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EXISTENCE OF MINIMIZERS FOR POLYCONVEX AND NONPOLYCONVEX PROBLEMS*

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Abstract. We study the existence of Lipschitz minimizers of integral functionals

$$\mathcal{I}(u) = \int_\Omega \varphi(x, \det Du(x))\, dx,$$

where Ω is an open subset of $\mathbb{R}^N$ with Lipschitz boundary, $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ is a continuous function, and $u \in W^{1,N}(\Omega, \mathbb{R}^N)$, $u(x) = x$ on $\partial\Omega$. We consider both the cases of φ convex and nonconvex with respect to the last variable. The attainment results are obtained passing through the minimization of an auxiliary functional and the solution of a prescribed Jacobian equation.

Key words. nonpolyconvex functional, existence of minimizers, Lipschitz regularity, prescribed Jacobian equation

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1. Introduction.

In this paper we consider integral functionals

$$\mathcal{I}(u) = \int_\Omega \varphi(x, \det Du(x))\, dx, \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with a Lipschitz boundary, $N \geq 2$, $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ is a continuous function, and $u \in W^{1,N}(\Omega, \mathbb{R}^N)$.

We aim at proving the existence of Lipschitz solutions to the variational problem

$$\min \left\{ \mathcal{I}(u) \,:\, u \in W^{1,N}(\Omega, \mathbb{R}^N), \quad \det Du > 0 \text{ a.e.}, \quad u(x) = x \text{ on } \partial\Omega \right\}. \tag{1.2}$$

Notice that even if a growth condition from below of the type $t^p \leq \varphi(x,t)$ (which is common in the theory of calculus of variations) is assumed, no coercivity of $\mathcal{I}$ follows in any Sobolev space, preventing us from establishing the existence of minimizers via the direct method. Nevertheless many problems of this type have a solution, and the question of fixing which conditions on φ ensure the existence of solutions is worthy of interest, as is its applications in physics, mainly in elasticity theory and in the problem of the equilibrium of gases (see [17], [5], [6], and [12]). For instance, (1.2) is the variational problem corresponding to a nonhomogeneous elastic material with reference configuration Ω whose stored energy φ is a nonnegative, continuous function depending on the position x in the reference configuration and the size of the deformation of the volume element $\det Du(x) > 0$.

It is well known that an important role is played by the convexity of φ with respect to the last variable: when φ is convex, then $\mathcal{I}$ is said to be a polyconvex functional; if not, then $\mathcal{I}$ is nonpolyconvex. The polyconvex case $\varphi = \varphi(t)$ has been studied by Dacorogna [5] and the nonpolyconvex case by Mascolo and Schianchi [14] and Cellina and Zagatti [4].

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In order to solve (1.2) our strategy is the following: The first step is to look for solutions to the following problem (from now on referred to as the *auxiliary* problem)

$$\min\left\{\mathcal{J}(v)=\int_\Omega \varphi(x,v(x))\,dx\,:\,v\in L^1(\Omega),\quad v>0\text{ a.e.},\quad \int_\Omega v(x)\,dx=|\Omega|\right\}, \tag{1.3}$$

where $|\Omega|$ stands for the N-dimensional Lebesgue measure of Ω. Then, if v is a solution to (1.3), the second step is to solve in $W^{1,N}(\Omega)$ the boundary value problem

$$\begin{cases}\det Du(x)=v(x) & \text{for a.e. } x \text{ in } \Omega,\\ u(x)=x & \text{on } \partial\Omega.\end{cases} \tag{1.4}$$

A solution u to (1.4) is a solution to (1.2), too. In fact, if $w\in W^{1,N}(\Omega)$, $w(x)=x$ on $\partial\Omega$, then $\det Dw\in L^1(\Omega)$ and $\int_\Omega \det Dw(x)\,dx=|\Omega|$; therefore, if $\det Dw>0$ a.e., then

$$\mathcal{I}(u)=\mathcal{J}(v)\le\mathcal{J}(\det Dw)=\mathcal{I}(w)\,.$$

Following the above scheme, Mascolo in [13] proves the existence of minimizers of (1.2) for smooth domains Ω and $\varphi\in C^2(\overline{\Omega}\times(0,+\infty))$ strictly convex in the last variable.

As far as problem (1.3) is concerned, Ekeland and Temam in [8] prove a relaxation result and Ball and Knowles in [1] obtain an attainment result with the tool of the Young measures; see also Friesecke [10] for related results. The boundary value problem (1.4) may have no solution unless v is sufficiently regular. For instance, the simple continuity of v is not a sufficient condition to get Lipschitz solutions; see the counterexamples independently given by Burago and Kleiner [2] and McMullen [15]. Thus, also the regularity properties of minimizers of the auxiliary problem have to be studied. The pioneering papers on (1.4) are due to Moser [16] and Dacorogna and Moser [7]. In particular, in [7] the authors prove that if v is in $C^{k,\alpha}(\overline{\Omega})$, $k\ge 0$, and $\partial\Omega\in C^{k+3,\alpha}$, then there exists a diffeomorphism of class $C^{k+1,\alpha}(\overline{\Omega})$ solution to (1.4). Later results are due to Rivière and Ye, who prove in [18, Theorem 4] the existence of a bi-Lipschitz homeomorphism u solution to (1.4) under less restrictive assumptions on Ω with v satisfying a Dini-type continuity property. In [19] Ye proves existence results in the framework of the Sobolev spaces.

The plan of the paper is the following. In section 2 we introduce a class of open sets, invariant under bi-Lipschitz homeomorphisms, which is slightly larger than that of open sets with Lipschitz boundaries; see Definition 2.1. In Theorem 2.4 we state the existence of Lipschitz solutions to (1.4) with Ω in this class of open sets and Hölder continuous datum v. It is a variant of the above-cited Theorem 4 in [18], and in the appendix we give the details of the proof. In section 3 we deal with polyconvex functionals. We consider the class of functions φ strictly convex in the last variable satisfying, as a substitute for the growth conditions,

$$\lim_{t\to 0^+} D_t\varphi(x,t)=\lambda_0 \text{ with } \lambda_0\in\mathbb{R}\cup\{-\infty\},\qquad \lim_{t\to+\infty} D_t\varphi(x,t)=+\infty, \tag{1.5}$$

uniformly with respect to x. In Proposition 3.1 we prove that a unique solution v to (1.3) exists and that v is in $L^\infty(\Omega)$. In Proposition 3.5, under more regularity assumptions on φ, we prove that v is Hölder continuous. Therefore, the Lipschitz solution u to (1.4), which exists by Theorem 2.4, is a minimizer of (1.2); see Theorem 3.6. In section 4 we deal with a function φ nonconvex with respect to t, satisfying (1.5).

Denoting φ^{**} the convex envelope of φ with respect to t, we assume that there exist $\alpha, \beta \in L^\infty(\Omega)$, $\beta(x) > \alpha(x)$, $\inf \alpha > 0$ such that for every $x \in \Omega$,

$$t \mapsto \varphi^{**}(x,t) \text{ is affine in } [\alpha(x), \beta(x)]$$

and

$$\varphi(x,\cdot) \equiv \varphi^{**}(x,\cdot) \text{ and } \varphi(x,\cdot) \text{ is strictly convex} \qquad \text{in } (0,\alpha(x)] \text{ and } [\beta(x),+\infty).$$

Under these assumptions in Theorem 4.1 we prove the existence of a bounded solution v to the auxiliary problem (1.3). In section 5 under regularity assumptions on φ we get that v is piecewise Hölder continuous; see Theorem 5.2. In section 6 first we prove that if in (1.4) the datum v is piecewise Hölder continuous, there exists a Lipschitz solution; see Proposition 6.2. Then, solving (1.4) with v the piecewise Hölder continuous solution to the auxiliary problem, in Theorems 6.3 and 6.4 we get a Lipschitz continuous minimizer of functional (1.1). In section 7 we consider special classes of nonpolyconvex functionals. First we consider the class of functionals with a nonconvex φ satisfying $\varphi(x,\alpha(x)) = \varphi(x,\beta(x)) = 0$. This class has been considered by Zagatti [20] (see also Celada and Perrotta [3] for the case $\varphi(x,u,t)$) with the assumption $\int_\Omega \alpha(x)\,dx < |\Omega| < \int_\Omega \beta(x)\,dx$. In [20] and [3] the attainment result is proved using different arguments: the Baire category method and the convex integration method, respectively. Theorems 7.1 and 7.2 are attainment results including the cases $\int_\Omega \alpha\,dx \ge |\Omega|$ and $\int_\Omega \beta(x)\,dx \le |\Omega|$. Theorem 7.4 deals with a *perturbation* of these functionals; see problem (7.2). We conclude the section considering functionals with φ satisfying the structure condition $\varphi(x,t) = \tilde\varphi(|x|,t)$. In this case the existence of bounded radial solutions to (1.3) directly implies the existence of Lipschitz solutions to (1.4).

2. Notation and preliminary results. In the following if Ω is a measurable subset of $\mathbb{R}^N$, then $|\Omega|$ stands for its N-dimensional Lebesgue measure. We write Q in place of $(0,1)^N$ and $B_r(x)$ denotes the ball in $\mathbb{R}^N$ with center at x and radius r. If $\varphi : \Omega \times (0,+\infty) \to [0,+\infty)$, then φ^{**} is the convex envelope of φ with respect to the second variable, i.e., $t \mapsto \varphi^{**}(x,t)$ is the greatest convex function lower than $t \mapsto \varphi(x,t)$. For the sake of simplicity we write $\varphi(x,\cdot)$ instead of $t \mapsto \varphi(x,t)$,

$$D_t^- \varphi(x,s) := \lim_{t\to s^-} \frac{\varphi(x,t)-\varphi(x,s)}{t-s}, \qquad D_t^+ \varphi(x,s) := \lim_{t\to s^+} \frac{\varphi(x,t)-\varphi(x,s)}{t-s},$$

and $\partial\varphi(x,s) := \{d \in \mathbb{R} : \varphi(x,t) \ge \varphi(x,s) + d(t-s) \text{ for every } t \in (0,+\infty)\}$.

We define a class of bounded open subsets of $\mathbb{R}^N$.

DEFINITION 2.1. *We say that a bounded open set Ω of $\mathbb{R}^N$ is of class (L) if $\overline{\Omega}$ has a covering of finitely many open sets Ω_j such that for every j there exists a bi-Lipschitz homeomorphism $\psi_j : \overline{\Omega_j \cap \Omega} \to \overline{Q}$ satisfying*

(a) *$\psi_j(\overline{\Omega}_j \cap \partial\Omega) = \{0\} \times [0,1]^{N-1}$, whenever $\overline{\Omega}_j \cap \partial\Omega$ is not empty;*

(b) *$\det D\psi_j$ is Lipschitz continuous and there exists $A \ge 1$ such that $\frac{1}{A} \le \det D\psi_j \le A$.*

The above definition describes a larger class than that of open sets with Lipschitz boundary, i.e., with the boundary which locally is the graph of a Lipschitz function. This result can be proved in a way similar to that of Proposition A.1 in [7].

LEMMA 2.2. *If a bounded open set Ω of $\mathbb{R}^N$ has a Lipschitz boundary, then it is of class (L).*

An easy consequence of the chain rule for Lipschitz functions is that Definition 2.1 is invariant under bi-Lipschitz homeomorphisms.

