ such that

(i) $v_{\lambda_0}(x) \notin (\alpha(x), \beta(x))$ for every $x \in \Omega_A$;

(ii) $\lambda \in \partial \varphi^{**}(x, v_{\lambda_0}(x))$ for every $x \in \Omega$;

(iii) $\int_{\Omega} v_{\lambda_0}(x) \, dx = a$.

In particular, $v_{\lambda_0}$ is a solution to (4.1). Moreover, if $\Omega = B_1(0)$ and $\varphi(x, t) = \tilde{\varphi}(|x|, t)$, then $v_{\lambda_0}$ is a radial function.

We postpone the proof of Theorem 4.1 to the following lemma.

**Lemma 4.2.** Let $O$ be a bounded measurable subset of $\mathbb{R}^N$. Let $\alpha, \beta \in L^1(O)$ be such that $\alpha(x) \leq \beta(x)$ for a.e. $x$ and suppose

$$\int_{\Omega} \alpha(x) \, dx < \kappa < \int_{\Omega} \beta(x) \, dx.$$ 

Then there exists $r > 0$ such that $\Theta: O \to \mathbb{R}$, $\Theta(x) := \alpha(x)$ if $x \in O \cap B_r(0)$ and $\Theta(x) := \beta(x)$ else, satisfying $\int_{\Omega} \Theta(x) \, dx = \kappa$.

**Proof.** Let $R$ be such that $O \subset B_R(0)$. Consider the functions $\theta_\rho: O \to \mathbb{R}$, $0 \leq \rho \leq R$, defined as follows: $\theta_0 := \beta$ and if $\rho > 0$, then $\theta_\rho(x) := \alpha(x)$, if $x \in O \cap B_\rho(0)$ and $\theta_\rho(x) := \beta(x)$ else. The continuity of $\rho \to \int_{\Omega} \theta_\rho(x) \, dx$ and (4.2) imply that there exists $0 < r < R$ such that $\int_{\Omega} \theta_r(x) \, dx = \kappa$. \hfill $\square$

We are now ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** We divide the proof into three steps. In Step 1 we define a family of functions $v^-_{\lambda_0}: \Omega \to (0, +\infty)$, $\lambda > \lambda_0$, such that

$$v^-_{\lambda_0}(x) \notin (\alpha(x), \beta(x)) \quad \forall x \in \Omega_A \quad \forall \lambda > \lambda_0$$

and

$$\lambda \in \partial \varphi^{**}(x, v^-_{\lambda_0}(x)) \quad \forall x \in \Omega \quad \forall \lambda > \lambda_0.$$ 

In Step 2 we define a function $v_{\lambda_0}$ satisfying (i), (ii), and (iii). Finally, in Step 3 we consider the case $\varphi(x, t) = \tilde{\varphi}(|x|, t)$. 


Step 1. The definition of $v^*_\lambda$. Let us define the function $\psi : \Omega \times (0, +\infty) \to [0, +\infty)$ such that $\psi \equiv \varphi$ in $(\Omega \setminus \Omega_A) \times (0, +\infty)$ and

\[
\psi(x, t) = \begin{cases} 
\varphi(x, t) & \text{if } x \in \Omega_A, 0 < t \leq \alpha(x), \\
\varphi(x, t + \beta(x) - \alpha(x)), & \text{if } x \in \Omega_A, t > \alpha(x).
\end{cases}
\]

(K1) and (K2) imply that for every $x \in \Omega_A$

\[
D^-_t \psi(x, \alpha(x)) \leq h(x) = \frac{\varphi(x, \beta(x)) - \varphi(x, \alpha(x))}{\beta(x) - \alpha(x)} \leq D^-_t \varphi(x, \beta(x))
\]
and that $\psi$ satisfies (H1). Moreover, for every $x \notin \Omega_A$ and every $t > 0$, we have $\partial\psi(x, t) = \partial\varphi(x, t) = \partial\varphi^{**}(x, t)$. If instead $x \in \Omega_A$, then

\[
\partial\psi(x, t) = \begin{cases} 
\partial\varphi(x, t) & \text{if } 0 < t < \alpha(x), \\
\partial\varphi^{**}(x, \alpha(x)) \cup \partial\varphi^{**}(x, \beta(x)) & \text{if } t = \alpha(x), \\
\partial\varphi(x, t + \beta(x) - \alpha(x)) & \text{if } t > \alpha(x).
\end{cases}
\]

We claim that (K3) implies that $\psi$ satisfies (H2).

The first limit in (K3) and the assumption $\inf \alpha > 0$ imply $\lim_{t \to 0^+} D^-_t \psi(x, t) = \lambda_0$, uniformly. Let us prove that $\psi$ satisfies the property on the second limit in (H2). Since $\alpha, \beta \in L^\infty(\Omega_A)$, then for every $x \in \Omega$ and $t > \|\alpha\|_{L^\infty(\Omega_A)}$,

\[
\inf_{y \in \Omega_A} D^-_t \varphi(y, t) \leq \min \left\{ \inf_{y \in \Omega_A} D^-_t \varphi(y, t + \beta(y) - \alpha(y)), \inf_{y \in \Omega_A} D^-_t \varphi(y, t) \right\}
\]

so that by (K3) as $t$ goes to $+\infty$, we get

\[
\lim_{t \to +\infty} \inf_{y \in \Omega} D^-_t \psi(y, t) = \lim_{t \to +\infty} D^-_t \psi(x, t) = +\infty \quad \forall x \in \Omega.
\]

Since $\psi$ satisfies the assumptions of Proposition 3.1, then for every $\lambda > \lambda_0$ there exists $u_\lambda \in L^\infty(\Omega)$, $\inf u_\lambda > 0$, satisfying (3.3). Moreover, for every $x \in \Omega_A$,

\[
\begin{aligned}
&u_\lambda(x) < \alpha(x) & \text{if } \lambda < D^-_t \varphi(x, \alpha(x)), \\
&u_\lambda(x) = \alpha(x) & \text{if } \lambda \in [D^-_t \varphi(x, \alpha(x)), D^-_t \varphi(x, \beta(x))], \\
&u_\lambda(x) > \alpha(x) & \text{if } \lambda > D^-_t \varphi(x, \beta(x)).
\end{aligned}
\]

Let us define $v^-_\lambda : \Omega \to (0, +\infty)$,

\[
v^-_\lambda(x) := u_\lambda(x) + (\beta(x) - \alpha(x)) \chi_{\{y \in \Omega_A : h(y) < \lambda\}}(x).
\]

Since $u_\lambda \in L^\infty(\Omega)$ and $\alpha, \beta \in L^\infty(\Omega_A)$, then $v^-_\lambda \in L^\infty(\Omega)$.

From (3.3), (4.6), (4.7), and (4.8) if $x \in \Omega_A$, the following implications hold:

- if $\lambda < D^-_t \varphi(x, \alpha(x))$, then $v^-_\lambda(x) = u_\lambda(x) < \alpha(x)$ and $\lambda \in \partial\psi(x, u_\lambda(x)) = \partial\varphi(x, v^-_\lambda(x))$;
- if $\lambda \in [D^-_t \varphi(x, \alpha(x)), h(x)]$, then $v^-_\lambda(x) = u_\lambda(x) = \alpha(x)$ and $\lambda \in \partial\varphi^{**}(x, \alpha(x))$;
- if $\lambda \in (h(x), D^-_t \varphi(x, \beta(x))]$, then $v^-_\lambda(x) = \beta(x)$ and $\lambda \in \partial\varphi^{**}(x, \beta(x))$;
- if $\lambda > D^-_t \varphi(x, \beta(x))$, then $v^-_\lambda(x) = u_\lambda(x) + \beta(x) - \alpha(x) > \beta(x)$ and $\lambda \in \partial\psi(x, u_\lambda(x)) = \partial\varphi(x, v^-_\lambda(x))$. 