LEMMA 2.3. *Let* $u_0 : \mathbb{R}^N \to \mathbb{R}^N$ *be a bi-Lipschitz homeomorphism, with* $\det Du_0$ *Lipschitz continuous,* $\frac{1}{A} \leq \det Du_0 \leq A$ *for some* A*. If* Ω *is of class* (L)*, then* $u_0(\Omega)$ *is of class* (L)*, too.*

On the contrary, there are examples of bounded open sets of $\mathbb{R}^N$ with Lipschitz boundary which are mapped by a bi-Lipschitz homeomorphism $u : \mathbb{R}^N \to \mathbb{R}^N$ onto sets with a not (Lipschitz) continuous boundary; see, e.g., [11, pp. 8–9]. Therefore, the converse of Lemma 2.2 is not true.

Now, we state an existence result of Lipschitz solutions to

$$\begin{cases} \det Du = f & \text{in } \Omega, \\ u(x) = x & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

with f Hölder continuous.

THEOREM 2.4. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded connected open set of class* (L)*. Let* f *be a Hölder continuous function,* $\inf f > 0$*,* $\int_\Omega f(x)\,dx = |\Omega|$*. Then there exists a bi-Lipschitz homeomorphism* $u : \overline{\Omega} \to \overline{\Omega}$ *solution to* (2.1)*.*

A similar result is proved in [18, Theorem 4], with a weaker assumption on v, which is assumed to satisfy a Dini-type continuity property, and a regular domain Ω. In [18] the proof is given for cubes only. The proof of Theorem 2.4, based upon the application to open sets of class (L) of the partition method due to Moser [16], is in the appendix.

3. Polyconvex problems: An attainment result. In this section we consider the variational problem

$$\min\left\{\int_\Omega \psi(x, \det Du(x))\,dx \;:\; u \in W^{1,N}(\Omega, \mathbb{R}^N), \quad \det Du > 0 \text{ a.e.}, \quad u(x) = x \text{ on } \partial\Omega\right\}, \tag{3.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with a Lipschitz boundary and $\psi : \Omega \times (0, +\infty) \to [0, +\infty)$ is a continuous function.

To get solutions to (3.1), we first consider the following variational problem:

$$\min\left\{\int_\Omega \psi(x, v(x))\,dx \;:\; v \in L^1(\Omega), \quad v > 0 \text{ a.e.}, \quad \int_\Omega v(x)\,dx = a\right\}, \quad a > 0. \tag{3.2}$$

As far as the problem (3.2) is concerned, the Lipschitz regularity of the boundary of Ω can be dropped.

We prove that there exists a (unique) bounded solution to (3.2) if

(H1) $t \mapsto \psi(x,t)$ is strictly convex for all $x \in \Omega$;

(H2) there exists $\lambda_0 \in \mathbb{R} \cup \{-\infty\}$ such that

$$\lim_{t \to 0^+} D_t^+ \psi(x,t) = \lambda_0, \qquad \lim_{t \to +\infty} D_t^- \psi(x,t) = +\infty, \quad \text{uniformly in } x.$$

PROPOSITION 3.1. *Assume that* $\psi : \Omega \times (0, +\infty) \to [0, +\infty)$ *is a continuous function satisfying* (H1) *and* (H2)*. Then for every* $\lambda > \lambda_0$ *there exists a unique* $u_\lambda \in L^\infty(\Omega)$*,* $\inf u_\lambda > 0$ *such that*

$$\lambda \in \partial\psi(x, u_\lambda(x)) \quad \forall\, x \in \Omega. \tag{3.3}$$

Moreover, there exists $\lambda_a > \lambda_0$ *such that* u_{λ_a} *is the unique solution to* (3.2).

Proof. We proceed as follows: At first we prove that for every $\lambda > \lambda_0$ there exists a function u_λ such that (3.3) holds. Then, we prove that u_λ is in $L^\infty(\Omega)$, $\inf u_\lambda > 0$, and there exists λ_a such that $\int_\Omega u_{\lambda_a}\, dx = a$. Thus, it turns out that u_{λ_a} is a solution to (3.2) and it is unique, because of the strict convexity of the functional.

Step 1. *The definition of* u_λ. Fixing $x \in \Omega$, we define the sets

$$C(x) := \left\{s \in (0,+\infty) \,:\, D_t^-\psi(x,s) < D_t^+\psi(x,s)\right\}, \quad \Omega_C := \{x \in \Omega \,:\, C(x) \neq \emptyset\}.$$

Notice that $\partial\psi(x,s) = [D_t^-\psi(x,s), D_t^+\psi(x,s)]$ for all $(x,s) \in \Omega \times (0,+\infty)$.

Suppose that $x \in \Omega \setminus \Omega_C$. From (H1) and the definition of Ω_C, the function $D_t\psi(x,\cdot) : (0,+\infty) \to (\lambda_0,+\infty)$ is well defined, continuous, and strictly increasing. Moreover, it is a surjective function because of (H2). Let $u(x,\cdot)$ be its inverse function, i.e., $u(x,\cdot) : (\lambda_0,+\infty) \to (0,+\infty)$ is such that $u(x,\lambda)$ (from now on denoted by $u_\lambda(x)$) is the unique positive number such that $\lambda = D_t\psi(x,u_\lambda(x))$. $u(x,\cdot)$ is a well defined, strictly increasing, and continuous function.

Now let us consider $x \in \Omega_C$. From (H1), $C(x)$ is (at most) a countable set, so that we denote $C(x) = \{t_n(x)\}_{n\in J(x)}$, where $J(x) \subseteq \mathbb{N}$. As in the above case, if $\lambda \notin \cup_{n\in J(x)}\partial\psi(x,t_n(x))$, we define $u_\lambda(x)$ as the unique positive number such that $D_t\psi(x,u_\lambda(x)) = \lambda$. If instead $\lambda \in \partial\psi(x,t_n(x))$ for some $n \in J(x)$, then we set $u_\lambda(x) = t_n(x)$. Notice that if $u_\lambda(x)$ is chosen greater (less) than $t_n(x)$, then $\lambda < D_t^-\psi(x,u_\lambda(x))$ ($\lambda > D_t^+\psi(x,u_\lambda(x))$). It is easy to prove that for each $x \in \Omega_C$ the function $u(x,\cdot) : (\lambda_0,+\infty) \to (0,+\infty)$ is well defined, increasing, and continuous.

Thus, $u_\lambda : \Omega \to (0,+\infty)$ is the unique function satisfying (3.3) and it is measurable, since

$$\{x \in \Omega \,:\, u_\lambda(x) < t\} = \left\{x \in \Omega \,:\, D_t^-\psi(x,t) > \lambda\right\}$$

and $D_t^-\psi(x,t) = \sup_{h<0}(\psi(x,t+h)-\psi(x,t))/h$. By the second limit in (H2) for every $\lambda > \lambda_0$ there exists $R > 0$ such that $D_t^-\psi(x,R) > \lambda$ for every $x \in \Omega$, which implies $u_\lambda(x) < R$ for every $x \in \Omega$. In fact, if $u_\lambda(x) \geq R$ for some x, then by the convexity of ψ with respect to the second variable it would be $D_t^-\psi(x,R) \leq D_t^-\psi(x,u_\lambda(x))$ and by (3.3) we would obtain $D_t^-\psi(x,R) \leq \lambda$, which is a contradiction. Thus, u_λ is in $L^\infty(\Omega)$. The first limit in (H2) implies that for each $\lambda > \lambda_0$ there exists $c(\lambda) > 0$ such that $\sup_{y\in\Omega} D_t^+\psi(y,t) < \lambda$ for every $t < c(\lambda)$. Therefore, it cannot be $u_\lambda(x) < c(\lambda)$, because $\lambda \leq D_t^+\psi(x,u_\lambda(x))$, so that $\inf u_\lambda > 0$.

Step 2. *The definition of* λ_a. Define $\Psi : (\lambda_0,+\infty) \to (0,+\infty)$, $\Psi(\lambda) := \int_\Omega u_\lambda(x)\, dx$, where $u_\lambda(x) = u(x,\lambda)$ is defined as in Step 1. By the monotonicity of u with respect to λ, Ψ is increasing. It holds true that $\lim_{\lambda\to\lambda_0^+} u_\lambda(x) = 0$. In fact, suppose that $\lim_{\lambda\to\lambda_0^+} u_\lambda(x) = \delta(x) > 0$. By (H1), the first limit in (H2), and (3.3), we get

$$\lambda_0 < D_t^-\psi(x,\delta(x)) \leq D_t^-\psi(x,u_\lambda(x)) \leq \lambda\,.$$

Therefore, letting λ go to λ_0^+ we get a contradiction. Analogously it can be proved that $\lim_{\lambda\to+\infty} u_\lambda(x) = +\infty$. Hence,

$$\lim_{\lambda\to\lambda_0^+} \Psi(\lambda) = 0, \qquad \lim_{\lambda\to+\infty} \Psi(\lambda) = +\infty. \tag{3.4}$$

From the previous step $\lambda \mapsto u_\lambda(x)$ is continuous and increasing for all x and $u_\lambda \in L^\infty(\Omega)$ for all λ, and therefore Ψ is a continuous function. Thus, there exists $\lambda_a > \lambda_0$

such that $\Psi(\lambda_a) = a$. We claim that u_{λ_a} is a solution to (3.2). In fact, from (H1) and (3.3) for every $w \in L^1(\Omega)$ such that $w > 0$ and $\int_\Omega w(x)\,dx = a$, we have that

$$\psi(x, w(x)) \geq \psi(x, u_{\lambda_a}(x)) + \lambda_a\big(w(x) - u_{\lambda_a}(x)\big) \quad \forall\, x \in \Omega.$$

Thus,

$$\int_\Omega \psi(x, w(x))\,dx \geq \int_\Omega \psi(x, u_{\lambda_a}(x))\,dx + \lambda_a \int_\Omega (w(x) - u_{\lambda_a}(x))\,dx$$
$$= \int_\Omega \psi(x, u_{\lambda_a}(x))\,dx\,. \qquad \square$$

REMARK 3.2. *The growth conditions*

$$\lim_{t\to 0^+} \inf_{y\in\Omega} \psi(y,t) = +\infty\,, \qquad \lim_{t\to+\infty} \inf_{y\in\Omega} \frac{\psi(y,t)}{t} = +\infty$$

imply (H2)*. If the first limit in* (H2) *is not uniform with respect to x, then maybe* $\inf u_\lambda = 0$. *Moreover, the proof of Proposition* 3.1 *works also if we replace* $\lim_{t\to+\infty} D_t^- \psi(x,t) = +\infty$ *with the more general*

$$\lim_{t\to+\infty} D_t^- \psi(x,t) = \lambda_\infty, \qquad \lambda_\infty \in \mathbb{R} \cup \{+\infty\}.$$

It is easy to prove the following refinement of Proposition 3.1.

PROPOSITION 3.3. *Let $\psi : \Omega \times (0,+\infty) \to [0,+\infty)$ be a continuous function, differentiable with respect to the last variable, $D_t\psi \in C(\Omega \times (0,+\infty))$. If* (H1) *and* (H2) *hold, then the functions u_λ in Proposition* 3.1 *are continuous for every $\lambda > \lambda_0$.*

Proof. For every $\lambda > \lambda_0$ let $u_\lambda \in L^\infty(\Omega)$ be as in Proposition 3.1. u_λ is lower semicontinuous. In fact, if

$$\liminf_{x\to x_0} u_\lambda(x) < \alpha < u_\lambda(x_0)\,, \tag{3.5}$$

then (H1) and (3.3) imply $D_t\psi(x_0, \alpha) < \lambda$. By continuity of $D_t\psi$ there exists $\delta > 0$ such that $D_t\psi(x, \alpha) < \lambda$ for every $x \in (x_0 - \delta, x_0 + \delta)$. Then, from (3.3) again we have that $D_t\psi(x, \alpha) < D_t\psi(x, u_\lambda(x))$ for every $x \in (x_0 - \delta, x_0 + \delta)$, which implies $\alpha < u_\lambda(x)$, in contradiction with (3.5). Analogously the upper semicontinuity of u_λ can be proved. $\square$

To get Hölder continuous solutions to (3.2) we require more regularity on ψ:

(H3) there exists $0 < \sigma \leq 1$ such that for every compact $K \subset (0,+\infty)$ and for every $t \in K$ the function $x \mapsto D_t\psi(x,t)$ is of class $C^{0,\sigma}(\Omega)$ with $[D_t\psi(\cdot,t)]_{0,\sigma} \leq k_K$;

(H4) for every $m > 0$ there exists $c_m > 0$ such that

$$\psi(x,t) \geq \psi(x,s) + D_t\psi(x,s)(t-s) + c_m|t-s|^{2+\varepsilon}$$

for every $t > s \geq m$, for every $x \in \Omega$, and for some $\varepsilon \geq 0$.

REMARK 3.4. *Assumption* (H4) *is equivalent to assuming that for every $m > 0$ there exists $\tilde{c}_m > 0$ such that*

$$D_t\psi(x,t) - D_t\psi(x,s) \geq \tilde{c}_m|t-s|^{1+\varepsilon} \quad \forall\, t > s \geq m \quad \forall\, x \in \Omega. \tag{3.6}$$

Roughly speaking, if $\psi \in C^2$ satisfies (H4)*, then $D_{tt}\psi$ may vanish provided that a suitable growth near the zeros is satisfied; see* (3)(a) *below.*

Notice that if ψ_0 satisfies (H4) *and $\psi_1 = \psi_1(x,t)$ is such that $\psi_1(x,\cdot)$ is convex and C^1, then $\psi = \psi_0 + \psi_1$ satisfies* (H4)*, too. Examples of functions ψ_0 satisfying* (H4) *are as follows.*

(1) $\psi_0(t) := (1+t^2)^{p/2}$, $p \geq 2$. *See* [9] *for details.*
(2) $\psi_0(x,t) := |t-a(x)|^p$ *with* $a : \Omega \to \mathbb{R}$ *and* $p \geq 2$.
(3) $\psi_0 : \overline{\Omega} \times (0,+\infty) \to [0,+\infty)$ *of class* C^2, *strictly convex with respect to* t *such that for every* x *there exist at most finitely many positive numbers* $\{s_i(x)\}$ *such that* $D_{tt}\psi_0(x, s_i(x)) = 0$ *and the following hold:*
   (a) *there exist* $\varepsilon, c > 0$ *such that* $D_{tt}\psi_0(x,t) \geq c|t - s_i(x)|^{\varepsilon}$ *for every* t *in a neighborhood of* $s_i(x)$;
   (b) *there exists* $M > 0$ *such that* $\inf\{D_{tt}\psi_0(x,t) : (x,t) \in \Omega \times [M,+\infty)\} > 0$.

Proposition 3.5. *Let* $\psi : \Omega \times (0,+\infty) \to [0,+\infty)$ *be a continuous function, differentiable with respect to the last variable, satisfying* (H1)–(H4). *Then for each* $\lambda > \lambda_0$, *the function* u_λ *in Proposition* 3.1 *is in* $C^{0,\sigma/(1+\varepsilon)}(\Omega)$. *In particular, for every* $a > 0$ *the unique solution* u_{λ_a} *to* (3.2) *is Hölder continuous.*

Proof. Fix λ and let u_λ, from now on referred to as u, be the correspondent function as described in Proposition 3.1. From the strict convexity of ψ with respect to the last variable and since $\lambda = D_t\psi(x, u(x))$ for every $x \in \Omega$ it is easy to check that u is γ-Hölder continuous with Hölder constant $[u]_\gamma$ if and only if

$$D_t\psi\big(y, u(x) + [u]_{0,\gamma}|x-y|^{\gamma}\big) - D_t\psi(x, u(x)) \geq 0 \quad \forall\, x, y \in \Omega. \tag{3.7}$$

Fix $x, y \in \Omega$. By (H4) and (3.6) there exist $\varepsilon \geq 0$ and $\tilde{c} > 0$ such that

$$D_t\psi(x,t) - D_t\psi(x,s) \geq \tilde{c}(t-s)^{1+\varepsilon} \quad \forall\, t > s \geq \inf u > 0 \quad \forall\, x \in \Omega. \tag{3.8}$$

Consider the compact interval $K = [\inf u, \|u\|_\infty]$ and let s and t be equal to $u(x)$ and $u(x) + (\frac{k}{\tilde{c}}|x-y|^{\sigma})^{1/(1+\varepsilon)}$, respectively, with σ and k_K as in (H3). Using (3.8) and (H3) to estimate $D_t\psi(y,t) - D_t\psi(y,s)$ and $D_t\psi(y,s) - D_t\psi(x,s)$, respectively, we get

$$D_t\psi(y,t) - D_t\psi(x,s) = D_t\psi(y,t) - D_t\psi(y,s) + D_t\psi(y,s) - D_t\psi(x,s) \geq 0.$$

Then u is γ-Hölder continuous with $\gamma = \frac{\sigma}{1+\varepsilon}$.

Thus, for fixed $a > 0$, the solution u_{λ_a} to (3.2), which exists by Proposition 3.1, is Hölder continuous. □

Now we are ready to state an existence result of Lipschitz solutions to the polyconvex problem (3.1).

Theorem 3.6. *Suppose that* Ω *is a bounded open subset of* $\mathbb{R}^N$ *with Lipschitz boundary and let* $\psi : \Omega \times (0,+\infty) \to [0,+\infty)$ *be a continuous function, differentiable with respect to the last variable, satisfying* (H1)–(H4). *Then there exists a Lipschitz continuous solution to* (3.1).

Proof. Set $a = |\Omega|$ and consider the variational problem (3.2). From Propositions 3.1 and 3.5 such a problem has a (unique) solution $u_{\lambda_a} \in C^{0,\gamma}(\Omega)$, $\gamma > 0$, and $\inf u_{\lambda_a} > 0$. Hence, from Theorem 2.4 there exists a bi-Lipschitz homeomorphism u solving

$$\begin{cases} \det Du = u_{\lambda_a} & \text{in } \Omega, \\ u(x) = x & \text{on } \partial\Omega, \end{cases}$$

and u is a solution to (3.1), too. □

4. Nonpolyconvex problems: Attainment result for the auxiliary problem. In this section we consider the variational problem

$$\min\left\{\int_\Omega \varphi(x, v(x))\,dx \,:\, v \in L^1(\Omega),\ v > 0 \text{ a.e.}, \quad \int_\Omega v(x)\,dx = a\right\}, \quad a > 0, \tag{4.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, and $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ is a continuous function, nonconvex with respect to the last variable t.

Let φ^{**} be the convex envelope of φ with respect to the second variable and define

$$\Omega_A := \{x \in \Omega : t \to \varphi(x,t) \text{ is not strictly convex}\}.$$

We assume that the following assumptions hold:

(K1) Ω_A is a (not empty) measurable set and there exist $\alpha, \beta \in L^\infty(\Omega_A)$, $\beta(x) > \alpha(x)$ for all x, $\inf \alpha > 0$, such that $\varphi(x, \cdot)$ and $\varphi^{**}(x, \cdot)$ both coincide and are strictly convex in $(0, \alpha(x)]$ and $[\beta(x), +\infty)$ for every $x \in \Omega_A$;

(K2) $\varphi^{**}(x, \cdot)$ is affine in $[\alpha(x), \beta(x)]$ for all $x \in \Omega_A$, i.e., for every $\alpha(x) \le t \le \beta(x)$,

$$\varphi^{**}(x,t) = h(x)t + q(x) \text{ with } h(x) = \frac{\varphi(x, \beta(x)) - \varphi(x, \alpha(x))}{\beta(x) - \alpha(x)};$$

(K3) there exists $\lambda_0 \in \mathbb{R} \cup \{-\infty\}$ such that

$$\lim_{t \to 0^+} D_t^+ \varphi(x,t) = \lambda_0, \quad \lim_{t \to +\infty} D_t^- \varphi(x,t) = +\infty, \quad \text{uniformly in } x.$$

THEOREM 4.1. *Assume* (K1), (K2), *and* (K3). *Then there exist* $\lambda_a > \lambda_0$ *and* $v_{\lambda_a} \in L^\infty(\Omega)$, $\inf v_{\lambda_a} > 0$ *such that*

(i) $v_{\lambda_a}(x) \notin (\alpha(x), \beta(x))$ *for every* $x \in \Omega_A$;

(ii) $\lambda_a \in \partial\varphi^{**}(x, v_{\lambda_a}(x))$ *for every* $x \in \Omega$;

(iii) $\int_\Omega v_{\lambda_a}(x)\, dx = a$.

In particular, v_{λ_a} *is a solution to* (4.1). *Moreover, if* $\Omega = B_1(0)$ *and* $\varphi(x,t) = \tilde{\varphi}(|x|, t)$, *then* v_{λ_a} *is a radial function.*

We postpone the proof of Theorem 4.1 to the following lemma.

LEMMA 4.2. *Let* O *be a bounded measurable subset of* $\mathbb{R}^N$. *Let* $\alpha, \beta \in L^1(O)$ *be such that* $\alpha(x) \le \beta(x)$ *for a.e.* x *and suppose*

$$\int_O \alpha(x)\, dx < \kappa < \int_O \beta(x)\, dx\,. \tag{4.2}$$

Then there exists $r > 0$ *such that* $\Theta : O \to \mathbb{R}$, $\Theta(x) := \alpha(x)$ *if* $x \in O \cap B_r(0)$ *and* $\Theta(x) := \beta(x)$ *else, satisfying* $\int_O \Theta(x)\, dx = \kappa$.

Proof. Let R be such that $O \subset B_R(0)$. Consider the functions $\theta_\rho : O \to \mathbb{R}$, $0 \le \rho \le R$, defined as follows: $\theta_0 := \beta$ and if $\rho \ne 0$, then $\theta_\rho(x) := \alpha(x)$, if $x \in O \cap B_r(0)$ and $\theta_\rho(x) := \beta(x)$ else. The continuity of $\rho \to \int_O \theta_\rho(x)\, dx$ and (4.2) imply that there exists $0 < r < R$ such that $\int_O \theta_r(x)\, dx = \kappa$. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. We divide the proof into three steps. In Step 1 we define a family of functions $v_\lambda^- : \Omega \to (0, +\infty)$, $\lambda > \lambda_0$, such that

$$v_\lambda^-(x) \notin (\alpha(x), \beta(x)) \quad \forall x \in \Omega_A \quad \forall \lambda > \lambda_0 \tag{4.3}$$

and

$$\lambda \in \partial\varphi^{**}\big(x, v_\lambda^-(x)\big) \quad \forall x \in \Omega \quad \forall \lambda > \lambda_0. \tag{4.4}$$

In Step 2 we define a function v_{λ_a} satisfying (i), (ii), and (iii). Finally, in Step 3 we consider the case $\varphi(x,t) = \tilde{\varphi}(|x|, t)$.