Thus (4.3) holds and

\begin{equation}
\lambda \in \partial \phi^{**}(x, v_{\lambda}^{-}(x))
\end{equation}

for every \( x \in \Omega_{A} \) and \( \lambda > \lambda_{0} \). When \( x \notin \Omega_{A} \), the equality \( v_{\lambda}^{-}(x) = u_{\lambda}(x) \) and (3.3) imply (4.9). Therefore, (4.4) holds true.

**Step 2.** The definition of \( \lambda_{a} \) and \( v_{\lambda_{a}}^{+} \). Let us define \( \Phi: (\lambda_{0}, +\infty) \rightarrow (0, +\infty) \),

\[ \Phi(\lambda) := \int_{\Omega^{\lambda}} v_{\lambda}^{-}(x) \, dx = \int_{\Omega} \left( u_{\lambda}(x) + (\beta(x) - \alpha(x))\chi_{\{y \in \Omega_{A} : h(y) < \lambda\}}(x) \right) \, dx. \]

As in the proof of (3.4) we have that \( \lim_{\lambda \rightarrow \lambda_{0}^{+}} \Phi(\lambda) = 0 \) and \( \lim_{\lambda \rightarrow +\infty} \Phi(\lambda) = +\infty. \)

For each \( \lambda > \lambda_{0} \), define \( v_{\lambda}^{+} : \Omega \rightarrow (0, +\infty) \),

\[ v_{\lambda}^{+}(x) := u_{\lambda}(x) + (\beta(x) - \alpha(x))\chi_{\{y \in \Omega_{A} : h(y) \leq \lambda\}}(x). \]

For every \( \mu > \lambda_{0} \),

\[ \lim_{\lambda \rightarrow \mu^{-}} \Phi(\lambda) = \Phi(\mu), \quad \lim_{\lambda \rightarrow \mu^{+}} \Phi(\lambda) = \int_{\Omega} v_{\mu}^{+}(x) \, dx. \]

Thus, \( \Phi \) is discontinuous at \( \mu \) if and only if \( |\{y \in \Omega_{A} : h(y) = \mu\}| > 0. \)

Only one of the following cases is possible:

1. there exists \( \lambda_{a} > \lambda_{0} \) such that \( \Phi(\lambda_{a}) = a; \)
2. there exists \( \lambda_{a} > \lambda_{0} \) such that \( \Phi(\lambda_{a}) < a = \lim_{\lambda \rightarrow \lambda_{a}^{-}} \Phi(\lambda); \)
3. there exists \( \lambda_{a} > \lambda_{0} \) such that \( \Phi(\lambda_{a}) < a < \lim_{\lambda \rightarrow \lambda_{a}^{+}} \Phi(\lambda). \)

**Case 1.** As proved in Step 1, \( v_{\lambda_{a}}^{-} \) satisfies (i), (ii), and \( \inf v_{\lambda_{a}}^{-} \geq \inf u_{\lambda_{a}}^{0} > 0. \)

Moreover, by definition of \( \lambda_{a}, \) (iii) holds. Thus, define \( v_{\lambda_{a}} = v_{\lambda_{a}}^{-}. \)

**Case 2.** As above, \( v_{\lambda_{a}}^{-} \) satisfies (i), (ii), and \( \inf v_{\lambda_{a}}^{-} \geq \inf u_{\lambda_{a}} > 0. \) It is easy to check that a property analogous to (i) is satisfied by \( v_{\lambda_{a}}^{+} \) and that \( \inf v_{\lambda_{a}}^{+} \geq \inf v_{\lambda_{a}}^{-} > 0. \)

By the very definition of \( v_{\lambda_{a}}^{+} \) we have also \( \int_{\Omega} v_{\lambda_{a}}^{+} \, dx = a. \)

Let us prove that \( \lambda_{a} \in \partial \phi^{**}(x, v_{\lambda_{a}}^{+}(x)) \) for every \( x \). If \( x \notin \Omega_{A} \) or if \( x \in \Omega_{A} \) and \( h(x) \neq \lambda_{a} \), then \( v_{\lambda_{a}}^{-}(x) = v_{\lambda_{a}}^{+}(x) \) and the above inclusion follows. Suppose that \( x \in \Omega_{A} \) and \( h(x) = \lambda_{a}. \) Then \( v_{\lambda_{a}}^{-}(x) = \alpha(x) < \beta(x) = v_{\lambda_{a}}^{+}(x) \) and (K2) implies \( \lambda_{a} \in \partial \phi^{**}(x, \beta(x)) = \partial \phi^{**}(x, v_{\lambda_{a}}^{+}(x)). \)

We have so proved that \( \lambda_{a} \in \partial \phi^{**}(x, v_{\lambda_{a}}^{+}(x)) \) for every \( x \in \Omega. \) Thus, define \( v_{\lambda_{a}} := v_{\lambda_{a}}^{+}. \)

**Case 3.** Define \( O := \{ x \in \Omega_{A} : \lambda_{a} = h(x) \} \) and \( \kappa := a - \int_{\Omega \setminus O} v_{\lambda_{a}}^{-}(x) \, dx. \) The assumption \( \Phi(\lambda_{a}) < a < \lim_{\lambda \rightarrow \lambda_{a}^{+}} \Phi(\lambda) \) implies

\[ \int_{O} \alpha(x) \, dx = \int_{\Omega} v_{\lambda_{a}}^{-}(x) \, dx < \kappa < \int_{\Omega} v_{\lambda_{a}}^{+}(x) \, dx - \int_{\Omega \setminus O} v_{\lambda_{a}}^{-}(x) \, dx = \int_{O} \beta(x) \, dx. \]

From Lemma 4.2, there exists \( \Theta : O \rightarrow \mathbb{R}, \) \( \Theta(x) \in \{\alpha(x), \beta(x)\} \) such that \( \int_{O} \Theta(x) \, dx = \kappa. \)

Define \( v_{\lambda_{a}} : \Omega \rightarrow \mathbb{R}, v_{\lambda_{a}}(x) = v_{\lambda_{a}}^{-}(x) \) if \( x \notin O \) and \( v_{\lambda_{a}}(x) = \Theta(x) \) else.

It is easy to prove that \( v_{\lambda_{a}} \) satisfies (i), (ii), (iii), and \( \inf v_{\lambda_{a}} > 0. \)

Since \( \varphi \geq \varphi^{**}, \) then for every \( v \in L^{1}(\Omega) \) such that \( v > 0 \) a.e. and \( \int_{\Omega} v \, dx = a, \) we have that

\begin{equation}
\int_{\Omega} \varphi(x, v(x)) \, dx \geq \int_{\Omega} \varphi^{**}(x, v(x)) \, dx \geq \int_{\Omega} \varphi^{**}(x, v_{\lambda_{a}}(x)) \, dx + \lambda_{a} \int_{\Omega} (v(x) - v_{\lambda_{a}}(x)) \, dx = \int_{\Omega} \varphi(x, v_{\lambda_{a}}(x)) \, dx.
\end{equation}
Thus, \( v_{\lambda_0} \) is a solution to (4.1).