Step 1. *The definition of* v_λ^-. Let us define the function $\psi : \Omega\times(0,+\infty) \to [0,+\infty)$ such that $\psi \equiv \varphi$ in $(\Omega \setminus \Omega_A) \times (0,+\infty)$ and

$$\psi(x,t) := \begin{cases} \varphi(x,t) & \text{if } x \in \Omega_A,\ 0 < t \le \alpha(x), \\ \varphi(x,t+\beta(x)-\alpha(x)), & \\ -\varphi(x,\beta(x)) + \varphi(x,\alpha(x)) & \text{if } x \in \Omega_A,\ t > \alpha(x). \end{cases} \tag{4.5}$$

(K1) and (K2) imply that for every $x \in \Omega_A$

$$D_t^-\varphi(x,\alpha(x)) \le h(x) = \frac{\varphi(x,\beta(x)) - \varphi(x,\alpha(x))}{\beta(x)-\alpha(x)} \le D_t^+\varphi(x,\beta(x)) \tag{4.6}$$

and that ψ satisfies (H1). Moreover, for every $x \notin \Omega_A$ and every $t > 0$ we have $\partial\psi(x,t) = \partial\varphi(x,t) = \partial\varphi^{**}(x,t)$. If instead $x \in \Omega_A$, then

$$\partial\psi(x,t) = \begin{cases} \partial\varphi(x,t) & \text{if } \ 0 < t < \alpha(x), \\ \partial\varphi^{**}(x,\alpha(x)) \cup \partial\varphi^{**}(x,\beta(x)) & \text{if } \ t = \alpha(x), \\ \partial\varphi(x,t+\beta(x)-\alpha(x)) & \text{if } \ t > \alpha(x). \end{cases} \tag{4.7}$$

We claim that (K3) implies that ψ satisfies (H2).

The first limit in (K3) and the assumption $\inf\alpha > 0$ imply $\lim_{t\to 0^+} D_t^+\psi(x,t) = \lambda_0$, uniformly. Let us prove that ψ satisfies the property on the second limit in (H2). Since $\alpha, \beta \in L^\infty(\Omega_A)$, then for every $x \in \Omega$ and $t > \|\alpha\|_{L^\infty(\Omega_A)}$,

$$\begin{aligned} \inf_{y\in\Omega} D_t^-\varphi(y,t) &\le \min\Big\{ \inf_{y\in\Omega_A} D_t^-\varphi(y,t+\beta(y)-\alpha(y)), \inf_{y\in\Omega\setminus\Omega_A} D_t^-\varphi(y,t)\Big\} \\ &= \inf_{y\in\Omega} D_t^-\psi(y,t) \le D_t^-\psi(x,t) \le D_t^-\varphi(x,t+\|\beta-\alpha\|_{L^\infty(\Omega_A)}) \end{aligned}$$

so that by (K3) as t goes to $+\infty$, we get

$$\lim_{t\to+\infty} \inf_{y\in\Omega} D_t^-\psi(y,t) = \lim_{t\to+\infty} D_t^-\psi(x,t) = +\infty \quad \forall x \in \Omega.$$

Since ψ satisfies the assumptions of Proposition 3.1, then for every $\lambda > \lambda_0$ there exists $u_\lambda \in L^\infty(\Omega)$, $\inf u_\lambda > 0$, satisfying (3.3). Moreover, for every $x \in \Omega_A$,

$$\begin{aligned} u_\lambda(x) < \alpha(x) \quad &\text{if} \quad \lambda < D_t^-\varphi(x,\alpha(x)), \\ u_\lambda(x) = \alpha(x) \quad &\text{if} \quad \lambda \in [D_t^-\varphi(x,\alpha(x)), D_t^+\varphi(x,\beta(x))], \\ u_\lambda(x) > \alpha(x) \quad &\text{if} \quad \lambda > D_t^+\varphi(x,\beta(x)). \end{aligned} \tag{4.8}$$

Let us define $v_\lambda^- : \Omega \to (0,+\infty)$,

$$v_\lambda^-(x) := u_\lambda(x) + (\beta(x)-\alpha(x))\chi_{\{y\in\Omega_A\,:\,h(y)<\lambda\}}(x).$$

Since $u_\lambda \in L^\infty(\Omega)$ and $\alpha, \beta \in L^\infty(\Omega_A)$, then $v_\lambda^- \in L^\infty(\Omega)$. From (3.3), (4.6), (4.7), and (4.8) if $x \in \Omega_A$, the following implications hold:

- if $\lambda < D_t^-\varphi(x,\alpha(x))$, then $v_\lambda^-(x) = u_\lambda(x) < \alpha(x)$ and $\lambda \in \partial\psi(x,u_\lambda(x)) = \partial\varphi(x,v_\lambda^-(x))$;
- if $\lambda \in [D_t^-\varphi(x,\alpha(x)), h(x)]$, then $v_\lambda^-(x) = u_\lambda(x) = \alpha(x)$ and $\lambda \in \partial\varphi^{**}(x,\alpha(x))$;
- if $\lambda \in (h(x), D_t^+\varphi(x,\beta(x))]$, then $v_\lambda^-(x) = \beta(x)$ and $\lambda \in \partial\varphi^{**}(x,\beta(x))$;
- if $\lambda > D_t^+\varphi(x,\beta(x))$, then $v_\lambda^-(x) = u_\lambda(x) + \beta(x) - \alpha(x) > \beta(x)$ and $\lambda \in \partial\psi(x,u_\lambda(x)) = \partial\varphi(x,v_\lambda^-(x))$.

Thus (4.3) holds and

$$\lambda \in \partial\varphi^{**}\big(x, v_\lambda^-(x)\big) \tag{4.9}$$

for every $x \in \Omega_A$ and $\lambda > \lambda_0$. When $x \notin \Omega_A$, the equality $v_\lambda^-(x) = u_\lambda(x)$ and (3.3) imply (4.9). Therefore, (4.4) holds true.

Step 2. *The definition of* λ_a *and* v_{λ_a}. Let us define $\Phi : (\lambda_0, +\infty) \to (0, +\infty)$,

$$\Phi(\lambda) := \int_\Omega v_\lambda^-(x)\,dx = \int_\Omega \Big(u_\lambda(x) + (\beta(x) - \alpha(x))\chi_{\{y\in\Omega_A\,:\,h(y)<\lambda\}}(x)\Big)\,dx.$$

As in the proof of (3.4) we have that $\lim_{\lambda\to\lambda_0^+}\Phi(\lambda) = 0$ and $\lim_{\lambda\to+\infty}\Phi(\lambda) = +\infty$.

For each $\lambda > \lambda_0$, define $v_\lambda^+ : \Omega \to (0, +\infty)$,

$$v_\lambda^+(x) := u_\lambda(x) + (\beta(x) - \alpha(x))\chi_{\{y\in\Omega_A\,:\,h(y)\le\lambda\}}(x).$$

For every $\mu > \lambda_0$,

$$\lim_{\lambda\to\mu^-}\Phi(\lambda) = \Phi(\mu), \qquad \lim_{\lambda\to\mu^+}\Phi(\lambda) = \int_\Omega v_\mu^+(x)\,dx.$$

Thus, Φ is discontinuous at μ if and only if $|\{y \in \Omega_A \,:\, h(y) = \mu\}| > 0$.

Only one of the following cases is possible:

1. there exists $\lambda_a > \lambda_0$ such that $\Phi(\lambda_a) = a$;
2. there exists $\lambda_a > \lambda_0$ such that $\Phi(\lambda_a) < a = \lim_{\lambda\to\lambda_a^+}\Phi(\lambda)$;
3. there exists $\lambda_a > \lambda_0$ such that $\Phi(\lambda_a) < a < \lim_{\lambda\to\lambda_a^+}\Phi(\lambda)$.

Case 1. As proved in Step 1, $v_{\lambda_a}^-$ satisfies (i), (ii), and $\inf v_{\lambda_a}^- \ge \inf u_{\lambda_a} > 0$. Moreover, by definition of λ_a, (iii) holds. Thus, define $v_{\lambda_a} = v_{\lambda_a}^-$.

Case 2. As above, $v_{\lambda_a}^-$ satisfies (i), (ii), and $\inf v_{\lambda_a}^- \ge \inf u_{\lambda_a} > 0$. It is easy to check that a property analogous to (i) is satisfied by $v_{\lambda_a}^+$ and that $\inf v_{\lambda_a}^+ \ge \inf v_{\lambda_a}^- > 0$. By the very definition of $v_{\lambda_a}^+$ we have also $\int_\Omega v_{\lambda_a}^+\,dx = a$.

Let us prove that $\lambda_a \in \partial\varphi^{**}(x, v_{\lambda_a}^+(x))$ for every x. If $x \notin \Omega_A$ or if $x \in \Omega_A$ and $h(x) \ne \lambda_a$, then $v_{\lambda_a}^-(x) = v_{\lambda_a}^+(x)$ and the above inclusion follows. Suppose that $x \in \Omega_A$ and $h(x) = \lambda_a$. Then $v_{\lambda_a}^-(x) = \alpha(x) < \beta(x) = v_{\lambda_a}^+(x)$ and (K2) implies $\lambda_a \in \partial\varphi^{**}(x, \beta(x)) = \partial\varphi^{**}(x, v_{\lambda_a}^+(x))$.

We have so proved that $\lambda_a \in \partial\varphi^{**}(x, v_{\lambda_a}^+(x))$ for every $x \in \Omega$. Thus, define $v_{\lambda_a} := v_{\lambda_a}^+$.

Case 3. Define $O := \{x \in \Omega_A \,:\, \lambda_a = h(x)\}$ and $\kappa := a - \int_{\Omega\setminus O} v_{\lambda_a}^-(x)\,dx$. The assumption $\Phi(\lambda_a) < a < \lim_{\lambda\to\lambda_a^+}\Phi(\lambda)$ implies

$$\int_O \alpha(x)\,dx = \int_O v_{\lambda_a}^-(x)\,dx < \kappa < \int_\Omega v_{\lambda_a}^+(x)\,dx - \int_{\Omega\setminus O} v_{\lambda_a}^-(x)\,dx = \int_O \beta(x)\,dx.$$

From Lemma 4.2, there exists $\Theta : O \to \mathbb{R}$, $\Theta(x) \in \{\alpha(x), \beta(x)\}$ such that $\int_O \Theta(x)\,dx = \kappa$. Define $v_{\lambda_a} : \Omega \to \mathbb{R}$, $v_{\lambda_a}(x) = v_{\lambda_a}^-(x)$ if $x \notin O$ and $v_{\lambda_a}(x) = \Theta(x)$ else.

It is easy to prove that v_{λ_a} satisfies (i), (ii), (iii), and $\inf v_{\lambda_a} > 0$.

Since $\varphi \ge \varphi^{**}$, then for every $v \in L^1(\Omega)$ such that $v > 0$ a.e. and $\int_\Omega v\,dx = a$, we have that

$$\begin{aligned}\int_\Omega \varphi(x, v(x))\,dx &\ge \int_\Omega \varphi^{**}(x, v(x))\,dx \\ &\ge \int_\Omega \varphi^{**}(x, v_{\lambda_a}(x))\,dx + \lambda_a \int_\Omega (v(x) - v_{\lambda_a}(x))\,dx = \int_\Omega \varphi(x, v_{\lambda_a}(x))\,dx.\end{aligned} \tag{4.10}$$

Thus, v_{λ_a} is a solution to (4.1).

Step 3. *The case* $\varphi(x,t) = \tilde{\varphi}(|x|,t)$. Assume that Ω is the unit ball $B_1(0)$ and that φ has the radial structure $\varphi(x,t) = \tilde{\varphi}(|x|,t)$. It is easy to prove that $\varphi^{**}(x,t) = (\tilde{\varphi})^{**}(|x|,t)$ and that α, β, h are radial functions. Moreover, the sets Ω_A, $\{y \in \Omega_A : h(y) < \lambda\}$ and $\{y \in \Omega_A : h(y) = \lambda\}$ are symmetric sets with respect to the origin. If ψ is defined as in Step 1 above, then it immediately follows that $\psi(x,t) = \tilde{\psi}(|x|,t)$. Looking at the first step of the proof of Proposition 3.1, it turns out that u_λ, satisfying $\partial\psi(x, u_\lambda(x)) = \lambda$, is a radial function for all λ. All these facts allow us to conclude that whenever Cases 1 or 2 in Step 2 hold, i.e., $\Phi(\lambda_a) = a$ or $\Phi(\lambda_a) < a = \lim_{\lambda\to\lambda_a^+}\Phi(\lambda)$, respectively, then v_{λ_a} is a radial function. To prove that v_{λ_a} is radial in the third case it is sufficient to notice that the sets O, $O \cap B_r(0)$, and $O \setminus B_r(0)$ are symmetric with respect to the origin and consequently the function Θ is radial. ◻

5. Nonpolyconvex problems: Regularity result for the auxiliary problem. In this section we prove a regularity result for solutions to the nonconvex variational problem (4.1). Let Ω be a bounded open subset of $\mathbb{R}^N$ and let $\varphi : \Omega\times(0,+\infty) \to [0,+\infty)$ be a continuous function, differentiable with respect to the last variable, $D_t\varphi \in C^{0,\delta}(\Omega\times K)$, $0 < \delta \le 1$, for every compact K in $(0,+\infty)$ such that

(A1) there exist $\alpha, \beta \in C^{0,\delta}(\Omega)$, $\beta(x) > \alpha(x)$ for every x, $\inf \alpha > 0$ such that $\varphi(x,\cdot)$ and $\varphi^{**}(x,\cdot)$ both coincide and are strictly convex in $(0,\alpha(x)]$ and $[\beta(x),+\infty)$ for every $x \in \Omega$;

(A2) $t \to \varphi^{**}(x,t)$ is affine in $[\alpha(x),\beta(x)]$ for every $x \in \Omega$, i.e., for every $\alpha(x) \le t \le \beta(x)$,

$$\varphi^{**}(x,t) = h(x)t + q(x) \text{ with } h(x) = \frac{\varphi(x,\beta(x)) - \varphi(x,\alpha(x))}{\beta(x)-\alpha(x)}.$$

Moreover,

$$|\partial\{x : h(x) = \lambda\}| = 0 \quad \forall\, \lambda \in \mathbb{R};$$

(A3) there exists $\lambda_0 \in \mathbb{R}\cup\{-\infty\}$ such that

$$\lim_{t\to 0^+} D_t\varphi(x,t) = \lambda_0, \quad \lim_{t\to+\infty} D_t\varphi(x,t) = +\infty, \quad \text{uniformly in } x;$$

(A4) for every $m > 0$ there exists $c_m > 0$ such that

$$\varphi(x,t) \ge \varphi(x,s) + D_t\varphi(x,s)(t-s) + c_m|t-s|^{2+\varepsilon}$$

for every $s,t \ge m$ such that $s < t \le \alpha(x)$ or $\beta(x) \le s < t$ for every $x \in \Omega$ and some $\varepsilon \ge 0$.

The following result is in the same spirit of Lemma 4.2.

LEMMA 5.1. *Let O be an open set in $\mathbb{R}^N$. Let $\alpha,\beta \in L^1(O)$ be such that $\alpha(x) \le \beta(x)$ for a.e. x and suppose that*

$$\int_O \alpha(x)\,dx < \kappa < \int_O \beta(x)\,dx. \tag{5.1}$$

Then there exists a finite number of balls $B_{\rho_j}(y_j)$, $j = 1,\dots,m$, satisfying

(1) $B_{\rho_j}(y_j) \subset\subset O$, $j = 1,\dots,m$;

(2) $\overline{B_{\rho_i}(y_i)} \cap \overline{B_{\rho_j}(y_j)} = \emptyset$ *for every* $i \ne j$;

(3) $\int_O \Theta(x)\,dx = \kappa$,

where $\Theta(x) := \alpha(x)$ *if* $x \in \cup_{1\le j\le m} B_{\rho_j}(y_j)$ *and* $\Theta(x) := \beta(x)$ *else.*

Proof. Since O is open, there exist (at most) countably many pairwise disjoint balls $\{B_{R_j}(y_j)\}_{j\in J}$ in O, and a negligible set $\mathcal{N}$ such that $O = \mathcal{N} \cup (\bigcup_{j\in J} B_{R_j}(y_j))$. Without loss of generality we assume $J = \{1, 2, \dots, m\}$ if card $J = m \in \mathbb{N}$ and $J = \mathbb{N}$ if J is countable. For every $n \in J$, let us define the function $\theta_n : O \to \mathbb{R}$,

$$\theta_n(x) := \begin{cases} \alpha(x) & \text{if } x \in \bigcup_{1\le j\le n} B_{R_j}(y_j), \\ \beta(x) & \text{else.} \end{cases}$$

If J is finite, then (5.1) implies $\int_O \theta_m(x)\,dx < \kappa$. If $J = \mathbb{N}$, it is easy to check that $\lim_{n\to+\infty} \int_O \theta_n(x)\,dx < \kappa$; thus, there exists $m \in \mathbb{N}$ such that

$$\int_O \theta_m(x)\,dx = \int_{\cup_{1\le j\le m} B_{R_j}(y_j)} \alpha(x)\,dx + \int_{O\setminus\cup_{1\le j\le m} B_{R_j}(y_j)} \beta(x)\,dx < \kappa\,.$$

Aiming at (1) and (2), we slightly reduce the radius of the previously selected balls $\{B_{R_j}(y_j)\}_{1\le j\le m}$. This can easily be done by noticing that

$$\lim_{\varepsilon\to 0^+} \int_{\cup_{j=1}^m B_{R_j}(y_j)\setminus B_{R_j-\varepsilon}(y_j)} (\beta(x) - \alpha(x))\,dx = 0.$$

Thus, there exists $0 < \varepsilon < \min\{R_j : 1 \le j \le m\}$ such that

$$\int_{\cup_{1\le j\le m} B_{R_j-\varepsilon}(y_j)} \alpha(x)\,dx + \int_{O\setminus\cup_{1\le j\le m} B_{R_j-\varepsilon}(y_j)} \beta(x)\,dx < \kappa\,. \tag{5.2}$$

Set $R := \max\{R_j - \varepsilon : 1 \le j \le m\}$ and define $\theta : O \times [0, R] \to \mathbb{R}$, $\theta(x, 0) := \beta(x)$ and

$$\theta(x,\rho) := \begin{cases} \alpha(x) & \text{if } x \in \bigcup_{1\le j\le m} (B_{R_j-\varepsilon}(y_j) \cap B_\rho(y_j)), \\ \beta(x) & \text{else} \end{cases}$$

for every $\rho > 0$. From (5.2) we have that

$$\int_O \theta(x, R)\,dx < \kappa < \int_O \theta(x, 0)\,dx = \int_O \beta(x)\,dx.$$

Since $\rho \to \int_O \theta(x,\rho)\,dx$ is a continuous function, there exists $\overline{\rho}$ such that $\int_O \theta(x,\overline{\rho})\,dx = \kappa$. The claim of the theorem follows by defining $\Theta(x) := \theta(x,\overline{\rho})$ and $\rho_j := \min\{R_j - \varepsilon, \overline{\rho}\}$, $1 \le j \le m$. ∎

Let h be as in (A2). For every $\lambda > \lambda_0$ we define

$$\Omega_\lambda^+ := \{x : h(x) > \lambda\}, \quad \Omega_\lambda^- := \{x : h(x) < \lambda\}, \quad \Omega_\lambda^= := \{x : h(x) = \lambda\}. \tag{5.3}$$

Under (A1)–(A4) there exists a piecewise Hölder continuous solution to (4.1).

THEOREM 5.2. *Let* $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ *be a continuous function, differentiable with respect to the last variable,* $D_t\varphi(x, t)$ *in* $C^{0,\delta}(\Omega \times K)$ *for every compact* $K \subset (0, +\infty)$. *Suppose that* (A1)–(A4) *hold. Then, with fixed* $a > 0$ *there exist* $\lambda_a > \lambda_0$ *and* $v_{\lambda_a} \in L^\infty(\Omega)$, $\inf v_{\lambda_a} > 0$, *satisfying the following properties:*

(i) $D_t\varphi^{**}(x, v_{\lambda_a}(x)) = \lambda_a$ *for every* $x \in \Omega$;

(ii) $\int_\Omega v_{\lambda_a}(x)\,dx = a$;
(iii) v_{λ_a} *is Hölder continuous in* $\Omega^+_{\lambda_a} \cup \Omega^-_{\lambda_a}$;
(iv) $v_{\lambda_a}(x) < \alpha(x)$ *for all* $x \in \Omega^+_{\lambda_a}$ *and* $v_{\lambda_a}(x) > \beta(x)$ *for all* $x \in \Omega^-_{\lambda_a}$;
(v) *in* $\Omega^=_{\lambda_a}$ *either* $v_{\lambda_a} \equiv \alpha$ *or* $v_{\lambda_a} \equiv \beta$ *or*

$$v_{\lambda_a}(x) = \begin{cases} \alpha(x) & \text{if } x \in \bigcup_{1\le j\le m} B_{\rho_j}(y_j), \\ \beta(x) & \text{if } x \in \Omega^=_{\lambda_a} \setminus \bigcup_{1\le j\le m} B_{\rho_j}(y_j) \end{cases} \tag{5.4}$$

with $B_{\rho_j}(y_j) \subset\subset \operatorname{int}\Omega^=_{\lambda_a}$, $j = 1,\dots,m$ *such that* $\overline{B_{\rho_i}(y_i)} \cap \overline{B_{\rho_j}(y_j)} = \emptyset$ *if* $i \neq j$.

Moreover, v_{λ_a} *is a solution to* (4.1).