*Step 3. The case \( \varphi(x,t) = \tilde{\varphi}(|x|,t) \).* Assume that \( \Omega \) is the unit ball \( B_1(0) \) and that \( \varphi \) has the radial structure \( \varphi(x,t) = \tilde{\varphi}(|x|,t) \). It is easy to prove that \( \varphi^{**}(x,t) = (\tilde{\varphi})^{**}(|x|,t) \) and that \( \alpha, \beta, h \) are radial functions. Moreover, the sets \( \Omega_A, \{ y \in \Omega_A : h(y) < \lambda \} \) and \( \{ y \in \Omega_A : h(y) = \lambda \} \) are symmetric sets with respect to the origin. If \( \psi \) is defined as in Step 1 above, then it immediately follows that \( \psi(x,t) = \tilde{\psi}(|x|,t) \).

Looking at the first step of the proof of Proposition 3.1, it turns out that \( u_{\lambda_0} \), satisfying \( \partial \psi(x,u_{\lambda_0}(x)) = \lambda \), is a radial function for all \( \lambda \). All these facts allow us to conclude that whenever Cases 1 or 2 in Step 2 hold, i.e., \( \Phi(\lambda_0) = a \) or \( \Phi(\lambda_0) < a = \lim_{\lambda \to \lambda_0^+} \Phi(\lambda) \), respectively, then \( v_{\lambda_0} \) is a radial function. To prove that \( v_{\lambda_0} \) is radial in the third case it is sufficient to notice that the sets \( O, O \cap B_r(0) \), and \( O \setminus B_r(0) \) are symmetric with respect to the origin and consequently the function \( \Theta \) is radial. \( \square \)

5. Nonpolyconvex problems: Regularity result for the auxiliary problem. In this section we prove a regularity result for solutions to the nonconvex variational problem (4.1). Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) and let \( \varphi : \Omega \times (0, +\infty) \to [0, +\infty) \) be a continuous function, differentiable with respect to the last variable, \( D_i \varphi \in C^{0,\delta}(\Omega \times K) \), \( 0 < \delta \leq 1 \), for every compact \( K \) in \( (0, +\infty) \) such that

(A1) there exist \( \alpha, \beta \in C^{0,\delta}(\Omega) \), \( \beta(x) > \alpha(x) \) for every \( x, \inf \alpha > 0 \) such that \( \varphi(x,\cdot) \) and \( \varphi^{**}(x,\cdot) \) both coincide and are strictly convex in \( [0, \alpha(x)] \) and \( [\beta(x), +\infty) \) for every \( x \in \Omega \);

(A2) \( t \to \varphi^{**}(x,t) \) is affine in \([\alpha(x), \beta(x)]\) for every \( x \in \Omega \), i.e., for every \( \alpha(x) \leq t \leq \beta(x) \),

\[
\varphi^{**}(x,t) = h(x)t + q(x) \text{ with } h(x) = \frac{\varphi(x,\beta(x)) - \varphi(x,\alpha(x))}{\beta(x) - \alpha(x)}.
\]

Moreover,

\[
|\partial \{ x : h(x) = \lambda \}| = 0 \quad \forall \lambda \in \mathbb{R};
\]

(A3) there exists \( \lambda_0 \in \mathbb{R} \cup \{ -\infty \} \) such that

\[
\lim_{t \to 0^+} D_i \varphi(x,t) = \lambda_0, \quad \lim_{t \to +\infty} D_i \varphi(x,t) = +\infty, \text{ uniformly in } x;
\]

(A4) for every \( m > 0 \) there exists \( c_m > 0 \) such that

\[
\varphi(x,t) \geq \varphi(x,s) + D_i \varphi(x,s)(t - s) + c_m|t - s|^{2+\varepsilon}
\]

for every \( s, t \geq m \) such that \( s < t \leq \alpha(x) \) or \( \beta(x) \leq s < t \) for every \( x \in \Omega \) and some \( \varepsilon \geq 0 \).

The following result is in the same spirit of Lemma 4.2.

**Lemma 5.1.** Let \( O \) be an open set in \( \mathbb{R}^N \). Let \( \alpha, \beta \in L^1(O) \) be such that \( \alpha(x) \leq \beta(x) \) for a.e. \( x \) and suppose that

\[
\int_O \alpha(x) \, dx < \kappa < \int_O \beta(x) \, dx.
\]

Then there exists a finite number of balls \( B_{r_j}(y_j), \, j = 1, \ldots, m \), satisfying

1. \( B_{r_j}(y_j) \subset O, \, j = 1, \ldots, m;\)
2. \( B_{r_i}(y_i) \cap B_{r_j}(y_j) = \emptyset \) for every \( i \neq j; \)
3. \( \int_O \Theta(x) \, dx = \kappa, \)

where

\[
\Theta(x) = \begin{cases} 
\alpha(x) & \text{if } \varphi^{**}(x,t) = \alpha(x), \\
\beta(x) & \text{if } \varphi^{**}(x,t) = \beta(x), \\
\varphi^{**}(x,t) & \text{otherwise}.
\end{cases}
\]
where $\Theta(x) := \alpha(x)$ if $x \in \bigcup_{1 \leq j \leq m} B_{\rho_j}(y_j)$ and $\Theta(x) := \beta(x)$ else.

Proof. Since $O$ is open, there exist (at most) countably many pairwise disjoint balls $\{B_{R_j}(y_j)\}_{j \in J}$ in $O$, and a negligible set $\mathcal{N}$ such that $O = \mathcal{N} \cup \left(\bigcup_{j \in J} B_{R_j}(y_j)\right)$. Without loss of generality we assume $J = \{1, 2, \ldots, m\}$ if card $J = m \in \mathbb{N}$ and $J = \mathbb{N}$ if $J$ is countable. For every $n \in J$, let us define the function $\theta_n : O \to \mathbb{R}$,

$$
\theta_n(x) := \begin{cases} \alpha(x) & \text{if } x \in \bigcup_{1 \leq j \leq n} B_{R_j}(y_j), \\ \beta(x) & \text{else}. \end{cases}
$$

If $J$ is finite, then (5.1) implies $\int_O \theta_m(x) \, dx < \kappa$. If $J = \mathbb{N}$, it is easy to check that \(\lim_{n \to \infty} \int_O \theta_n(x) \, dx < \kappa\); thus, there exists $m \in \mathbb{N}$ such that

$$
\int_O \theta_m(x) \, dx = \int_{\bigcup_{1 \leq j \leq m} B_{R_j}(y_j)} \alpha(x) \, dx + \int_{O \setminus \bigcup_{1 \leq j \leq m} B_{R_j}(y_j)} \beta(x) \, dx < \kappa.
$$

Aiming at (1) and (2), we slightly reduce the radius of the previously selected balls $\{B_{R_j}(y_j)\}_{1 \leq j \leq m}$. This can easily be done by noticing that

$$
\lim_{\varepsilon \to 0^+} \int_{\bigcup_{j=1}^m B_{R_j}(y_j) \setminus B_{R_j-\varepsilon}(y_j)} (\beta(x) - \alpha(x)) \, dx = 0.
$$

Thus, there exists $0 < \varepsilon < \min\{R_j : 1 \leq j \leq m\}$ such that

$$
(5.2) \quad \int_{\bigcup_{1 \leq j \leq m} B_{R_j-\varepsilon}(y_j)} \alpha(x) \, dx + \int_{O \setminus \bigcup_{1 \leq j \leq m} B_{R_j-\varepsilon}(y_j)} \beta(x) \, dx < \kappa.
$$

Set $R := \max\{R_j - \varepsilon : 1 \leq j \leq m\}$ and define $\theta : O \times [0, R] \to \mathbb{R}$, $\theta(x, 0) := \beta(x)$ and

$$
\theta(x, \rho) := \begin{cases} \alpha(x) & \text{if } x \in \bigcup_{1 \leq j \leq m} (B_{R_j-\varepsilon}(y_j) \cap B_{\rho}(y_j)), \\ \beta(x) & \text{else}. \end{cases}
$$

for every $\rho > 0$. From (5.2) we have that

$$
\int_O \theta(x, R) \, dx < \kappa < \int_O \theta(x, 0) \, dx = \int_O \beta(x) \, dx.
$$

Since $\rho \to \int_O \theta(x, \rho) \, dx$ is a continuous function, there exists $\overline{\rho}$ such that $\int_O \theta(x, \overline{\rho}) \, dx = \kappa$. The claim of the theorem follows by defining $\Theta(x) := \theta(x, \overline{\rho})$ and $\rho_j := \min\{R_j - \varepsilon, \overline{\rho}\}, 1 \leq j \leq m$.