Proof. Let $\psi : \Omega \times (0,+\infty) \to [0,+\infty)$ be defined as

$$\psi(x,t) := \begin{cases} \varphi(x,t) & \text{if } \ 0 < t \le \alpha(x)\,,\ x \in \Omega, \\ \varphi(x, t+\beta(x)-\alpha(x)), & \\ -\varphi(x,\beta(x)) + \varphi(x,\alpha(x)) & \text{if } \ t > \alpha(x)\,,\ x \in \Omega. \end{cases} \tag{5.5}$$

It holds true that ψ is a continuous function, differentiable with respect to the last variable, satisfying (H1)–(H4) in section 3, with possibly different constants. By Proposition 3.5 for every $\lambda > \lambda_0$, there exists u_λ such that $u_\lambda \in C^{0,\gamma}(\Omega)$ for some $0 < \gamma \le 1$, $\inf u_\lambda > 0$, and

$$D_t\psi(x, u_\lambda(x)) = \lambda \quad \forall\, x \in \Omega. \tag{5.6}$$

Moreover (see (4.6) and (4.8)),

$$u_\lambda < \alpha \text{ in } \Omega^+_\lambda, \quad u_\lambda = \alpha \text{ in } \Omega^=_\lambda, \quad u_\lambda > \alpha \text{ in } \Omega^-_\lambda. \tag{5.7}$$

Let $\Phi : (\lambda_0, +\infty) \to \mathbb{R}$ be the left-continuous function defined as

$$\Phi(\lambda) := \int_\Omega \Big(u_\lambda(x) + (\beta(x) - \alpha(x))\chi_{\Omega^-_\lambda}(x) \Big)\, dx, \quad \lambda > \lambda_0.$$

We have three different cases:

1. there exists $\lambda_a > \lambda_0$ such that $\Phi(\lambda_a) = a$;
2. there exists $\lambda_a > \lambda_0$ such that $\Phi(\lambda_a) < a = \lim_{\lambda\to\lambda_a^+}\Phi(\lambda)$;
3. there exists $\lambda_a > \lambda_0$ such that $\Phi(\lambda_a) < a < \lim_{\lambda\to\lambda_a^+}\Phi(\lambda)$.

Let us consider the first two cases: since (A1)–(A3) imply (K1)–(K3), then by proceeding as in Theorem 4.1 there exists $v_{\lambda_a} \in L^\infty(\Omega)$, $\inf v_{\lambda_a} > 0$, which satisfies (i) and (ii). Moreover, if case 1 holds, then $v_{\lambda_a} := u_{\lambda_a} + (\beta - \alpha)\chi_{\{h<\lambda_a\}}$, i.e.,

$$v_{\lambda_a} := u_{\lambda_a} \text{ in } \Omega^+_{\lambda_a}, \quad v_{\lambda_a} := \alpha \text{ in } \Omega^=_{\lambda_a}, \quad v_{\lambda_a} := u_{\lambda_a} + \beta - \alpha \text{ in } \Omega^-_{\lambda_a};$$

if instead case 2 holds, then $v_{\lambda_a} := u_{\lambda_a} + (\beta - \alpha)\chi_{\{h\le\lambda\}}$, i.e.,

$$v_{\lambda_a} := u_{\lambda_a} \text{ in } \Omega^+_{\lambda_a}, \quad v_{\lambda_a} := \beta \text{ in } \Omega^=_{\lambda_a}, \quad v_{\lambda_a} := u_{\lambda_a} + \beta - \alpha \text{ in } \Omega^-_{\lambda_a}.$$

Therefore, from the Hölder continuity of α and β, (5.6) and (5.7) it follows that v_{λ_a} satisfies (iii), (iv), and (v). Moreover, reasoning as in (4.10) we get that v_{λ_a} is a solution to (4.1).

Suppose the third case holds. We define v_{λ_a} as in the proof of Theorem 4.1, but using Lemma 5.1 instead of Lemma 4.2. Precisely, since

$$\int_{\Omega^=_{\lambda_a}} \alpha(x)\,dx < \kappa < \int_{\Omega^=_{\lambda_a}} \beta(x)\,dx$$

with

$$\kappa := a - \int_{\Omega\setminus\Omega^{=}_{\lambda_a}} \Big(u_{\lambda_a}(x) + (\beta(x)-\alpha(x))\chi_{\Omega^{-}_{\lambda_a}}(x)\Big)\, dx,$$

then from Lemma 5.1 there exist m balls $B_{\rho_j}(y_j) \subset\subset \operatorname{int}\Omega^{=}_{\lambda_a}$, $j=1,\ldots,m$, $\overline{B_{\rho_i}(y_i)} \cap \overline{B_{\rho_j}(y_j)} = \emptyset$ for every $i \neq j$ such that $\Theta : \operatorname{int}\Omega^{=}_{\lambda_a} \to \mathbb{R}$,

$$\Theta := \alpha \quad \text{in } \textstyle\bigcup_{1\le j\le m} B_{\rho_j}(y_j), \qquad \Theta := \beta \quad \text{in } \operatorname{int}\Omega^{=}_{\lambda_a} \setminus \textstyle\bigcup_{1\le j\le m} B_{\rho_j}(y_j)$$

satisfies $\int_{\operatorname{int}\Omega^{=}_{\lambda_a}} \Theta(x)\, dx = \kappa$.

Define v_{λ_a} as follows:

$$v_{\lambda_a}(x) := \begin{cases} u_{\lambda_a}(x) & \text{if } x \in \Omega^{+}_{\lambda_a}, \\ \alpha(x) & \text{if } x \in \bigcup_{1\le j\le m} B_{\rho_j}(y_j), \\ \beta(x) & \text{if } x \in \Omega^{=}_{\lambda_a} \setminus \bigcup_{1\le j\le m} B_{\rho_j}(y_j), \\ u_{\lambda_a}(x) + \beta(x) - \alpha(x) & \text{if } x \in \Omega^{-}_{\lambda_a}. \end{cases}$$

We have that $v_{\lambda_a} \in L^\infty(\Omega)$, $\inf v_{\lambda_a} > 0$, and it satisfies (i)–(v). Moreover, v_{λ_a} is a solution to (4.1). □

6. Nonpolyconvex problems: Attainment result in a general setting.

In this section we consider the variational problem

$$\min\left\{ \int_\Omega \varphi(x, \det Du(x))\, dx \;:\; u \in W^{1,N}(\Omega,\mathbb{R}^N), \quad \det Du > 0 \text{ a.e.}, \; u(x) = x \text{ on } \partial\Omega \right\}, \tag{6.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary and $\varphi : \Omega \times (0,+\infty) \to [0,+\infty)$ is a nonconvex function with respect to the second variable.

Before stating an attainment result for (6.1), we need some preliminary results.

LEMMA 6.1. *Let Ω be a bounded open set with Lipschitz boundary and let $\overline{\Omega} = \cup_{i=1}^m \overline{\Omega}_i$ with $\{\Omega_i\}$ pairwise disjoint open connected sets with Lipschitz boundary.*

Consider $\alpha_i > 0$, $i = 1,\ldots,m$, with $\sum_{i=1}^m \alpha_i = |\Omega|$. Then there exists a bi-Lipschitz homeomorphism $u_0 : \overline{\Omega} \to \overline{\Omega}$ such that $\det Du_0 \in C^\infty(\overline{\Omega})$, $\inf \det Du_0 > 0$, and

$$u_0(x) = x \quad \text{on } \partial\Omega, \quad |u_0(\Omega_i)| = \alpha_i, \quad i = 1,\ldots,m. \tag{6.2}$$

Moreover, $u_0(\Omega_i)$ is an open set of class (L) for every i.

Proof. Fix $0 < \delta < \min\{\alpha_i/|\Omega_i| \;:\; i = 1,\ldots,m\}$. For every $1 \le i \le m$ let $\eta_i \in C_c^\infty(\Omega_i)$ be such that $\int_{\Omega_i} \eta_i(x)\, dx = 1$. Define

$$f(x) = \delta + \sum_{i=1}^m (\alpha_i - \delta|\Omega_i|)\eta_i(x), \quad x \in \overline{\Omega}.$$

Hence, $f \in C^\infty(\overline{\Omega})$, $\inf f > 0$, $\int_{\Omega_i} f(x)\, dx = \alpha_i$ for every i, and $\int_\Omega f(x)\, dx = |\Omega|$. From Theorem 2.4 there exists a bi-Lipschitz homeomorphism $u_0 : \overline{\Omega} \to \overline{\Omega}$ such that

$$\det Du_0 = f \text{ in } \Omega, \qquad u_0(x) = x \text{ on } \partial\Omega.$$

Therefore,

$$|u_0(\Omega_i)| = \int_{\Omega_i} \det Du_0(x)\, dx = \int_{\Omega_i} f(x)\, dx = \alpha_i, \quad i = 1,\ldots,m;$$

moreover, Lemma 2.3 implies that $u_0(\Omega_i)$ is an open set of class (L) for each i. □

PROPOSITION 6.2. *Let Ω and Ω_i, $i = 1, \dots, m$, be as in Lemma 6.1. Suppose that $g_i : \overline{\Omega_i} \to [c_0, +\infty)$, with $c_0 > 0$, $i = 1, \dots, m$, are Hölder continuous functions satisfying*

$$\sum_{i=1}^{m} \int_{\Omega_i} g_i(x)\, dx = |\Omega|.$$

Then there exists a Lipschitz continuous function $u : \overline{\Omega} \to \overline{\Omega}$ such that

$$u(x) = x \text{ on } \partial\Omega, \quad \det Du(x) = g_i(x) \quad \forall x \in \Omega_i \quad \forall i = 1, \dots, m. \tag{6.3}$$

Proof. By Lemma 6.1 there exists a bi-Lipschitz homeomorphism $u_0 : \overline{\Omega} \to \overline{\Omega}$ such that

$$u_0(x) = x \text{ on } \partial\Omega, \qquad |u_0(\Omega_i)| = \int_{\Omega_i} g_i(x)\, dx$$

and $u_0(\Omega_i)$ is of class (L) for each $i = 1, \dots, m$. Moreover, $f := \det Du_0$ is of class $C^\infty(\overline{\Omega})$ and $\inf f > 0$. Since $\frac{g_i}{f} \circ u_0^{-1}$ is Hölder continuous in $\overline{u_0(\Omega_i)}$ and it satisfies

$$\int_{u_0(\Omega_i)} \frac{g_i}{f} \circ u_0^{-1}(y)\, dy = \int_{\Omega_i} g_i(x)\, dx = |u_0(\Omega_i)|,$$

then from Theorem 2.4 there exists a bi-Lipschitz homeomorphism $z_i : \overline{u_0(\Omega_i)} \to \overline{u_0(\Omega_i)}$ such that

$$\begin{cases} \det Dz_i = \dfrac{g_i}{f} \circ u_0^{-1} & \text{in } u_0(\Omega_i), \\ z_i(y) = y & \text{on } \partial u_0(\Omega_i). \end{cases}$$

Thus, $u_i = z_i \circ u_0$ is a Lipschitz homeomorphism such that

$$\begin{cases} \det Du_i = g_i & \text{in } \Omega_i, \\ u_i = u_0 & \text{on } \partial\Omega_i. \end{cases}$$

Hence, the Lipschitz continuous function $u : \overline{\Omega} \to \overline{\Omega}$ such that $u(x) = u_i(x)$ for every $x \in \overline{\Omega}_i$, $i = 1, \dots, m$, satisfies (6.3). □

We are in position to state an existence result for the nonpolyconvex problem (6.1). The sets Ω_λ^+, Ω_λ^-, and $\Omega_\lambda^=$ are defined in (5.3).

THEOREM 6.3. *Let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary and let $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ be a continuous function, differentiable with respect to the last variable, $D_t\varphi \in C^{0,\delta}(\Omega \times K)$, $0 < \delta \leq 1$, for every compact $K \subset (0, +\infty)$.*

Suppose that (A1)–(A4) *hold and assume that, for every $\lambda > \lambda_0$, Ω_λ^+, Ω_λ^-, and* $\operatorname{int} \Omega_\lambda^=$ *are either empty or connected open sets with Lipschitz boundary. Then the variational problem* (6.1) *has a Lipschitz continuous solution.*

Proof. From Theorem 5.2, applied with $a = |\Omega|$, there exist $\lambda_a > \lambda_0$ and a solution v_{λ_a} to (4.1) with $\inf v_{\lambda_a} > 0$. Throughout we write v instead of v_{λ_a}.

From Theorem 5.2 v is Hölder continuous in $\Omega_{\lambda_a}^+ \cup \Omega_{\lambda_a}^-$. If $\operatorname{int} \Omega_{\lambda_a}^=$ is empty, we get the thesis applying Proposition 6.2 with $\Omega_1 = \Omega_{\lambda_a}^+$, $\Omega_2 = \Omega_{\lambda_a}^-$, and replacing g_1 and g_2 with the continuous extension of v to $\Omega_{\lambda_a}^+$ and to $\Omega_{\lambda_a}^-$, respectively.

If $\operatorname{int}\Omega^{=}_{\lambda_a}$ is not empty, correspondingly to (v) of Theorem 5.2 we have to consider three cases.

If $v = \alpha$ in $\Omega^{=}_{\lambda_a}$, the thesis follows by applying Proposition 6.2 with $m = 3$, choosing $\Omega_1 = \Omega^{+}_{\lambda_a}$, $\Omega_2 = \Omega^{-}_{\lambda_a}$, $\Omega_3 = \operatorname{int}\Omega^{=}_{\lambda_a}$, and replacing, as above, g_1 and g_2 with the continuous extension of v to $\Omega^{+}_{\lambda_a}$ and $\Omega^{-}_{\lambda_a}$, respectively, and g_3 with α. Analogously, we proceed if $v = \beta$ in $\Omega^{=}_{\lambda_a}$, but defining $g_3 = \beta$.

Now suppose that (5.4) holds. In this case the thesis follows by Proposition 6.2 choosing $\Omega_1 = \Omega^{+}_{\lambda_a}$, $\Omega_2 = \Omega^{-}_{\lambda_a}$, $\Omega_3 = \operatorname{int}\Omega^{=}_{\lambda_a} \setminus \cup_{1\le j\le n} B_{\rho_j}(y_j)$, $\Omega_{3+i} = B_{\rho_i}(y_i)$ for every $i = 1, \dots, n$ and $g_1 = v$, $g_2 = v$, $g_3 = \beta$, $g_{3+i} = \alpha$, for every $i = 1, \dots, n$. ∎

With obvious changes in the proof above, we get the following theorem.

THEOREM 6.4. *Let Ω and φ be as in Theorem 6.3. Suppose that (A1)–(A4) hold and assume that for every $\lambda > \lambda_0$,*

$$\overline{\Omega^{+}_{\lambda}} = \bigcup_{i=1}^{h} \overline{A_i}, \quad \overline{\Omega^{-}_{\lambda}} = \bigcup_{i=h+1}^{k} \overline{A_i}, \quad \operatorname{int}\Omega^{=}_{\lambda} = \bigcup_{i=k+1}^{l} A_i \tag{6.4}$$

with A_i either empty or pairwise disjoint open connected sets with Lipschitz boundary. Then the variational problem (6.1) has a Lipschitz continuous solution.

REMARK 6.5. *The following are examples of sets Ω and functions $h : \Omega \to \mathbb{R}$ such that for every $\lambda \in \mathbb{R}$ (6.4) holds with either empty or disjoint open sets $\{A_i\}$ with Lipschitz boundary:*

(a) *Ω is a bounded and convex set and h is strictly convex in Ω and constant on $\partial\Omega$;*

(b) *$\Omega = B_1(0)$ and h is a radial function, $h(x) = \tilde{h}(|x|)$, with $\tilde{h}$ piecewise monotone, i.e., there exists $0 = s_0 < s_1 < \cdots < s_m = 1$ such that $\tilde{h}|_{[s_i,s_{i+1}]}$ is monotone for all i.*

7. Nonpolyconvex problems: Some special cases. In this section we consider particular classes of the variational problem (6.1), where Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary and $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ is a continuous function satisfying (A1) and (A2). We begin considering the case of functions φ such that h in (A2) is a constant. See [20] and [3] for related results.

THEOREM 7.1. *Let $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ be a continuous function satisfying (A1) and (A2) with h constant. If $\int_\Omega \alpha(x)\,dx \le |\Omega| \le \int_\Omega \beta(x)\,dx$, then (6.1) has a Lipschitz continuous solution.*

Proof. Consider the auxiliary problem (4.1) with $a = |\Omega|$. If $\int_\Omega \alpha(x)\,dx$ is equal to $|\Omega|$, then α solves (4.1). Then from Theorem 2.4 there exists a Lipschitz homeomorphism u solution to (2.1) with $f = \alpha$. Moreover, u is a solution of (6.1). The same argument works if $\int_\Omega \beta(x)\,dx$ is equal to $|\Omega|$. Of course in this case choose $f = \beta$.

Suppose $\int_\Omega \alpha(x)\,dx < |\Omega| < \int_\Omega \beta(x)\,dx$. Then using Lemma 5.1 with $O = \Omega$, we get that a Lipschitz continuous solution u to (4.1) exists with $u \equiv \alpha$ on pairwise disjoint balls $B_{\rho_j}(y_j) \subset\subset \Omega$, $j = 1, \dots, n$, and with $u \equiv \beta$ outside these balls. The thesis follows by Proposition 6.2 with $m = n + 1$, $\Omega_j = B_{\rho_j}(y_j)$, and $g_j = \alpha$ if $j = 1, \dots, m - 1$ and with $\Omega_m = \Omega \setminus \bigcup_{j=1}^{n} B_{\rho_j}(y_j)$, $g_m = \beta$. ∎

THEOREM 7.2. *Let $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ be a continuous function, differentiable with respect to the last variable, $D_t\varphi \in C^{0,\delta}(\Omega \times K)$, $0 < \delta \le 1$, for every compact $K \subset (0, +\infty)$. Suppose that (A1), (A2) with h constant, (A3), and (A4) hold. If $\int_\Omega \alpha(x)\,dx > |\Omega|$ or $\int_\Omega \beta(x)\,dx < |\Omega|$, then (6.1) has a Lipschitz continuous solution.*

Proof. Let $a = |\Omega|$. From Theorem 5.2 there exist $\lambda_a > \lambda_0$ and $v_{\lambda_a} \in L^\infty(\Omega)$ satisfying

$$v_{\lambda_a}(x) \notin (\alpha(x), \beta(x)), \quad D_t\varphi^{**}(x, v_{\lambda_a}(x)) = \lambda_a, \quad \int_\Omega v_{\lambda_a}(x)\, dx = |\Omega|. \tag{7.1}$$

(A1), (A2), and (A3) imply $h = D_t\varphi(x, \alpha(x)) = D_t\varphi(x, \beta(x))$ and the definition of $\{v_\lambda\}$ (see the proofs of Theorems 4.1 and 5.2) gives that $\lambda < h$ if and only if $v_\lambda(x) < \alpha(x)$ for all x, $\lambda > h$ if and only if $v_\lambda(x) > \beta(x)$ for all x. Therefore, if $\int_\Omega \alpha(x)\, dx > |\Omega|$, then $\lambda_a < h$ and $v_{\lambda_a}(x) < \alpha(x)$. Thus, using the notation in (5.3), $\Omega^+_{\lambda_a} = \Omega$. Analogously, if $\int_\Omega \beta(x)\, dx < |\Omega|$, then $\lambda_a > h$ and $v_{\lambda_a}(x) > \beta(x)$, so that $\Omega^-_{\lambda_a} = \Omega$. Therefore, Theorem 5.2 implies that v_{λ_a} is Hölder continuous in Ω. A Lipschitz continuous solution u to

$$\begin{cases} \det Du = v_{\lambda_a} & \text{in } \Omega, \\ u(x) = x & \text{on } \partial\Omega, \end{cases}$$

solution also to (6.1), exists because of Theorem 2.4. □

In Propositions 7.3 and 7.4 we deal with a variant of functionals considered above, precisely

$$\min\left\{\int_\Omega \Phi(x, \det Du(x))\, dx : u \in W^{1,N}(\Omega, \mathbb{R}^N),\ \det Du > 0 \text{ a.e.}, \quad u(x) = x \text{ on } \partial\Omega\right\} \tag{7.2}$$

with $\Phi(x,t) = \varphi(x,t) + f(x)t$.

PROPOSITION 7.3. *Let Ω be a bounded open convex set in $\mathbb{R}^N$ and let $\varphi : \Omega \times (0, +\infty) \to [0, +\infty)$ satisfy the assumptions of Theorem* 7.2 *with $\lambda_0 = -\infty$ in* (A3). *Suppose that $f : \Omega \to (0, +\infty)$ is a strictly convex function, constant on $\partial\Omega$. Then there exists a Lipschitz solution to* (7.2).

Proof. It is easy to see that Φ satisfies the assumptions of Theorem 6.3. Since $\Phi^{**}(x,t) = \varphi^{**}(x,t) + f(x)t$ for every $x \in \Omega$, then in $(0, \alpha(x)]$ and in $[\beta(x), +\infty)$ we have that $\Phi(x,\cdot) = \Phi^{**}(x,\cdot)$. Moreover, for every $t \in [\alpha(x), \beta(x)]$ it holds true that $\Phi^{**}(x,t) = H(x)t + q(x)$ with $H(x) := \mu + f(x)$ and the superlevel, sublevel, and level sets of H satisfy the assumptions in Theorem 6.3. (A3) implies that $D_t\Phi(x,t) = D_t\varphi(x,t) + f(x)$ goes to $-\infty$ as $t \to -\infty$ and goes to $+\infty$ as $t \to +\infty$, uniformly with respect to x. The thesis easily follows from Theorem 6.3. □

From now on, Ω is the unit ball B in $\mathbb{R}^N$ centered at the origin.

PROPOSITION 7.4. *Let $\varphi : B \times (0, +\infty) \to [0, +\infty)$ satisfy the assumptions of Theorem* 7.2 *with $\lambda_0 = -\infty$ in* (A3). *Let $f \in C^{0,\gamma}([0,1])$, $0 < \gamma \le 1$, $f(s) > 0$ for every s, f piecewise monotone. Then there exists a Lipschitz continuous solution to* (7.2) *with $\Phi(x,t) = \varphi(x,t) + f(|x|)t$.*

Proof. Proceeding as in the proof of Proposition 7.3, the thesis easily follows from Remark 6.5(b) and from Theorem 6.4 applied to $\Phi(x,t) = \varphi(x,t) + f(|x|)t$. □

Now, we deal with one more class of nonpolyconvex functionals, characterized by an integrand φ with radial structure $\varphi(x,t) = \tilde{\varphi}(|x|, t)$. Precisely, we deal with the variational problem

$$\min\left\{\int_B \tilde{\varphi}(|x|, \det Du(x))\, dx \,:\, u \in W^{1,N}(B, \mathbb{R}^N),\ \det Du > 0 \text{ a.e.},\ u(x) = x \text{ on } \partial B\right\} \tag{7.3}$$

and $\tilde{\varphi} : [0, 1) \times (0, +\infty) \to [0, +\infty)$ is a continuous function.