Let $h$ be as in (A2). For every $\lambda > \lambda_0$ we define

$$
(5.3) \quad \Omega^+ \lambda := \{x : h(x) > \lambda\}, \quad \Omega^- \lambda := \{x : h(x) < \lambda\}, \quad \Omega \lambda := \{x : h(x) = \lambda\}.
$$

Under (A1)-(A4) there exists a piecewise Hölder continuous solution to (4.1).

Theorem 5.2. Let $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ be a continuous function, differentiable with respect to the last variable, $D_t \varphi(x, t)$ in $C^{0,\delta}(\Omega \times K)$ for every compact $K \subset (0, +\infty)$. Suppose that (A1)-(A4) hold. Then, with fixed $a > 0$ there exist $\lambda_0 > 0$ and $v_{\lambda_0} \in L^\infty(\Omega)$, inf $v_{\lambda_0} > 0$, satisfying the following properties:

(i) $\left.D_t \varphi^a(x, v_{\lambda_0}(x)) = \lambda_a \right.$ for every $x \in \Omega$.
Moreover (see (4.6) and (4.8)), \( v_{\lambda_a} \) satisfies (iii), (iv), and (v). Moreover, reasoning as in (4.10) we get that

$$\int_{\Omega^+} v_{\lambda_a} (x) \, dx = a;$$

(ii) \( v_{\lambda_a} \) is Hölder continuous in \( \Omega^+_a \cup \Omega^-_a \); (iii) \( v_{\lambda_a} \) is Hölder continuous in \( \Omega^+_a \cup \Omega^-_a \);

(iv) \( v_{\lambda_a} (x) < \alpha (x) \) for all \( x \in \Omega^+_a \) and \( v_{\lambda_a} (x) > \beta (x) \) for all \( x \in \Omega^-_a \);

(v) in \( \Omega^-_a \) either \( v_{\lambda_a} \equiv \alpha \) or \( v_{\lambda_a} \equiv \beta \) or

$$v_{\lambda_a} (x) = \begin{cases} \alpha (x) & \text{if } x \in \bigcup_{1 \leq j \leq m} B_{\rho_j} (y_j), \\ \beta (x) & \text{if } x \in \Omega^-_a \setminus \bigcup_{1 \leq j \leq m} B_{\rho_j} (y_j) \end{cases}$$

with \( B_{\rho_j} (y_j) \subset \subset \text{int } \Omega^-_a \), \( j = 1, \ldots, m \) such that \( \overline{B}_{\rho_i} (y_i) \cap \overline{B}_{\rho_j} (y_j) = \emptyset \) if \( i \neq j \).

Moreover, \( v_{\lambda_a} \) is a solution to (4.1).

Proof. Let \( \psi : \Omega \times (0, +\infty) \rightarrow [0, +\infty) \) be defined as

$$\psi (x, t) = \begin{cases} \varphi (x, t) & \text{if } 0 < t \leq \alpha (x), \ x \in \Omega, \\ \varphi (x, t + \beta (x) - \alpha (x)) & \text{if } t > \alpha (x), \ x \in \Omega. \end{cases}$$

It holds true that \( \psi \) is a continuous function, differentiable with respect to the last variable, satisfying (H1)–(H4) in section 3, with possibly different constants. By Proposition 3.5 for every \( \lambda > \lambda_0 \), there exists \( u_\lambda \) such that \( u_\lambda \in C^{0, \gamma} (\Omega) \) for some \( 0 < \gamma \leq 1 \), inf \( u_\lambda > 0 \), and

$$D_t \psi (x, u_\lambda (x)) = \lambda \quad \forall \ x \in \Omega.$$
with
\[ \kappa := a - \int_{\Omega \setminus \Omega^+_{\lambda_a}} \left( u_{\lambda_a}(x) + (\beta(x) - \alpha(x))\chi_{\Omega^+_{\lambda_a}}(x) \right) \, dx, \]
then from Lemma 5.1 there exist \( m \) balls \( B_{\rho_i}(y_j) \subset \subset \text{int} \, \Omega^+_{\lambda_a}, \ j = 1, \ldots, m, \) \( \overline{B}_{\rho_i}(y_j) \cap \overline{B}_{\rho_j}(y_j) = \emptyset \) for every \( i \neq j \) such that \( \Theta : \text{int} \, \Omega^+_{\lambda_a} \to \mathbb{R}, \)
\[ \Theta := \alpha \quad \text{in } \bigcup_{1 \leq j \leq m} B_{\rho_i}(y_j), \quad \Theta := \beta \quad \text{in } \text{int} \, \Omega^+_{\lambda_a} \setminus \bigcup_{1 \leq j \leq m} B_{\rho_j}(y_j) \]
satisfies \( \int_{\text{int} \, \Omega^+_{\lambda_a}} \Theta(x) \, dx = \kappa. \)
Define \( v_{\lambda_a} \) as follows:
\[ v_{\lambda_a}(x) := \begin{cases} 
\alpha(x) & \text{if } x \in \Omega^+_{\lambda_a}, \\
\beta(x) & \text{if } x \in \bigcup_{1 \leq j \leq m} B_{\rho_j}(y_j), \\
u_{\lambda_a}(x) + \beta(x) - \alpha(x) & \text{if } x \in \Omega^+_{\lambda_a}. 
\end{cases} \]
We have that \( v_{\lambda_a} \in L^\infty(\Omega), \inf \nu_{\lambda_a} > 0, \) and it satisfies (i)–(v). Moreover, \( v_{\lambda_a} \) is a solution to (4.1). \( \square \)

6. Nonpolyconvex problems: Attainment result in a general setting.
In this section we consider the variational problem
\[ \min \left\{ \int_{\Omega} f(x, \det Du(x)) \, dx : u \in W^{1,N}(\Omega, \mathbb{R}^N), \ \det Du > 0 \text{ a.e.}, \ u(x) = x \text{ on } \partial \Omega \right\}, \tag{6.1} \]
where \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \) with Lipschitz boundary and \( f : \Omega \times (0, +\infty) \to [0, +\infty) \) is a nonconvex function with respect to the second variable.

Before stating an attainment result for (6.1), we need some preliminary results.

**Lemma 6.1.** Let \( \Omega \) be a bounded open set with Lipschitz boundary and let \( \Omega = \bigcup_{i=1}^m \Omega_i \) with \( \{ \Omega_i \} \) pairwise disjoint open connected sets with Lipschitz boundary.

Consider \( \alpha_i > 0, \ i = 1, \ldots, m, \) with \( \sum_{i=1}^m \alpha_i = |\Omega|. \) Then there exists a bi-Lipschitz homeomorphism \( u_0 : \Omega \to \Omega \) such that \( \det Du_0 \in C^\infty(\overline{\Omega}), \) \( \inf \det Du_0 > 0, \) and
\[ u_0(x) = x \text{ on } \partial \Omega, \quad |u_0(\Omega_i)| = \alpha_i, \quad i = 1, \ldots, m. \tag{6.2} \]
Moreover, \( u_0(\Omega_i) \) is an open set of class (L) for every \( i. \)

**Proof.** Fix \( 0 < \delta < \min\left\{ \alpha_i/|\Omega_i| : i = 1, \ldots, m \right\}. \) For every \( 1 \leq i \leq m \) let \( \eta_i \in C^\infty_c(\Omega_i) \) be such that \( \int_{\Omega_i} \eta_i(x) \, dx = 1. \) Define
\[ f(x) = \delta + \sum_{i=1}^m (\alpha_i - \delta |\Omega_i|) \eta_i(x), \quad x \in \overline{\Omega}. \]
Hence, \( f \in C^\infty(\overline{\Omega}), \inf \ f > 0, \) \( \int_{\Omega_i} f(x) \, dx = \alpha_i \) for every \( i, \) and \( \int_{\Omega} f(x) \, dx = |\Omega|. \)
From Theorem 2.4 there exists a bi-Lipschitz homeomorphism \( u_0 : \overline{\Omega} \to \overline{\Omega} \) such that
\[ \det Du_0 = f \text{ in } \Omega, \quad u_0(x) = x \text{ on } \partial \Omega. \]
Therefore,
\[ |u_0(\Omega_i)| = \int_{\Omega_i} \det Du_0(x) \, dx = \int_{\Omega_i} f(x) \, dx = \alpha_i, \quad i = 1, \ldots, m; \]
Proposition 6.2. Let \( \Omega \) and \( \Omega_i \), \( i = 1, \ldots, m \), be as in Lemma 6.1. Suppose that \( g_i : \overline{\Omega_i} \to [c_0, +\infty) \), with \( c_0 > 0 \), \( i = 1, \ldots, m \), are Hölder continuous functions satisfying

\[
\sum_{i=1}^{m} \int_{\Omega_i} g_i(x) \, dx = |\Omega|.
\]

Then there exists a Lipschitz continuous function \( u : \overline{\Omega} \to \overline{\Omega} \) such that

\[
u(x) = x \quad \text{on} \quad \partial \Omega, \quad \det Du(x) = g_i(x) \quad \forall x \in \Omega_i \quad \forall i = 1, \ldots, m.
\]

Proof. By Lemma 6.1 there exists a bi-Lipschitz homeomorphism \( u_0 : \overline{\Omega} \to \overline{\Omega} \) such that

\[
\quad u_0(x) = x \quad \text{on} \quad \partial \Omega, \quad |u_0(\Omega_i)| = \int_{\Omega_i} g_i(x) \, dx
\]

and \( u_0(\Omega_i) \) is of class \( (L) \) for each \( i = 1, \ldots, m \). Moreover, \( f := \det Du_0 \) is of class \( C^\infty(\overline{\Omega}) \) and \( \inf f > 0 \). Since \( \frac{i}{f} \circ u_0^{-1} \) is Hölder continuous in \( u_0(\Omega_i) \) and it satisfies

\[
\int_{u_0(\Omega_i)} \frac{g_i}{f} \circ u_0^{-1}(y) \, dy = \int_{\Omega_i} g_i(x) \, dx = |u_0(\Omega_i)|,
\]

then from Theorem 2.4 there exists a bi-Lipschitz homeomorphism \( z_i : \overline{u_0(\Omega_i)} \to \overline{u_0(\Omega_i)} \) such that

\[
\begin{cases}
\det Dz_i = \frac{g_i}{f} \circ u_0^{-1} & \text{in} \quad u_0(\Omega_i), \\
z_i(y) = y & \text{on} \quad \partial u_0(\Omega_i).
\end{cases}
\]

Thus, \( u_i = z_i \circ u_0 \) is a Lipschitz homeomorphism such that

\[
\begin{cases}
\det Du_i = g_i & \text{in} \quad \Omega_i, \\
u_i = u_0 & \text{on} \quad \partial \Omega_i.
\end{cases}
\]

Hence, the Lipschitz continuous function \( u : \overline{\Omega} \to \overline{\Omega} \) such that \( u(x) = u_i(x) \) for every \( x \in \overline{\Omega}_i, \) \( i = 1, \ldots, m \), satisfies (6.3). \( \square \)

We are in position to state an existence result for the nonpolyconvex problem (6.1). The sets \( \Omega^+_\lambda, \Omega^-_\lambda \), and \( \Omega^0_\lambda \) are defined in (5.3).

Theorem 6.3. Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) with Lipschitz boundary and let \( \varphi : \Omega \times (0, +\infty) \to [0, +\infty) \) be a continuous function, differentiable with respect to the last variable, \( D_1 \varphi \in C^{0, \delta} \left( \Omega \times K \right), 0 < \delta \leq 1 \), for every compact \( K \subset (0, +\infty) \).

Suppose that (A1)–(A4) hold and assume that, for every \( \lambda > \lambda_0, \Omega^+_\lambda, \Omega^-_\lambda, \) and \( \text{int} \Omega^0_\lambda \) are either empty or connected open sets with Lipschitz boundary. Then the variational problem (6.1) has a Lipschitz continuous solution.

Proof. From Theorem 5.2, applied with \( \alpha = |\Omega| \), there exist \( \lambda_0 > \lambda_0 \) and a solution \( v_{\lambda_0} \) to (4.1) with \( \inf v_{\lambda_0} > 0 \). Throughout we write \( v \) instead of \( v_{\lambda_0} \).

From Theorem 5.2 \( v \) is Hölder continuous in \( \Omega^+_\lambda \cup \Omega^-_\lambda \). If \( \text{int} \Omega^0_\lambda \) is empty, we get the thesis applying Proposition 6.2 with \( \Omega_1 = \Omega^+_\lambda, \Omega_2 = \Omega^-_\lambda, \) and replacing \( g_1 \) and \( g_2 \) with the continuous extension of \( v \) to \( \Omega^+_\lambda \) and to \( \Omega^-_\lambda \), respectively.
If \( \text{int } \Omega^a_{\lambda_0} \) is not empty, correspondingly to (v) of Theorem 5.2 we have to consider three cases.

If \( v = \alpha \) in \( \Omega^a_{\lambda_0} \), the thesis follows by applying Proposition 6.2 with \( m = 3 \), choosing \( \Omega_1 = \Omega^a_{\lambda_0} \), \( \Omega_2 = \Omega^a_{\lambda_0} \), \( \Omega_3 = \int \Omega^a_{\lambda_0} \), and replacing, as above, \( g_1 \) and \( g_2 \) with the continuous extension of \( v \) to \( \Omega^a_{\lambda_0} \) and \( \Omega^a_{\lambda_0} \), respectively, and \( g_3 \) with \( \alpha \). Analogously, we proceed if \( v = \beta \) in \( \Omega^a_{\lambda_0} \), but defining \( g_3 = \beta \).