THEOREM 7.5. *Let $\tilde{\varphi} : [0, 1) \times (0, +\infty) \to [0, +\infty)$ be a continuous function satisfying the following assumptions:*

(i) *there exist $a, b \in L^\infty(0, 1)$, $b(s) > a(s) > 0$ for every s, $\inf a > 0$, such that $\tilde{\varphi}(s, \cdot)$ and $\tilde{\varphi}^{**}(s, \cdot)$ both coincide and are strictly convex in $(0, a(s)]$ and $[b(s), +\infty)$ for all $s \in [0, 1)$;*

(ii) *$\tilde{\varphi}^{**}(x, \cdot)$ is affine in $[a(s), b(s)]$ for all $s \in [0, 1)$;*

(iii) *there exists $\lambda_0 \in \mathbb{R} \cup \{-\infty\}$ such that*

$$\lim_{t\to 0^+} D_t^+ \tilde{\varphi}(s, t) = \lambda_0, \quad \lim_{t\to +\infty} D_t^- \tilde{\varphi}(s, t) = +\infty, \quad \textit{uniformly in } s.$$

Then there exists a Lipschitz solution to (7.3).

Proof. Let us define $\varphi(x, t) := \tilde{\varphi}(|x|, t)$ for every $x \in B$. Notice that $\varphi^{**}(x, t) = \tilde{\varphi}^{**}(|x|, t)$ and that assumptions (K1), (K2), and (K3) of Theorem 4.1 holds with $\Omega = \Omega_A = B$, $\alpha(x) = a(|x|)$, and $\beta(x) = b(|x|)$. Let $v \in L^\infty(B)$, $\inf v > 0$, be the radial solution of (4.1). It is a known fact (see, e.g., [15]) that there exists a bi-Lipschitz solution u to (2.1) with $f = v$ and $\Omega = B$. Thus, u is a solution to (7.3), too. ∎

Appendix. Proof of Theorem 2.4. In the following we use the arguments of the proof of Lemma 1 in [16] and the fact, proved in [18], that if $\Omega = (0, 1)^N$ and f is Hölder continuous, then there exists a bi-Lipschitz homeomorphism solution to (2.1). We divide the proof into steps.

Step 1. Let Ω be a bounded open connected subset of $\mathbb{R}^N$ of class (L). Thus, there exist m open sets Ω_j such that $\overline{\Omega} \subset \cup_j \Omega_j$ and m bi-Lipschitz homeomorphisms $\psi_j : \overline{\Sigma}_j \to \overline{Q}$, with $\Sigma_j = \Omega \cap \Omega_j$ and $Q = (0, 1)^N$ such that $\det D\psi_j \in \mathrm{Lip}(\overline{\Sigma}_j)$ and $\frac{1}{A} < \det D\psi_j < A$ for some $A \geq 1$. Consider a partition of unity $\{\phi_j\}_{j=1}^m$ subordinate to such a covering of $\overline{\Omega}$: $\{\phi_j\}_{j=1}^m$ is a family of smooth and nonnegative functions, $\sum_j \phi_j(x) = 1$ for every $x \in \overline{\Omega}$ and

$$\operatorname{supp} \phi_j \subset\subset \Omega_j \quad \forall\, j = 1, \dots, m. \tag{7.4}$$

Since $\Omega = \cup_{j=1}^m \Sigma_j$ and Ω is connected, we can assume that for every $k = 2, \dots, m$ there exists $\rho(k) < k$ such that $\Sigma_k \cap \Sigma_{\rho(k)}$ is not empty. Define the matrix (α_{hk}), $1 \leq h \leq m$, $2 \leq k \leq m$,

$$\alpha_{hk} = \begin{cases} 1 & \text{if}\, h = k, \\ -1 & \text{if}\, h = \rho(k), \\ 0 & \text{else.} \end{cases}$$

Each of the $m - 1$ columns contains exactly one pair $+1$, -1 so that $\sum_{k=2}^m \alpha_{hk} = 0$ for every h.

Define $\eta_k \in C_c^\infty(\Sigma_k \cap \Sigma_{\rho(k)})$ such that $\int_\Omega \eta_k(x)\, dx = 1$. Let $g \in C^{0,\alpha}(\overline{\Omega})$ be such that $\int_\Omega g(x)\, dx = 0$. Define the Hölder continuous functions $g_h : \overline{\Omega} \to \mathbb{R}$, $1 \leq h \leq m$,

$$g_h := g\phi_h|_{\overline{\Omega}} - \sum_{k=2}^m \lambda_k \alpha_{hk} \eta_k,$$

where $\lambda_2, \dots, \lambda_m$ are real numbers solutions of the following system of m equations

$$\sum_{k=2}^m \lambda_k \alpha_{hk} = \int_\Omega g\phi_h\, dx, \qquad h = 1, \dots, m. \tag{7.5}$$

Since the rank of (α_{hk}) is $m-1$ and both $\sum_{h=1}^m \sum_{k=2}^m \lambda_k \alpha_{hk}$ and $\sum_{h=1}^m \int_\Omega g\phi_h\,dx$ are equal to 0, then system (7.5) is uniquely solvable.

We claim that $\operatorname{supp} g_h \subseteq \overline{\Sigma}_h$. In fact $\operatorname{supp}\ \phi_h|_{\overline{\Omega}} \subseteq \overline{\Sigma}_h$ and, since $\alpha_{hk} \neq 0$ if and only if $h = k$ or $h = \rho(k)$,

$$\operatorname{supp} \lambda_k \alpha_{hk} \eta_k \subset \Sigma_k \cap \Sigma_{\rho(k)} \subseteq \Sigma_h$$

for every $k = 2, \ldots, m$. Moreover, from (7.5) there exists $M > 0$ depending on Ω, $\{\phi_j\}_j$, and $\{\eta_j\}_j$ only such that $\sup|g_h| \leq M \sup|g|$.

Step 2. Let Ω, $\{\Sigma_j\}_j$, $\{\psi_j\}_j$, $\{\phi_j\}_j$, $\{\eta_j\}_j$, m, and M be as above. Let f in (2.1) be such that $\sup|f-1| < m^{-1}M^{-1}$. Define m Hölder continuous functions g_h reasoning as in the previous step with g replaced by $f-1$. For every $j = 1, \ldots, m+1$ define $f_j : \overline{\Omega} \to (0, +\infty)$,

$$f_j(x) := \begin{cases} 1 & \text{if } j = 1, \\ 1 + \sum_{h=1}^{j-1} g_h(x) & \text{if } j > 1. \end{cases}$$

In particular $f_{m+1} = f$. Notice that each f_j is a Hölder continuous function, and since $\sup|f-1| < m^{-1}M^{-1}$, then $\inf f_j > 0$. Fixed $j = 1, \ldots, m$, we have that

$$f_{j+1} - f_j = 0 \quad \text{in } \overline{\Omega} \setminus \overline{\Sigma}_j, \qquad \int_\Omega f_j(x)\,dx = |\Omega|, \qquad \int_{\Sigma_j} f_{j+1}(x)\,dx = \int_{\Sigma_j} f_j(x)\,dx. \tag{7.6}$$

Define $f_j^*, f_{j+1}^* : \overline{Q} \to (0, +\infty)$,

$$f_j^* := f_j\big(\psi_j^{-1}\big)\det D\psi_j^{-1}, \quad f_{j+1}^* := f_{j+1}\big(\psi_j^{-1}\big)\det D\psi_j^{-1},$$

so that $f_j^*, f_{j+1}^* \in C^{0,\alpha}(\overline{Q})$ and $\int_Q f_j^*\,dx = \int_Q f_{j+1}^*\,dx$.

As proved in [18] there exist two bi-Lipschitz homeomorphisms $v_j, w_j : \overline{Q} \to \overline{Q}$ solutions to

$$\begin{cases} \det Dv_j = \dfrac{f_j^*}{\int_Q f_j^*\,dx} & \text{in } Q, \\ v_j(y) = y & \text{on } \partial Q, \end{cases} \qquad \text{and} \qquad \begin{cases} \det Dw_j = \dfrac{f_{j+1}^*}{\int_Q f_j^*\,dx} & \text{in } Q, \\ w_j(y) = y & \text{on } \partial Q, \end{cases}$$

respectively. Let us consider $\varphi_j : \overline{Q} \to \overline{Q}$, $\varphi_j(y) := (v_j^{-1} \circ w_j)(y)$. Then

$$\det D\varphi_j(y) = \det Dv_j^{-1}(w_j(y)) \det Dw_j(y) = \frac{f_{j+1}^*(y)}{f_j^*(\varphi_j(y))} \quad \forall\, y \in \overline{Q}$$

so that

$$f_j\big((\psi_j^{-1} \circ \varphi_j)(y)\big)\det D\psi_j^{-1}(\varphi_j(y)) \det D\varphi_j(y) = f_{j+1}\big(\psi_j^{-1}(y)\big)\det D\psi_j^{-1}(y) \quad \forall y \in \overline{Q}.$$

Using the invertibility of ψ_j the equality above implies that

$$f_j(u_j(x)) \det Du_j(x) = f_{j+1}(x) \quad \forall\, x \in \overline{\Sigma}_j, \tag{7.7}$$

where $u_j : \overline{\Sigma}_j \to \overline{\Sigma}_j$ is the Lipschitz continuous function defined as $u_j(x) := (\psi_j^{-1} \circ \varphi_j \circ \psi_j)(x)$.

Since $\varphi_j(\psi_j(x)) = \psi_j(x)$ for all $x \in \partial\Sigma_j$, we have that $u_j(x) = x$ for every $x \in \partial\Sigma_j$. Then $\tilde{u}_j : \overline{\Omega} \to \mathbb{R}$, $j = 1, \ldots, m$,

$$\tilde{u}_j(x) := \begin{cases} u_j(x) & \text{if } x \in \overline{\Sigma}_j, \\ x & \text{else} \end{cases}$$

is Lipschitz continuous and from (7.6) and (7.7)

$$f_j(\tilde{u}_j(x)) \det D\tilde{u}_j(x) = f_{j+1}(x) \quad \forall\, x \in \overline{\Omega}.$$

Iterating this argument on j and recalling that $f_1 = 1$ and $f_{m+1} = f$, we get that $\tilde{u}_1 \circ \cdots \circ \tilde{u}_m$ is a Lipschitz solution to (2.1).

Step 3. Now we suppose that f in (2.1) satisfies $\sup|f-1| \geq m^{-1}M^{-1}$. There exists $c_1 > 0$ and $0 < t_1 < 1$ such that $\int_\Omega c_1 f^{t_1}(x)\,dx = |\Omega|$ and $\sup|c_1 f^{t_1} - 1| < m^{-1}M^{-1}$. Applying the same arguments described in Step 2 to $g := c_1 f^{t_1} - 1$, we obtain a Lipschitz function u_1 satisfying (2.1) with f replaced by $c_1 f^{t_1}$. Applying again this procedure to $g := c_2 f^{t_2} - c_1 f^{t_1}$, with a suitable choice of c_2 and t_2 in such a way that $t_1 < t_2 \leq 1$, $\int_\Omega c_2 f^{t_2}\,dx = |\Omega|$ and $\sup|c_2 f^{t_2} - c_1 f^{t_1}| < m^{-1}M^{-1}$, we get u_2 Lipschitz solution to

$$\begin{cases} c_1 f^{t_1}(u_2) \det Du_2 = c_2 f^{t_2} & \text{in } \Omega, \\ u_2(x) = x & \text{on } \partial\Omega. \end{cases}$$

Hence, $u_1 \circ u_2$ solves (2.1) with f replaced by $c_2 f^{t_2}$. It can be proved that the exponents $\{t_i\}$ can be chosen such that in finitely many steps, say n, we get $t_n = 1$. The existence of a Lipschitz continuous solution to (2.1) follows.

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