Now suppose that (5.4) holds. In this case the thesis follows by Proposition 6.2 choosing \( \Omega_1 = \Omega^a_{\lambda_0} \), \( \Omega_2 = \Omega^a_{\lambda_0} \), \( \Omega_3 = \int \Omega^a_{\lambda_0} \), \( \Omega_3 + i = B_{\rho_j}(y_j) \), for every \( i = 1, \ldots, n \) and \( g_1 = v \), \( g_2 = v \), \( g_3 = \beta \), \( g_{3+i} = \alpha \), for every \( i = 1, \ldots, n \).

With obvious changes in the proof above, we get the following theorem.

**Theorem 6.4.** Let \( \Omega \) and \( \varphi \) be as in Theorem 6.3. Suppose that (A1)–(A4) hold and assume that for every \( \lambda > \lambda_0 \),

\[
(6.4) \quad \Omega^a_{\lambda} = \bigcup_{i=1}^{h} \mathcal{A}_i, \quad \Omega^a_{\lambda^+} = \bigcup_{i=h+1}^{k} \mathcal{A}_i, \quad \text{int } \Omega^a_{\lambda^+} = \bigcup_{i=k+1}^{l} A_i
\]

with \( A_i \) either empty or pairwise disjoint open connected sets with Lipschitz boundary.

Then the variational problem (6.1) has a Lipschitz continuous solution.

**Remark 6.5.** The following are examples of sets \( \Omega \) and functions \( h : \Omega \to \mathbb{R} \) such that for every \( \lambda \in \mathbb{R} \) (6.4) holds with either empty or disjoint open sets \( \{A_i\} \) with Lipschitz boundary:

(a) \( \Omega \) is a bounded and convex set and \( h \) is strictly convex in \( \Omega \) and constant on \( \partial \Omega \);

(b) \( \Omega = B_1(0) \) and \( h \) is a radial function, \( h(x) = \tilde{h}(|x|) \), with \( \tilde{h} \) piecewise monotone, i.e., there exists \( 0 = s_0 < s_1 < \cdots < s_m = 1 \) such that \( \tilde{h}[s_i, s_{i+1}] \) is monotone for all \( i \).

7. Nonpolyconvex problems: Some special cases. In this section we consider particular classes of the variational problem (6.1), where \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \) with Lipschitz boundary and \( \varphi : \Omega \times (0, +\infty) \to [0, +\infty) \) is a continuous function satisfying (A1) and (A2). We begin considering the case of functions \( \varphi \) such that \( h \) in (A2) is a constant. See [20] and [3] for related results.

**Theorem 7.1.** Let \( \varphi : \Omega \times (0, +\infty) \to [0, +\infty) \) be a continuous function satisfying (A1) and (A2) with \( h \) constant. If \( \int_{\Omega} \alpha(x) dx \leq |\Omega| \leq \int_{\Omega} \beta(x) dx \), then (6.1) has a Lipschitz continuous solution.

**Proof.** Consider the auxiliary problem (4.1) with \( a = |\Omega| \). If \( \int_{\Omega} \alpha(x) dx \) is equal to \( |\Omega| \), then \( \alpha \) solves (4.1). Then from Theorem 2.4 there exists a Lipschitz homeomorphism \( u \) solution to (2.1) with \( f = \alpha \). Moreover, \( u \) is a solution of (6.1). The same argument works if \( \int_{\Omega} \beta(x) dx \) is equal to \( |\Omega| \). Of course in this case choose \( f = \beta \).

Suppose \( \int_{\Omega} \alpha(x) dx < |\Omega| < \int_{\Omega} \beta(x) dx \). Then using Lemma 5.1 with \( O = \Omega \), we get that a Lipschitz continuous solution \( u \) to (4.1) exists with \( u \equiv \alpha \) on pairwise disjoint balls \( B_{\rho_j}(y_j) \subset \Omega \), \( j = 1, \ldots, n \), and with \( u \equiv \beta \) outside these balls. The thesis follows by Proposition 6.2 with \( m = n + 1 \), \( \Omega_j = B_{\rho_j}(y_j) \), and \( g_j = \alpha \) if \( j = 1, \ldots, m-1 \) and with \( \Omega_m = \Omega \setminus \bigcup_{j=1}^{m} B_{\rho_j}(y_j) \), \( g_m = \beta \).

**Theorem 7.2.** Let \( \varphi : \Omega \times (0, +\infty) \to [0, +\infty) \) be a continuous function, differentiable with respect to the last variable, \( \partial_\varphi \in C^{0,\delta}(\Omega \times K) \), \( 0 < \delta \leq 1 \), for every compact \( K \subset (0, +\infty) \). Suppose that (A1), (A2) with \( h \) constant, (A3), and (A4) hold. If \( \int_{\Omega} \alpha(x) dx \geq |\Omega| \) or \( \int_{\Omega} \beta(x) dx \leq |\Omega| \), then (6.1) has a Lipschitz continuous solution.
Proof. Let \( a = |\Omega| \). From Theorem 5.2 there exist \( \lambda_a > \lambda_0 \) and \( v_{\lambda_a} \in L^\infty(\Omega) \) satisfying

\[
(7.1) \quad v_{\lambda_a}(x) \notin (\alpha(x), \beta(x)), \quad D\varphi^{**}(x, v_{\lambda_a}(x)) = \lambda_a, \quad \int_\Omega v_{\lambda_a}(x) \, dx = |\Omega|.
\]

(A1), (A2), and (A3) imply \( h = D\varphi(x, \alpha(x)) = D\varphi(x, \beta(x)) \) and the definition of \( \{v_a\} \) (see the proofs of Theorems 4.1 and 5.2) gives that \( \lambda < h \) if and only if \( v_\lambda(x) < \alpha(x) \) for all \( x \), \( \lambda > h \) if and only if \( v_\lambda(x) > \beta(x) \) for all \( x \). Therefore, if \( \int_\Omega \alpha(x) \, dx > |\Omega| \), then \( \lambda_a < h \) and \( v_{\lambda_a}(x) < \alpha(x) \). Thus, using the notation in (5.3), \( \Omega_{\lambda_a}^\alpha = \Omega \). Analogously, if \( \int_\Omega \beta(x) \, dx < |\Omega| \), then \( \lambda_a > h \) and \( v_{\lambda_a}(x) > \beta(x) \), so that \( \Omega_{\lambda_a}^\alpha = \Omega \). Therefore, Theorem 5.2 implies that \( v_{\lambda_a} \) is Hölder continuous in \( \Omega \). A Lipschitz continuous solution to

\[
\begin{aligned}
\{\int_\Omega & \Phi(x, \det Du(x)) \, dx : u \in W^{1,N}(\Omega, \mathbb{R}^N), \ det Du > 0 \ \text{a.e.}, \ u(x) = x \ \text{on} \ \partial\Omega, \\
\det Du = v_{\lambda_a} & \ \text{in} \ \Omega,
\end{aligned}
\]

solution also to (6.1), exists because of Theorem 2.4. \( \square \)

In Propositions 7.3 and 7.4 we deal with a variant of functionals considered above, precisely

\[
(7.2) \quad \min \left\{ \int_\Omega \Phi(x, \det Du(x)) \, dx : u \in W^{1,N}(\Omega, \mathbb{R}^N), \ det Du > 0 \ \text{a.e.}, \ u(x) = x \ \text{on} \ \partial\Omega \right\}
\]

with \( \Phi(x, t) = \varphi(x, t) + f(x) t \).

**Proposition 7.3.** Let \( \Omega \) be a bounded open convex set in \( \mathbb{R}^N \) and let \( \varphi : \Omega \times (0, +\infty) \to [0, +\infty) \) satisfy the assumptions of Theorem 7.2 with \( \lambda_0 = -\infty \) in (A3). Suppose that \( f : \Omega \to (0, +\infty) \) is a strictly convex function, constant on \( \partial\Omega \). Then there exists a Lipschitz solution to (7.2).

**Proof.** It is easy to see that \( \Phi \) satisfies the assumptions of Theorem 6.3. Since \( \Phi^{**}(x, t) = \varphi^{**}(x,t) + f(x) t \) for every \( x \in \Omega \), then in \((0,\alpha(x)]\) and in \([\beta(x),+\infty)\) we have that \( \Phi(x,.) = \Phi^{**}(x,.) \). Moreover, for every \( t \in [\alpha(x), \beta(x)] \) it holds true that \( \Phi^{**}(x,t) = H(x) t + q(x) \) with \( H(x) := \mu + f(x) \) and the superlevel, sublevel, and level sets of \( H \) satisfy the assumptions in Theorem 6.3. (A3) implies that \( D\Phi(x,t) = D\varphi(x,t) + f(x) \) goes to \(-\infty \) as \( t \to -\infty \) and goes to \(+\infty \) as \( t \to +\infty \), uniformly with respect to \( x \). The thesis easily follows from Theorem 6.3. \( \square \)

From now on, \( \Omega \) is the unit ball \( B \) in \( \mathbb{R}^N \) centered at the origin.

**Proposition 7.4.** Let \( \varphi : B \times (0, +\infty) \to [0, +\infty) \) satisfy the assumptions of Theorem 7.2 with \( \lambda_0 = -\infty \) in (A3). Let \( f \in C^0,\gamma([0,1]), 0 < \gamma \leq 1, f(s) > 0 \) for every \( s \), \( f \) piecewise monotone. Then there exists a Lipschitz continuous solution to (7.2) with \( \Phi(x,t) = \varphi(x,t) + f(|x|) t \).

**Proof.** Proceeding as in the proof of Proposition 7.3, the thesis easily follows from Remark 6.5(b) and from Theorem 6.4 applied to \( \Phi(x,t) = \varphi(x,t) + f(|x|) t \). \( \square \)

Now, we deal with one more class of nonpolyconvex functionals, characterized by an integrand \( \varphi \) with radial structure \( \varphi(x,t) = \tilde{\varphi}(|x|,t) \). Precisely, we deal with the variational problem

\[
(7.3) \quad \min \left\{ \int_B \tilde{\varphi}(|x|, \det Du(x)) \, dx : u \in W^{1,N}(B, \mathbb{R}^N), \ det Du > 0 \ \text{a.e.}, \ u(x) = x \ \text{on} \ \partial B \right\}
\]
and $\tilde{\varphi} : [0,1) \times (0, +\infty) \rightarrow [0, +\infty)$ is a continuous function.

**Theorem 7.5.** Let $\tilde{\varphi} : [0,1) \times (0, +\infty) \rightarrow [0, +\infty)$ be a continuous function satisfying the following assumptions:

(i) there exist $a, b \in L^\infty(0,1), b(s) > a(s) > 0$ for every $s$, $\inf a > 0$, such that $\tilde{\varphi}(s, \cdot)$ and $\tilde{\varphi}^*(s, \cdot)$ both coincide and are strictly convex in $(0, a(s))$ and $[b(s), +\infty)$ for all $s \in [0,1]$;

(ii) $\tilde{\varphi}^*(x, \cdot)$ is affine in $[a(s), b(s)]$ for all $s \in [0,1]$;

(iii) there exists $\lambda_0 \in \mathbb{R} \cup \{-\infty\}$ such that

$$\lim_{t \to 0^+} D^+_t \tilde{\varphi}(s, t) = \lambda_0, \quad \lim_{t \to +\infty} D^-_t \tilde{\varphi}(s, t) = +\infty,$$

uniformly in $s$.

Then there exists a Lipschitz solution to (7.3).

**Proof.** Let us define $\varphi(x, t) := \tilde{\varphi}(|x|, t)$ for every $x \in B$. Notice that $\varphi^*(x, t) = \tilde{\varphi}^*(|x|, t)$ and that assumptions (K1), (K2), and (K3) of Theorem 4.1 holds with $\Omega = \Omega_A = B$, $\alpha(x) = a(|x|)$, and $\beta(x) = b(|x|)$. Let $v \in L^\infty(B)$, $\inf v > 0$, be the radial solution of (4.1). It is a known fact (see, e.g., [15]) that there exists a bi-Lipschitz solution $u$ to (2.1) with $f = v$ and $\Omega = B$. Thus, $u$ is a solution to (7.3), too.

**Appendix. Proof of Theorem 2.4.** In the following we use the arguments of the proof of Lemma 1 in [16] and the fact, proved in [18], that if $\Omega = (0,1)^N$ and $f$ is Hölder continuous, then there exists a bi-Lipschitz homeomorphism solution to (2.1).

We divide the proof into steps.

**Step 1.** Let $\Omega$ be a bounded open connected subset of $\mathbb{R}^N$ of class $(L)$. Thus, there exist $m$ open sets $\Omega_j$ such that $\Omega \subset \bigcup_j \Omega_j$ and $m$ bi-Lipschitz homeomorphisms $\psi_j : \Sigma_j \to \overline{Q}$, with $\Sigma_j = \Omega \cap \Omega_j$ and $Q = (0,1)^N$ such that $\det D \psi_j \in \mathrm{Lip}(\Sigma_j)$ and $\frac{1}{A} < \det D \psi_j < A$ for some $A \geq 1$. Consider a partition of unity $\{\phi_j\}_{j=1}^m$ subordinate to such a covering of $\overline{\Omega}$: $\{\phi_j\}_{j=1}^m$ is a family of smooth and nonnegative functions, $\sum_j \phi_j(x) = 1$ for every $x \in \overline{\Omega}$ and

$$\text{supp } \phi_j \subset \subset \Omega_j \quad \forall j = 1, \ldots, m.$$

Since $\Omega = \bigcup_{j=1}^m \Sigma_j$ and $\Omega$ is connected, we can assume that for every $k = 2, \ldots, m$ there exists $\rho(k) < k$ such that $\Sigma_k \cap \Sigma_{\rho(k)}$ is not empty. Define the matrix $(\alpha_{hk})$, $1 \leq h \leq m$, $2 \leq k \leq m$,

$$\alpha_{hk} = \begin{cases} 1 & \text{if } h = k, \\ -1 & \text{if } h = \rho(k), \\ 0 & \text{else}. \end{cases}$$

Each of the $m-1$ columns contains exactly one pair $+1$, $-1$ so that $\sum_{k=2}^m \alpha_{hk} = 0$ for every $h$.

Define $\eta_k \in C_0^\infty(\Sigma_k \cap \Sigma_{\rho(k)})$ such that $\int_{\Sigma_k} \eta_k(x) \, dx = 1$. Let $g \in C^{0,\alpha}(\overline{\Omega})$ be such that $\int_{\Omega} g(x) \, dx = 0$. Define the Hölder continuous functions $g_h : \overline{\Omega} \to \mathbb{R}, 1 \leq h \leq m$,

$$g_h := g \phi_h \big|_{\overline{\Omega}} - \sum_{k=2}^m \lambda_k \alpha_{hk} \eta_k,$$

where $\lambda_2, \ldots, \lambda_m$ are real numbers solutions of the following system of $m$ equations

$$\sum_{k=2}^m \lambda_k \alpha_{hk} = \int_{\Omega} g \phi_h \, dx, \quad h = 1, \ldots, m.$$
Since the rank of \((\alpha_hk)\) is \(m-1\) and both \(\sum_{k=1}^{m} \sum_{h=1}^{m} \lambda_k \alpha_hk\) and \(\sum_{h=1}^{m} \int_{\Omega} g \phi_h \, dx\) are equal to 0, then system (7.5) is uniquely solvable.

We claim that supp \(g_h \subseteq \Sigma_h\). In fact supp \(\phi_h|_{\Omega} \subseteq \Sigma_h\) and, since \(\alpha_hk \neq 0\) if and only if \(h = k\) or \(h = \rho(k)\),

\[
\text{supp} \lambda \alpha_hk \eta_k \subseteq \Sigma_k \cap \Sigma_{\rho(k)} \subseteq \Sigma_h
\]

for every \(k = 2, \ldots, m\). Moreover, from (7.5) there exists \(M \geq 0\) depending on \(\Omega\), \(\{\phi_j\}_{j},\) and \(\{\eta_j\}_{j}\) only such that \(\sup |g_n| \leq M \sup |g|\).

**Step 2.** Let \(\Omega, \{\Sigma_j\}_{j}, \{\psi_j\}_{j}, \{\phi_j\}_{j}, \{\eta_j\}_{j}, M,\) and \(M\) be as above. Let \(f\) in (2.1) be such that \(\sup |f - 1| < m^{-1}M^{-1}\). Define \(m\) Hölder continuous functions \(g_h\) reasoning as in the previous step with \(g\) replaced by \(f - 1\). For every \(j = 1, \ldots, m + 1\) define \(f_j : \overline{\Omega} \rightarrow (0, +\infty)\),

\[
f_j(x) = \begin{cases} 1 & \text{if } j = 1, \\ 1 + \sum_{k=1}^{m} g_h(x) & \text{if } j > 1. \end{cases}
\]

In particular \(f_{m+1} = f\). Notice that each \(f_j\) is a Hölder continuous function, and since \(\sup |f - 1| < m^{-1}M^{-1}\), then \(\inf \ f_j > 0\). Fixed \(j = 1, \ldots, m\), we have that

\[
(7.6) \quad f_{j+1} - f_j = 0 \quad \text{in } \overline{\Omega} \setminus \Sigma_j, \quad \int_{\Omega} f_j(x) \, dx = |\Omega|, \quad \int_{\Sigma_j} f_{j+1}(x) \, dx = \int_{\Sigma_j} f_j(x) \, dx.
\]

Define \(f_j^*, f_{j+1}^* : \overline{Q} \rightarrow (0, +\infty)\),

\[
f_j^* := f_j(\psi_j^{-1}) \det D\psi_j^{-1}, \quad f_{j+1}^* := f_{j+1}(\psi_j^{-1}) \det D\psi_j^{-1},
\]

so that \(f_j^*, f_{j+1}^* \in C^{0,\alpha}(\overline{Q})\) and \(\int_Q f_j^* \, dx = \int_Q f_{j+1}^* \, dx\).

As proved in [18] there exist two bi-Lipschitz homeomorphisms \(v_j, w_j : \overline{Q} \rightarrow \overline{Q}\) solutions to

\[
\begin{cases}
\det Dv_j = \frac{f_j^*}{\int_Q f_j^* \, dx} \quad \text{in } Q, \\
\text{on } \partial Q, \\
v_j(y) = y,
\end{cases}
\quad \text{and} \quad \begin{cases}
\det Dw_j = \frac{f_{j+1}^*}{\int_Q f_j^* \, dx} \quad \text{in } Q, \\
\text{on } \partial Q, \\
w_j(y) = y,
\end{cases}
\]

respectively. Let us consider \(\varphi_j : \overline{Q} \rightarrow \overline{Q}, \varphi_j(y) := (v_j^{-1} \circ w_j)(y)\). Then

\[
\det D\varphi_j(y) = \det Dw_j^{-1}(w_j(y)) \det Dw_j(y) = \frac{f_{j+1}^*}{f_j^*(\varphi_j(y))} \quad \forall y \in \overline{Q}
\]

so that

\[
f_j((\psi_j^{-1} \circ \varphi_j)(y)) \det D\psi_j^{-1}(\varphi_j(y)) \det D\varphi_j(y) = f_{j+1}(\psi_j^{-1}(y)) \det D\psi_j^{-1}(y) \quad \forall y \in \overline{Q}.
\]

Using the invertibility of \(\psi_j\) the equality above implies that

\[
(7.7) \quad f_j(u_j(x)) \det Du_j(x) = f_{j+1}(x) \quad \forall x \in \Sigma_j,
\]

where \(u_j : \Sigma_j \rightarrow \overline{\Sigma}_j\) is the Lipschitz continuous function defined as \(u_j(x) := (\psi_j^{-1} \circ \varphi_j \circ \psi_j)(x)\).

Since \(\varphi_j(\psi_j(x)) = \psi_j(x)\) for all \(x \in \partial \Sigma_j\), we have that \(u_j(x) = x\) for every \(x \in \partial \Sigma_j\). Then \(u_j : \Omega \rightarrow \mathbb{R}, j = 1, \ldots, m,\)

\[
u_j(x) := \begin{cases} u_j(x) & \text{if } x \in \Sigma_j, \\ x & \text{else.} \end{cases}
\]
is Lipschitz continuous and from (7.6) and (7.7)
\[
f_j(\tilde{u}_j(x)) \det D\tilde{u}_j(x) = f_{j+1}(x) \quad \forall x \in \Omega.
\]

Iterating this argument on \( j \) and recalling that \( f_1 = 1 \) and \( f_{m+1} = f \), we get that \( \tilde{u}_1 \circ \cdots \circ \tilde{u}_m \) is a Lipschitz solution to (2.1).

**Step 3.** Now we suppose that \( f \) in (2.1) satisfies \( \sup |f - 1| \geq m^{-1}M^{-1} \). There exists \( c_1 > 0 \) and \( 0 < t_1 < 1 \) such that \( \int_\Omega c_1 f^{t_1} \, dx = |\Omega| \) and \( \sup |c_1 f^{t_1} - 1| < m^{-1}M^{-1} \). Applying the same arguments described in Step 2 to \( g := c_1 f^{t_1} - 1 \), we obtain a Lipschitz function \( u_1 \) satisfying (2.1) with \( f \) replaced by \( c_1 f^{t_1} \). Applying again this procedure to \( g := c_2 f^{t_2} - c_1 f^{t_1} \), with a suitable choice of \( c_2 \) and \( t_2 \) in such a way that \( t_1 < t_2 \leq 1 \), \( \int_\Omega c_2 f^{t_2} \, dx = |\Omega| \) and \( \sup |c_2 f^{t_2} - c_1 f^{t_1}| < m^{-1}M^{-1} \), we get \( u_2 \) Lipschitz solution to
\[
\begin{align*}
&\{c_1 f^{t_1}(u_2)\det Du_2 = c_2 f^{t_2} & \text{in } \Omega, \\
&u_2(x) = x & \text{on } \partial \Omega.
\end{align*}
\]

Hence, \( u_1 \circ u_2 \) solves (2.1) with \( f \) replaced by \( c_2 f^{t_2} \). It can be proved that the exponents \( t_i \) can be chosen such that in finitely many steps, say \( n \), we get \( t_n = 1 \). The existence of a Lipschitz continuous solution to (2.1) follows.

**REFERENCES**


